ON A WEAK VARIANT OF THE GEOMETRIC TORSION CONJECTURE

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Abstract. A consequence of the geometric torsion conjecture for abelian varieties over function fields is the following. Let k be an algebraically closed field of characteristic 0. For any integers $d,\ g \geq 0$ there exists an integer $N := N(k, d, g) \geq 1$ such that for any function field L/k with transcendence degree 1 and genus $\leq g$ and any d-dimensional abelian variety $A \to L$ containing no nontrivial k-isotrivial abelian subvariety, $A(L)_{tors} \subset A[N]$. In this paper, we deal with a weak variant of this statement, where $A \to L$ runs only over abelian varieties obtained from a fixed (d-dimensional) abelian variety by base change. More precisely, let K/k be a function field with transcendence degree 1 and $A \to K$ an abelian variety containing no nontrivial k-isotrivial abelian subvariety. Then we show that if K has genus ≥ 1 or if $A \to K$ has semistable reduction over all but possibly one place, then, for any integer $g \geq 0$, there exists an integer $N := N(A, g) \geq 1$ such that for any finite extension L/K with genus $\leq g$, $A(L)_{tors} \subset A[N]$. Previous works of the authors show that this holds – without any restriction on K – for the ℓ-primary torsion (with ℓ a fixed prime). So, it is enough to prove that there exists an integer $N := N(A, g) \geq 1$ such that for any finite extension L/K with genus $\leq g$, the prime divisors of $|A(L)_{tors}|$ are all $\leq N$.

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1. Introduction

The torsion conjecture for abelian varieties over finitely generated fields of characteristic 0 asserts that *for any finitely generated field F of characteristic 0 and integer $d \geq 1$ there exists an integer $N := N(F, d) \geq 1$ such that for any d-dimensional abelian variety $A \to F$, $A(F)_{tors} \subset A[N]$.* One can state a geometric variant of this conjecture over function fields.

Conjecture 1.1. *Let k be an algebraically closed field of characteristic 0. Then, for any function field L/k and any integer $d \geq 0$, there exists an integer $N := N(L/k, d) \geq 1$ such that for any d-dimensional abelian variety $A \to L$ containing no nontrivial k-isotrivial abelian subvariety, $A(L)_{tors} \subset A[N]$.*

Classical arguments (hyperplane section, Weil restriction) show[1] that conjecture 1.1 (for all d) is equivalent to conjecture 1.1 for $L = k(\mathbb{P}^1_k)$ (and for all d), and that conjecture 1.1 implies the following uniform version: *For any integers $d,\ g \geq 0$ there exists an integer $N := N(k, d, g) \geq 1$ such that for any function field L/k with transcendence degree 1 and genus $\leq g$ and any d-dimensional abelian variety $A \to L$ containing no nontrivial k-isotrivial abelian subvariety,*

[1]More precisely, assume conjecture 1.1 for $L = k(\mathbb{P}^1_k)$ (and for all d). If $L = k(C)$ with $C \to k$ a smooth, proper, connected curve, let $C \to \mathbb{P}^1_k$ be a non-constant morphism of degree, say, γ. Then the Weil restriction $Res_{k(C)/k(\mathbb{P}^1_k)}(A) \to k(\mathbb{P}^1_k)$ is a γd-dimensional abelian variety containing no nontrivial k-isotrivial abelian subvariety and $Res_{k(C)/k(\mathbb{P}^1_k)}(A)(k(\mathbb{P}^1_k)) \simeq A(k(C))$. Now, since a curve of genus $\leq g$ has gonality $\leq \frac{g+3}{2}$, one gets the desired uniform version of conjecture 1.1, by setting $N(k, d, g) := N(k(\mathbb{P}^1_k), \left[\frac{g+3}{2}\right] d)$. If $L = k(S)$ with $S \to k$ a smooth, projective, connected scheme, fix a closed embedding $S \hookrightarrow \mathbb{P}^r_k$. Then any curve obtained by cutting S with $(\dim(S) - 1)$ hyperplanes has same (arithmetic) genus, say, g. Given a d-dimensional abelian variety $A \to k(S)$ with zero section ϵ and a $k(S)$-rational torsion point P of order, say, N, there exists a non-empty open subscheme $U \subset S$ such that the smooth, projective morphism $A \to k(S)$ and the sections ϵ, $P : \mathrm{Spec}(k(S)) \to A$ extend to a smooth, projective morphism $\mathcal{A} \to U$ and sections ε, $\mathcal{P} : U \to \mathcal{A}$, respectively. By Grothendieck's rigidity theorem [MF82, Th. 6.14, Chap 6 §3], $\mathcal{A} \to U$ is an abelian scheme with zero section ε. Now, by considering suitable hyperplane sections, one gets a curve C of genus $\leq g$ on S, such that $C \cap U \neq \emptyset$ and that $\mathcal{A}_{k(C)} := \mathcal{A} \times_U k(C) \to k(C)$ contains no nontrivial k-isotrivial abelian subvariety. Since $\mathcal{A}_{k(C)}$ has a $k(C)$-rational torsion point $\mathcal{P}_{k(C)} := \mathcal{P} \times_U k(C)$ of order N, $N(L/k, d) := N(k, d, g)$ has the desired property.

$A(L)_{tors} \subset A[N]$. In this note, we deal with a weak variant of this statement, where $A \to L$ runs only over abelian varieties obtained from a fixed (d-dimensional) abelian variety by base change.

More precisely, let k be an algebraically closed field of characteristic 0 and let X be a smooth, separated and connected curve over k with generic point η. Let $\tilde{X}$ denote the smooth compactification of X, and g_X the genus of $\tilde{X}$. Write $\pi_1(X)$ for the etale fundamental group of X. Let $A \to X$ be an abelian scheme such that A_η contains no nontrivial k-isotrivial abelian subvariety. For any prime ℓ, let $\rho_{A,\ell} : \pi_1(X) \to \mathrm{GL}(A_\eta[\ell])$ denote the canonical representation of $\pi_1(X)$ on the group of (generic) ℓ-torsion points and let $X[\ell] \to X$ be the finite etale cover corresponding to the inclusion of open subgroups $\ker(\rho_{A,\ell}) \subset \pi_1(X)$. For any $v \in A_\eta[\ell]$, write $X_v \to X$ for the finite etale cover corresponding to the inclusion of open subgroups $\mathrm{Stab}_{\pi_1(X)}(v) \subset \pi_1(X)$. Set:

$$g(n) := \min\{g_{X_v}\}_{v \in A_\eta[n]^\times}.$$

(Here, given an integer $n \geq 0$, we will write $A_\eta[n]^\times$ for the set of torsion points of order exactly n). We consider the following:

Conjecture 1.2. $\lim\limits_{n \mapsto \infty} g(n) = +\infty.$

Previous works of the authors show that the "vertical" part of conjecture 1.2 holds, that is, for any prime ℓ, $\lim\limits_{n \mapsto \infty} g(\ell^n) = +\infty$ [CT08, Th. 1.1]. So, here, we focus on the "horizontal" part of conjecture 1.2. Namely, we show:

Theorem 1.3. *Assume either that $g_X \geq 1$ or that $A \to X$ has semistable reduction over all except possibly one point of $\tilde{X} \setminus X$. Then:*

$$\lim_{\ell \mapsto \infty;\ \ell:\ \text{prime}} g(\ell) = +\infty.$$

So, the only problem to complete the proof of conjecture 1.2 is to remove, in theorem 1.3, the semistability assumption when $g_X = 0$.

There is also an arithmetic motivation for this work, namely, the torsion conjecture for fibers of abelian schemes. More precisely, let F be a finitely generated field of characteristic 0, X a smooth, separated and geometrically connected curve over F, and $A \to X$ an abelian scheme. Then this amounts to showing (cf. [CT08, Lemma 4.4]) that $X_v(F) = \emptyset$, $v \in A_\eta[N]^\times$, $N \gg 0$ (depending on A). For example, when applied to the "universal" elliptic scheme $\mathcal{E} \to X := \mathbb{P}^1 \setminus \{0, 1728, \infty\}$ defined by:

$$\mathcal{E}_j : y^2 + xy = x^3 - \frac{36}{j - 1728}x - \frac{1}{j - 1728},$$

this assertion is closely related to the celebrated theorem of Mazur, Kamienny, Merel and others establishing the torsion conjecture for elliptic curves.

Recall that, from Mordell's conjecture [FW92], $X_v(F)$ is finite if $g_{X_v} \geq 2$. In the "vertical" situation of [CT08, Th. 1.1], one can use this combined with a projective system argument to show that $X_v(F) = \emptyset$, $v \in A_\eta[\ell^n]^\times$, $n \gg 0$ [CT08, Cor. 1.2]. Unfortunately, such an argument is not available in the "horizontal" situation. However, combining [CT08, Cor. 1.2] and Mordell's conjecture applied to theorem 1.3, one can state the following arithmetic result:

Corollary 1.4. *Let F be a finitely generated field of characteristic 0, X a smooth, separated and geometrically connected curve over F and $A \to X$ an abelian scheme. Assume either that X has genus ≥ 1 or that $A \to X$ has semistable reduction over all except possibly one (geometric) point of $\tilde{X} \setminus X$. Then, for each prime ℓ there exists an integer $n(\ell) \geq 1$ such that:*

(i) $n(\ell) = 1$ for $\ell \gg 0$;

(ii) the set of $x \in X(F)$ such that $\ell^{n(\ell)} || A_x(F)_{tors}|$ is finite for any $\ell \geq 0$.

The present paper is organized as follows. In section 2, we perform two reductions. In subsection 2.1, we show that theorem 1.3 for $g_X \geq 2$ follows from the geometric Lang-Néron theorem and, in subsection 2.2, we invoke a semisimplicity argument to show that, when $g_X = 1$, it is enough to prove that $g(\ell) \geq 2$ for $\ell \gg 0$. Section 3 is devoted to the proof of theorem 1.3. In subsection 3.1 we complete the proof of theorem 1.3 when $g_X = 1$. The heart of this subsection is corollary 3.6, which asserts that for any integer $B \geq 1$ and $\ell \gg 0$ (depending on B) the image of $\pi_1(X)$ acting on a nonzero $\pi_1(X)$-submodule of $A_\eta[\ell]$ contains no abelian subgroups of index $\leq B$; the proof of this statement involves several arguments of arithmetic, geometric and group-theoretic nature. In subsection 3.2, we carry out the proof of theorem 1.3 when $g_X = 0$. The argument here, based on the Riemann-Hurwitz formula and the specific structure of $\pi_1(X)$ when $g_X = 0$, is rather of combinatorial nature. Eventually, subsection 3.3 is devoted to the proof of corollary 1.4.

2. Reduction steps

In the rest of this paper, we follow the notations of section 1, unless otherwise stated. In particular, k denotes an algebraically closed field of characteristic 0, X denotes a smooth, separated and connected curve over k with generic point η, and $A \to X$ denotes an abelian scheme such that A_η contains no nontrivial k-isotrivial abelian subvariety. Let $K = k(\eta)$ denote the function field of X.

For each prime ℓ, let G_ℓ denote the image of $\rho_{A,\ell} : \pi_1(X) \to \mathrm{GL}(A_\eta[\ell])$. More generally, given a $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, we will write $\rho_{A,M} : \pi_1(X) \to \mathrm{GL}(M)$ for the corresponding representation and denote by G_M and K_M its image and kernel respectively. We will consider, in particular, $\pi_1(X)$-submodules of the form $M(v) := \mathbb{F}_\ell[G_\ell v] \subset A_\eta[\ell]$, $v \in A_\eta[\ell]$.

