UNIVERSITÉ DE BORDEAUX
M2, p-adic Hodge Theory
2024-2025

Midterm homework assignment
Solutions

Exercise 1. Let K be the splitting field of the polynomial $f(X) = X^p - p \in \mathbf{Q}_p[X]$ (*i.e.* K is generated over $\mathbf{Q}_p$ by the roots of $f(X)$). Set $G = \mathrm{Gal}(K/\mathbf{Q}_p)$.

1) Show that $K = \mathbf{Q}_p[\alpha, \zeta_p]$ where α is a root of $f(X)$ and ζ_p is a primitive pth root of unity.

Solution. The roots of $f(X)$ are of the form $\alpha\zeta_p^i$ $(0 \leqslant i \leqslant p-1)$, where α is a fixed root and ζ is a fixed primitive pth root of unity. Therefore the field generated by the roots is $\mathbf{Q}_p[\alpha, \zeta_p]$.

2) Show that $[K : \mathbf{Q}_p] = p(p-1)$ and that $H = \mathrm{Gal}(K/\mathbf{Q}_p[\zeta_p])$ is a normal subgroup of $G = \mathrm{Gal}(K/\mathbf{Q}_p)$ of index $(p-1)$.

Solution. Since K contains the intermediate subfield $\mathbf{Q}_p[\zeta_p]$, the degree $[\mathbf{Q}_p[\zeta_p] : \mathbf{Q}_p] = p-1$ divides $[K : \mathbf{Q}_p]$. Also, K contains the subfield $\mathbf{Q}_p[\alpha]$. Since $f(X)$ is an Eisenstein polynomial, it is irreducible and $[\mathbf{Q}_p[\alpha] : \mathbf{Q}_p] = p$. Therefore p divides $[K : \mathbf{Q}_p]$. Since p and $p-1$ are coprime, $p(p-1)$ divides $[K : \mathbf{Q}_p]$. On the other hand,

$$[K : \mathbf{Q}_p] = [K : \mathbf{Q}_p[\zeta_p]] \cdot [\mathbf{Q}_p[\zeta_p] : \mathbf{Q}_p] \leqslant p(p-1).$$

Hence $[K : \mathbf{Q}_p] = p(p-1)$.

Since $\mathbf{Q}_p[\zeta_p]/\mathbf{Q}_p$ is a Galois subextension of $K/\mathbf{Q}_p$, the group $H = \mathrm{Gal}(K/\mathbf{Q}_p[\zeta_p])$ is normal in G and G/H is isomorphic to $\mathrm{Gal}(\mathbf{Q}_p[\zeta_p]/\mathbf{Q}_p)$.

3) Show that $K/\mathbf{Q}_p$ is a totally ramified extension and give an uniformizer of K.

Solution. For any extension A/B of local fields, let $e(A/B)$ denote its ramification index. Then

$$e(K/\mathbf{Q}_p) = e(K/\mathbf{Q}_p[\zeta_p]) \cdot e(\mathbf{Q}_p[\zeta_p]/\mathbf{Q}_p) = e(K/\mathbf{Q}_p[\alpha]) \cdot e(\mathbf{Q}_p[\alpha]/\mathbf{Q}_p).$$

Since $\mathbf{Q}_p[\alpha]/\mathbf{Q}_p$ and $\mathbf{Q}_p[\zeta_p]/\mathbf{Q}_p$ are totally ramified, we obtain that both $p-1$ and p divide $e(K/\mathbf{Q}_p)$. Therefore $e(K/\mathbf{Q}_p) = p(p-1) = [K : \mathbf{Q}_p]$, and $K/\mathbf{Q}_p$ is totally ramified.

Let v_K denote the normalized discrete valuation on K, i.e. such that $v_K(p) = e(K/\mathbf{Q}_p)$. Since $\pi_0 = \zeta_p - 1$ is a uniformizer of $\mathbf{Q}_p[\zeta_p]$, and $\mathbf{Q}_p[\zeta_p]/\mathbf{Q}_p$ is totally ramified of degree $p-1$, we have $v_K(\pi_0) = v_K(p)/(p-1) = p$. The same argument shows that $v_K(\alpha) = v_K(p)/p = p-1$. Set $\pi := \pi_0/\alpha$. Then $v_K(\pi) = v_K(\pi_0) - v_K(\alpha) = 1$. Therefore π is a uniformizer of K.

