STABLE STANDING WAVES FOR NONLINEAR SCHRÖDINGER-POISSON SYSTEM WITH A DOPING PROFILE

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Abstract. This paper is devoted to the study of the nonlinear Schrödinger-Poisson system with a doping profile. We are interested in the existence of stable standing waves by considering the associated L^2-minimization problem. The presence of a doping profile causes a difficulty in the proof of the strict sub-additivity. A key ingredient is to establish the strict sub-additivity by adapting a scaling and an iteration argument, which is inspired by [32]. When the doping profile is a characteristic function supported on a bounded smooth domain, smallness of some geometric quantity related to the domain ensures the existence of stable standing waves.

1. Introduction

In this paper, we are concerned with the following nonlinear Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + \omega u + e\phi u = |u|^{p-1}u \\ -\Delta \phi = \frac{e}{2}\left(|u|^2 - \rho(x)\right) \end{cases} \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\omega \in \mathbb{R}$, $e > 0$ and $1 < p < \frac{7}{3}$. Equation (1.1) appears as a stationary problem for the time-dependent nonlinear Schrödinger-Poisson system:

$$\begin{cases} i\psi_t + \Delta\psi - e\phi\psi + |\psi|^{p-1}\psi = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ -\Delta\phi = \frac{e}{2}\big(|\psi|^2 - \rho(x)\big) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \psi(0,x) = \psi_0. \end{cases} \tag{1.2}$$

Indeed when we look for a standing wave of the form: $\psi(t,x) = e^{i\omega t}u(x)$, we are led to the elliptic problem (1.1). In this paper, we are interested in the existence of stable standing waves for (1.2) by considering the solvability of the associated L^2-constraint minimization problem.

The Schrödinger-Poisson system appears in various fields of physics, such as quantum mechanics, black holes in gravitation and plasma physics. Especially, the Schrödinger-Poisson system plays an important role in the study of semi-conductor theory; see [21, 25, 27], and then the function $\rho(x)$ is referred as *impurities* or a *doping profile*. The doping profile comes from the difference of the number densities of positively charged donor ions and negatively charged acceptor ions, and the most typical examples are characteristic functions, step functions or Gaussian functions. Equation (1.1) also appears as a stationary problem for the Maxwell-Schrödinger system. We refer to [6, 13, 14] for the physical background and the stability result of standing waves for the Maxwell-Schrödinger system. In this case, the constant e describes the strength of the interaction between a particle and an external electromagnetic field.

The nonlinear Schrödinger-Poisson system with $\rho \equiv 0$:

$$\begin{cases} -\Delta u + \omega u + e\phi u = |u|^{p-1}u \\ -\Delta \phi = \frac{e}{2}|u|^2 \end{cases} \text{in } \mathbb{R}^3 \tag{1.3}$$

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has been studied widely in the last two decades. Especially, the existence of L^2-constraint minimizers depending on p and the size of the mass, the existence of ground state solutions of (1.3) and their stability have been investigated in detail; see [2, 3, 4, 5, 10, 13, 19, 20, 22, 26, 28, 29, 31] and references therein. On the other hand, the nonlinear Schrödinger-Poisson system with a doping profile is less studied. In [16, 17], the corresponding 1D problem has been considered. Moreover, the linear Schrödinger-Poisson system (that is, the problem (1.1) without $|u|^{p-1}u$) with a doping profile in $\mathbb{R}^3$ has been studied in [7, 8]. As far as we known, there is no literature concerning with (1.1) and the existence of stable standing waves, which is exactly the motivation of this paper.

To state our main results, let us give some notation. For $u \in H^1(\mathbb{R}^3, \mathbb{C})$, the energy functional associated with (1.1) is given by

$$
\begin{aligned}
I(u) &= \mathcal{E}(u) + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2\,dx, \\
\mathcal{E}(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx + e^2 A(u).
\end{aligned} \tag{1.4}
$$

Here we denote the nonlocal term by $S(u) = S_1(u) + S_2$ with

$$
\begin{aligned}
S_1(u)(x) &:= (-\Delta)^{-1}\left(\frac{|u|^2}{2}\right) = \frac{1}{8\pi|x|} * |u|^2, \\
S_2(x) &:= (-\Delta)^{-1}\left(\frac{-\rho}{2}\right) = -\frac{1}{8\pi|x|} * \rho(x),
\end{aligned}
$$

and the functional corresponding to the nonlocal term by

$$
A(u) := \frac{1}{4}\int_{\mathbb{R}^3} S(u)\big(|u|^2 - \rho(x)\big)\,dx = \frac{1}{32\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\big(|u(x)|^2 - \rho(x)\big)\big(|u(y)|^2 - \rho(y)\big)}{|x-y|}\,dx\,dy.
$$

For $\mu > 0$, let us consider the minimization problem:

$$
\mathcal{C}(\mu) = \inf_{u \in B(\mu)} \mathcal{E}(u), \tag{1.5}
$$

where $B(\mu) = \{u \in H^1(\mathbb{R}^3, \mathbb{C})\ ;\ \|u\|^2_{L^2(\mathbb{R}^3)} = \mu\}$. We also define the set of minimizers by

$$
\mathcal{M}(\mu) := \{u \in B(\mu)\ ;\ \mathcal{E}(u) = \mathcal{C}(\mu)\}.
$$

In this setting, the constant ω in (1.1) appears as a Lagrange multiplier.

Let us define the energy associated with (1.3):

$$
E_\infty(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx + \frac{e^2}{4}\int_{\mathbb{R}^3} S_1(u)|u|^2\,dx.
$$

Indeed if we assume $\rho(x) \to 0$ as $|x| \to \infty$, (1.3) can be seen as a problem at infinity. We define the minimum energy associated with (1.3) by

$$
c_{e,\infty}(\mu) = c_\infty(\mu) := \inf_{u \in B(\mu)} E_\infty(u). \tag{1.6}
$$

The existence of minimizers for $c_{e,\infty}(\mu)$ has been studied widely and is summarized as follows.

(i) In the case $2 < p < \frac{7}{3}$, $c_{e,\infty}(\mu)$ is attained if and only if $c_{e,\infty}(\mu) < 0$. Moreover $c_{e,\infty}(\mu) < 0$ when μ is large for fixed e or e is small for fixed μ.

(ii) In the case $p = 2$, $c_{e,\infty}(\mu)$ is attained if and only if $c_{e,\infty}(\mu) < 0$. Moreover $c_{e,\infty}(\mu) < 0$ when e is small for fixed μ.

(iii) In the case $1 < p < 2$, $c_{e,\infty}(\mu) < 0$ for all μ and e. Moreover $c_{e,\infty}(\mu)$ is attained when μ is small for fixed e or e is small for fixed μ.

For the proof, we refer to [4, 5, 10, 13, 20, 22, 19, 29].

For the doping profile ρ, we assume that

$$\rho(x) \in L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^q_{loc}(\mathbb{R}^3) \text{ for some } q > 3 \quad \text{and} \quad x \cdot \nabla\rho(x) \in L^{\frac{6}{5}}(\mathbb{R}^3). \tag{1.7}$$

Our first main result is the following.

Theorem 1.1 (Existence of a minimizer). *Under the assumption (1.7), we have the followings.*

(i) *Suppose that $2 < p < \frac{7}{3}$ and choose $\mu > 0$ so that $c_\infty(\mu) < 0$. Then there exists $\rho_0 = \rho_0(e,\mu) > 0$ such that if $\|\rho\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|x \cdot \nabla\rho\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \leq \rho_0$, the minimization problem (1.5) admits a minimizer u_μ.*

Moreover the associated Lagrange multiplier $\omega = \omega(\mu)$ is positive.

(ii) *Suppose that $1 < p \leq 2$. Then there exists $e_0 = e_0(\mu,\rho) > 0$ such that if $0 < e \leq e_0$, the minimization problem (1.5) admits a minimizer u_μ.*

In the statement of (i), we may choose $e > 0$ so that $c_\infty(\mu) < 0$ for fixed $\mu > 0$. The assumption (1.7) rules out the case ρ is a characteristic function supported on a bounded smooth domain. Even in this case, we are still able to obtain the existence of minimizers under a smallness condition on some geometric quantity related to the domain; See Section 6.

The positivity of the Lagrange multiplier $\omega(\mu)$ will be useful to establish the relation between L^2-constraint minimizers and ground state solutions, which we leave for a future work. We also refer to [15, 18] for this direction.

The next result states the orbital stability of standing waves corresponding to minimizers.

Theorem 1.2 (Orbital stability of standing wave). *Under the assumption of Theorem 1.1, the standing wave $\psi(t,x) = e^{i\omega t}u_\mu(x)$ is orbitally stable in the following sense: For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if an initial value ψ_0 satisfies $\|\psi_0 - u_\mu\|_{H^1(\mathbb{R}^3)} < \delta$, then the corresponding solution ψ of (1.2) satisfies*

$$\sup_{t>0} \inf_{u \in M(\mu)} \left\{ \left\|\psi(t,\cdot) - u(\cdot)\right\|_{H^1(\mathbb{R}^3)} + \left\|\phi(t,\cdot) - \frac{e}{2}(-\Delta)^{-1}|u(\cdot)|^2\right\|_{D^{1,2}(\mathbb{R}^3)} \right\} < \varepsilon.$$

Here $D^{1,2}(\mathbb{R}^3) = \dot{H}(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm: $\|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2\,dx$. As for the global well-posedness of the Cauchy problem for (1.2), see Section 4 below.

Here we briefly explain our strategy and its difficulty. The existence of L^2-constraint minimizer can be shown by applying the concentration compactness principle. A key of the proof is to establish the *strict sub-additivity*:

$$\mathcal{C}(\mu) < \mathcal{C}\left(\mu'\right) + \mathcal{C}\left(\mu - \mu'\right) \quad \text{for all} \quad 0 < \mu' < \mu. \tag{1.8}$$

In the case $2 < p < \frac{7}{3}$ and $\rho \equiv 0$, (1.8) can be readily obtained by adapting a suitable scaling. However this scaling does not work straightforwardly if a doping profile is present, because of the loss of spatial homogeneity. In order to overcome this difficulty, we perform an iteration argument inspired by [32]. By imposing the smallness of ρ and $x \cdot \nabla\rho$, it is possible to prove (1.8) if $2 < p < \frac{7}{3}$. In the case $1 < p \leq 2$, we need to assume that, not ρ itself is small, but e is sufficiently small. We refer to Remark 3.5 below why we have to distinguish into the cases $2 < p < \frac{7}{3}$ and $1 < p \leq 2$.

When ρ is a characteristic function, further consideration is required because ρ cannot be weakly differentiable. In this case, a key of the proof is the *sharp boundary trace inequality* which was developed in [1]. Then by imposing a smallness condition of some geometric quantity related to the support of ρ, we are able to obtain the existence of stable standing waves.

This paper is organized as follows. In Section 2, we introduce several properties of the energy functional and some lemmas which will be used in this paper. We establish the existence of a minimizer and prove Theorem 1.1 in Section 3. Section 4 is devoted to the solvability of the Cauchy problem, and the stability of standing waves will be investigated in Section 5. In Section 6, we consider the case ρ is a characteristic function and present the existence of standing waves for this case. Finally in Section 7, we finish this paper by providing a concluding remark and one open question.

Hereafter in this paper, unless otherwise specified, we write $\|u\|_{L^p(\mathbb{R}^3)} = \|u\|_p$.