2.1. **Proof of theorem 1.3 – $g_X \geq 2$.**

From the following geometric variant of the Lang-Néron theorem [LN59]:

Theorem 2.1. *The abelian group $A_\eta(K)$ is finitely generated. In particular, its torsion subgroup $A_\eta(K)_{tors}$ is finite.*

one can deduce:

Lemma 2.2. (1) $A_\eta[\ell]^{G_\ell} = 0$ for $\ell \gg 0$.

(2) $\lim_{\ell \mapsto \infty} \min\{|G_\ell v|\}_{v \in A_\eta[\ell]^\times} = +\infty$. *In particular,* $\lim_{\ell \mapsto \infty} \min\{|G_M|\}_{0 \neq M \subset A_\eta[\ell]} = +\infty$.

Proof. (1) is straightforward, as $A_\eta[\ell]^{G_\ell} = A_\eta(K)[\ell]$. As for the first assertion of (2), suppose that for some integer $B \geq 1$ and infinitely many primes ℓ, there exists $v \in A_\eta[\ell]^\times$ such that $|G_\ell v| \leq B$. From Riemann's existence theorem, there are only finitely many possibilities for finite etale covers of X with degree $\leq B$. So, up to replacing X by a finite etale cover, one may assume that for infinitely many primes ℓ there exists $v \in A_\eta[\ell]^\times$ such that $|G_\ell v| = 1$, which contradicts (1). The second assertion of (2) follows from the first, since $|G_M| \geq |G_\ell v|$ holds for any $v \in M \smallsetminus \{0\}$. □

For each $P \in \tilde{X} \smallsetminus X$, let $I_{P,\ell} \subset G_\ell$ be the inertia group at P (well-defined up to conjugacy).

Lemma 2.3. *Let $v \in A_\eta[\ell]$. For each $Q \in \tilde{X}_v \smallsetminus X_v$, let $e(Q) \geq 1$ be the ramification index at Q in the cover $\pi_v : \tilde{X}_v \to \tilde{X}$. Then one has:*

$$\begin{aligned} 2g_{X_v} - 2 &= |G_\ell v|(2g_X - 2) + \sum_{P \in \tilde{X} \smallsetminus X} \sum_{Q \in \pi_v^{-1}(P)} (e(Q) - 1) \\ &= |G_\ell v|(2g_X - 2) + \sum_{P \in \tilde{X} \smallsetminus X} (|G_\ell v| - |I_{P,\ell} \backslash G_\ell v|) \end{aligned}$$

Proof. This is the Riemann-Hurwitz formula for the (ramified) cover $\pi_v : \tilde{X}_v \to \tilde{X}$. For the second equality, observe that $\pi_v^{-1}(P)$ is identified with $I_{P,\ell}\backslash G_\ell v$. □

Now, one obtains:

Corollary 2.4. *Conjecture 1.2 holds for $g_X \geq 2$.*

Proof. By lemma 2.3, one has $2g_{X_v} - 2 \geq |G_\ell v|(2g_X - 2)$, hence $g_{X_v} \geq |G_\ell v|(g_X - 1) + 1$. Now, the assertion follows from lemma 2.2 (2). □

So, we will now focus on the cases when X has genus 0 or 1. Also, without loss of generality, one may and will assume that $\tilde{X} \smallsetminus X$ is exactly the set of places where $A \to X$ has bad reduction.

When $g_X = 1$, one can make a further reduction: to prove theorem 1.3 when $g_X = 1$, it is enough to prove that $g(\ell) \geq 2$ for $\ell \gg 0$. We establish this result in the next subsection.

2.2. **Semisimplicity.**

Lemma 2.5. *Let O be a noetherian integral domain and set $S := \mathrm{Spec}(O)$. Let F be the field of fractions of O and assume that F is perfect. Let R be an (a not necessarily commutative) O-algebra, and M a left R-module which is finitely generated as an O-module. Assume that $M_F := M \otimes_O F$ is semisimple as a left R_F-module, where $R_F := R \otimes_O F$. Then there exists a non-empty open subset $U \subset S$, such that, for each $\mathfrak{p} \in U$, $M_{\kappa(\mathfrak{p})} := M \otimes_O \kappa(\mathfrak{p})$ is semisimple as a left $R_{\kappa(\mathfrak{p})}$-module, where $R_{\kappa(\mathfrak{p})} := R \otimes_O \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{p})$ denotes the residue field at $\mathfrak{p}$.*

Proof. One may write $M_F = \oplus_{i=1}^r M_{i,F}$, where $M_{i,F}$ is a simple R_F-submodule for each $i = 1, \ldots, r$. Define M_i to be the inverse image of $M_{i,F}$ in M, which is an R-submodule of M and is finitely generated as an O-module, since O is noetherian. It is easy to check that the natural map $M_i \otimes_O F \to M_{i,F}$ is an isomorphism. Accordingly, the natural map $j : \oplus_{i=1}^r M_i \to M$ becomes an isomorphism after tensored with F over O. Since both the source and the target of j are finitely generated O-modules, j already becomes an isomorphism after tensored with $O[1/f]$ over O for some $f \in O \smallsetminus \{0\}$. So, up to replacing O by such $O[1/f]$, one may assume that $M = \oplus_{i=1}^r M_i$. Thus, by considering each factor M_i one by one, one may assume that M_F is a simple R_F-module. Similarly, up to replacing O by $O[1/f]$ for some $f \in O \smallsetminus \{0\}$, one may assume that M is a free O-module. In particular, the natural map $\mathrm{End}_O(M) \to \mathrm{End}_O(M) \otimes_O F \xrightarrow{\sim} \mathrm{End}_F(M_F)$ is injective.

Next, up to replacing R by the image of R in $\mathrm{End}_O(M)$, one may assume that $R \hookrightarrow \mathrm{End}_O(M)$. In particular, R is finitely generated as an O-module, and $R \hookrightarrow R_F \hookrightarrow \mathrm{End}_O(M) \otimes_O F \xrightarrow{\sim} \mathrm{End}_F(M_F)$. Let Z and Z_F denote the centers of R and R_F, respectively. Then Z coincides with the inverse image of Z_F in R, and the natural map $Z \otimes_O F \to Z_F$ is an isomorphism.

Since M_F is a faithful, simple R_F-module, Z_F is a field and R_F is a central simple algebra over Z_F. Observe that Z is an integral domain and that Z_F is identified with the field of fractions of Z. Let R^{opp} and R_F^{opp} denote the opposite algebras of R and R_F, respectively, and consider the natural O-algebra homomorphism $m : R \otimes_Z R^{opp} \to \mathrm{End}_{Z\text{-module}}(R)$ defined by $m(a \otimes b)(x) = axb$. This map tensored with F over O is identified with the natural F-algebra homomorphism $R_F \otimes_{Z_F} R_F^{opp} \to \mathrm{End}_{Z_F\text{-vector space}}(R_F)$, which is an isomorphism, as R_F is a central simple algebra over Z_F. Since both the source and the target of m are finitely generated O-modules, the map m already becomes an isomorphism after tensored with $O[1/f]$ over O for some $f \in O \smallsetminus \{0\}$. So, up to replacing O by such $O[1/f]$, one may assume that m is an isomorphism.

Since F is perfect, the finite extension Z_F/F is separable. In other words, the finite morphism $\pi : \mathrm{Spec}(Z) \to \mathrm{Spec}(O) = S$ obtained by the natural homomorphism $O \hookrightarrow Z$ is generically etale, hence there exists a non-empty open subset U of S over which π is etale. Let $\mathfrak{p} \in U$. Then $Z_{\kappa(\mathfrak{p})} := Z \otimes_O \kappa(\mathfrak{p})$ is a finite direct product of finite separable extensions of $\kappa(\mathfrak{p})$. This fact, together with the fact that the natural map $R_{\kappa(\mathfrak{p})} \otimes_{Z_{\kappa(\mathfrak{p})}} R_{\kappa(\mathfrak{p})}^{opp} \to \mathrm{End}_{Z_{\kappa(\mathfrak{p})}\text{-module}}(R_{\kappa(\mathfrak{p})})$, which is identified with $m \otimes_O \kappa(\mathfrak{p})$, is an isomorphism, implies that $R_{\kappa(\mathfrak{p})}$ is a semisimple algebra. In particular, $M_{\kappa(\mathfrak{p})}$ is a semisimple $R_{\kappa(\mathfrak{p})}$-module, as desired. □

Proposition 2.6. *$A_\eta[\ell]$ is a semisimple $\mathbb{F}_\ell[G_\ell]$-module for $\ell \gg 0$.*

Proof. First, by taking a suitable model of $A \to X \to k$, one may reduce the problem to the case where k is of finite transcendence degree over $\mathbb{Q}$. Second, by considering the base change of $A \to X \to k$ with respect to any embedding $k \hookrightarrow \mathbb{C}$, one may reduce the problem to the case where $k = \mathbb{C}$. Now, consider the complex-analytification $A^{an} \to X^{an}$ of $A \to X$. The (singular) homology groups $H_1(A^{an}_x, \mathbb{Z})$, $x \in X^{an}$, form a local system on X^{an}, or, equivalently, a $\pi_1^{top}(X^{an})$-module M, which is free of rank $2\dim(A_\eta)$ as a $\mathbb{Z}$-module. By definition, $M_{\mathbb{F}_\ell}$ is identified with $A_\eta[\ell]$ as a $\pi_1^{top}(X^{an})$-module. (Here, $\pi_1^{top}(X^{an})$ acts on $A_\eta[\ell]$ via the comparison isomorphism $\pi_1^{top}(X^{an})^\wedge \xrightarrow{\sim} \pi_1(X)$.) In particular, the image of $\pi_1^{top}(X^{an})$ in $\mathrm{GL}(M_{\mathbb{F}_\ell})$ is identified with G_ℓ. Set $R := \mathbb{Z}[\pi_1^{top}(X^{an})]$. Then, by [D71, Th. (4.2.6)], $M_\mathbb{Q}$ is a semisimple $R_\mathbb{Q}$-module. Thus, the assertion follows from lemma 2.5. (See also [FW92, Chap. VI].) □

Remark 2.7. As the proof shows, proposition 2.6 remains true when X is a smooth, connected k-scheme of arbitrary dimension and $A \to X$ is an arbitrary abelian scheme (without the non-isotriviality assumption).