4) Describe the ramification subgroups G_i of G. Let $g \in \mathrm{Gal}(K/\mathbf{Q}_p)$. Then $g(\zeta_p) = \zeta_p^{a_g}$ and $g(\alpha) = \alpha\zeta_p^{b_g}$ for some $a_g \in (\mathbf{Z}/p\mathbf{Z})^*$ and $b_g \in \mathbf{Z}/p\mathbf{Z}$. Then $g(\pi_0) = (1+\pi_0)^{a_g} - 1$ and

$$g(\pi) - \pi = \frac{g(\pi_0}{\alpha\zeta_p^{b_g}} - \frac{\pi_0}{\alpha} = \frac{(1+\pi_0)^{a_g} - 1 - \pi_0\zeta_p^{b_g}}{\alpha\zeta_p^{b_g}} = \frac{(1+\pi_0)^{a_g} - 1 - \pi_0(1+\pi_0)^{b_g}}{\alpha\zeta_p^{b_g}}.$$

Here $v_K(\alpha\zeta_p^{b_g}) = v_K(\alpha) = p-1$. For the numerator, we have

$$(1+\pi_0)^{a_g} - 1 - \pi_0(1+\pi_0)^{b_g} \equiv a_g\pi_0 - \pi_0 = (a_g - 1)\pi_0 \pmod{\pi_0^2}.$$

Therefore the valuation of the numerator is $v_K(\pi_0) = p$ if $a_g \neq 1$ i.e. if $g \notin H$. This gives:

$$v_K(g(\pi) - \pi) = 1 \Leftrightarrow g \notin H.$$

Assume that $g \in H \setminus \{1\}$. Then $a_g = 1$, $b_g \neq 0$ and

$$(1+\pi_0)^{a_g} - 1 - \pi_0(1+\pi_0)^{b_g} = \pi_0 - \pi_0(1+\pi_0)^{b_g} \equiv b_g\pi_0^2 \pmod{\pi_0^3}.$$

Therefore the valuation of the numerator is $v_K(\pi_0^2) = 2p$, and we obtain that

$$v_K(g(\pi) - \pi) = p + 1 \Leftrightarrow g \in H \setminus \{1\}.$$

These computations give:

$$G_0 = G, \quad G_1 = G_2 = \ldots = G_p = H, \quad G_{p+1} = \{1\}.$$

Exercise 2. Fix a finite extension K of $\mathbf{Q}_p$. Let $\mathbf{C}$ denote the completion of the algebraic closure of K. Fix a deeply ramified extension L of K and set $H = \mathrm{Gal}(\overline{K}/L)$.

Part I.
Let V be a finite-dimensional vector space over $\mathbf{C}$. We say that an action of H on V is semi-linear, if it satisfies the following properties:

$$h(v_1 + v_2) = h(v_1) + h(v_2), \qquad h \in H, \quad v_1, v_2 \in V,$$
$$h(\alpha v) = h(\alpha)h(v), \qquad h \in H, \quad \alpha \in \mathbf{C}, \quad v \in V,$$

where H acts naturally on $\mathbf{C}$.

Choose a basis $(v_1, \ldots, v_n)$ of V, and denote by $A(h) \in \mathrm{GL}_n(\mathbf{C})$ the unique matrix such that

$$(h(v_1), h(v_2), \ldots, h(v_n)) = (v_1, v_2, \ldots, v_n)A(h).$$

($A(h)$ can be seen as the matrix of h in the basis $(v_1, \ldots, v_n)$, but the action of h is not more linear.) The group H acts on the elements of $\mathrm{GL}_n(\mathbf{C})$ coordinatewisely.

1) Show that $A(h_1h_2) = A(h_1)(h_1A(h_2))$ for all $h_1, h_2 \in H$.