2. Variational formulation and preliminaries

The aim of this section is to prepare several properties of the energy functional and present intermediate lemmas which will be used later on.

2.1. Reduction to a single equation.

First we observe that the energy functional defined in (1.4) actually corresponds to the system (1.1). Let us consider the functional of two variables, which is associated with (1.1):

$$J(u,\phi) := \frac{1}{2}\int_{\mathbb{R}^3} \left\{ |\nabla u|^2 + \omega |u|^2 + e\phi\left(|u|^2 - \rho(x)\right) - |\nabla \phi|^2 \right\} dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx,$$

for $(u,\phi) \in H^1(\mathbb{R}^3,\mathbb{C}) \times D^{1,2}(\mathbb{R}^3,\mathbb{R})$. A direct computation shows that the identity $\frac{\partial J}{\partial u} = 0$ is equivalent to the first equation of (1.1). Moreover one finds that

$$\frac{\partial J}{\partial \phi}\psi = \int_{\mathbb{R}^3} \left\{ -\nabla\phi\cdot\nabla\psi + \frac{e}{2}\left(|u|^2 - \rho(x)\right)\psi \right\} dx \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^3,\mathbb{R}),$$

from which we deduce that

$$-\Delta\phi = \frac{e}{2}\left(|u|^2 - \rho(x)\right). \tag{2.1}$$

Since $|u|^2 - \rho \in L^{\frac{6}{5}}(\mathbb{R}^3) = (L^6(\mathbb{R}^3))^*$, by arguing similarly as in [6], the Poisson equation (2.1) has a unique solution:

$$\phi = eS(u) = \frac{e}{2}(-\Delta)^{-1}\left(|u|^2 - \rho(x)\right) \in D^{1,2}(\mathbb{R}^3,\mathbb{R}).$$

Moreover multiplying (2.1) by ϕ, we have

$$\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx = \frac{e}{2}\int_{\mathbb{R}^3} \phi\left(|u|^2 - \rho(x)\right) dx.$$

This implies that

$$\begin{aligned} J(u,eS(u)) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \omega|u|^2)\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx + \frac{e^2}{4}\int_{\mathbb{R}^3} S(u)\left(|u|^2 - \rho(x)\right) dx \\ &= I(u). \end{aligned}$$

2.2. Decomposition of the energy.

In this subsection, we rewrite the energy functional $\mathcal{E}$ in a more convenient way. First we decompose $\mathcal{E}$ in the following way:

$$\begin{aligned} \mathcal{E}(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx, \\ &+ \frac{e^2}{4}\int_{\mathbb{R}^3} S_1(u)|u|^2\, dx + \frac{e^2}{4}\int_{\mathbb{R}^3} S_2|u|^2\, dx - \frac{e^2}{4}\int_{\mathbb{R}^3} S_1(u)\rho(x)\, dx - \frac{e^2}{4}\int_{\mathbb{R}^3} S_2\rho(x)\, dx \end{aligned}$$

and define four nonlocal terms:

$$\begin{aligned} A_1(u) &= \frac{1}{4}\int_{\mathbb{R}^3} S_1(u)|u|^2\,dx, \\ A_2(u) &= -\frac{1}{4}\int_{\mathbb{R}^3} S_1(u)\rho(x)\,dx, \\ A_{2'}(u) &= \frac{1}{4}\int_{\mathbb{R}^3} S_2|u|^2\,dx, \\ A_0 &= -\frac{1}{4}\int_{\mathbb{R}^3} S_2\rho(x)\,dx. \end{aligned}$$

Note that A_0 is independent of u. One can also see that

$$A_2(u) = -\frac{1}{32\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(y)|^2\rho(x)}{|x-y|}\,dx\,dy = A_{2'}(u).$$

Then we are able to write $\mathcal{E}$ in the following form:

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx + e^2A_1(u) + 2e^2A_2(u) + e^2A_0.$$

Recalling that

$$S_1(u)(x) = (-\Delta)^{-1}\left(\frac{|u|^2}{2}\right) \geq 0,$$

we find that

$$A_1(u) \geq 0 \quad \text{for all } u \in H^1(\mathbb{R}^3,\mathbb{C}). \tag{2.2}$$

Now it is convenient to put

$$E(u) := \mathcal{E}(u) - e^2A_0. \tag{2.3}$$

Since A_0 is independent of u, we have only to consider the minimization problem for E.

2.3. **Estimates of nonlocal terms.**

This subsection is devoted to present estimates for the nonlocal terms of the functional E. For later use, let us define

$$A_3(u) = \frac{1}{2}\int_{\mathbb{R}^3} S_1(u)x\cdot\nabla\rho(x)\,dx$$

which is well-defined for $u \in H^1(\mathbb{R}^3,\mathbb{C})$, and a constant

$$A_4 = \frac{1}{2}\int_{\mathbb{R}^3} S_2 x\cdot\nabla\rho(x)\,dx.$$

Then we have the following.

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3,\mathbb{C})$, S_1, A_1, A_2 and A_3 satisfy the estimates:*

$$\begin{aligned} \|S_1(u)\|_6 &\leq C\|\nabla S_1(u)\|_2 \leq C\|u\|_{\frac{12}{5}}^2 \leq C\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}}, \\ 0 \leq A_1(u) &\leq C\|S_1(u)\|_6\|u\|_{\frac{12}{5}}^2 \leq C\|u\|_2^3\|\nabla u\|_2, \\ |A_2(u)| &\leq \frac{1}{4}\|S_1(u)\|_6\|\rho\|_{\frac{6}{5}} \leq C\|\rho\|_{\frac{6}{5}}\|u\|_{\frac{12}{5}}^2 \leq C\|\rho\|_{\frac{6}{5}}\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}}, \\ |A_3(u)| &\leq \frac{1}{2}\|S_1(u)\|_6\|x\cdot\nabla\rho\|_{\frac{6}{5}} \leq C\|x\cdot\nabla\rho\|_{\frac{6}{5}}\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}}. \end{aligned}$$

Moreover S_2, the constants A_0 and A_4 can be estimated as follows.

$$\begin{aligned} \|S_2\|_6 &\leq C\|\nabla S_2\|_2 \leq C\|\rho\|_{\frac{6}{5}}, \\ |A_0| &\leq \frac{1}{4}\|S_2\|_6\|\rho\|_{\frac{6}{5}} \leq C\|\rho\|_{\frac{6}{5}}^2, \\ |A_4| &\leq \frac{1}{2}\|S_2\|_6\,\|x\cdot\nabla\rho\|_{\frac{6}{5}} \leq C\|\rho\|_{\frac{6}{5}}\|x\cdot\nabla\rho\|_{\frac{6}{5}}. \end{aligned}$$

For the proof of the inequality on $S_1(u)$, we refer to [26]. The other estimates can be obtained by the Hödler inequality and the Sobolev inequality.

Lemma 2.2. *Assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Then it follows that*

$$\lim_{n\to\infty} \{A_1(u_n-u) - A_1(u_n) + A_1(u)\} = 0,$$
$$\lim_{n\to\infty} \{A_2(u_n-u) - A_2(u_n) + A_2(u)\} = 0.$$

Moreover if $u_n \to u$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$, we also have

$$\lim_{n\to\infty} A_1(u_n) = A_1(u) \quad \text{and} \quad \lim_{n\to\infty} A_2(u_n) = A_2(u).$$

Proof. The proof for A_1 can be found in [31, Lemma 2.2]. Since $\rho \in L^{\frac{6}{5}}(\mathbb{R}^3) = (L^6(\mathbb{R}^3))^*$, the property for A_2 can be established in a similar way. ☐

2.4. **Scaling properties.**

In this subsection, we collect scaling properties of the nonlocal terms A_1 and A_2. For a, $b \in \mathbb{R}$ and $\lambda > 0$, let us adapt the scaling $u_\lambda(x) := \lambda^a u\left(\lambda^b x\right)$. We first recall that

$$S_1(u)(x) = (-\Delta)^{-1}\left(\frac{|u(x)|^2}{2}\right) = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\,dy.$$

Putting $y = \lambda^{-b} z$, we have

$$\begin{aligned} S_1(u_\lambda)(x) &= \frac{1}{8\pi}\int_{\mathbb{R}^3}\frac{|u_\lambda(y)|^2}{|x-y|}\,dy = \frac{\lambda^{2a}}{8\pi}\int_{\mathbb{R}^3}\frac{|u(\lambda^b y)|^2}{|x-y|}\,dy \\ &= \frac{\lambda^{2a+b}}{8\pi}\int_{\mathbb{R}^3}\frac{|u(\lambda^b y)|^2}{|\lambda^b x - \lambda^b y|}\,dy \\ &= \frac{\lambda^{2a-2b}}{8\pi}\int_{\mathbb{R}^3}\frac{|u(z)|^2}{|\lambda^b x - z|}\,dz. \end{aligned}$$

Thus one finds that

$$\begin{aligned} S_1(u_\lambda)(x) &= \lambda^{2a-2b} S_1(u)(\lambda^b x), \\ A_1(u_\lambda) &= \lambda^{4a-5b} A_1(u), \end{aligned} \tag{2.4}$$

$$\begin{aligned} A_2(u_\lambda) &= -\frac{1}{4}\int_{\mathbb{R}^3} S_1(u_\lambda)\rho(x)\,dx = -\frac{\lambda^{2a-2b}}{4}\int_{\mathbb{R}^3} S_1(u)(\lambda^b x)\rho(x)\,dx \\ &= -\frac{\lambda^{2a-5b}}{4}\int_{\mathbb{R}^3} S_1(u)\rho\left(\lambda^{-b}x\right)\,dx. \end{aligned} \tag{2.5}$$

By the Hölder inequality, it follows that

$$|A_2(u_\lambda)| \le \frac{\lambda^{2a-5b}}{4}\|S_1(u)\|_6 \|\rho(\lambda^{-b}\cdot)\|_{\frac{6}{5}} \le C\lambda^{2a-\frac{5}{2}b}\|\rho\|_{\frac{6}{5}}\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}}. \tag{2.6}$$

2.5. **Nehari and Pohozaev identities.**

This subsection is devoted to establish the Nehari identity and the Pohozaev identity associated with (1.1). First we observe that

$$\begin{aligned} I'(u)\varphi &= \int_{\mathbb{R}^3} \nabla u\cdot\nabla\bar{\varphi}\,dx + \omega\int_{\mathbb{R}^3} u\bar{\varphi}\,dx - \int |u|^{p-2}u\bar{\varphi}\,dx \\ &\quad + \frac{e^2}{4}\int_{\mathbb{R}^3} S'(u)\varphi\left(|u|^2-\rho(x)\right)dx + \frac{e^2}{2}\int_{\mathbb{R}^3} S(u)u\bar{\varphi}\,dx, \end{aligned}$$

for any $\varphi \in H^1(\mathbb{R}^3,\mathbb{C})$. The definition $S(u) = S_1(u) + S_2$ shows that $S'(u) = S_1'(u)$, and moreover

$$S_1'(u)\varphi = (-\Delta)^{-1} * (u\bar{\varphi}).$$

This yields that

$$S'(u)u = S_1'(u)u = (-\Delta)^{-1} * |u|^2 = 2S_1(u) = 2S(u) - 2S_2$$

and hence

$$\begin{aligned}
&\frac{e^2}{4}\int_{\mathbb{R}^3} S'(u)u\left(|u|^2-\rho(x)\right)dx + \frac{e^2}{2}\int_{\mathbb{R}^3} S(u)|u|^2\,dx \\
&= \frac{e^2}{2}\int_{\mathbb{R}^3} (S(u)-S_2)\left(|u|^2-\rho(x)\right)dx + \frac{e^2}{2}\int_{\mathbb{R}^3} S(u)|u|^2\,dx \\
&= e^2\int_{\mathbb{R}^3} S(u)\left(|u|^2-\rho(x)\right)dx - \frac{e^2}{2}\int_{\mathbb{R}^3} S_2\left(|u|^2-\rho(x)\right)dx + \frac{e^2}{2}\int_{R} S(u)\rho(x)\,dx \\
&= e^2\int_{\mathbb{R}^3} S(u)\left(|u|^2-\rho(x)\right)dx + \frac{e^2}{2}\int_{\mathbb{R}^3} \left\{S_1(u)\rho(x) - S_2|u|^2 + 2S_2\rho(x)\right\}dx \\
&= e^2\int_{\mathbb{R}^3} S(u)\left(|u|^2-\rho(x)\right)dx + e^2\int_{\mathbb{R}^3} S(u)\rho(x)\,dx.
\end{aligned}$$