Lemma 2.8. *Let F be a field. Let G be a finite group and M an $F[G]$-module of finite dimension over F. Let $v \in M \smallsetminus \{0\}$ and set $M(v) := F[Gv] \subset M$. Let $\mathcal{L} : M(v) \twoheadrightarrow F$ be a nonzero F-linear form. Assume that $M(v)$ is a simple $F[G]$-module. Then:*

$$|Gv| \leq |\mathcal{L}(Gv)|^{\dim_F(M(v))}.$$

Proof. Set $r := \dim_F(M(v))$. Consider the first case $\mathcal{L}(v) \neq 0$ and the second case $\mathcal{L}(v) = 0$ separately. In the first case, one has $M(v) = Fv \oplus \ker(\mathcal{L})$. In this case, set $e_1 := v$ and let $e_2, \dots, e_r$ be an F-basis of $\ker(\mathcal{L})$. In the second case, one has $Fv \subset \ker(\mathcal{L})$ and $r \geq 2$. In this case, set $e_1 := v$, take $e_2 \in M(v) \smallsetminus \ker(\mathcal{L})$ and take an F-basis of $\ker(\mathcal{L})$ in the form of $e_1, e_3, \dots, e_r$. Then, in both cases, $\underline{e} := (e_1, \dots, e_r)$ forms an F-basis of $M(v)$. Consider the dual F-basis $e_1^\vee, \dots, e_r^\vee$ of $M(v)^\vee := \mathrm{Hom}_F(M(v), F)$. Then, by definition, $\mathcal{L} = a e_k^\vee$ for some $a \in F^\times$, where $k = 1$ (resp. $k = 2$) in the first (resp. second) case. Given $g \in G$, write $C_{g,i}$ (resp. $R_{g,i}$) for the ith column (resp. row) of the matrix of g written in $\underline{e}$, $i = 1, \dots, r$. Then:

$$\mathcal{E} := \mathcal{L}(Gv) = \{\mathcal{L}(gv)\}_{g \in G} = \{\mathcal{L}(gg'v)\}_{g,g' \in G} = \{aR_{g,k}C_{g',1}\}_{g,g' \in G}$$

Since $M(v)$ is a simple $F[G]$-module, $M(v)^\vee$ is a simple $F[G]$-module as well. In particular, the $g^{-1}\mathcal{L} = \mathcal{L}(g-) = aR_{g,k}$, $g \in G$ generate $M(v)^\vee$ as an F-vector space. Hence, one can fix an F-basis of the form $aR_{g_1,k}, \dots, aR_{g_r,k}$ for $M(v)^\vee$. The matrix A whose rows are the $aR_{g_i,k}$, $i = 1, \dots, r$ is in $\mathrm{GL}_r(F)$ with the property that $AC_{g,1} \in \mathcal{E}^r$, $g \in G$. Hence:

$$Gv = \{C_{g,1}\}_{g \in G} \subset A^{-1}\mathcal{E}^r,$$

from which the desired inequality follows. □.

Proposition 2.9. *Assume that $g_X = 1$ and that $g(\ell) \geq 2$ for $\ell \gg 0$. Then $\lim\limits_{\ell \mapsto \infty} g(\ell) = +\infty$.*

Proof. Let ℓ be a prime and $v \in A_\eta[\ell]^\times$. From proposition 2.6, $A_\eta[\ell]$ is a semisimple $\mathbb{F}_\ell[G_\ell]$-module for $\ell \gg 0$, hence $M(v)$ can be written as a direct sum:

$$M(v) = \bigoplus_{1 \leq i \leq r} M_i$$

with M_i a simple $\mathbb{F}_\ell[G_\ell]$-module, $i = 1, \dots, r$. For each $i = 1, \dots, r$ let v_i denote the projection of v onto M_i, so that $M_i = M(v_i)$. Then, since $\mathrm{Stab}_{\pi_1(X)}(v) \subset \mathrm{Stab}_{\pi_1(X)}(v_i)$, the etale cover $X_v \to X$ factors through $X_v \to X_{v_i}$, hence $g_{X_v} \geq g_{X_{v_i}}$. Thus, up to replacing v by, say, v_1, one may assume that $M(v)$ is a simple $\mathbb{F}_\ell[G_\ell]$-module.

By assumption and lemma 2.3, one has

$$0<(2g(\ell)-2\leq)\ 2g_{X_v}-2=\sum_{P\in\tilde{X}\smallsetminus X}(|G_\ell v|-|I_{P,\ell}\backslash G_\ell v|)=\sum_{P\in S}(|G_\ell v|-|I_{P,\ell}\backslash G_\ell v|)$$

for $\ell\gg 0$, where $S:=\{P\in\tilde{X}\smallsetminus X\mid I_{P,\ell}$ acts nontrivially on $G_\ell v\}$. In particular, S is nonempty. Further, since

$$\begin{aligned}|I_{P,\ell}\backslash G_\ell v| &= |(G_\ell v)^{I_{P,\ell}}|+|I_{P,\ell}\backslash(G_\ell v\smallsetminus(G_\ell v)^{I_{P,\ell}})| \\ &\leq |(G_\ell v)^{I_{P,\ell}}|+\frac{1}{2}|G_\ell v\smallsetminus(G_\ell v)^{I_{P,\ell}}| \\ &= \frac{1}{2}|G_\ell v|+\frac{1}{2}|(G_\ell v)^{I_{P,\ell}}|,\end{aligned}$$

one has

$$2g_{X_v}-2\geq\sum_{P\in S}\frac{1}{2}(|G_\ell v|-|(G_\ell v)^{I_{P,\ell}}|).$$

For each $P\in S$, one has $M(v)^{I_{P,\ell}}\subsetneq M(v)$, hence one can choose a nonzero $\mathbb{F}_\ell$-linear form:

$$\mathcal{L}=\mathcal{L}_{\ell,v,P}:M(v)\twoheadrightarrow M(v)/M(v)^{I_{P,\ell}}\twoheadrightarrow\mathbb{F}_\ell.$$

By construction, $(G_\ell v)^{I_{P,\ell}}\subset\mathcal{L}^{-1}(0)$ so:

$$|G_\ell v|-|(G_\ell v)^{I_{P,\ell}}|\geq|\mathcal{L}(G_\ell v)|-1.$$

Now, since $M(v)$ is a simple $\mathbb{F}_\ell[G_\ell]$-module with $\mathbb{F}_\ell$-dimension $\leq\dim(A_\eta[\ell])=2\dim(A_\eta)$, one has $|\mathcal{L}(G_\ell v)|\geq|G_\ell v|^{\frac{1}{2\dim(A_\eta)}}$ by lemma 2.8. Thus, the assertion follows from lemma 2.2 (2). $\square$

Remark 2.10. The first step of the proof of proposition 2.9 shows that, for $\ell\gg 0$, there exists $v\in A_\eta[\ell]^\times$ such that $g_{X_v}=g(\ell)$ and that $M(v)$ is a simple $\mathbb{F}_\ell[G_\ell]$-module.

3. Proof of theorem 1.3

3.1. **Proof of theorem 1.3 – $g_X=1$.** The technical core is the following general fact:

Proposition 3.1. *There exists an integer $B=B(A)\geq 1$, such that for any prime ℓ, any $\pi_1(X)$-submodule $M\subset A_\eta[\ell]$, and any abelian normal subgroup $C\subset G_M$, one has: $|C|\leq B$.*

Proof. Set $d:=\dim(A_\eta)$. Consider the following weaker assertion:

Claim 3.2. *There exists an integer $B'=B'(A)\geq 1$, such that for any prime ℓ and any $\pi_1(X)$-submodule $M\subset A_\eta[\ell]$, one has: $|Z(G_M)|\leq B'$, where $Z(G)$ stands for the center of a given group G.*

We shall first prove proposition 3.1, assuming claim 3.2. For this, one may ignore finitely many ℓ. So, by proposition 2.6, one may assume that $A_\eta[\ell]$ is a semisimple $\pi_1(X)$-module, hence so is $M\subset A_\eta[\ell]$. Set $E:=\mathbb{F}_\ell[C]\subset\mathrm{End}_{\mathbb{F}_\ell}(M)$. Then E is a commutative algebra of finite dimension, say, r over $\mathbb{F}_\ell$. Observe that the action by conjugation of G_M on C (*via* group automorphisms) extends by $\mathbb{F}_\ell$-linearity to an action on E (*via* $\mathbb{F}_\ell$-algebra automorphisms). Further, E is reduced. Indeed, set $J:=\sqrt{0_E}$, the radical of E, so that $J^N=\{0\}$ for some $N\geq 0$. The action of G_M on E preserves J. So, the filtration $M=J^0M\supset JM\supset J^2M\supset\cdots\supset J^NM=\{0\}$ is G_M-stable. Since M is semisimple as G_M-module, this implies that $M\simeq\oplus_{i=1}^N(J^{i-1}M/J^iM)$ as $\mathbb{F}_\ell[G_M]$-modules, hence, in particular, as E-modules. As J acts trivially on the right-hand side, it also acts trivially on M. Since E acts faithfully on M by definition, this implies $J=\{0\}$, as desired. Accordingly, E is a finite direct product of finite extensions of $\mathbb{F}_\ell$. As $\mathbb{F}_\ell$ is perfect, $E\otimes_{\mathbb{F}_\ell}\overline{\mathbb{F}}_\ell$ is isomorphic to $\overline{\mathbb{F}}_\ell^r$ as $\overline{\mathbb{F}}_\ell$-algebra and, in particular:

$$\mathrm{Aut}_{\mathbb{F}_\ell\text{-}alg}(E)\subset\mathrm{Aut}_{\overline{\mathbb{F}}_\ell\text{-}alg}(E\otimes_{\mathbb{F}_\ell}\overline{\mathbb{F}}_\ell)\simeq\mathcal{S}_r.$$

Also, since $E \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell \simeq \overline{\mathbb{F}}_\ell^r$ acts faithfully on $M \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell$, one gets

$$r \leq \dim_{\mathbb{F}_\ell}(M) \leq \dim_{\mathbb{F}_\ell}(A_\eta[\ell]) = 2d.$$

(To see the first inequality, consider the canonical decomposition $M \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell \simeq \oplus_{i=1}^r M_i$ corresponding to the decomposition $E \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell \simeq \overline{\mathbb{F}}_\ell^r$. Since $E \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell$ acts faithfully on $M \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell$, M_i must be nonzero, or, equivalently, $\dim_{\overline{\mathbb{F}}_\ell}(M_i) \geq 1$, for each $i = 1, \ldots, r$. Therefore, $\dim_{\mathbb{F}_\ell}(M) = \dim_{\overline{\mathbb{F}}_\ell}(M \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell) \geq r$.) Let H_C and N_C be the image and the kernel of $G_M \to \mathrm{Aut}_{\mathbb{F}_\ell\text{-}alg}(E)$, respectively. By definition, N_C coincides with the centralizer of C in G_M. Let $Y_C \to X$ be the Galois cover corresponding to the quotient $\pi_1(X)(\twoheadrightarrow G_M) \twoheadrightarrow H_C$. By definition, the image of $\pi_1(Y_C)$ in G_M coincides with N_C. As $C \subset Z(N_C)$, one concludes: $|C| \leq |Z(N_C)| \leq B'(A \times_X Y_C)$ by claim 3.2. Since $[Y_C : X] = |H_C| \leq r! \leq (2d)!$ is bounded, there are only finitely many (non-isomorphic) Galois covers $Y_C \to X$ by Riemann's existence theorem. Thus, proposition 3.1 follows.

Next, we shall prove claim 3.2. For this, fix a model $A_1 \to X_1 \to k_1$ of $A \to X \to k$ over a finitely generated field k_1 (of characteristic 0). Up to enlarging k_1, one may assume that $X_1(k_1) \neq \emptyset$. Fix $x_1 \in X_1(k_1)$, which gives a splitting of the canonical short exact sequence:

$$1 \to \pi_1(X) \to \pi_1(X_1) \to \Gamma_{k_1} \to 1.$$

(Here, we identify $\pi_1(X) = \pi_1((X_1)_{\overline{k}_1})$, as the characteristic is 0, and $\Gamma_F = \pi_1(\mathrm{Spec}(F))$ stands for the absolute Galois group of a given field F.) In particular, Γ_{k_1} acts on $\pi_1(X)$ by conjugation. For each $\ell \geq 0$, write $\rho_{A_1,\ell} : \pi_1(X_1) \to \mathrm{GL}(A_\eta[\ell])$ for the corresponding representation (here, we identify $A_\eta[\ell] = (A_1)_{\eta_1}[\ell]$). Then, $\rho_{A,\ell} = \rho_{A_1,\ell}|_{\pi_1(X)}$. So, writing $G_{1,\ell}$ for the image of $\rho_{A_1,\ell}$, one gets $G_\ell \lhd G_{1,\ell}$.