Solution. We have

$$\begin{aligned}(h_1h_2(v_1), h_1h_2(v_2), \ldots, h_1h_2(v_n)) &= h_1((h_2(v_1), h_2(v_2), \ldots, h_2(v_n)) \\ &= h_1\big((v_1, v_2, \ldots, v_n)A(h_2)\big) = (h_1(v_1), h_1(v_2), \ldots, h_1(v_n))(h_1A(h_2)) \\ &= (v_1, v_2, \ldots, v_n)A(h_1)(h_1A(h_2)).\end{aligned}$$

On the other hand, $(h_1h_2(v_1), h_1h_2(v_2), \ldots, h_1h_2(v_n)) = (v_1, v_2, \ldots, v_n)A(h_1h_2)$. Therefore $A(h_1h_2) = A(h_1)(h_1A(h_2))$.

2) Show that the following conditions are equivalent :
a) V has a basis formed by H-invariant vectors (i.e. stable under the action of H).
b) There exists $B \in \mathrm{GL}_n(\mathbf{C})$ such that

$$A(h) = Bh(B)^{-1}, \qquad \forall h \in H.$$

Solution. For any basis $\{w_i\}_{i=1}^n$ of V, we denote by $B \in \mathrm{GL}_n(\mathbf{C})$ the transition matrix i.e.

$$(w_1, \ldots, w_n) = (v_1, \ldots, v_n)B.$$

The basis $\{w_i\}_{i=1}^n$ is H-invariant if and only if

$$(h(w_1), \ldots, h(w_n)) = (w_1, \ldots, w_n), \quad \forall h \in H.$$

This condition means that

$$(h(v_1),\dots,h(v_n))h(B) = (v_1,\dots,v_n)B,$$

or equivalently that

$$A(h)h(B) = B, \qquad \forall h \in H.$$

Part II.

Recall that $\mathbf{C}$ is equipped with the canonical topology provided by the absolute value. This topology induces a topology on $\mathrm{GL}_n(\mathbf{C})$. We say that a continuous map $f : H \to \mathrm{GL}_n(\mathbf{C})$ is a cocycle if it satisfies the condition

$$f(h_1h_2) = f(h_1)(h_1 f(h_2)), \qquad h_1, h_2 \in H.$$

In particular, the map $h \mapsto A(h)$ from question 1) is a cocycle.

3) Show that there exists a normal subgroup H' of H of finite index such that

$$f(H') \subset 1 + p^2\mathrm{M}_n(O_{\mathbf{C}}),$$

where $\mathrm{M}_n(O_{\mathbf{C}})$ is the set of all square matrices of order n. (Hint : in a Galois group, open subgroups are of finite index.)

Solution. From the cocycle condition, it's easy to see that $f(e) = I_n$ (write $f(e) = f(e^e) = f(e)(ef(e)) = f(e)f(e)$ and use the fact that $f(e)$ is an invertible matrix). Let U be any open subgroup of $\mathrm{GL}_n(\mathbf{C})$ containing $1 + p^2\mathrm{M}_n(O_{\mathbf{C}})$. (For example, one can take $1 + p^2\mathrm{M}_n(\mathfrak{m}_{\mathbf{C}})$, where $\mathfrak{m}_{\mathbf{C}}$ is the maximal ideal of $O_{\mathbf{C}}$ because $\mathfrak{m}_{\mathbf{C}}$ is open in $\mathbf{C}$.) Since the inverse image of an open subset under the continuous map f is open and open subgroups of H are of finite index and form a neighborhood base at 1, there exists a subgroup of finite index $S \subset H$ such that $f(S) \subset 1+p^2\mathrm{M}_n(O_{\mathbf{C}})$. Write $H = \bigcup_{i=1}^{m} h_i S$. Then $H' := \bigcap_{i=1}^{m} h_i S h_i^{-1}$ is a normal subgroup of finite index and such that $f(H') \subset 1+p^2\mathrm{M}_n(O_{\mathbf{C}})$.

4) Set $F = \overline{K}^{H'}$. Assume that a cocycle f satisfies the condition $f(H') \subset 1 + p^m\mathrm{M}_n(O_{\mathbf{C}})$.