Here we used the fact $\int_{\mathbb{R}^3} S_1(u)\rho\,dx = -\int_{\mathbb{R}^3} S_2|u|^2\,dx$. Thus we find that the Nehari identity corresponding to (1.1) is given by

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} \left\{|\nabla u|^2 + \omega|u|^2 - |u|^{p+1} + e^2S(u)\left(|u|^2-\rho(x)\right) + e^2S(u)\rho(x)\right\}dx \\
&= \|\nabla u\|_2^2 + \omega\|u\|_2^2 - \|u\|_{p+1}^{p+1} + 4e^2A(u) + e^2\int_{\mathbb{R}^3} S(u)\rho(x)\,dx. \qquad (2.7)
\end{aligned}$$

Next, we show that the Pohozaev identity associated with (1.1) is described as

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} \left\{\frac{1}{2}|\nabla u|^2 + \frac{3\omega}{2}|u|^2 - \frac{3}{p+1}|u|^{p+1} + \frac{5e^2}{4}S(u)\left(|u|^2-\rho(x)\right) - \frac{e^2}{2}S(u)x\cdot\nabla\rho(x)\right\}dx \\
&= \frac{1}{2}\|\nabla u\|_2^2 + \frac{3\omega}{2}\|u\|_2^2 - \frac{3}{p+1}\|u\|_{p+1}^{p+1} + 5e^2A(u) - \frac{e^2}{2}\int_{\mathbb{R}^3} S(u)x\cdot\nabla\rho(x)\,dx. \qquad (2.8)
\end{aligned}$$

First we derive (2.8) by a formal calculation. Let us consider $u_\lambda(x) = u\left(\frac{x}{\lambda}\right)$, that is, take $a = 0$ and $b = -1$. Then from (2.4) and (2.5), one has

$$\begin{aligned}
I(u_\lambda) &= \frac{1}{2}\|\nabla u_\lambda\|_2^2 + \frac{\omega}{2}\|u_\lambda\|_2^2 - \frac{1}{p+1}\|u_\lambda\|_{p+1}^{p+1} + e^2A_1(u_\lambda) + 2e^2A_2(u_\lambda) + e^2A_0 \\
&= \frac{\lambda}{2}\|\nabla u\|_2^2 + \frac{\lambda^3\omega}{2}\|u\|_2^2 - \frac{\lambda^3}{p+1}\|u\|_{p+1}^{p+1} + e^2\lambda^5A_1(u) - \frac{\lambda^5e^2}{2}\int_{\mathbb{R}^3} S_1(u)\rho(\lambda x)\,dx + e^2A_0.
\end{aligned}$$

Now we suppose that u is a solution of (1.1). Since $A = A_1 + 2A_2 + A_0$, it follows that

$$\begin{aligned}
0 &= \frac{d}{d\lambda}I(u_\lambda)\Big|_{\lambda=1} \qquad (2.9) \\
&= \frac{1}{2}\|\nabla u\|_2^2 + \frac{3\omega}{2}\|u\|_2^2 - \frac{3}{p+1}\|u\|_{p+1}^{p+1} + 5e^2A_1(u) + 10e^2A_2(u) - \frac{e^2}{2}\int_{\mathbb{R}^3} S_1(u)x\cdot\nabla\rho(x)\,dx \\
&= \frac{1}{2}\|\nabla u\|_2^2 + \frac{3\omega}{2}\|u\|_2^2 - \frac{3}{p+1}\|u\|\|_{p+1}^{p+1} + 5e^2A(u) - 5e^2A_0 \\
&\quad - \frac{e^2}{2}\int_{\mathbb{R}^3} S(u)x\cdot\nabla\rho(x)\,dx + \frac{e^2}{2}\int_{\mathbb{R}^3} S_2x\cdot\nabla\rho(x)\,dx.
\end{aligned}$$

We put

$$\begin{aligned}
R &= -5e^2A_0 + \frac{e^2}{2}\int_{\mathbb{R}^3} S_2(x)x\cdot\nabla\rho(x)\,dx \\
&= \frac{5e^2}{4}\int_{\mathbb{R}^3} S_2(x)\rho(x)\,dx + \frac{e^2}{2}\int_{\mathbb{R}^3} S_2(x)x\cdot\nabla\rho(x)\,dx.
\end{aligned}$$

Recalling that

$$-\Delta S_2 = \frac{-\rho(x)}{2} \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla S_2|^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}^3} S_2 \rho(x) \, dx,$$

one finds that

$$\begin{aligned}
\int_{\mathbb{R}^3} S_2 x \cdot \nabla \rho \, dx &= -\int_{\mathbb{R}^3} \nabla S_2 \cdot x \rho \, dx - 3 \int_{\mathbb{R}^3} S_2 \rho \, dx \\
&= -2 \int_{\mathbb{R}^3} \nabla S_2 \cdot x \Delta S_2 \, dx - 3 \int_{\mathbb{R}^3} S_2 \rho \, dx \\
&= 2 \int_{\mathbb{R}^3} \nabla \left(\nabla S_2 \cdot x \right) \cdot \nabla S_2 \, dx - 3 \int_{\mathbb{R}^3} S_2 \rho \, dx \\
&= 2 \int_{\mathbb{R}^3} |\nabla S_2|^2 \, dx + 2 \int_{\mathbb{R}^3} x \cdot \nabla \left(\frac{1}{2} |\nabla S_2|^2 \right) dx - 3 \int_{\mathbb{R}^3} S_2 \rho \, dx \\
&= 2 \int_{\mathbb{R}^3} |\nabla S_2|^2 \, dx - 3 \int_{\mathbb{R}^3} |\nabla S_2|^2 \, dx - 3 \int_{\mathbb{R}^3} S_2 \rho \, dx \\
&= -\int_{\mathbb{R}^3} |\nabla S_2|^2 \, dx - 3 \int_{\mathbb{R}^3} S_2 \rho \, dx = -\frac{5}{2} \int_{\mathbb{R}^3} S_2 \rho dx.
\end{aligned}$$

This means that $R = 0$. Thus from (2.9), we obtain (2.8).

A rigorous proof can be done by establishing the $C^{1,\alpha}$-regularity of any weak solution of (1.1) for some $\alpha \in (0,1)$. Note that since $\rho \in L^q(\mathbb{R}^3)$ for some $q > 3$, it follows by the elliptic regularity theory that $S_2 \in W^{2,q}_{loc}(\mathbb{R}^3) \hookrightarrow C^{1,\alpha}_{loc}(\mathbb{R}^3)$. The smoothness of u can be shown similarly by applying the elliptic regularity theory. Then multiplying $x \cdot \nabla \bar{u}$ and $ex \cdot \nabla S(u)$ by (1.1) respectively, integrating over $B_R(0)$ and passing to a limit $R \to \infty$, we are able to prove (2.8) as in [9, 11].

Lemma 2.3. *Any nontrivial solution u of* (1.1) *satisfies the following identity.*

$$\begin{aligned}
(5p-7)\mathcal{E}(u) &= 2(p-2)\|\nabla u\|_2^2 - \frac{(3p-5)\omega}{2}\|u\|_2^2 \\
&\quad - 2e^2 \int_{\mathbb{R}^3} S(u)\rho(x) \, dx - \frac{(3-p)e^2}{2} \int_{\mathbb{R}^3} S(u) x \cdot \nabla \rho(x) \, dx.
\end{aligned}$$

Proof. From (2.7) and (2.8), we find that

$$\begin{aligned}
\frac{5p-7}{p+1}\|u\|_{p+1}^{p+1} &= 3\|\nabla u\|_2^2 - \omega \|u\|_2^2 + 5e^2 \int_{\mathbb{R}^3} S(u)\rho(x) \, dx + 2e^2 \int_{\mathbb{R}^3} S(u) x \cdot \nabla \rho(x) \, dx, \\
(5p-7)e^2 A(u) &= \frac{5-p}{2}\|\nabla u\|_2^2 - \frac{3(p-1)\omega}{2}\|u\|_2^2 \\
&\quad + 3e^2 \int_{\mathbb{R}^3} S(u)\rho(x) \, dx + \frac{(p+1)e^2}{2} \int_{\mathbb{R}^3} S(u) x \cdot \nabla \rho(x) \, dx.
\end{aligned}$$

Thus one deduces that

$$\begin{aligned}
(5p-7)\mathcal{E}(u) &= \frac{5p-7}{2}\|\nabla u\|_2^2 + (5p-7)e^2 A(u) - \frac{5p-7}{p+1}\|u\|_{p+1}^{p+1} \\
&= 2(p-2)\|\nabla u\|_2^2 - \frac{(3p-5)\omega}{2}\|u\|_2^2 - 2e^2 \int_{\mathbb{R}^3} S(u)\rho(x) \, dx - \frac{(3-p)e^2}{2} \int_{\mathbb{R}^3} S(u) x \cdot \nabla \rho(x) \, dx.
\end{aligned}$$

This ends the proof. □

3. Existence of a minimizer

In this section, we aim to prove that the minimization problem (1.5) admits a solution, provided that the minimum energy for $\rho \equiv 0$ is negative and $\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}}$ is small. First we begin with the following.

Lemma 3.1. *Suppose that* $1 < p < \frac{7}{3}$. *Then for any* $\mu > 0$, E *is bounded from below on* $B(\mu)$.

Proof. We use the fact that $A_1 \geq 0$ and $\frac{3}{2}(p-1) < 2$. The Gagliardo-Nirenberg inequality, the Young inequality and Lemma 2.1 yield that

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \, dx + e^2 A_1(u) + 2e^2 A_2(u) \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - C\|u\|_2^{\frac{5-p}{2}}\|\nabla u\|_2^{\frac{3}{2}(p-1)} - Ce^2\|\rho\|_{\frac{6}{5}}\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}} \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{3(p-1)\varepsilon}{4}\|\nabla u\|_2^2 - \frac{(7-3p)C}{4\varepsilon}\|u\|_2^{\frac{2(5-p)}{7-3p}} - \frac{\varepsilon}{4}\|\nabla u\|_2^2 - \frac{4Ce^{\frac{8}{3}}}{3\varepsilon}\|\rho\|_{\frac{6}{5}}^{\frac{4}{3}}\|u\|_2^2 \\
&\geq \frac{1}{4}\|\nabla u\|_2^2 - C\mu^{\frac{5-p}{7-3p}} - Ce^{\frac{8}{3}}\mu\|\rho\|_{\frac{6}{5}}^{\frac{4}{3}} \geq -C\mu^{\frac{5-p}{7-3p}} - Ce^{\frac{8}{3}}\mu\|\rho\|_{\frac{6}{5}}^{\frac{4}{3}},
\end{aligned}$$

from which we conclude. □

Next we define

$$c(\mu) = \inf_{\|u\|_2^2=\mu} E(u). \tag{3.1}$$

Instead of $C(\mu)$ defined in (1.5), it suffices to show that $c(\mu)$ is attained because of (2.3).