For each $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, set $M^{sat} := A_\eta[\ell]^{K_M}$. Then one has $M \subset M^{sat}$, $K_{M^{sat}} = K_M$ (hence $G_{M^{sat}} = G_M$), and $(M^{sat})^{sat} = M^{sat}$. Let us say that M is saturated if $M^{sat} = M$. Now, up to replacing M by M^{sat} if necessary, one may assume that M is saturated when one proves the assertion of claim 3.2.

Also, by proposition 2.6, there exists an integer $N = N(A) \geq 1$, such that for any prime $\ell > N$ $A_\eta[\ell]$ is a semisimple G_ℓ-module. In particular, $P := \mathbb{F}_\ell[G_\ell] \subset \mathrm{End}_{\mathbb{F}_\ell}(A_\eta[\ell])$ is a semisimple algebra of finite dimension over $\mathbb{F}_\ell$. Let F be the center of P. Thus, one has a canonical decomposition $P = \prod_{i \in I} P_i$ and $F = \prod_{i \in I} F_i$, where I is a finite set and P_i is a central simple algebra over F_i for each $i \in I$. Since the Brauer group of the finite field F_i is trivial, one has $P_i \simeq M_{s_i}(F_i)$ for some $s_i \geq 1$. Further, according to the above decomposition of P, the P-module $A_\eta[\ell]$ is also decomposed canonically: $A_\eta[\ell] = \oplus_{i \in I} T_i$ (sometimes called the canonical isotypical decomposition). More concretely, $T_i \simeq S_i^{\oplus m_i} = P_i A_\eta[\ell]$ for each $i \in I$, where $m_i \geq 1$ and S_i is a simple G_ℓ-submodule of $A_\eta[\ell]$ on which P acts via the projection $P \to P_i$ and which is of dimension s_i over F_i. (Note that $S_i \not\simeq S_j$ if $i \neq j$.) In particular, $|I| \leq 2d$.

Claim 3.3. *There exists an integer $B_1 = B_1(A)$ (independent of the choice of the model $A_1 \to X_1 \to k_1$ of $A \to X \to k$) satisfying the following property: For any prime ℓ, there exists a finite Galois extension $k_2 = k_2(\ell)/k_1$ with $[k_2 : k_1] \leq B_1$, such that any saturated $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$ is $\pi_1(X_1 \times_{k_1} k_2)$-stable and that the image $G_{2,M}$ of $\pi_1(X_1 \times_{k_1} k_2)$ in $\mathrm{GL}(M)$ commutes with $Z(G_M)$.*

First, consider a prime $\ell > N$. Observe that the action by conjugation of $G_{1,\ell}$ on G_ℓ (*via* group automorphisms) extends by $\mathbb{F}_\ell$-linearity to an action on P (*via* $\mathbb{F}_\ell$-algebra automorphisms), which induces an action on F (*via* $\mathbb{F}_\ell$-algebra automorphisms). One has $F \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell \simeq \overline{\mathbb{F}}_\ell^r$ as $\overline{\mathbb{F}}_\ell$-algebras for some $r \geq 0$, and, in particular:

$$\mathrm{Aut}_{\mathbb{F}_\ell\text{-}alg}(F) \subset \mathrm{Aut}_{\overline{\mathbb{F}}_\ell\text{-}alg}(F \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell) \simeq \mathcal{S}_r.$$

Also, since $F \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell \simeq \overline{\mathbb{F}}_\ell^r$ acts faithfully on $A_\eta[\ell] \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell$, one gets $r \leq \dim_{\mathbb{F}_\ell}(A_\eta[\ell]) = 2d$ (see above). Consider the homomorphism $\rho : G_{1,\ell} \to \mathrm{Aut}_{\mathbb{F}_\ell\text{-}alg}(F)$ given by the above action. Let H denote the image of ρ. As $G_\ell \subset P$ and F is the center of P, the homomorphism ρ factors through $G_{1,\ell} \twoheadrightarrow G_{1,\ell}/G_\ell$. Define k_2 to be the Galois extension corresponding to the quotient $\Gamma_{k_1} = \pi_1(X_1)/\pi_1(X) \twoheadrightarrow G_{1,\ell}/G_\ell \twoheadrightarrow H$. By definition, $[k_2 : k_1] = |H| \leq r! \leq (2d)!$, and the image of $\pi_1(X_1 \times_{k_1} k_2)$ in $\mathrm{GL}(A_\eta[\ell])$ commutes with F. Now, let M be a saturated $\pi_1(X)$-submodule of $A_\eta[\ell]$. Then there exists a subset $I_M \subset I$, such that $M = \oplus_{i\in I_M} T_i$. (Indeed, one has $M \simeq \oplus_{i\in I} S_i^{\oplus e_i}$, where $0 \leq e_i \leq m_i$, $i \in I$. Now, since M is saturated, $e_i \geq 1$ if and only if $T_i \subset M$.) Consider the idempotent $e_M := (e_{M,i})_{i\in I} \in F = \prod_{i\in I} F_i$, where $e_{M,i} = 1$ (resp. $e_{M,i} = 0$) for $i \in I_M$ (resp. $i \in I \smallsetminus I_M$). Then one gets $M = e_M(A_\eta[\ell])$, which implies that M is $\pi_1(X_1 \times_{k_1} k_2)$-stable, as $\pi_1(X_1 \times_{k_1} k_2)$ commutes with $e_M \in F$. Further, set $P_M := \prod_{i\in I_M} P_i$ and $F_M := \prod_{i\in I_M} F_i$. Then F_M is the center of P_M. Since $P_M = \mathbb{F}_\ell[G_M]$ in $\mathrm{End}_{\mathbb{F}_\ell}(M)$, one has $Z(G_M) = F_M \cap G_M \subset F_M$. Now, since $G_{2,M}$ commutes with F_M, it commutes with $Z(G_M)$, as desired.

Second, consider a prime $\ell \leq N$. Let k_2 be the Galois extension of k_1 corresponding to the quotient $\Gamma_{k_1} \twoheadrightarrow G_{1,\ell}/G_\ell$. By definition, $[k_2 : k_1] = [G_{1,\ell} : G_\ell] \leq |G_{1,\ell}| \leq |\mathrm{GL}(A_\eta[\ell])| \leq \ell^{4d^2} \leq N^{4d^2}$, and the image of $\pi_1(X_1 \times_{k_1} k_2)$ in $\mathrm{GL}(A_\eta[\ell])$ coincides with G_ℓ. Thus, any G_ℓ-submodule $M \subset A_\eta[\ell]$ is $\pi_1(X_1 \times_{k_1} k_2)$-stable, and the image $G_{2,M}$ of $\pi_1(X_1 \times_{k_1} k_2)$ in $\mathrm{GL}(M)$ coincides with G_M. In particular, $G_{2,M}$ commutes with $Z(G_M)$. Now, $B_1 := \max((2d)!, N^{4d^2})$ satisfies the desired property, which completes the proof of claim 3.3.

Claim 3.4. *There exists an integer $B'' = B''(A)$ satisfying the following property: For any prime ℓ and any $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, one has: $|Z(G_M)| \leq B''|\overline{Z}(G_M)|$, where $\overline{Z}(G_M)$ denotes the image of $Z(G_M)$ in $(G_M)^{ab}$.*

Indeed, to prove claim 3.4, one may ignore finitely many primes ℓ and assume that $A_\eta[\ell]$ is semisimple by proposition 2.6. Also, as $G_M = G_{M^{sat}}$, one may assume that M is saturated. Then, as in the proof of claim 3.3, $G_M \subset P_M^\times$ and $Z(G_M) \subset F_M^\times$. Consider the determinant map $\delta_M : P_M^\times \to F_M^\times$ induced by the determinant maps $P_i^\times(\simeq \mathrm{GL}_{s_i}(F_i)) \to F_i^\times$ for $i \in I_M$. Note that $\ker(\delta_M|_{F_M^\times}) = \prod_{i\in I_M} \mu_{s_i}(F_i^\times)$ has cardinality $\leq \prod_{i\in I_M} s_i \leq (2d)^{2d}$, as $s_i = \dim_{F_i}(S_i) \leq \dim_{\mathbb{F}_\ell}(A_\eta[\ell]) = 2d$ and $|I_M| \leq |I| \leq 2d$. As $\delta_M(G_M) \subset F_M^\times$ is abelian, it is a quotient of $(G_M)^{ab}$. Accordingly, $\delta_M(Z(G_M))$ is a quotient of $\overline{Z}(G_M)$. Now, one gets:

$$|Z(G_M)| = |\ker(\delta_M|_{Z(G_M)})||\delta_M(Z(G_M))| \leq (2d)^{2d}|\overline{Z}(G_M)|.$$

This completes the proof of claim 3.4.

Now, turn to the proof of claim 3.2. Let $k_2 = k_2(\ell)$ be as in claim 3.3. Then it follows from the various definitions that, for each saturated $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, one has the following morphisms of Γ_{k_2}-modules:

$$Z(G_M) \twoheadrightarrow \overline{Z}(G_M) \hookrightarrow (G_M)^{ab} \twoheadleftarrow \pi_1(X)^{ab},$$

where Γ_{k_2} acts trivially on $Z(G_M)$, hence also on $\overline{Z}(G_M)$. Now, to conclude, one needs one more specialization step. From now on, write $\overline{Z} = \overline{Z}(G_M)$ for simplicity.

Consider a model $(\mathcal{X} \to \mathrm{Spec}(R),\ x : \mathrm{Spec}(R) \to \mathcal{X})$ of $(X_1 \to k_1,\ x_1 : \mathrm{Spec}(k_1) \to X_1)$. More precisely, R is a finitely generated normal integral $\mathbb{Z}$-algebra with fraction field k_1 (hence $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$ is dominant); $\mathcal{X} \to R$ is a smooth curve, that is, a proper, smooth, geometrically connected curve over R minus a relatively finite etale divisor, such that $\mathcal{X} \times_R k_1$ is isomorphic to (and will be identified with) X_1 over k_1; and $x : \mathrm{Spec}(R) \to \mathcal{X}$ is an (a unique) extension of $x_1 : \mathrm{Spec}(k_1) \to X_1$ (under the identification $\mathcal{X} \times_R k_1 = X_1$). Fix two primes $p \neq q$ in the image of $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$. Choose any closed point $s \in \mathrm{Spec}(R)$ lying above p, then one gets a canonical specialization isomorphism for the prime-to-p part of the etale fundamental groups ([SGA1, Exp. XIII]):

$$\pi_1^{(p')}(X) \tilde{\to} \pi_1^{(p')}(\mathcal{X}_{\overline{s}}),$$

which is compatible with the actions of

$$\Gamma_{k_1} \supset D_s \twoheadrightarrow \Gamma_{\kappa(s)},$$

where D_s stands for the decomposition group at s. Further, let R_2 be the integral closure of R in k_2 and let s_2 be the closed point of $\mathrm{Spec}(R_2)$ above s such that $D_{s_2} \subset D_s$. Now, one gets homomorphisms

$$\overline{Z}^{(p')} \hookrightarrow (G_M)^{ab,(p')} \twoheadleftarrow \pi_1^{(p')}(X)^{ab} \tilde{\to} \pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab},$$

which are compatible with the actions of $\Gamma_{k_2} \supset D_{s_2} \twoheadrightarrow \Gamma_{\kappa(s_2)}$. In particular, the action of D_{s_2} on $\overline{Z}^{(p')}$ factors through $\Gamma_{\kappa(s_2)}$, as $\overline{Z}^{(p')}$ is a subquotient of the $\Gamma_{\kappa(s_2)}$-module $\pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab}$.

