UNIVERSITÉ DE BORDEAUX
M2, p-adic Hodge Theory
2024-2025

Midterm homework assignment

due on March 26th 2025

Exercise 1. Let K be the splitting field of the polynomial $f(X) = X^p - p \in \mathbf{Q}_p[X]$ (*i.e.* K is generated over $\mathbf{Q}_p$ by the roots of $f(X)$). Set $G = \mathrm{Gal}(K/\mathbf{Q}_p)$.

1) Show that $K = \mathbf{Q}_p[\alpha, \zeta_p]$ where α is a root of $f(X)$ and ζ_p is a primitive pth root of unity.

2) Show that $[K : \mathbf{Q}_p] = p(p-1)$ and that $H = \mathrm{Gal}(K/\mathbf{Q}_p[\zeta_p])$ is a normal subgroup of $G = \mathrm{Gal}(K/\mathbf{Q}_p)$ of index $(p-1)$.

3) Show that $K/\mathbf{Q}_p$ is a totally ramified extension and give an uniformizer of K.

4) Describe the ramification subgroups G_i of G.

Exercise 2. Fix a finite extension K of $\mathbf{Q}_p$. Let $\mathbf{C}$ denote the completion of the algebraic closure of K. Fix a deeply ramified extension L of K and set $H = \mathrm{Gal}(\overline{K}/L)$.

Part I.
Let V be a finite-dimensional vector space over $\mathbf{C}$. We say that an action of H on V is semi-linear, if it satisfies the following properties:

$$h(v_1 + v_2) = h(v_1) + h(v_2), \qquad h \in H, \quad v_1, v_2 \in V,$$
$$h(\alpha v) = h(\alpha) h(v), \qquad h \in H, \quad \alpha \in \mathbf{C}, \quad v \in V,$$

where H acts naturally on $\mathbf{C}$.

Choose a basis $(v_1, \dots, v_n)$ of V, and denote by $A(h) \in \mathrm{GL}_n(\mathbf{C})$ the unique matrix such that

$$(h(v_1), h(v_2), \dots, h(v_n)) = (v_1, v_2, \dots, v_n) A(h).$$

($A(h)$ can be seen as the matrix of h in the basis $(v_1, \dots, v_n)$, but the action of h is not more linear.) The group H acts on the elements of $\mathrm{GL}_n(\mathbf{C})$ coordinatewisely.

1) Show that $A(h_1 h_2) = A(h_1)(h_1 A(h_2))$ for all $h_1, h_2 \in H$.

2) Show that the following conditions are equivalent :
a) V has a basis formed by H-invariant vectors (i.e. stable under the action of H).
b) There exists $B \in \mathrm{GL}_n(\mathbf{C})$ such that

$$A(h) = B h(B)^{-1}, \qquad \forall h \in H.$$

Part II.
Recall that $\mathbf{C}$ is equipped with the canonical topology provided by the absolute value. This topology induces a topology on $\mathrm{GL}_n(\mathbf{C})$. We say that a continuous map $f : H \to \mathrm{GL}_n(\mathbf{C})$ is a cocycle if it satisfies the condition

$$f(h_1 h_2) = f(h_1)(h_1 f(h_2)), \qquad h_1, h_2 \in H.$$

In particular, the map $h \mapsto A(h)$ from question 1) is a cocycle.

3) Show that there exists a normal subgroup H' of H of finite index such that

$$f(H') \subset 1 + p^2 \mathrm{M}_n(O_{\mathbf{C}}),$$

where $\mathrm{M}_n(O_{\mathbf{C}})$ is the set of all square matrices of order n. (Hint : in a Galois group, open subgroups are of finite index.)

4) Set $F = \overline{K}^{H'}$. Assume that a cocycle f satisfies the condition $f(H') \subset 1 + p^m \mathrm{M}_n(O_{\mathbf{C}})$.

4a) Show that there exists a finite Galois extension E/F such that

$$f(N) \subset 1 + p^{m+2} \mathrm{M}_n(O_{\mathbf{C}}), \qquad N := \mathrm{Gal}(\overline{K}/E).$$

Show that there exists $y \in O_E$ such that

$$\sum_{\sigma \in \mathrm{Gal}(E/F)} \sigma(y) = p.$$

For each $\sigma \in \mathrm{Gal}(E/F)$ choose a lift $\widehat{\sigma} \in \mathrm{Gal}(\overline{K}/F)$, and set

$$B_m := \frac{1}{p} \sum_{\sigma \in \mathrm{Gal}(E/F)} f(\widehat{\sigma})\widehat{\sigma}(y).$$

4b) Show that $B_m \in 1 + p^{m-1} \mathrm{M}_n(O_{\mathbf{C}})$.

4c) Show that for any $h \in H'$,

$$h(B_m) \equiv f(h)^{-1} B_m \pmod{p^{m+1}},$$

and therefore $B_m^{-1} f(h) h(B_m) \equiv 1 \pmod{p^{m+1}}$.

5) By successive approximation, show that for any cocycle $f : H \to \mathrm{GL}_n(\mathbf{C})$ there exists a normal subgroup of finite index H' and a matrix $B \in \mathrm{GL}_n(O_{\mathbf{C}})$ such that

$$f(h) = Bh(B)^{-1}, \qquad \forall h \in H'.$$

Part III.
In this part, we apply the results of Part II to the study of semi-linear action of H. We use the notations and conventions of Part I. Let V be a finite-dimensional $\mathbf{C}$-vector space equipped with a semi-linear action of H.

6) Show that there exists a basis $(w_1, \ldots, w_n)$ of V invariant under the action of a normal subgroup of finite index H'.

7) For any $h \in H$, write $(h(w_1), \ldots, h(w_n)) = (w_1, \ldots, w_n)C(h)$. Show that $C(h) \in \mathrm{GL}_n(\widehat{F})$, where $\widehat{F}$ is the completion of the field $F = \overline{K}^{H'}$.

8) Let G be a *finite group* of automorphisms of a field E. Hilbert's theorem 90 asserts that any finite-dimensional E-vector space equipped with a semi-linear action of G, has a G-invariant basis. Using this result, show that V has a H-invariant basis.