Lemma 3.2. *Suppose that* $1 < p < \frac{7}{3}$. *Then* $c(\mu) \leq 0$ *for all* $\mu > 0$.

Proof. Let us consider $u_\lambda(x) = \lambda^{\frac{3}{2}}u(\lambda x)$ for $\|u\|_2^2 = \mu$. Note that $\|u_\lambda\|_2^2 = \mu$ for any $\lambda > 0$. Using (2.4) and (2.6), we have

$$\begin{aligned}
E(u_\lambda) &= \frac{\lambda^2}{2}\|\nabla u\|_2^2 - \frac{\lambda^{\frac{3(p-1)}{2}}}{p+1}\|u\|_{p+1}^{p+1} + e^2\lambda A_1(u) + 2e^2 A_2(u_\lambda) \\
&\leq \frac{\lambda^2}{2}\|\nabla u\|_2^2 - \frac{\lambda^{\frac{3(p-1)}{2}}}{p+1}\|u\|_{p+1}^{p+1} + e^2\lambda A_1(u) + Ce^2\lambda^{\frac{1}{2}}\|\rho\|_6\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}} \to 0 \text{ as } \lambda \to +0.
\end{aligned}$$

This implies that $c(\mu) \leq 0$, as claimed. □

Lemma 3.3. *Suppose that* $1 < p < \frac{7}{3}$. *Then* $c(\mu)$ *satisfies the weak sub-additive condition:*

$$c(\mu) \leq c(\mu') + c(\mu - \mu') \quad \textit{for all} \quad 0 < \mu' < \mu.$$

Proof. We take $u_1, u_2 \in C_0^\infty(\mathbb{R}^3)$ such that $\|u_1\|_2^2 = \mu'$, $\|u_2\|_2^2 - \mu - \mu'$,

$$\operatorname{supp} u_1 \cap \operatorname{supp} u_2 = \emptyset, \tag{3.2}$$

$$E(u_1) \leq c(\mu') + \frac{\varepsilon}{2} \quad \text{and} \quad E(u_2) \leq c(\mu - \mu') + \frac{\varepsilon}{2} \tag{3.3}$$

for arbitrary $\varepsilon > 0$. From (3.2), one finds that

$$\begin{aligned}
S_1(u_1 + u_2) &= \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|u_1(y) + u_2(y)|^2}{|x-y|} \, dy \\
&= \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|u_1(y)|^2}{|x-y|} \, dy + \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|u_2(y)|^2}{|x-y|} \, dy \\
&= S_1(u_1) + S_1(u_2).
\end{aligned}$$

Then it holds that

$$\begin{aligned}A_1(u_1+u_2) &= \frac{1}{4}\int_{\mathbb{R}^3} S_1(u_1+u_2)\,|u_1+u_2|^2\,dx\\ &= \frac{1}{4}\int_{\mathbb{R}^3} (S_1(u_1)+S_1(u_2))\left(|u_1|^2+|u_2|^2\right)dx\\ &= \frac{1}{4}\int_{\mathbb{R}^3} S_1(u_1)\,|u_1|^2\,dx+\frac{1}{4}\int_{\mathbb{R}^3} S_1(u_2)\,|u_2|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^3} S_1(u_1)|u_2|^2\,dx\\ &= A_1(u_1)+A_2(u_2)+\frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_1(y)|^2|u_2(x)|^2}{|x-y|}\,dx\,dy,\end{aligned}$$

and

$$A_2(u_1+u_2)=\frac{1}{4}\int_{R^3} S_1(u_1+u_2)\,\rho(x)\,dx = A_2(u_1)+A_2(u_2).$$

Thus we have

$$E(u_1+u_2)=E(u_1)+E(u_2)+\frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_1(y)|^2|u_2(x)|^2}{|x-y|}\,dx\,dy.$$

Now we replace u_2 by $u_2(\cdot-k)$ for $k\in\mathbb{R}^3$ and put

$$\begin{aligned}R(k) &:= \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_1(y)|^2|u_2(x-k)|^2}{|x-y|}\,dx\,dy\\ &= \frac{1}{16\pi}\int_{\operatorname{supp}u_1}\int_{\operatorname{supp}u_2}\frac{|u_1(y)|^2|u_2(x-k)|^2}{|x-y|}\,dx\,dy.\end{aligned}$$

If $x\in\operatorname{supp}u_2+k$ and $y\in\operatorname{supp}u_1$, it follows that $|x-y|\geq \operatorname{dist}(\operatorname{supp}u_1,\operatorname{supp}u_2)+|k|$, provided that $|k|$ is sufficiently large. Then one has

$$R(k)\leq\frac{1}{16\pi\operatorname{dist}(\operatorname{supp}u_1,\operatorname{supp}u_2)+|k|}\,\|u_1\|_2^2\,\|u_2\|_2^2\to 0 \text{ as } |k|\to\infty.$$

Since $\|u_1+u_2(\cdot-k)\|_{L^2}^2=\mu$, we have from (3.3) that

$$\begin{aligned}c(\mu) &\leq E(u_1+u_2(\cdot-k)) = E(u_1)+E(u_2)+R(k)\\ &\leq c(\mu')+c(\mu-\mu')+\varepsilon+R(k)\end{aligned}$$

and hence

$$\begin{aligned}c(\mu) &\leq \limsup_{|k|\to\infty}\left\{c(\mu')+c(\mu-\mu')+\varepsilon+R(k)\right\}\\ &= c(\mu')+c(\mu-\mu')+\varepsilon.\end{aligned}$$

Passing a limit $\varepsilon\to+0$, we obtain $c(\mu)\leq c(\mu')+c(\mu-\mu')$. □

It is important to mention that $c(\mu)$ is non-increasing in μ by Lemmas 3.2 and 3.3. In order to prove the strict sub-additivity, we recall that

$$E_\infty(u)=\frac{1}{2}\|\nabla u\|_2^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}+e^2A_1(u).$$

By Lemma 2.1, one finds that

$$\begin{aligned}c(\mu) &\leq E(u)=E_\infty(u)+2e^2A_2(u)\\ &\leq E_\infty(u)+Ce^2\mu^{\frac{3}{4}}\|\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}} \quad \text{for any } u\in B(\mu).\end{aligned}$$

If $c_\infty(\mu)$ admits a minimizer $u=u_{e,\mu}$, it holds that

$$c(\mu)\leq c_\infty(\mu)+Ce^2\mu^{\frac{3}{4}}\|\rho\|_{\frac{6}{5}}\|\nabla u_{e,\mu}\|_2^{\frac{1}{2}}.$$

As we have mentioned in the introduction, $c_\infty(\mu)$ is attained when

$$c_\infty(\mu)<0 \text{ for } 2<p<\frac{7}{3} \quad \text{or} \quad e\ll 1 \text{ for } 1<p\le 2.$$

Especially it follows that

$$c(\mu)\le \frac{1}{2}c_\infty(\mu)<0 \quad \text{if } 2<p<\frac{7}{3},\ c_\infty(\mu)<0 \text{ and } \|\rho\|_{\frac{6}{5}} \text{ is sufficiently small.} \tag{3.4}$$

The next lemma is the most important in the proof of the existence of a minimizer for $c(\mu)$.

Lemma 3.4. *Suppose that $2<p<\frac{7}{3}$ and choose $\mu>0$ so that $c_\infty(\mu)<0$. Then there exists a constant $\rho_0=\rho_0(e,\mu)>0$ such that if $\|\rho\|_{\frac{6}{5}}+\|x\cdot\nabla\rho\|_{\frac{6}{5}}\le\rho_0$, it follows that*

$$c(\lambda\mu)<\lambda c(\mu) \quad \textit{for all } \lambda>1.$$

We note that Lemma 3.4 implies that

$$c(\mu)<c\left(\mu'\right)+c\left(\mu-\mu'\right) \quad \text{for all} \quad 0<\mu'<\mu. \tag{3.5}$$

(See e.g. [23, Lemma II.1].)

Proof. The proof consists of two steps.
Step 1: We show that there exists $\delta>0$ such that

$$c\left(\lambda\mu\right)<\lambda c(\mu) \quad \text{for all } \lambda\in(1,1+\delta].$$

We take any $u\in H^1(\mathbb{R}^3)$ with $\|u\|_2^2=\mu$ and for $2a-3b=1$, put $u_\lambda(x)=\lambda^a u\left(\lambda^b x\right)$ so that

$$\|u_\lambda\|_2^2=\lambda^{2a-3b}\|u\|_2^2=\mu.$$

Then from (2.4) and (2.5), it follows that

$$E(u_\lambda)=\frac{\lambda^{1+2b}}{2}\|\nabla u\|_2^2-\frac{\lambda^{\frac{p+1+3(p-1)b}{2}}}{p+1}\|u\|_{p+1}^{p+1}+e^2\lambda^{2+b}A_1(u)-\frac{\lambda^{1-2b}}{2}e^2\int_{\mathbb{R}^3}S_1(u)\rho\left(\lambda^{-b}x\right)dx.$$

Let us consider

$$\begin{aligned}&\frac{1}{\lambda^{1+2b}}E(u_\lambda)-E(u)\\&=\frac{\|u\|_{p+1}^{p+1}}{p+1}\left(1-\lambda^{\frac{p-1+(3p-7)b}{2}}\right)-e^2A_1(u)\left(1-\lambda^{1-b}\right)-2e^2A_2(u)-\frac{\lambda^{-4b}e^2}{2}\int_{\mathbb{R}^3}S_1(u)\rho(\lambda^{-b}x)\,dx\\&=:f(\lambda).\end{aligned}$$

We claim that

$$f'(1)<0. \tag{3.6}$$

For this purpose, one computes

$$\begin{aligned}f'(\lambda)=&-\frac{p-1+(3p-7)b}{2}\cdot\lambda^{\frac{p-3+(3p-7)b}{2}}\cdot\frac{\|u\|_{p+1}^{p+1}}{p+1}+(1-b)\lambda^{-b}e^2A_1(u)\\&+2b\lambda^{-1-4b}e^2\int_{\mathbb{R}^3}S_1(u)\rho(\lambda^{-b}x)\,dx+\frac{b}{2}\lambda^{-1-5b}e^2\int_{\mathbb{R}^3}S_1(u)x\cdot\nabla\rho(\lambda^{-b}x)\,dx,\end{aligned}$$

from which we deduce that

$$f'(1)=-\frac{p-1+(3p-7)b}{2}\cdot\frac{\|u\|_{p+1}^{p+1}}{p+1}+(1-b)e^2A_1(u)-8be^2A_2(u)+be^2A_3(u).$$

Now we assume that

$$0<b<\frac{p-1}{7-3p}. \tag{3.7}$$

By the definition of E, one has

$$-\frac{1}{p+1}\|u\|_{p+1}^{p+1} = E(u) - \frac{1}{2}\|\nabla u\|_2^2 - e^2 A_1(u) - 2e^2 A_2(u),$$

and hence

$$f'(1) = \frac{p-1+(3p-7)b}{2}E(u) - \frac{p-1+(3p-7)b}{4}\|\nabla u\|_2^2 + \frac{1}{2}\big(3-p+(5-3p)b\big)e^2 A_1(u) - \big(p-1+(3p+1)b\big)e^2 A_2(u) + be^2 A_3(u).$$