Note that $[\Gamma_{\kappa(s)} : \Gamma_{\kappa(s_2)}] \leq [D_s : D_{s_2}] \leq [\Gamma_{k_1} : \Gamma_{k_2}] \leq B_1$. Since $\Gamma_{\kappa(s)} \simeq \hat{\mathbb{Z}}$ is a finitely generated profinite group, the intersection Γ of all open subgroups $\Gamma' \subset \Gamma_{\kappa(s)}$ with $[\Gamma_{\kappa(s)} : \Gamma'] \leq B_1$ is again an open subgroup. (The index $[\Gamma_{\kappa(s)} : \Gamma]$ is equal to the least common multiple of $1, \dots, B_1$, which is independent of ℓ.) Write κ for the finite extension of $\kappa(s)$ corresponding to $\Gamma \subset \Gamma_{\kappa(s)}$, and let ϕ denote the $|\kappa|$-th power Frobenius element, which is a generator of $\Gamma = \Gamma_\kappa$. By construction, ϕ acts trivially on the subquotient $\overline{Z}^{(p')}$ of $\pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab}$. This implies that $|\overline{Z}^{(p')}| \leq B(s, B_1, \mathcal{X})$ for some constant $B(s, B_1, \mathcal{X})$ independent of ℓ. More precisely, recall that the $\Gamma_{\kappa(s)}$-module $\pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab}$ can be written canonically as an extension:

$$1 \to \overline{I} \to \pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab} \to \prod_{a:\text{prime}\neq p} T_a(J_{\tilde{\mathcal{X}}_s}) \to 1,$$

where $J_{\tilde{\mathcal{X}}_s}$ is the jacobian of the smooth compactification $\tilde{\mathcal{X}}_s$ of $\mathcal{X}_s$ and $\overline{I}$ is the subgroup generated by the images of inertia subgroups at the points of $\tilde{\mathcal{X}}_{\overline{s}} \setminus \mathcal{X}_{\overline{s}}$. Denote by $P_\phi(t) \in \prod_{a\neq p} \mathbb{Z}_a[t]$ the characteristic polynomial of ϕ acting on $\pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab}$ by conjugation. Then, from the above exact sequence, one sees that P_ϕ has coefficients in $\mathbb{Z}$ and that the (complex) absolute values of the roots of P_ϕ are $|\kappa|^{\frac{1}{2}}$ ($2g$ times) and $|\kappa|$ ($\max(r-1, 0)$ times), where g is the genus of $\tilde{\mathcal{X}}_s$ and r is the number of points of $\tilde{\mathcal{X}}_{\overline{s}} \setminus \mathcal{X}_{\overline{s}}$. In particular, $P_\phi(1)$ is a nonzero integer, which is independent of ℓ.

Let T be the inverse image of $\overline{Z}^{(p')}$ in $\pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab}$ under the map $\pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab} \twoheadrightarrow (G_M)^{ab,(p')}$. Then T is a $\Gamma_{\kappa(s_2)}$-submodule of $\pi_1^{(p')}(\mathcal{X}_{\overline{s}})^{ab}$ of finite index. In particular, the characteristic polynomial of ϕ acting on T coincides with P_ϕ. The surjective map $T \twoheadrightarrow \overline{Z}^{(p')}$ factors through $T \twoheadrightarrow T_\Gamma$, where T_Γ is the maximal Γ-coinvariant (or, equivalently, ϕ-coinvariant) quotient of T. Thus, one concludes:

$$|\overline{Z}^{(p')}| \leq |T_\Gamma| = |P_\phi(1)|' =: B(s, B_1, \mathcal{X}),$$

where N' stands for the prime-to-p part of a given positive integer N. (Here, to get the equality $|T_\Gamma| = |P_\phi(1)|'$, consider the elementary divisors of $\phi - Id : T_a \to T_a$ for each prime $a \neq p$, where T_a stands for the a-adic part of T.) Similarly, considering a closed point $t \in \mathrm{Spec}(R)$ lying above q, one gets $|\overline{Z}^{(q')}| \leq B(t, B_1, \mathcal{X})$. Set $B''' = B(s, B_1, \mathcal{X})B(t, B_1, \mathcal{X})$, then, for any prime ℓ, one gets $|\overline{Z}| \leq B'''$. This, together with claim 3.4, completes the proof of claim 3.2. □

Corollary 3.5. *Conjecture 1.2 holds for $g_X = 1$.*

Proof. By proposition 2.9, it is enough to prove that $g(\ell) \geq 2$ for $\ell \gg 0$. Suppose otherwise, then there exist infinitely many primes ℓ and $v \in A_\eta[\ell]^\times$ such that $g_X = g_{X_v} = 1$. Then the finite etale cover $X_v \to X$ is automatically Galois and abelian. So $C_v := G_{M(v)}$ is abelian but, as well, $|C_v| = |G_\ell v| \to +\infty$, by lemma 2.2 (2), which contradicts proposition 3.1. □

Corollary 3.6. *For any integer $b \geq 1$ there exists an integer $N(b, A) \geq 0$ such that for any nontrivial $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, G_M contains no abelian subgroup of index $\leq b$ for any $\ell \geq N(b, A)$.*

Proof. Else, there exist $b \geq 1$ and infinitely many primes $\ell \geq 0$ such that there exists a $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$ with G_M containing an abelian subgroup C_0 of index $\leq b$. Set $C := \cap_{g \in G_M} gC_0g^{-1}$, which is an abelian normal subgroup of G_M of index $\leq b!$. Now, by proposition 3.1, one gets: $|G_M| = [G_M : C]|C| \leq b!B$, which contradicts lemma 2.2 (2). □

Remark 3.7. The argument of [CT09, Remark 5.8] shows that proposition 3.1 and corollary 3.6 remain true when X is a smooth, connected k-scheme of arbitrary dimension.

We conclude this subsection with an application of corollary 3.6. For any nontrivial $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, write $X_M \to X$ for the etale cover corresponding to the inclusion of open subgroups $K_M = \ker(\rho_{A,M}) \subset \pi_1(X)$ and define:

$$g_{tot}(\ell) := \min\{g_{X_M}\}_{0 \neq M \subset A_\eta[\ell]}.$$

Corollary 3.8.

$$\lim_{\ell \mapsto \infty} g_{tot}(\ell) = +\infty.$$

Proof. The main point is that $X_M \to X$ is Galois with group G_M.

Claim 3.9. $\lim_{\ell \mapsto \infty} g_{tot}(\ell) = +\infty$ *does not hold if and only if there exists a nontrivial* $\pi_1(X)$-*submodule* $M \subset A_\eta[\ell]$ *such that* $g_{X_M} = 0,\ 1$ *for infinitely many* $\ell \geq 0$.

Indeed, the "if" implication is straightforward. For the "only if" implication, assume that $g_{tot}(\ell) \geq 2$, $\ell \gg 0$. Then, for $\ell \gg 0$ and for any nontrivial $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, $g_{X_M} \geq g_{tot}(\ell) \geq 2$ so,

$$|G_M| \leq |\mathrm{Aut}(X_M)| \leq 84(g_{X_M} - 1)$$

by the Hurwitz bound. Whence $\min\{|G_M|\}_{0 \neq M \subset A_\eta[\ell]} \leq 84(g_{tot}(\ell) - 1)$. Now, from lemma 2.2 (2), one has $\lim_{\ell \mapsto \infty} g_{tot}(\ell) = +\infty$. This completes the proof of claim 3.9.

As a result, the only cases to rule out are:

(i) $g_X = 0$ and $g_{X_M} = 0$, for infinitely many $\ell \geq 0$;
(ii) $g_X = 0$ and $g_{X_M} = 1$, for infinitely many $\ell \geq 0$;
(iii) $g_X = 1$ and $g_{X_M} = 1$, for infinitely many $\ell \geq 0$.

For (i), it follows from the classification of finite subgroups of $\mathrm{PGL}_2(k)$ and $\lim_{\ell \mapsto \infty} |G_M| = +\infty$ that the group G_M is either cyclic or dihedral for $\ell \gg 0$. In both cases, G_M contains an abelian normal subgroup A_ℓ ($\twoheadleftarrow \mathbb{Z}$) with $[G_M : A_\ell] \leq 2$, which contradicts corollary 3.6.

For (ii) and (iii), G_M is a finite subgroup of the automorphism group of a genus 1 curve. But such a group contains an abelian normal subgroup A_ℓ ($\twoheadleftarrow \mathbb{Z}^2$) with $[G_M : A_\ell] \leq 6$, which, again, contradicts corollary 3.6. □

Remark 3.10. When $k = \mathbb{C}$ and and A_η is principally polarized, J.-M. Hwang and W.-K. To proved that a uniform bound (i.e., depending only on $\dim(A_\eta)$) for the growth of $g_{X[\ell]}$ ($\geq g_{tot}(\ell)$) exists [HT06]. By classical arguments (Zarhin's trick and specialization), such a uniform bound also exists only under the assumption that k has characteristic 0.

3.2. **Proof of theorem 1.3** – $g_X = 0$. From now on, we will write PG, SS, $PSS \subset \tilde{X} \smallsetminus X$ for the subsets corresponding to the places of potentially good (but not good), semistable (but not good), potentially semistable (but neither semistable nor potentially good) reduction respectively. Since we have assumed that $\tilde{X} \smallsetminus X$ is exactly the set of places where $A \to X$ has bad reduction, one has $\tilde{X} \smallsetminus X = PG \sqcup SS \sqcup PSS$. For each place $P \in \tilde{X} \smallsetminus X$ and prime ℓ, we will write $I_{P,\ell}$ for the image of the corresponding inertia group in G_ℓ, which is a finite cyclic group (as the characteristic of k is 0). From the semistable reduction theorem [SGA7, Exp. IX]:
- If $P \in PG$ then there exists an integer $N_P \geq 2$ such that $I_{P,\ell}^{N_P} = 1$ for any ℓ and that $I_{P,\ell}^{N} \neq 1$ for $N < N_P$ and $\ell \gg 0$.

- If $P \in SS$ then $I_{P,\ell}$ is unipotent of echelon 2.
- If $P \in PSS$ then there exists an integer $N_P \geq 2$ such that $I_{P,\ell}^{N_P}$ is unipotent of echelon 2 for any ℓ and that $I_{P,\ell}^{N}$ is not unipotent for $N < N_P$ and $\ell \gg 0$.

We will sometimes say that $A \to X$ has reduction type $(n_P)_{P\in\tilde{X}\smallsetminus X}$, where

$$n_P := \begin{array}{ll} N_P, & P \in PG; \\ \infty, & P \in SS; \\ N_P\infty, & P \in PSS. \end{array}$$

Before carrying out the proof of theorem 1.3 when $g_X = 0$, we describe briefly the strategy.