4a) Show that there exists a finite Galois extension E/F such that

$$f(N) \subset 1 + p^{m+2}\mathrm{M}_n(O_{\mathbf{C}}), \qquad N := \mathrm{Gal}(\overline{K}/E).$$

Show that there exists $y \in O_E$ such that

$$\sum_{\sigma\in\mathrm{Gal}(E/F)} \sigma(y) = p.$$

Solution. The existence of N can be proved by the same argument as in the proof of 3). The existence of y follows directly from Theorem 2.2, Chapter 2. Here we use the assumption that L is deeply ramified.

For each $\sigma \in \mathrm{Gal}(E/F)$ choose a lift $\widehat{\sigma} \in \mathrm{Gal}(\overline{K}/F)$, and set

$$B_m := \frac{1}{p}\sum_{\sigma\in\mathrm{Gal}(E/F)} f(\widehat{\sigma})\widehat{\sigma}(y).$$

4b) Show that $B_m \in 1 + p^{m-1}\mathrm{M}_n(O_{\mathbf{C}})$.

Solution. One has

$$B_m - 1 = \frac{1}{p}\sum_{\sigma\in\mathrm{Gal}(E/F)} f(\widehat{\sigma})\widehat{\sigma}(y) - \frac{1}{p}\sum_{\sigma\in\mathrm{Gal}(E/F)} \widehat{\sigma}(y) = \frac{1}{p}\sum_{\sigma\in\mathrm{Gal}(E/F)} (f(\widehat{\sigma}) - 1)\widehat{\sigma}(y).$$

Since $f(\widehat{\sigma}) - 1 \in p^m \mathrm{M}_n(O_{\mathbf{C}})$, this implies that $B_m - 1 \in p^{m-1}\mathrm{M}_n(O_{\mathbf{C}})$.

4c) Show that for any $h \in H'$,

$$h(B_m) \equiv f(h)^{-1} B_m \pmod{p^{m+1}},$$

and therefore $B_m^{-1} f(h) h(B_m) \equiv 1 \pmod{p^{m+1}}$.

Solution. We have

$$h(B_m) = \frac{1}{p} \sum_{\sigma \in \mathrm{Gal}(E/F)} h(f(\widehat{\sigma}))h\widehat{\sigma}(y).$$

Since $h(f(\widehat{\sigma})) = f(h)^{-1} f(h\widehat{\sigma})$ (cocycle property), we have

$$h(B_m) = \frac{1}{p} \sum_{\sigma \in \mathrm{Gal}(E/F)} f(h)^{-1} f(h\widehat{\sigma}) h\widehat{\sigma}(y) = f(h)^{-1} \cdot \frac{1}{p} \sum_{\sigma \in \mathrm{Gal}(E/F)} f(h\widehat{\sigma}) h\widehat{\sigma}(y)$$

We want to compare $\frac{1}{p} \sum_{\sigma \in \mathrm{Gal}(E/F)} f(h\widehat{\sigma}) h\widehat{\sigma}(y)$ with B_m. In general, the multiplication by h does not permute the elements of the set $\{\widehat{\sigma} \mid \sigma \in \mathrm{Gal}(E/F)\}$, but the set $\{h\widehat{\sigma} \mid \sigma \in \mathrm{Gal}(E/F)\}$ is another family of lifts of the elements of $\mathrm{Gal}(E/F)$ in $\mathrm{Gal}(\overline{K}/F)$. If τ_1 and $\tau_2 \in \mathrm{Gal}(\overline{K}/F)$ are two lifts of the same element $\sigma \in \mathrm{Gal}(E/F)$, then $\tau_1 = \tau_2 s$ with $s \in \mathrm{Gal}(\overline{K}/E)$, and therefore

$$f(\tau_1) = f(\tau_2 s) = f(\tau_2)(\tau_2 f(s)).$$

But $f(s) \in 1 + p^{m+2}\mathrm{M}_n(O_{\mathbf{C}})$ by the choice of E (question 4a)), and therefore $f(\tau_1) \equiv f(\tau_2) \pmod{p^{m+2}\mathrm{M}_n(O_{\mathbf{C}})}$. This observation shows that

$$\frac{1}{p} \sum_{\sigma \in \mathrm{Gal}(E/F)} f(h\widehat{\sigma}) h\widehat{\sigma}(y) \equiv \frac{1}{p} \sum_{\sigma \in \mathrm{Gal}(E/F)} f(\widehat{\sigma})\widehat{\sigma}(y) = B_m \pmod{p^{m+1}\mathrm{M}_n(O_{\mathbf{C}})}.$$