Next since $p>2$, we can take $b>0$ so that

$$3-p+(5-3p)b \le 0. \tag{3.8}$$

Indeed one can choose $b=1$ because $\frac{p-1}{7-3p} > 1$. Then from (2.2), one has

$$f'(1) \le \frac{p-1+(3p-7)b}{2}E(u) - \frac{p-1+(3p-7)b}{4}\|\nabla u\|_2^2 - \big(p-1+(3p+1)b\big)e^2 A_2(u) + be^2 A_3(u).$$

Moreover by Lemma 2.1 and the Young inequality, for any $\varepsilon' > 0$, we obtain

$$\begin{aligned} f'(1) &\le \frac{p-1+(3p-7)b}{2}E(u) - \frac{p-1+(3p-7)b}{4}\|\nabla u\|_2^2 \\ &\quad + Ce^2\mu^{\frac{3}{4}}\|\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}} + Ce^2\mu^{\frac{3}{4}}\|x\cdot\nabla\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}} \\ &\le \frac{p-1+(3p-7)b}{2}E(u) - \left(\frac{p-1+(3p-7)b}{2} - 2\varepsilon'\right)\|\nabla u\|_2^2 \\ &\quad + \frac{C}{\varepsilon'}e^{\frac{8}{3}}\mu\left(\|\rho\|_{\frac{6}{5}}^{\frac{4}{3}} + \|x\cdot\nabla\rho\|_{\frac{6}{5}}^{\frac{4}{3}}\right), \end{aligned}$$

where C is a positive constant independent of e, μ, ρ and u.

For any $\varepsilon \in \left(0, -\frac{1}{4}c_\infty(\mu)\right)$, we take $u_\varepsilon \in H^1(\mathbb{R}^3)$ with $\|u_\varepsilon\|_2^2 = \mu$ so that $E(u_\varepsilon) \le c(\mu) + \varepsilon$. Then from (3.4), one gets

$$E(u_\varepsilon) \le \frac{1}{2}c_\infty(\mu) + \varepsilon \le \frac{1}{4}c_\infty(\mu) < 0. \tag{3.9}$$

Putting $u = u_\varepsilon$ into the previous inequality, we have from (3.9) that

$$f'(1) \le \frac{p-1+(3p-7)b}{8}c_\infty(\mu) + Ce^{\frac{8}{3}}\mu\left(\|\rho\|_{\frac{6}{5}}^{\frac{4}{3}} + \|x\cdot\nabla\rho\|_{\frac{6}{5}}^{\frac{4}{3}}\right).$$

Since $p-1+(3p-7)b>0$, $c_\infty(\mu)$ is negative and independent of ρ, we find that

$$\|\rho\|_{\frac{6}{5}} + \|x\cdot\nabla\rho\|_{\frac{6}{5}} \le \rho_0 \ \Rightarrow\ f'(1) < 0 \quad \text{for some } \rho_0 = \rho_0(e,\mu) > 0,$$

which ends the proof of (3.6).

Now from (3.6) and $f(1)=0$, there exists $\delta>0$ independent of the choice of u_ε such that $f(\lambda)<0$ for $\lambda\in(1,1+\delta]$. Recalling that $E(u_\lambda) - \lambda^{1+2b}E(u) = \lambda^{1+2b}f(\lambda)$, one finds that

$$c(\lambda\mu) \le E\left((u_\varepsilon)_\lambda\right) < \lambda^{1+2b}E(u_\varepsilon) \le \lambda^{1+2b}(c(\mu)+\varepsilon).$$

Taking a limit $\varepsilon\to 0$, we get

$$c(\lambda\mu) \le \lambda^{1+2b}c(\mu).$$

Since $\lambda>1$, $b>0$ and from (3.4), it holds that

$$c(\lambda\mu) < \lambda c(\mu) \quad \text{for } \lambda\in(1,1+\delta].$$

Step 2: We prove that $c(\lambda\mu) < \lambda c(\mu)$ for any $\lambda>1$. For this purpose, we fix $\lambda>1$ and choose $t_0\in(0,\delta)$, $m\in\mathbb{N}$ so that

$$(1+t_0)^m \le \lambda < (1+t_0)^{m+1}.$$

This implies that

$$1 \le \frac{\lambda}{(1+t_0)^m} < 1+t_0 < 1+\delta.$$

By Step 1, we have

$$\begin{aligned} c(\lambda\mu) &= c\left((1+t_0)\,\frac{\lambda\mu}{1+t_0}\right) < (1+t_0)\,c\left(\frac{\lambda\mu}{1+t_0}\right) \\ &= (1+t_0)\,c\left((1+t_0)\,\frac{\lambda\mu}{(1+t_0)^2}\right) < (1+t_0)^2\,c\left(\frac{\lambda\mu}{(1+t_0)^2}\right) \\ &< (1+t_0)^m\,c\left(\frac{\lambda}{(1+t_0)^m}\cdot\mu\right) \\ &< (1+t_0)^m\cdot\frac{\lambda}{(1+t_0)^m}c(\mu) = \lambda c(\mu), \end{aligned}$$

from which we conclude. □

Remark 3.5. *In the case $1 < p \le 2$, (3.7) yields that $0 < b < 1$. On the other hand if $1 < p \le \frac{5}{3}$, one finds that $3 - p + (5-3p)b > 0$ and hence (3.8) cannot hold. When $\frac{5}{3} < p \le 2$, we also have*

$$3 - p + (5-3p)b > 4(2-p) \ge 0,$$

implying that (3.8) is impossible. Note that (3.8) was used to remove $A_1(u)$ which is independent of ρ. We also mention that the choice $b = 1$ corresponds to the scaling $u_\lambda(x) = \lambda^2 u(\lambda x)$.

In the case $1 < p \le 2$, another strategy is needed to establish the strict sub-additivity. We show that the strict sub-additivity for $1 < p \le 2$ holds if the coupling constant e is sufficiently small. For this purpose, we first establish the following asymptotic behavior.

Lemma 3.6. *Suppose that $1 < p < \frac{7}{3}$. Let $c_0(\mu)$ be the minimum energy defined by*

$$c_0(\mu) := \inf_{u\in B(\mu)} E_0(u), \quad E_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Writing $c(\mu) = c_e(\mu)$ to emphasize the dependence of e, it holds that

$$c_e(\mu) \to c_0(\mu) < 0 \quad \textit{for all } \mu > 0 \textit{ as } e \to +0.$$

Especially it follows that

$$c_e(\mu) \le \frac{1}{2}c_0(\mu) < 0 \quad \textit{provided that } e > 0 \textit{ is sufficiently small.}$$

Proof. By Lemma 2.1, we know that

$$\begin{aligned} c_e(\mu) &\le E(u) = E_0(u) + e^2 A_1(u) + 2e^2 A_2(u) \\ &\le E_0(u) + Ce^2\mu^{\frac{3}{2}}\|\nabla u\|_2 + Ce^2\mu^{\frac{3}{4}}\|\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}} \quad \text{for any } u \in B(\mu). \end{aligned}$$

Since $c_0(\mu)$ is negative and has a minimizer u_0 for all $\mu > 0$ (see [12]), it follows that

$$\limsup_{e\to+0} c_e(\mu) \le c_0(\mu) + \lim_{e\to+0}\left(Ce^2\mu^{\frac{3}{2}}\|\nabla u_0\|_2 + Ce^2\mu^{\frac{3}{4}}\|\rho\|_{\frac{6}{5}}\|\nabla u_0\|_2^{\frac{1}{2}}\right) = c_0(\mu).$$

On the other hand, let $\{u_j\} \subset B(\mu)$ be a minimizing sequence for $c_e(\mu)$. By Lemma 3.1, we find that $\|\nabla u_j\|_2$ is bounded and hence

$$c_0(\mu) - Ce^2 \le E_0(u_j) - Ce^2 \le E_0(u_j) + e^2 A_1(u_j) + 2e^2 A_2(u_j) = E(u_j).$$

Passing a limit $j \to \infty$, one obtains

$$c_0(\mu) - Ce^2 \le c_e(\mu)$$

and

$$c_0(\mu) \le \liminf_{e\to+0} c_e(\mu).$$

This competes the proof. □

Now we are ready to prove the strict sub-additivity when $1 < p \le 2$. In this case, we use the scaling $u_\lambda(x) = u(\lambda^{-\frac{1}{3}}x)$.

Lemma 3.7. *Suppose that $1 < p \le 2$. Then there exists $e_0 = e_0(\mu,\rho) > 0$ such that if $0 < e \le e_0$, the following properties hold.*

(i) *$c(\lambda\mu) \le \lambda c(\mu)$ for all $\lambda > 1$. Moreover if $c(\mu)$ is attained, the inequality is strict.*

(ii) *For $0 < \mu' < \mu$, suppose that either $c(\mu')$ or $c(\mu-\mu')$ is attained. Then (3.5) holds.*

Proof. (i) Under the same notation as the proof Lemma 3.4, let us consider

$$\begin{aligned} g(\lambda) &:= E(u_\lambda) - \lambda E(u) \\ &= \frac{\lambda^{1+2b}-\lambda}{2}\|\nabla u\|_2^2 - \frac{\lambda^{\frac{p+1+3(p-1)b}{2}}-\lambda}{p+1}\|u\|_{p+1}^{p+1} \\ &\quad + e^2(\lambda^{2+b}-\lambda)A_1(u) - \frac{\lambda^{1-2b}e^2}{2}\int_{\mathbb{R}^3} S_1(u)\rho(\lambda^{-b}x)\,dx - 2\lambda e^2 A_2(u). \end{aligned}$$

Then one has $g(1) = 0$ and

$$g'(1) = b\|\nabla u\|_2^2 - \frac{(p-1)(1+3b)}{2(p+1)}\|u\|_{p+1}^{p+1} + (1+b)e^2A_1(u) - 4be^2A_2(u) + be^2A_3(u).$$

Choosing $b = -\frac{1}{3}$, we obtain

$$g'(1) = -\frac{1}{3}\|\nabla u\|_2^2 + \frac{2}{3}e^2A_1(u) + \frac{4}{3}e^2A_2(u) - \frac{1}{3}e^2A_3(u).$$

By Lemma 2.1, we can estimate $g'(1)$ as

$$\begin{aligned} g'(1) &\le -\frac{1}{3}\|\nabla u\|_2^2 + Ce^2\mu^{\frac{3}{2}}\|\nabla u\|_2 + Ce^2\mu^{\frac{3}{4}}\|\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}} + Ce^2\mu^{\frac{3}{4}}\|x\cdot\nabla\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}} \\ &\le -\left(\frac{1}{3}-3\varepsilon'\right)\|\nabla u\|_2^2 + C\left(e^4\mu^3 + e^{\frac{8}{3}}\mu\|\rho\|_{\frac{6}{5}}^{\frac{4}{3}} + e^{\frac{8}{3}}\mu\|x\cdot\nabla\rho\|_{\frac{6}{5}}^{\frac{4}{3}}\right), \end{aligned} \tag{3.10}$$

where $\varepsilon' \in \left(0,\frac{1}{9}\right)$ and C is a positive constant independent of e, μ, ρ and u.