3.2.1. *Reduction to a combinatorial problem.* For each ℓ let $v_\ell \in A_\eta[\ell]^\times$ such that $g(\ell) = g_{X_{v_\ell}}$. (If $\ell \gg 0$, one can even assume that $M(v_\ell)$ is a simple $\mathbb{F}_\ell[G_\ell]$-module (cf. remark 2.10), though this fact will not be used in the following.) By lemma 2.3, one has

$$2g_{X_{v_\ell}} - 2 = -2|G_\ell v_\ell| + \sum_{P\in\tilde{X}\smallsetminus X} |G_\ell v|(1-\epsilon_P(v_\ell)),$$

with

$$\epsilon_P(v_\ell) = \frac{|I_{P,\ell}\backslash G_\ell v_\ell|}{|G_\ell v_\ell|},\ P \in \tilde{X} \smallsetminus X.$$

Set

$$\lambda_{v_\ell} := \frac{2g_{X_{v_\ell}} - 2}{|G_\ell v_\ell|} = r - 2 - \sum_{P\in\tilde{X}\smallsetminus X} \epsilon_P(v_\ell),$$

where $r := |\tilde{X} \smallsetminus X|$. Then: $g(\ell) \geq 2$ for $\ell \gg 0$ if and only if $\lambda_{v_\ell} > 0$ (or, equivalently $\sum_{P\in\tilde{X}\smallsetminus X} \epsilon_P(v_\ell) < r-2$) for $\ell \gg 0$; and, by lemma 2.2 (2), $\lim_{\ell\mapsto\infty} g(\ell) = +\infty$ if there exists $\epsilon > 0$ such that:

(*) $\lambda_{v_\ell} > \epsilon$ (or, equivalently, $\sum_{P\in\tilde{X}\smallsetminus X} \epsilon_P(v_\ell) < r-2-\epsilon$) for $\ell \gg 0$.

Thus, the problem amounts to estimating the size of the "local term" $\sum_{P\in\tilde{X}\smallsetminus X} \epsilon_P(v_\ell)$.

Under the semistability assumption, this can be done by combinatorial manipulations based on the specific structure of $\pi_1(X)$ when $g_X = 0$ to complete the proof of theorem 1.3. We postpone this issue to the next subsection and conclude this one by illustrating another idea, successfully exploited in [CT08] and [CT09]. Namely, we compare λ_{v_ℓ} with:

$$\lambda_\ell := \frac{2g_{X[\ell]} - 2}{|G_\ell|} = r - 2 - \sum_{P\in\tilde{X}\smallsetminus X} \frac{1}{|I_{P,\ell}|}.$$

For $\ell \gg 0$, one has:

$$\lambda_\ell = r - 2 - \sum_{P\in PG}\frac{1}{N_P} - \sum_{P\in SS}\frac{1}{\ell} - \sum_{P\in PSS}\frac{1}{\ell N_P},$$

which shows that:

$$\lim_{\ell\mapsto\infty} \lambda_\ell = \lambda := r - 2 - \sum_{P\in PG}\frac{1}{N_P}.$$

Now, corollary 3.8, together with the fact that $\lambda_\ell \leq \lambda_{\ell'}$ for $0 \ll \ell < \ell'$, implies that $\lambda > 0$ so it is enough to prove that:

$$\lim_{\ell\mapsto\infty} \lambda_{v_\ell} = \lambda.$$

As $\epsilon_P(v_\ell) \geq \frac{1}{|I_{P,\ell}|}$ by definition, this is equivalent to:

$$\lim_{\ell\mapsto\infty} (\epsilon_P(v_\ell) - \frac{1}{|I_{P,\ell}|}) = 0,\ \forall P \in \tilde{X} \smallsetminus X.$$

To go further, write $\mathcal{M}(F)$ for the set of nontrivial minimal subgroups of a given finite group F (equivalently, this is the set of cyclic subgroups of F with prime order) and, for $P \in \tilde{X} \setminus X$, set:

$$(G_\ell v_\ell)'_P := \bigcup_{H \in \mathcal{M}(I_{P,\ell})} (G_\ell v_\ell)^H.$$

Then one has:

$$\frac{1}{|I_{P,\ell}|}\left(1 - \frac{|(G_\ell v_\ell)'_P|}{|G_\ell v_\ell|}\right) \leq \epsilon_P(v_\ell) \leq \frac{1}{|I_{P,\ell}|}\left(1 - \frac{|(G_\ell v_\ell)'_P|}{|G_\ell v_\ell|}\right) + \frac{|(G_\ell v_\ell)'_P|}{|G_\ell v_\ell|}$$

So, it would be enough to prove that:

$$\lim_{\ell \mapsto \infty} \frac{|(G_\ell v_\ell)'_P|}{|G_\ell v_\ell|} = 0,\ P \in \tilde{X} \setminus X.$$

Let $\gamma_{P,\ell}$ be a generator of $I_{P,\ell}$, and, when $P \in PG \cup PSS$, let $\mathcal{P}_P$ be the set of prime divisors of N_P. Then one has, for $\ell \gg 0$:

$$0 \leq \frac{|(G_\ell v)'_P|}{|G_\ell v_\ell|} \leq \sum_{q \in \mathcal{P}_P} \frac{|(G_\ell v)^{\gamma_{P,\ell}^{N_P/q}}|}{|G_\ell v_\ell|},\ P \in PG,$$

$$0 \leq \frac{|(G_\ell v_\ell)'_P|}{|G_\ell v_\ell|} = \frac{|(G_\ell v_\ell)^{\gamma_{P,\ell}}|}{|G_\ell v_\ell|},\ P \in SS,$$

and

$$0 \leq \frac{|(G_\ell v_\ell)'_P|}{|G_\ell v_\ell|} \leq \left(\sum_{q \in \mathcal{P}_P} \frac{|(G_\ell v_\ell)^{\gamma_{P,\ell}^{\ell N_P/q}}|}{|G_\ell v_\ell|}\right) + \frac{|(G_\ell v_\ell)^{\gamma_{P,\ell}^{N_P}}|}{|G_\ell v_\ell|},\ P \in PSS.$$

Applying this method, one gets:

Proposition 3.11. *Conjecture 1.2 holds for* $\dim(A_\eta) = 1$.

Proof. First, $M(v_\ell) := \mathbb{F}_\ell[G_\ell v_\ell] \subset A_\eta[\ell]$ coincides with $A_\eta[\ell]$ for $\ell \gg 0$ and $v_\ell \in A_\eta[\ell]^\times$. Indeed, else, $M(v_\ell)$ is 1-dimensional, which contradicts corollary 3.6. In particular, G_ℓ acts faithfully on $G_\ell v_\ell$. So, one may apply lemma 3.12 below and deduce that, in any case,

$$\frac{|(G_\ell v_\ell)'_P|}{|G_\ell v_\ell|} \leq C_P \epsilon(\ell) \to 0,$$

where $C_P \geq 1$ is an integer depending only of the reduction type at $P \in \tilde{X} \setminus X$.

Lemma 3.12. *For each prime ℓ, there exists $\epsilon(\ell) \geq 0$ depending only on A and ℓ, such that $\epsilon(\ell) \to 0$ ($\ell \to \infty$) and that $\frac{|(G_\ell v)^\gamma|}{|G_\ell v|} \leq \epsilon(\ell)$ for any ℓ, any $v \in A_\eta[\ell]^\times$, and any $\gamma \in G_\ell$ acting nontrivially on $M(v)$.*

Proof. For any $\gamma \in G_\ell$ acting nontrivially on $M(v)$, set $M_\gamma(v) := \mathbb{F}_\ell[(G_\ell v)^\gamma] \subset M(v)^\gamma \subset M(v)$. Since γ acts nontrivially on $M(v)$ and $\dim(M(v)) \leq 2$, the only possibilities are $\dim(M_\gamma(v)) = 0$ or $(\dim(M_\gamma(v)), \dim(M(v)^\gamma), \dim(M(v))) = (1,1,2)$. In the former case, $(G_\ell v)^\gamma = \emptyset$, so there is nothing to do. In the latter case, up to replacing v by an element of $(G_\ell v)^\gamma \neq \emptyset$, one may assume that $\gamma v = v$ hence $M_\gamma(v) = \mathbb{F}_\ell v$. Set $U_{\gamma,v} := \{g \in G_\ell \mid g(M_\gamma(v)) = M_\gamma(v)\} \subset G_\ell$. Then, by definition, one has a surjective map $U_{\gamma,v} \twoheadrightarrow (G_\ell v)^\gamma$, $g \mapsto gv$, which is 1-to-$|G_v|$, where $G_v := \mathrm{Stab}_{G_\ell}(v)$. Whence $|(G_\ell v)^\gamma| = [U_{\gamma,v} : G_v]$ and $\dfrac{|(G_\ell v)^\gamma|}{|G_\ell v|} = \dfrac{1}{[G_\ell : U_{\gamma,v}]}$.

Now, assume that the statement of lemma 3.12 does not hold, that is there exists $N \geq 1$ such that for any integer $n \geq 0$ there exists a prime $\ell_n \geq n$, $v_n \in A_\eta[\ell_n]^\times$ and $\gamma_n \in G_{\ell_n}$ acting nontrivially on $M(v_n)$ such that $\dim(M_{\gamma_n}(v_n)) = 1$ and $[G_{\ell_n} : U_{\gamma_n,v_n}] \leq N$. By Riemann's existence theorem, there are only finitely many isomorphism classes of etale covers of X with degree $\leq N$. So, up to replacing X by such a cover, one may assume that $G_{\ell_n} = U_{\gamma_n,v_n}$ for infinitely many $n \geq 0$. But, then, $\mathbb{F}_{\ell_n} v_n$ is a G_{ℓ_n}-submodule of $\mathbb{F}_{\ell_n}$-dimension 1, which contradicts corollary 3.6

for $\ell_n \geq N(1, A)$. $\square$

This completes the proof of proposition 3.11. $\square$

Remark 3.13. Proposition 3.11 is also a direct consequence of the fact that the genus of modular curves $X_1(\ell)$ goes to ∞ with ℓ but our proof does not resort to this specific argument.

In fact, since corollary 3.8 takes into account any nontrivial $\pi_1(X)$-submodule $M \subset A_\eta[\ell]$, the proof of corollary 3.11 shows the following when $\dim(A_\eta)$ is arbitrary. For any $v \in A_\eta[\ell]^\times$, set (when it is defined):

$$g_2(\ell) := \min\{g_{X_v}\}_{v \in A_\eta[\ell]^\times,\, \dim(M(v)) \leq 2}.$$

Then $g_2(\ell) \to +\infty$.

3.2.2. *Proof of theorem 1.3 – $g_X = 0$.* From now on, write $\tilde{X} \smallsetminus X = \{P_1, \dots, P_r\}$ and recall that $\pi_1(X)$ is the profinite completion of the group given by the generators $\gamma_1, \dots, \gamma_r$ and the single relation $\gamma_1 \cdots \gamma_r = 1$, where γ_i is a distinguished generator of inertia at P_i, $i = 1, \dots, r$. Also, let $\gamma_{i,\ell}$ denote the image of γ_i in G_ℓ (hence $I_{P_i,\ell} = \langle \gamma_{i,\ell} \rangle$). Eventually, write $\mathcal{O}_{i,n}$ for the set of all $\omega \in G_\ell v$ such that $|\langle \gamma_{i,\ell} \rangle \omega| = n$. So, in particular, $\mathcal{O}_{i,1} = (G_\ell v)^{I_{P_i,\ell}}$, and $\mathcal{O}_{i,n} = \emptyset$ unless $n \mid |I_{P_i,\ell}|$.

3.2.2.1. A general computation. For any subset $I \subset \{1, \dots, r\}$, set

$$E_I := \bigcap_{i \in I} \mathcal{O}_{i,1} = (G_\ell v)^{\langle \gamma_i | i \in I \rangle}$$

(thus, in particular, $E_\emptyset = G_\ell v$) and, for each $0 \leq i \leq r$, set:

$$\Sigma_i := \sum_{I \subset \{1,\dots r\},\, |I| = i} |E_I|,$$

$$\overline{\Sigma}_i := |\bigcup_{I \subset \{1,\dots r\},\, |I| = i} E_I|.$$

Similarly, define the $*$-variants: for any subset $I \subset \{1, \dots, r\}$,

$$E_I^* := E_I \smallsetminus \bigcup_{I \subsetneq J} E_J$$

and, for each $0 \leq i \leq r$,

$$\Sigma_i^* := \sum_{I \subset \{1,\dots r\},\, |I| = i} |E_I^*|,$$

$$\overline{\Sigma}_i^* := |\bigcup_{I \subset \{1,\dots r\},\, |I| = i} E_I^*|.$$

Note that, actually, $\overline{\Sigma}_i^* = \Sigma_i^*$, $i = 0, \dots, r$.