Hence

$$h(B_m) \equiv f(h)^{-1} B_m \pmod{p^{m+1}\mathrm{M}_n(O_{\mathbf{C}})},$$

and we are done.

5) By successive approximation, show that for any cocycle $f : H \to \mathrm{GL}_n(\mathbf{C})$ there exists a normal subgroup of finite index H' and a matrix $B \in \mathrm{GL}_n(O_{\mathbf{C}})$ such that

$$f(h) = Bh(B)^{-1}, \qquad \forall h \in H'.$$

Solution. We proceed by successive approximation. Assume that

$$f(h) \equiv B_m h(B_m)^{-1} \pmod{p^{m+1}}, \qquad \forall h \in H'.$$

Set $\alpha(h) = B_m^{-1} f(h) h(B_m)$. A direct computation shows that α is a cocycle. Since $\alpha(h) \equiv 1 \pmod{p^{m+1}}$, by question 4), there exists a matrix $C_m \equiv 1 \pmod{p^m}$ such that $\alpha(h) \equiv C_m h(C_m)^{-1} \pmod{p^{m+2}}$. Set $B_{m+1} = B_m C_m$. Then

$$f(h) \equiv B_{m+1} h(B_{m+1})^{-1} \pmod{p^{m+2}}, \qquad \forall h \in H'.$$

It's easy to check that the sequence of matrices $(B_m)_m$ converges to some matrix B. Then $f(h) = Bh(B)^{-1}$.

Part III.

In this part, we apply the results of Part II to the study of semi-linear action of H. We use the notations and conventions of Part I. Let V be a finite-dimensional $\mathbf{C}$-vector space equipped with a semi-linear action of H.

6) Show that there exists a basis $(w_1, \ldots, w_n)$ of V invariant under the action of a normal subgroup of finite index H'.

Solution. This follows immediately from questions 2) and 5).

7) For any $h \in H$, write $(h(w_1), \ldots, h(w_n)) = (w_1, \ldots, w_n)C(h)$. Show that $C(h) \in \mathrm{GL}_n(\widehat{F})$, where $\widehat{F}$ is the completion of the field $F = \overline{K}^{H'}$.

Solution. Let $s \in H'$. Then for any $h \in H$ we have $sh = hs'$ for some $s' \in H'$. Hence

$$(sh(w_1), \ldots, sh(w_n)) = (hs'(w_1), \ldots, hs'(w_n)) = (h(w_1), \ldots, h(w_n)) = (w_1, \ldots, w_n)C(h),$$

and

$$(sh(w_1), \ldots, sh(w_n)) = s(w_1, \ldots, w_n)s(C(h)) = (w_1, \ldots, w_n)s(C(h)).$$

Therefore $s(C(h)) = C(h)$ for all $s \in H'$, and the coefficients of $C(h)$ belong to the field $\mathbf{C}^{H'} = \widehat{F}$ by Theorem 3.3, Chapter III.

8) Let G be a *finite group* of automorphisms of a field E. Hilbert's theorem 90 asserts that any finite-dimensional E-vector space equipped with a semi-linear action of G, has a G-invariant basis. Using this result, show that V has a H-invariant basis.

Solution. The finite group $G := H/H'$ acts on $\widehat{F}$ and $\widehat{F}^H = \widehat{L}$. Therefore $\widehat{F}/\widehat{L}$ is a Galois extension with the Galois group G. Let W denote the $\widehat{F}$-vector space generated by $(w_1, \ldots, w_n)$. By 7) the group G acts semi-linearly on W. Applying Hilbert's theorem 90 to W, we obtain that W has a G-invariant basis. Therefore V has a H-invariant basis.