For any $\varepsilon \in \left(0, -\frac{1}{4}c_0(\mu)\right)$, let $u_\varepsilon \in H^1(\mathbb{R}^3)$ be such that $\|u_\varepsilon\|_2^2 = \mu$ and $E(u_\varepsilon) \le c(\mu) + \varepsilon$. We claim that there exists $\delta_0 > 0$ independent of ε such that $\|\nabla u_\varepsilon\|_2 \ge \delta_0$. Indeed suppose by contradiction that there is a sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$ and $\{u_j\} \subset H^1(\mathbb{R}^3)$ with $\|u_j\|_2^2 = \mu$ such that $E(u_j) \le c(\mu) + \varepsilon_j$ but $\|\nabla u_j\|_2 \to 0$ as $j \to \infty$. Then it follows that $\|u_j\|_q \to 0$ for any $q \in (2,6)$ and hence by Lemma 2.1,

$$E(u_j) = \frac{1}{2}\|\nabla u_j\|_2^2 - \frac{1}{p+1}\|u_j\|_{p+1}^{p+1} + e^2A_1(u_j) + 2e^2A_2(u_j) \to 0.$$

This contradicts Lemma 3.6. We also mention that δ_0 is independent of e.

Now from (3.10), we find that there exists $e_0 > 0$ independent of ε such that

$$g'(1) < 0 \quad \text{for } 0 < e \le e_0 \text{ and for } \varepsilon \in \left(0, -\frac{1}{4}c_0(\mu)\right).$$

Since $g(1) = 0$, it follows that $g(\lambda) < 0$ for $\lambda \in (1, 1+\delta]$, where $\delta > 0$ is independent of ε. By the definition of $g(\lambda)$, one has

$$c(\lambda\mu) \le E\big((u_\varepsilon)_\lambda\big) < \lambda E(u_\varepsilon) \le \lambda(c(\mu)+\varepsilon).$$

Passing to a limit $\varepsilon \to 0$, we obtain

$$c(\lambda\mu) \le \lambda c(\mu) \quad \text{for } \lambda \in (1, 1+\delta].$$

Arguing similarly as in the proof of Lemma 3.4, we deduce that $c(\lambda\mu) \le \lambda c(\mu)$ for all $\lambda > 1$.

If $c(\mu)$ admits a minimizer u_μ, we choose u_μ as a test function. By Lemma 3.6, we can show that $\|\nabla u_\mu\|_2 \ge \delta_0$ for some $\delta_0 > 0$ independent of e, from which we obtain

$$c(\lambda\mu) \le E\big((u_\mu)_\lambda\big) < \lambda E(\mu) = \lambda c(\mu).$$

This ends the proof of (i).

(ii) If $\frac{\mu}{2} < \mu' < \mu$, one has from (i) that

$$c(\mu) = c\left(\frac{\mu}{\mu'}\mu'\right) \le \frac{\mu}{\mu'}c(\mu') = c(\mu') + \frac{\mu-\mu'}{\mu'}c\left(\frac{\mu'}{\mu-\mu'}(\mu-\mu')\right) \le c(\mu') + c(\mu-\mu'),$$

and the inequality is strict if either $c(\mu')$ or $c(\mu-\mu')$ is achieved. In the case $0 < \mu' < \frac{\mu}{2}$, we also have

$$c(\mu) = c\left(\frac{\mu}{\mu-\mu'}(\mu-\mu')\right) \le \frac{\mu}{\mu-\mu'}c(\mu-\mu') = c(\mu-\mu') + \frac{\mu'}{\mu-\mu'}c\left(\frac{\mu-\mu'}{\mu'}\cdot\mu'\right) \le c(\mu')+c(\mu-\mu').$$

When $\mu' = \frac{\mu}{2}$, it follows that

$$c(\mu) = c\left(2\cdot\frac{\mu}{2}\right) \le 2c\left(\frac{\mu}{2}\right) = 2c(\mu') = c(\mu') + c(\mu-\mu'),$$

from which we conclude. □

The next lemma deals with the compactness of any minimizing sequence for (3.1).

Lemma 3.8. *Suppose that $1 < p < \frac{7}{3}$. Assume that $c(\mu) < 0$ and $c(\mu)$ satisfies (3.5). Let $\{u_j\} \subset H^1(\mathbb{R}^3,\mathbb{C})$ be a sequence satisfying $\|u_j\|_2^2 \to \mu$ and $E(u_j) \to c(\mu)$.*

Then there exist a subsequence of $\{u_j\}$ which is still denoted by the same, a sequence $\{y_j\} \subset \mathbb{R}^3$ and $u_\mu \in H^1(\mathbb{R}^3,\mathbb{C})$ such that $u_j(\cdot - y_j) \to u_\mu$ in $H^1(\mathbb{R}^3,\mathbb{C})$ and $E(u_\mu) = c(\mu)$.

At first sight, Lemma 3.8 seems to be rather standard, once we have established the strict sub-additivity. But it is not straightforward in the case $1 < p \le 2$, because we have assumed the attainability of $c(\mu)$ to guarantee (3.5) in the statement of Lemma 3.7.

Proof. First we observe by the proof of Lemma 3.1 that $\|u_j\|_{H^1}$ is bounded. Moreover by replacing u_j by $\frac{\sqrt{\mu}}{\|u_j\|_2^2}u_j$, we may assume that $\{u_j\}$ is a minimizing sequence of $c(\mu)$.

Now we apply the concentration compactness principle [23, Lemma I.1, p. 115] to the sequence $\rho_j(x) = |u_j(x)|^2$. It is well-known that the behavior of the sequence $(\rho_j)_{j\in\mathbb{N}}$ is governed by the three possibilities: Compactness, Vanishing and Dichotomy. Our goal is to show that Compactness occurs.

If Vanishing occurs, there exists a subsequence of $\{\rho_j\}$, still denoted by $\{\rho_j\}$, such that

$$\lim_{j\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}\rho_j(x)\,dx = 0 \quad \text{for all } R > 0.$$

Here $B_R(y)$ describes a ball of radius R with the center at $y \in \mathbb{R}^3$. Then by [24, Lemma I.1, P. 231], it follows that $u_j \to 0$ in $L^q(\mathbb{R}^3)$ for any $q \in (2,6)$. On the other hand since $\{u_j\}$ is a minimizing sequence for $c(\mu)$, one has by Lemma 2.1 that

$$\begin{aligned} c(\mu) + o(1) = E(u_j) &= \frac{1}{2}\|\nabla u_j\|_2^2 - \frac{1}{p+1}\|u_j\|_{p+1}^{p+1} + e^2A_1(u_j) + 2e^2A_2(u_j) \\ &\ge -\frac{1}{p+1}\|u_j\|_{p+1}^{p+1} - C\|\rho\|_{\frac{6}{5}}\|u_j\|_{\frac{12}{5}}^2. \end{aligned}$$

Passing a limit $j \to \infty$, we get $0 > c(\mu) \ge 0$. This is a contradiction, which rules out Vanishing.

Next we assume that Dichotomy occurs. Then by a standard argument (see [24, Section I.2] or [11, Proposition 1.7.6, P. 23]), there exist $\mu' \in (0,\mu)$ and $\{u_{j,1}\}$, $\{u_{j,2}\} \subset H^1(\mathbb{R}^3,\mathbb{C})$ such that

$$\|u_{j,1}\|_{L^2}^2 \to \mu', \quad \|u_{j,2}\|_{L^2}^2 \to \mu-\mu',$$

$$\mathrm{supp}(u_{j,1}) \cap \mathrm{supp}(u_{j,2}) = \emptyset, \ \ \delta_j := \mathrm{dist}\big(\mathrm{supp}(u_{j,1}),\, \mathrm{supp}(u_{j,2})\big) \to \infty, \tag{3.11}$$

$$\|u_j - u_{j,1} - u_{j,2}\|_q \to 0 \quad \text{for all } 2 \le q < 6, \tag{3.12}$$

$$\int_{\mathbb{R}^3} \big(|\nabla u_j|^2 - |\nabla u_{j,1}|^2 - |\nabla u_{j,2}|^2\big)\, dx \ge o(1). \tag{3.13}$$

Moreover replacing $u_{j,1}$, $u_{j,2}$ by $\frac{\sqrt{\mu'}}{\|u_{j,1}\|_2^2} u_{j,1}$, $\frac{\sqrt{\mu-\mu'}}{\|u_{j,2}\|_2^2} u_{j,2}$ respectively, we may assume that $\|u_{j,1}\|_2^2 = \mu'$, $\|u_{j,2}\|_2^2 = \mu-\mu'$ and (3.11)-(3.13) hold. Now from (3.11), one has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_{j,1}(x)|^2 |u_{j,2}(y)|^2}{|x-y|}\, dx\, dy = \int_{\mathrm{supp}(u_{j,2})}\int_{\mathrm{supp}(u_{j,1})} \frac{|u_{j,1}(x)|^2 |u_{j,2}(y)|^2}{|x-y|}\, dx\, dy$$
$$\le \frac{1}{\delta_j}\|u_{j,1}\|_2^2 \|u_{j,2}\|_2^2 \to 0 \quad \text{as } j \to \infty.$$

Using (3.12) and arguing as in the proof of Lemma 2.2 in [31], a direct computation yields that

$$A_1(u_j) - A_1(u_{j,1}) - A_1(u_{j,2}) = \int_{\mathbb{R}^3} S_1(u_j)|u_j|^2 - S_1(u_{j,1})|u_{j,1}|^2 - S_1(u_{j,2})|u_{j,2}|^2\, dx$$
$$= \int_{\mathbb{R}^3} \Big\{ \big(S_1(u_j)|u_j| + S_1(u_{j,1})|u_{j,1}| + S_1(u_{j,2})|u_{j,2}|\big)\big(|u_j| - |u_{j,1}| - |u_{j,2}|\big)$$
$$+ |u_j|\big(|u_{j,1}| + |u_{j,2}|\big)\big(S_1(u_j) - S_1(u_{j,1}) - S_1(u_{j,2})\big)$$
$$+ |u_{j,1}||u_{j,2}|\big(S_1(u_{j,1}) + S_1(u_{j,2})\big) + |u_j|\big(|u_{j,1}|S_1(u_{j,2}) + |u_{j,2}|S_1(u_{j,1})\big)\Big\}\, dx$$
$$\to 0 \quad (n \to \infty).$$

Similarly, one finds that

$$A_2(u_j) - A_2(u_{j,1}) - A_2(u_{j,2}) \to 0.$$

Thus from (3.12) and (3.13), we obtain

$$c(\mu) = E(u_j) + o(1)$$
$$\ge E(u_{j,1}) + E(u_{j,2}) + o(1)$$
$$\ge c(\mu') + c(\mu-\mu') + o(1).$$

Taking $\liminf_{j\to\infty}$ in both sides, one finds that

$$c(\mu) \ge c(\mu') + c(\mu-\mu'). \tag{3.14}$$

When $2 < p < \frac{7}{3}$, this readily leads a contradiction by Lemma 3.4.

Next we consider the case $1 < p \le 2$. By Lemma 3.2, one knows that $c(\mu') \le 0$ and $c(\mu-\mu') \le 0$. Since $c(\mu) < 0$, it follows from (3.14) that either $c(\mu')$ or $c(\mu-\mu')$ must be negative. We suppose that $c(\mu') < 0$ and distinguish into four steps to derive a contradiction.

Step 1: If $\liminf_{j\to\infty} E(u_{j,1}) > c(\mu')$, we have

$$c(\mu) \ge \liminf_{j\to\infty} E(u_{j,1}) + c(\mu-\mu') > c(\mu') + c(\mu-\mu'),$$

which contradicts Lemma 3.3.