Now, consider the map $\nu : G_\ell v \to \{0, \dots, r\}$ which sends $\omega \in G_\ell v$ to

$$\nu(\omega) := |\{1 \leq i \leq r \mid \omega \in E_{\{i\}}\}|.$$

Then,

$$\Sigma_1 = \sum_{1 \leq i \leq r} |E_{\{i\}}| = \sum_{\omega \in G_\ell v} \nu(\omega) = \sum_{0 \leq i \leq r} i|\nu^{-1}(i)| = \sum_{0 \leq i \leq r} i\overline{\Sigma}_i^* = \sum_{0 \leq i \leq r} i\Sigma_i^*.$$ [2]

But, on the other hand, one has:

$$\overline{\Sigma}_i = \sum_{i \leq j \leq r} \Sigma_j^*,\ i = 1, \dots, r.$$

[2] More generally, one has $\Sigma_i = \sum_{i \leq j \leq r} C_j^i \Sigma_j^*$

So, one eventually gets:

$$\Sigma_1 = \sum_{1\leq i\leq r} \overline{\Sigma}_i.$$

Now, from lemma 2.2 (1), for any $\ell \gg 0$ and any $I \subset \{1,\dots,r\}$ with $|I| = r,\ r-1$, one has $A_\eta[\ell]^{\langle \gamma_{i,\ell}|i\in I\rangle} = A_\eta[\ell]^{G_\ell} = 0$, hence, in particular, $E_I = \emptyset$. As a result:

$$\begin{aligned}\overline{\Sigma}_r &= \Sigma_r = 0;\\ \overline{\Sigma}_{r-1} &= \Sigma_{r-1} = 0;\\ \overline{\Sigma}_{r-2} &= \Sigma^*_{r-2} = \Sigma_{r-2}\end{aligned}$$

and $\overline{\Sigma}_i \leq |G_\ell v|$, $i = 1,\dots,r-3$. Whence:

$$\Sigma_1 \leq (r-3)|G_\ell v| + \Sigma_{r-2}.$$

3.2.2.2. Estimate for Σ_{r-2}. We will now make use of the semistable reduction theorem [SGA7, Exp. IX] which implies that for any $1 \leq i \leq r$ with $P_i \in SS$ and any $\ell \gg 0$, the element $\gamma_{i,\ell}$ is unipotent of echelon exactly 2, that is, $\gamma_{i,\ell} = Id + \nu_{i,\ell}$ with $\nu^2_{i,\ell} = 0$ and $\nu_{i,\ell} \neq 0$; in particular, $\gamma_{i,\ell}$ has order exactly ℓ.

(1) Everywhere semistable reduction: Fix $I \subset \{1,\dots,r\}$ such that $|I| = r-2$ and let $\omega \neq \omega' \in E_I$. Then, for any $j \in \{1,\dots,r\}\smallsetminus I$ one has $\langle\gamma_{j,\ell}\rangle\omega \cap \langle\gamma_{j,\ell}\rangle\omega' = \emptyset$. Indeed, else, there would exist an integer $1 \leq k \leq \ell-1$ such that $\gamma^k_{j,\ell}\omega = \omega'$. So, as $\gamma^k_{j,\ell}\omega = \omega + k\nu_{j,\ell}(\omega)$, one gets: $0 \neq \omega' - \omega = k\nu_{j,\ell}(\omega) \in \ker(\nu_{j,\ell})$. But, by assumption, $\omega, \omega' \in \ker(\nu_{i,\ell})$, $i \in I$. Hence:

$$0 \neq \omega' - \omega \in \bigcap_{i\in I\cup\{j\}} \ker(\nu_{i,\ell}),$$

which contradicts the fact that $A_\eta[\ell]^{\langle\gamma_{i,\ell}|i\in I\cup\{j\}\rangle} = A_\eta[\ell]^{G_\ell} = 0$.

But, for any $\omega \in E_I$ and any $j \in \{1,\dots,r\}\smallsetminus I$ one has $|\langle\gamma_{j,\ell}\rangle\omega| = \ell$ hence:

$$\ell|E_I| \leq |G_\ell v| - |E_{\{j\}}|.$$

So, with $\{1,\dots,r\}\smallsetminus I = \{j,j'\}$, one has:

$$|E_I| \leq \frac{|G_\ell v|}{\ell} - \frac{|E_{\{j\}}| + |E_{\{j'\}}|}{2\ell}$$

and summing the above over all $I \subset \{1,\dots,r\}$ with $|I| = r-2$, one eventually obtains:

$$\Sigma_{r-2} \leq \frac{C^{r-2}_r}{\ell}|G_\ell v| - \frac{r-1}{2\ell}\Sigma_1 \leq \frac{r(r-1)}{2\ell}|G_\ell v|.$$

(2) Semistable reduction over all but one point: Assume that $A \to X$ has semistable reduction over $P_1, \dots, P_{r-1}$. Let $I \subset \{1,\dots,r\}$ such that $|I| = r-2$. Then, if $r \in I$ one has, again, with $\{1,\dots,r\}\smallsetminus I = \{j,j'\}$:

$$|E_I| \leq \frac{|G_\ell v|}{\ell} - \frac{|E_{\{j\}}| + |E_{\{j'\}}|}{2\ell} \leq \frac{|G_\ell v|}{\ell}.$$

If $r \notin I$ then, with $\{1,\dots,r\}\smallsetminus I = \{j,r\}$, one only has:

$$|E_I| \leq \frac{|G_\ell v|}{\ell} - \frac{|E_{\{j\}}|}{\ell} \leq \frac{|G_\ell v|}{\ell}.$$

Thus, summing the above over all $I \subset \{1,\dots,r\}$ with $|I| = r-2$, one obtains, again,

$$\Sigma_{r-2} \leq \frac{r(r-1)}{2\ell}|G_\ell v|.$$

3.2.2.3. Conclusion.

(1) Everywhere semistable reduction: First, observe that $G_\ell v$ can be written as the disjoint union of $\mathcal{O}_{i,1}$ and $G_\ell v \smallsetminus \mathcal{O}_{i,1} = \mathcal{O}_{i,\ell}$. Whence, one obtains:

$$\epsilon_{P_i}(v) = \frac{|\mathcal{O}_{i,1}|}{|G_\ell v|} + \frac{1}{\ell}(1 - \frac{|\mathcal{O}_{i,1}|}{|G_\ell v|}).$$

Thus, one gets:

$$\lambda_v = r(1 - \frac{1}{\ell}) - 2 - \frac{1}{|G_\ell v|}(1 - \frac{1}{\ell})\Sigma_1.$$

So, (*) is equivalent to:

$$(^{**}) \quad \Sigma_1 < (r - (2+\epsilon)\frac{\ell}{\ell - 1}))|G_\ell v| \text{ for } v = v_\ell,\ \ell \gg 0.$$

But, from the above computation, one has:

$$\Sigma_1 \leq (r-3)|G_\ell v| + \Sigma_{r-2} \leq (r - 3 + \epsilon(\ell))|G_\ell v|,$$

where $\epsilon(\ell) = \frac{r(r-1)}{2\ell} = O(\frac{1}{\ell})$. So, it is enough to show that $r - 3 + \epsilon(\ell) < r - (2+\epsilon)\frac{\ell}{\ell-1}$ for $\ell \gg 0$. But this is always valid for $0 < \epsilon < 1$ since the left-hand term goes to $r-3$ whereas the right-hand term goes to $r - 2 - \epsilon$.

(2) Semistable reduction over all but one point: Assume again that $A \to X$ has semistable reduction over $P_1, \ldots, P_{r-1}$ and non-semistable bad reduction over P_r. Then one has:

$$\epsilon_{P_r}(v) = \frac{1}{|G_\ell v|}(|\mathcal{O}_{r,1}| + \sum_{n\geq 2}\frac{1}{n}|\mathcal{O}_{r,n}|)$$

Thus, one gets:

$$\lambda_v = r(1-\frac{1}{\ell}) - 2 + \frac{1}{\ell} - \frac{1}{|G_\ell v|}(1 - \frac{1}{\ell})\Sigma_1 - \frac{1}{\ell}\frac{|\mathcal{O}_{r,1}|}{|G_\ell v|} - \frac{1}{|G_\ell v|}\sum_{n\geq 2}\frac{1}{n}|\mathcal{O}_{r,n}|.$$

So, (*) is equivalent to:

$$(^{***}) \quad \Sigma_1 + \frac{|\mathcal{O}_{r,1}|}{\ell - 1} + \frac{\ell}{\ell-1}\sum_{n\geq 2}\frac{1}{n}|\mathcal{O}_{r,n}| < (r - \frac{\ell - 1}{\ell}(2 + \epsilon - \frac{1}{\ell}))|G_\ell v| \text{ for } v = v_\ell,\ \ell \gg 0.$$

Let q denote the minimal prime divisor of N_{P_r}. One may assume that $q < \ell$ for $\ell \gg 0$. Now, observe that:

$$\begin{aligned}\Sigma_1 + \frac{|\mathcal{O}_{r,1}|}{\ell - 1} + \frac{\ell}{\ell-1}\sum_{n\geq 2}\frac{1}{n}|\mathcal{O}_{r,n}| &\leq \Sigma_1 + \frac{|\mathcal{O}_{r,1}|}{\ell - 1} + \frac{\ell}{\ell-1}\frac{1}{q}\sum_{n\geq 2}|\mathcal{O}_{r,n}| \\ &\leq \Sigma_1 + \frac{|\mathcal{O}_{r,1}|}{\ell - 1} + \frac{\ell}{\ell-1}\frac{1}{q}(|G_\ell v| - |\mathcal{O}_{r,1}|) \\ &\leq \Sigma_1 + (\frac{1}{q} + \frac{1}{\ell - 1})|G_\ell v|.\end{aligned}$$

So, it is enough to prove that:

$$\Sigma_1 + (\frac{1}{q} + \frac{1}{\ell-1})|G_\ell v| \leq (r - \frac{\ell-1}{\ell}(2 + \epsilon - \frac{1}{\ell}))|G_\ell v|$$

But, from the above computation, one still has:

$$\Sigma_1 \leq (r-3)|G_\ell v| + \Sigma_{r-2} \leq (r - 3 + \epsilon(\ell))|G_\ell v|,$$

where $\epsilon(\ell) = \frac{r(r-1)}{2\ell} = O(\frac{1}{\ell})$. So, it is enough to show that $r - 3 + \epsilon(\ell) + (\frac{1}{q} + \frac{1}{\ell-1}) < r - \frac{\ell-1}{\ell}(2 + \epsilon - \frac{1}{\ell})$ for $\ell \gg 0$. But this is always valid for $0 < \epsilon < 1 - \frac{1}{q}$ since the left-hand term goes to $r - 3 + \frac{1}{q}$ whereas the right-hand term goes to $r - 2 - \epsilon$.