Step 2: Passing to a subsequence, we may assume that $E(u_{j,1}) \to c(\mu')$ as $j \to \infty$. From (3.13) and the boundedness of $\|u_j\|_{H^1}$, there exists $u_1 \in H^1(\mathbb{R}^3,\mathbb{C})$ such that $u_{j,1} \rightharpoonup u_1$ in $H^1(\mathbb{R}^3)$. Since $\{u_{j,1}\}$ is a minimizing sequence for $c(\mu')$ and $c(\mu') < 0$, Vanishing does not occur and hence $u_1 \neq 0$. If $u_{j,1} \to u_1$ in $H^1(\mathbb{R}^3)$, it holds that $E(u_1) = c(\mu')$, yielding that $c(\mu')$ has a minimizer. Then from (3.14) and Lemma 3.7 (ii), we arrive at a contradiction.

Step 3: When $u_{j,1} \not\to u_1$ in $H^1(\mathbb{R}^3)$, we define $v_j := u_{j,1} - u_1$ so that $v_j \rightharpoonup 0$ but $v_j \not\to 0$ in $H^1(\mathbb{R}^3)$. Putting $\mu'' = \|u_1\|_2^2$, one finds that $0 < \mu'' < \mu'$ because $\|v_j\|_2^2 = \mu' - \mu'' + o(1)$ and $v_j \not\to 0$. Normalizing v_j again, we may assume that $\|v_j\|_2^2 = \mu' - \mu''$. Then by Lemma 2.2 and the Brezis-Lieb lemma, we obtain

$$c(\mu') = E(u_{j,1}) + o(1) = E(v_j) + E(u_1) + o(1) \geq c(\mu' - \mu'') + E(u_1) + o(1).$$

Passing to a limit $j \to \infty$, we deduce that

$$c(\mu') \geq c(\mu' - \mu'') + E(u_1). \tag{3.15}$$

If u_1 is a minimizer for $c(\mu'')$, then (3.15) and Lemma 3.7 (ii) lead to a contradiction.

Step 4: If u_1 is not a minimizer for $c(\mu'')$, we have from (3.15) and Lemma 3.3 that

$$c(\mu'') + c(\mu' - \mu'') \geq c(\mu') \geq c(\mu' - \mu'') + E(u_1) > c(\mu' - \mu'') + c(\mu'')$$

and deduce a contradiction.

In any cases, we arrive at a contradiction. If $c(\mu') = 0$ and $c(\mu - \mu') < 0$, we argue similarly for $u_{j,2}$. Thus Dichotomy does not occur.

The only remaining possibility is Compactness, that is, there exists $\{y_j\} \subset \mathbb{R}^3$ such that for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ satisfying

$$\int_{B_{R_\varepsilon}(y_j)} |u_j(x)|^2 \, dx \geq \mu - \varepsilon. \tag{3.16}$$

Since $\|u_j\|_{H^1}$ is bounded, there exists $u_\mu \in H^1(\mathbb{R}^3, \mathbb{C})$ such that up to a subsequence, $u_j(\cdot - y_j) \rightharpoonup u_\mu$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Then from (3.16), it follows that $u_j(\cdot - y_j) \to u_\mu$ in $L^q(\mathbb{R}^3, \mathbb{C})$ for any $2 \leq q < 6$. Thus by the weak lower semi-continuity of $\|\nabla \cdot\|_2$ and by Lemma 2.2, we get

$$c(\mu) = \liminf_{j \to \infty} E\big(u_j(\cdot - y_j)\big) \geq E(u_\mu) \geq c(\mu).$$

This implies that $E(u_\mu) = c(\mu)$ and $\|\nabla u_j(\cdot - y_j)\|_2 \to \|\nabla u_\mu\|_2$. Thus we obtain $u_j(\cdot - y_j) \to u_\mu$ in $H^1(\mathbb{R}^3, \mathbb{C})$ and hence the proof is complete. □

Remark 3.9. *By the relation between E and $\mathcal{E}$ in (2.3), the relative compactness of minimizing sequences for (1.5) also holds true, which will be applied to show the orbital stability later on.*

Now suppose that $u \in H^1(\mathbb{R}^3, \mathbb{C})$ is a minimizer of (3.1), that is, $E(u) = c(\mu)$ and $\|u\|_2^2 = \mu$. Up to a phase shift, we may assume that u is real-valued. Indeed by the well-known pointwise inequality $\big|\nabla |u|\big| \leq |\nabla u|$, one can see that $|u|$ is also a minimizer and hence u can be chosen to be real-valued.

Furthermore there exists a Lagrange multiplier $\omega = \omega(\mu) \in \mathbb{R}$ such that u satisfies (1.1) with some constant $\omega(\mu)$.

Lemma 3.10. *Suppose that $2 < p < \frac{7}{3}$ and choose $\mu > 0$ so that $c_\infty(\mu) < 0$. Then there exists $\rho_0 = \rho_0(e, \mu) > 0$ such that if $\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \leq \rho_0$, the Lagrange multiplier $\omega = \omega(\mu)$ is positive.*

Proof. Let u be a minimizer for $c(\mu)$. From (2.3) and (3.4), we first note that

$$\mathcal{E}(u) = E(u) + e^2 A_0 = c(\mu) + e^2 A_0 \leq \frac{1}{2} c_\infty(\mu) + e^2 A_0.$$

Then by Lemma 2.3, it follows that

$$\begin{aligned}\frac{(3p-5)\omega\mu}{2} &\geq -\frac{5p-7}{2}c_\infty(\mu)+2(p-2)\|\nabla u\|_2^2-(5p-7)e^2A_0\\&\quad-2e^2\int_{\mathbb{R}^3}S(u)\rho(x)\,dx-\frac{(3-p)e^2}{2}\int_{\mathbb{R}^3}S(u)x\cdot\nabla\rho(x)\,dx\\&=-\frac{5p-7}{2}c_\infty(\mu)+2(p-2)\|\nabla u\|_2^2\\&\quad+8e^2A_2(u)+5(3-p)e^2A_0-(3-p)e^2A_3(u)-(3-p)e^2A_4.\end{aligned}$$

Since $p>2$, one can choose $\varepsilon\in\big(0,2(p-2)\big)$. Lemma 2.1 and the Young inequality yield that

$$\begin{aligned}\frac{(3p-5)\omega\mu}{2} &\geq -\frac{5p-7}{2}c_\infty(\mu)+2(p-2)\|\nabla u\|_2^2-Ce^2\|\rho\|_{\frac{6}{5}}^2\\&\quad-Ce^2\mu^{\frac{3}{4}}\|\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}}-Ce^2\mu^{\frac{3}{4}}\|x\cdot\nabla\rho\|_{\frac{6}{5}}\|\nabla u\|_2^{\frac{1}{2}}-Ce^2\|\rho\|_{\frac{6}{5}}\|x\cdot\nabla\rho\|_{\frac{6}{5}}\\&\geq -\frac{5p-7}{2}c_\infty(\mu)+\big(2(p-2)-\varepsilon\big)\|\nabla u\|_2^2-Ce^2\|\rho\|_{\frac{6}{5}}^2\\&\quad-Ce^{\frac{8}{3}}\mu\|\rho\|_{\frac{6}{5}}^{\frac{4}{3}}-Ce^{\frac{8}{3}}\mu\|x\cdot\nabla\rho\|_{\frac{6}{5}}^{\frac{4}{3}}-Ce^2\|\rho\|_{\frac{6}{5}}\|x\cdot\nabla\rho\|_{\frac{6}{5}},\end{aligned}$$

where C is a positive constant independent of e, μ and ρ. Since $(5p-7)c_\infty(\mu)<0$, there exists $\rho_0>0$ such that if $\|\rho\|_{\frac{6}{5}}+\|x\cdot\nabla\rho\|_{\frac{6}{5}}\leq\rho_0$, it follows that

$$\frac{(3p-5)}{2}\omega\mu\geq-\frac{5p-7}{4}c_\infty(\mu)>0,$$

from which we conclude. □

Proof of Theorem 1.1. It is a direct consequence of Lemmas 3.1, 3.4, 3.7, 3.8 and 3.10. □

4. Global well-posedness of the Cauchy problem

In this section, we consider the solvability of the Cauchy problem:

$$\begin{cases}i\psi_t+\Delta\psi-e\phi\psi+|\psi|^{p-1}\psi=0 & \text{in } \mathbb{R}_+\times\mathbb{R}^3,\\-\Delta\phi=\frac{e}{2}\big(|\psi|^2-\rho(x)\big) & \text{in } \mathbb{R}_+\times\mathbb{R}^3,\\\psi(0,x)=\psi_0,\end{cases}\tag{4.1}$$

where $e>0$, $1<p<5$ and $\psi_0\in H^1(\mathbb{R}^3,\mathbb{C})$. For the doping profile ρ, we only assume that $\rho\in L^{\frac{6}{5}}(\mathbb{R}^3)$. Then we have the following result on the local well-posedness.

Proposition 4.1. *There exists $T=T(\|\psi_0\|_{H^1(\mathbb{R}^3)})>0$ such that (4.1) has a unique solution $\psi\in X$, where*

$$X=\left\{\psi\in C\left([0,T],H^1(\mathbb{R}^3)\right)\cap L^\infty\left((0,T),H^1(\mathbb{R}^3)\right)\right\}.$$

Furthermore, ψ satisfies the energy conservation law and the charge conservation law:

$$\mathcal{E}\big(\psi(t)\big)=\mathcal{E}(\psi_0)\quad\text{and}\quad\|\psi(t)\|_2=\|\psi_0\|_2\quad\text{for all } t\in(0,T).$$

Although it seems that Proposition 4.1 can be obtained in the framework of [11, Proposition 3.2.9, Theorem 4.3.1 and Corollary 4.3.3], we give the proof for the sake of completeness and reader's convenience. For this purpose, we first recall the following inequalities in $\mathbb{R}^3$.

Lemma 4.2 (Strichartz's estimates)**.** *Let $e^{it\Delta}$ be the linear propagator generated by the free Schrödinger equation $i\psi_t+\Delta\psi=0$. Assume that pairs (q_0,r_0), (q_1,r_1), (q_2,r_2) are admissible, namely, they fulfill the relation:*

$$\frac{2}{q_i}=3\left(\frac{1}{2}-\frac{1}{r_i}\right),\quad 2\leq r_i\leq 6\quad(i=0,1,2).$$

(i) *There exists* $C > 0$ *such that*

$$\|e^{it\Delta}f\|_{L^{q_0}(\mathbb{R},L^{r_0}(\mathbb{R}^3))} \le C\|f\|_{L^2(\mathbb{R}^3)} \quad \text{for all } f \in L^2(\mathbb{R}^3).$$

(ii) *Let* $I \subset \mathbb{R}$ *be an interval,* $J = \bar{I}$ *and* $t_0 \in J$*. Then there exists* $C > 0$ *independent of* I *such that*

$$\left\|\int_{t_0}^t e^{i(t-s)\Delta}f(s)\,ds\right\|_{L^{q_1}(I,L^{r_1}(\mathbb{R}^3))} \le C\|f\|_{L^{q_2'}(I,L^{r_2'}(\mathbb{R}^3))} \quad \text{for all } f \in L^{q_2'}(I, L^{r_2'}(\mathbb{R}^3)),$$

where q_2' *and* r_2' *are Hölder conjugate exponents of* q_2 *and* r_2 *respectively.*

Lemma 4.3 (Hardy-Littlewood-Sobolev inequality)**.** *Let* $0 < \alpha < 3$ *and* $1 < q < r < \infty$ *satisfy* $\frac{1}{q} - \frac{1}{r} = 1 - \frac{\alpha}{3}$*, and define* $I_\alpha f$ *by*

$$I_\alpha f(x) := \int_{\mathbb{R}^3} |x-y|^{-\alpha} f(y)\,dy.$$

Then there exists $C > 0$ *such that*

$$\|I_\alpha f\|_r \le C\|f\|_q \quad \text{for all } f \in L^q(\mathbb{R}^3).$$

For simplicity, we write $L^q(I, L^r(\mathbb{R}^3))$ as $L^q_t L^r_x(I \times \mathbb{R}^3)$ and use a notation $A \lesssim B$ if there is a positive constant independent of A and B such that $A \le CB$.