3.2.3. *Semistable abelian schemes over $\mathbb{P}^1_k$ minus three points.* Using the same idea as in the proof of theorem 1.3, one gets:

Proposition 3.14. *There is no abelian scheme over $X = \mathbb{P}^1_k \smallsetminus \{P_1,\ P_2,\ P_3\}$ with semistable reduction at $P_1,\ P_2,\ P_3$ whose generic fiber is non-isotrivial.*

Proof. Suppose that $A \to X$ is an abelian scheme which has semistable reduction over P_i and whose generic fiber is non-isotrivial. Then, up to replacing $A \to X$ by the Néron model of a suitable (nontrivial) quotient of the generic fiber A_η, one may assume that A_η contains no nontrivial isotrivial abelian subvariety. Then $A_\eta[\ell]^{G_\ell} = 0$ for $\ell \gg 0$ by lemma 2.2 (1). Also, by the semistability condition, one may write $\gamma_{i,\ell} = Id + \nu_{i,\ell}$ with $\nu_{i,\ell}^2 = 0$. Now, the relation $\gamma_{1,\ell}\gamma_{2,\ell}\gamma_{3,\ell} = Id$ is equivalent to: $\nu_{1,\ell} + \nu_{2,\ell} + \nu_{3,\ell} + \nu_{1,\ell}\nu_{2,\ell} = 0$. Composing this relation with $\nu_{1,\ell}$, one obtains: $\nu_{1,\ell}\nu_{2,\ell} + \nu_{1,\ell}\nu_{3,\ell} = 0$. Since $\ker(\nu_{1,\ell}) \cap \ker(\nu_{2,\ell}) = 0$ and $\mathrm{im}(\nu_{2,\ell}) \subset \ker(\nu_{2,\ell})$, one has: $\ker(\nu_{1,\ell}\nu_{2,\ell}) = \ker(\nu_{2,\ell})$. Similarly, $\ker(\nu_{1,\ell}\nu_{3,\ell}) = \ker(\nu_{3,\ell})$. Whence $\ker(\nu_{2,\ell}) = \ker(\nu_{3,\ell}) \subset \ker(\nu_{2,\ell}) \cap \ker(\nu_{3,\ell}) = 0$. But this contradicts the fact that $\nu_{2,\ell},\ \nu_{3,\ell}$ are nilpotent. □

Remark 3.15. Let $Y \to X$ be a non-isotrivial curve with generic fiber of genus ≥ 2 or of genus 1 with a rational point. If $Y \to X$ has semistable reduction over $\tilde{X} \smallsetminus X$ then $\mathrm{Pic}^0_{Y|X}$ has semistable reduction as well over $\tilde{X} \smallsetminus X$. Thus, proposition 3.14, together with Torelli's theorem, implies [B81, Thm., p.100].

Example 3.16. Consider the abelian scheme given by the Legendre family $\mathcal{E} \to \mathbb{P}^1_\lambda \smallsetminus \{0, 1, \infty\}$ of elliptic curves defined by:

$$\mathcal{E}_\lambda : y^2 = x(x-1)(x-\lambda).$$

Then a straightforward computation shows that $\gamma_0 = \gamma_1 = \infty$ and $\gamma_\infty = 2\infty$. So, in some sense, the result of proposition 3.14 is optimal.

Corollary 3.17. *There is no abelian scheme $A \to X$ with X of genus zero and with reduction type:*

(i) $(2,2,n),\ (2,3,4),\ (2,3,5)$;
(ii) $(3,3,3),\ (2,4,4),\ (2,3,6),\ (2,2,2,2)$;
(iii) $(2,2,n\infty),\ (2,2\infty,\infty),\ (3,3,\infty)$;
(iv) $(2,3,3),\ (2,3,\infty)$

whose generic fiber is non-isotrivial.

Proof. We resort to an elementary base-change argument together with the following facts:

(1) If X has genus 0, there is no abelian scheme $A \to X$ with good reduction everywhere except possibly over two points of $\tilde{X} \smallsetminus X$ whose generic fiber is non-isotrivial;
(2) If X has genus 1, there is no abelian scheme $A \to X$ with good reduction everywhere whose generic fiber is non-isotrivial; and
(3) Proposition 3.14.

Here, (1) and (2) follow straightforwardly from corollary 3.6. (Or, one may also resort to [CT08, Cor. 2.5] or [CT09, Th. 5.1].)

For (i), make the base change by the Galois cover from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$ ramified over three points and with the same type of inertia to contradict (1). For (ii), make the base change by the Galois cover from a genus 1 curve to $\mathbb{P}^1_k$ ramified over three or four points and with the same type of inertia to contradict (2). For (iii) make the base change by cyclic Galois covers from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$ ramified over P_1 and P_2 with degree 2, 2 and 3, respectively, to contradict (1), (3) and (3), respectively. For (iv), make first the base change by the degree 2 cyclic Galois cover from $\mathbb{P}^1_k$ to $\mathbb{P}^1_k$ ramified over P_1 and P_3. Then it is reduced to the first case of (ii) and the last case of (iii), respectively. □

3.3. **Proof of corollary 1.4.** Let η be the generic point of X. For each integer $n \geq 1$, let $\rho_{A,n} : \pi_1(X) \to \mathrm{GL}(A_\eta[n])$ denote the canonical representation of the etale fundamental group $\pi_1(X)$ on the group of (generic) n-torsion points. First, let us start with the isotrivial case:

Proposition 3.18. *Assume that the generic fiber A_η is F-isotrivial, and let d be a positive integer. Then there exists a positive integer $N = N(A, d)$ such that, for any closed point $x \in X$ and any finite extension $\kappa/\kappa(x)$ with $([\kappa(x) : F] \leq)[\kappa : F] \leq d$, one has $|A_x(\kappa)_{tors}| \leq N$.*

Proof. Up to replacing F by a finite extension, one may assume that $X(F) \neq \emptyset$ and fix $b \in X(F)$. Write $\rho_A := \varprojlim \rho_{A,n} : \pi_1(X) \to \mathrm{GL}(T(A))$, where $T(A) := \varprojlim A_\eta[n]$, and set $G := \rho_A(\pi_1(X))$ and $G^{geo} := \rho_A(\pi_1(X_{\overline{F}}))$. Since A_η is isotrivial, $B := |G^{geo}| < \infty$.

For each closed point $x \in X$, write $s_x : \Gamma_{\kappa(x)} \to \pi_1(X_{\kappa(x)}) \subset \pi_1(X)$ for the corresponding section. Then $\rho_A \circ s_b$ induces a representation $c_b : \Gamma_F \to \mathrm{Aut}(G^{geo})$ via conjugation. Let $F_1 = F_1(b)/F$ be the finite (Galois) extension corresponding to $\ker(c_b) \subset \Gamma_F$. Then $[F_1 : F] \leq B!$. For any closed point $x \in X$ and any finite extension $\kappa/\kappa(x)$,

$$\begin{array}{rccl} c_{x,b,\kappa} : & \Gamma_\kappa & \to & G^{geo} \\ & \sigma & \mapsto & \rho_A(s_x(\sigma)s_b(\sigma)^{-1}) \end{array}$$

is a 1-cocycle with values in G^{geo} equipped with the Γ_F-action defined by $c_b : \Gamma_F \to \mathrm{Aut}(G^{geo})$. In particular, $c_{x,b,\kappa}|_{\Gamma_{F_1\kappa}} : \Gamma_{F_1\kappa} \to G^{geo}$ is a group homomorphism, hence, writing $F_2 = F_2(x, b, \kappa)/F_1\kappa$ for the finite (Galois) extension corresponding to $\ker(c_{x,b,\kappa}) \subset \Gamma_{F_1\kappa}$, one has $[F_2 : F] \leq B!Bd$ and $\rho_A \circ s_x|_{\Gamma_{F_2}} = \rho_A \circ s_b|_{\Gamma_{F_2}}$.

Now, suppose that $A_x(\kappa)[n]^\times \neq \emptyset$ for some positive integer n. Then, a fortiori, $A_x(F_2)[n]^\times \neq \emptyset$, hence the above equality implies $A_b(F_2)[n]^\times \neq \emptyset$. Since $[F_2 : F] \leq B!Bd$, the claim now follows from lemma 3.19 below. □

Lemma 3.19. *For any abelian variety $A \to F$ and integer $d \geq 1$, $A(\overline{F})^{\leq d} \cap A_{tors}$ is finite, where $A(\overline{F})^{\leq d} := \{\overline{v} \in A(\overline{F}) \mid [\kappa(v) : F] \leq d$, where v is the image of $\overline{v}$ in $A.\}$.*

Proof. Consider a model $\mathcal{A} \to R$ of $A \to F$ where R is a normal integral domain finitely generated over $\mathbb{Z}$ with fraction field F, then, by the same specialization argument as in the proof of claim 3.2, for any prime p in the image of $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$ and any closed point $s \in \mathrm{Spec}(R)$ above p, any point of $A(\overline{F})^{\leq d} \cap A[n]^\times$ $(p \nmid n)$ specializes to a point of $\mathcal{A}_s(\overline{\kappa(s)})^{\leq d} \cap \mathcal{A}_s[n]^\times \subset \mathcal{A}_s(\kappa(s)_d)[n]^\times$, where $\kappa(s)_d/\kappa(s)$ denotes the finite (Galois) extension of $\kappa(s)$ corresponding to the open subgroup $\Gamma \subset \Gamma_{\kappa(s)}$ defined to be the intersection of all $\Gamma' \subset \Gamma_{\kappa(s)}$ with $[\Gamma_{\kappa(s)} : \Gamma'] \leq d$. Now, from the Weil bound, this is possible only for finitely many n. Considering two distinct primes in the image of $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$, one deduces the desired finiteness eventually. □

Remark 3.20. As the proof shows, proposition 3.18 remains true when X is a smooth, connected F-scheme of arbitrary dimension.

For $n \geq 1$ and $v \in A_\eta[n]$, write $X_v \to X$ for the finite etale cover (defined over a finite extension F_v/F) corresponding to the inclusion of open subgroups $\mathrm{Stab}_{\pi_1(X)}(v) \subset \pi_1(X)$. For each $n \geq 1$, set $X_n := \sqcup_{v \in A_\eta[n]^\times} X_v$. Then, as in [CT08, 4.2], the image of $X_n(F) \to X(F)$ coincides with the set of points $x \in X(F)$ such that $A_x(F)[n]^\times \neq \emptyset$. Now, the assertion of corollary 1.4 is equivalent to (i) $|X_{\ell^n}(F)| < \infty$, ℓ: prime, $n \gg 0$ and (ii) $|X_\ell(F)| < \infty$, ℓ: prime $\gg 0$. Here, (i) follows from [CT08, Cor. 1.2]. Indeed, a special case ($\chi = 1$) of [CT08, Cor. 1.2] implies the following assertion (stronger than (i)): $|X_{\ell^n}(F)| = \emptyset$, ℓ: prime, $n \gg 0$. To prove (ii), let $(A_\eta)_0$ denote the largest isotrivial abelian subvariety of A_η (cf. [CT08, 2.1]), and, for any $v \in A_\eta$, write v^0 for the image of v in $A^0_\eta := A_\eta/(A_\eta)_0$. Then, for any $v \in A_\eta[\ell]^\times$, $g_{X_v} \geq g_{X_{v^0}}$. If $v^0 \neq 0$, then it follows from theorem 1.3 applied to (the Néron model over X of) A^0_η that $g_{X_v} \geq g_{X_{v^0}} \geq 2$, $\ell \gg 0$, so, from Mordell's conjecture, one gets the desired finiteness $|X_v(F)| < \infty$, $\ell \gg 0$. If $v^0 = 0$, *i.e.* $v \in (A_\eta)_0[\ell]$, then proposition 3.18 applied to (the Néron model over X of) $(A_\eta)_0$ implies the following assertion (stronger than the desired finiteness $|X_v(F)| < \infty$, $\ell \gg 0$): $X_v(F) = \emptyset$, $\ell \gg 0$. This completes the proof of corollary 1.4.

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