Proof of Proposition 4.1. First as we have seen in Subsection 2.1, the Poisson equation (2.1) has a unique solution

$$\psi = eS(\psi) \in D^{1,2}_x(\mathbb{R}^3) \quad \text{for any } \psi \in H^1_x(\mathbb{R}^3).$$

Let us consider the Duhamel formula associated with (4.1) and the solution map $\mathcal{H}(\psi)$ which is defined by

$$\mathcal{H}(\psi) := e^{it\Delta}\psi_0 + i\int_0^t e^{i(t-s)\Delta}|\psi|^{p-1}\psi\,ds - e^2 i\int_0^t e^{i(t-s)\Delta}S(\psi)\psi\,ds. \tag{4.2}$$

For $T > 0$ and $M = 3\|\psi_0\|_{H^1(\mathbb{R}^3)}$, we also define a complete metric space

$$\begin{aligned} X_T := \Big\{ &\psi \in L^\infty_t H^1_x\left((-T,T)\times\mathbb{R}^3\right) \cap L_t^{\frac{4(p+1)}{3(p-1)}} W_x^{1,p+1}\left((-T,T)\times\mathbb{R}^3\right) \ ; \\ &\|\psi\|_{L^\infty_t H^1_x} \le M,\ \|\psi\|_{L_t^{\frac{4(p+1)}{3(p-1)}} W_x^{1,p+1}} \le M \Big\} \end{aligned}$$

equipped with the distance

$$d(\psi_1,\psi_2) = \|\psi_1 - \psi_2\|_{L_t^{\frac{4(p+1)}{3(p-1)}} L_x^{p+1}} + \|\psi_1 - \psi_2\|_{L^\infty_t L^2_x}.$$

Note that $\left(\frac{4(p+1}{3(p-1)}, p+1\right)$ and $(\infty, 2)$ are both admissible. For simplicity, we write $q = \frac{4(p+1)}{3(p-1)}$. It suffices to show that $\mathcal{H}$ is a contraction mapping on X_T provided that T is sufficiently small.

First we establish that $\mathcal{H}$ maps X_T into itself. To this aim, we apply Lemma 4.2 to find that

$$\|\mathcal{H}(\psi)\|_{L^\infty_t H^1_x([0,T]\times\mathbb{R}^3)} \lesssim \|\psi_0\|_{H^1(\mathbb{R}^3)} + \|S(\psi)\psi\|_{L_t^{\frac{4}{3}} W_x^{1,\frac{3}{2}}([0,T]\times\mathbb{R}^3)} + \||\psi|^{p-1}\psi\|_{L_t^{q'} W_x^{1,\frac{p+1}{p}}([0,T]\times\mathbb{R}^3)},$$

$$\|\mathcal{H}(\psi)\|_{L^q_t W^{1,p+1}_x([0,T]\times\mathbb{R}^3)} \lesssim \|\psi_0\|_{H^1(\mathbb{R}^3)} + \|S(\psi)\psi\|_{L_t^{\frac{4}{3}} W_x^{1,\frac{3}{2}}([0,T]\times\mathbb{R}^3)} + \||\psi|^{p-1}\psi\|_{L_t^{q'} W_x^{1,\frac{p+1}{p}}([0,T]\times\mathbb{R}^3)}.$$

Here we mention that $\left(\frac{4}{3}\right)'=4$, $\left(\frac{3}{2}\right)'=3$ and the pair $(4,3)$ is admissible. By the Hölder inequality and the Sobolev inequality, it follows that

$$\begin{aligned}\||\psi|^{p-1}\psi\|_{L_t^{q'}L_x^{\frac{p+1}{p}}} &= \left(\int_0^T \|\psi\|_{L_x^{p+1}}^{q'}\|\psi\|_{L_x^{p+1}}^{(p-1)q'}\,dt\right)^{\frac{1}{q'}} \le \|\psi\|_{L_t^qL_x^{p+1}}\left(\int_0^T \|\psi\|_{L_x^{p+1}}^{\frac{2(p-1)(p+1)}{5-p}}\,dt\right)^{\frac{5-p}{2(p+1)}}\\ &\lesssim T^{\frac{5-p}{2(p+1)}}\|\psi\|_{L_t^qL_x^{p+1}}\|\psi\|_{L_t^\infty H_x^1}^{p-1} \le M^pT^{\frac{5-p}{2(p+1)}},\end{aligned} \tag{4.3}$$

where we used the fact $q'\left(1-\frac{q'}{q}\right)^{-1}=\frac{2(p+1)}{5-p}$. Similarly one has

$$\begin{aligned}\|\nabla(|\psi|^{p-1}\psi)\|_{L_t^{q'}L_x^{\frac{p+1}{p}}} &\lesssim \||\psi|^{p-1}\nabla\psi\|_{L_t^{q'}L_x^{\frac{p+1}{p}}}\\ &\lesssim T^{\frac{5-p}{2(p+1)}}\|\nabla\psi\|_{L_t^qL_x^{p+1}}\|\psi\|_{L_t^\infty H_x^1}^{p-1} \le M^pT^{\frac{5-p}{2(p+1)}}.\end{aligned} \tag{4.4}$$

Next by Lemma 2.1 and the fact $S(\psi)=S_1(\psi)+S_2$, we have

$$\begin{aligned}\|S(\psi)\psi\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}} &\le \left(\int_0^T \|S(\psi)\|_{L_x^6}^{\frac{4}{3}}\|\psi\|_{L_x^2}^{\frac{4}{3}}\,dt\right)^{\frac{3}{4}} \lesssim T^{\frac{3}{4}}\|S(\psi)\|_{L_t^\infty L_x^6}\|\psi\|_{L_t^\infty L_x^2}\\ &\lesssim T^{\frac{3}{4}}\left(\|\psi\|_{L_t^\infty H_x^1}^2+\|\rho\|_{\frac{6}{5}}\right)\|\psi\|_{L_t^\infty L_x^2} \le M(M^2+1)T^{\frac{3}{4}},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\|\nabla(S(\psi)\psi)\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}} &\le \|\psi\nabla S(\psi)\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}}+\|S(\psi)\nabla\psi\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}}\\ &\lesssim T^{\frac{3}{4}}\|\nabla S(\psi)\|_{L_t^\infty L_x^2}\|\psi\|_{L_t^\infty L_x^6}+T^{\frac{3}{4}}\|S(\psi)\|_{L_t^\infty L_x^6}\|\nabla\psi\|_{L_t^\infty L_x^2}\\ &\lesssim T^{\frac{3}{4}}(\|\psi^2_{L_t^\infty H_x^1}+\|\rho\|_{\frac{6}{5}})\|\psi\|_{L_t^\infty H_x^1} \le M(M^2+1)T^{\frac{3}{4}}.\end{aligned} \tag{4.6}$$

Thus from (4.3)-(4.6), one finds that

$$\begin{aligned}\|\mathcal{H}(\psi)\|_{L_t^\infty H_x^1([0,T]\times\mathbb{R}^3)} &\lesssim \frac{M}{3}+M^pT^{\frac{5-p}{2(p+1)}}+M(M^2+1)T^{\frac{3}{4}},\\ \|\mathcal{H}(\psi)\|_{L_t^qW_x^{1,p+1}([0,T]\times\mathbb{R}^3)} &\lesssim \frac{M}{3}+M^pT^{\frac{5-p}{2(p+1)}}+M(M^2+1)T^{\frac{3}{4}}.\end{aligned}$$

Choosing T sufficiently small, it follows that

$$\|\mathcal{H}(\psi)\|_{L_t^\infty H_x^1([0,T]\times\mathbb{R}^3)} \le M, \quad \|\mathcal{H}(\psi)\|_{L_t^qW_x^{1,p+1}([0,T]\times\mathbb{R}^3)} \le M$$

and hence $\mathcal{H}$ maps X_T into itself.

Finally we prove that $\mathcal{H}$ is a contraction mapping on X_T. Applying Lemma 4.2 again, we first observe that

$$d\big(\mathcal{H}(\psi_1),\mathcal{H}(\psi_2)\big) \lesssim \||\psi_1|^{p-1}\psi_1-|\psi_2|^{p-1}\psi_2\|_{L_t^{q'}L_x^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^3)}+\|S(\psi_1)\psi_1-S(\psi_2)\psi_2\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}([0,T]\times\mathbb{R}^3)}.$$

By the Hölder inequality, it follows that

$$\begin{aligned}\||\psi_1|^{p-1}\psi_1-|\psi_2|^{p-1}\psi_2\|_{L_t^{q'}L_x^{\frac{p+1}{p}}} &\lesssim \left(\int_0^T \|\psi_1-\psi_2\|_{L_x^{p+1}}^{q'}\left(\|\psi_1\|_{L_x^{p+1}}^{p-1}+\|\psi_2\|_{L_x^{p+1}}^{p-1}\right)^{q'}\,dt\right)^{\frac{1}{q'}}\\ &\lesssim T^{\frac{5-p}{2(p+1)}}\|\psi_1-\psi_2\|_{L_t^qL_x^{p+1}}\left(\|\psi\|_{L_t^\infty H_x^1}^{p-1}+\|\psi_2\|_{L_t^\infty H_x^1}^{p-1}\right)\\ &\le M^{p-1}T^{\frac{5-p}{2(p+1)}}d(\psi_1,\psi_2).\end{aligned} \tag{4.7}$$

Moreover one has

$$\begin{aligned}&\|S(\psi_1)\psi_1-S(\psi_2)\psi_2\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}} \le \|S(\psi_1)(\psi_1-\psi_2)\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}}+\|(S(\psi_1)-S_(\psi_2))\psi_2\|_{L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}}\\ &\lesssim T^{\frac{3}{4}}\|S(\psi_1)\|_{L_t^\infty L_x^6}\|\psi_1-\psi_2\|_{L_t^\infty L_x^2}+T^{\frac{3}{4}}\|S(\psi_1)-S(\psi_2)\|_{L_t^\infty L_x^6}\|\psi_2\|_{L_t^\infty L_x^2}.\end{aligned}$$

Here we recall that

$$S(\psi_1) - S(\psi_2) = S_1(\psi_1) + S_2 - S_1(\psi_2) - S_2 = \frac{1}{8\pi|x|} * (|\psi_1|^2 - |\psi_2|^2)$$
$$= \frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y|^{-1}(|\psi_1(y)|^2 - |\psi_2(y)|^2)\, dy.$$

Applying Lemma 4.3 with $\alpha = 1$, $q = \frac{6}{5}$ and $r = 6$, one gets

$$\|S(\psi_1) - S(\psi_2)\|_{L^6_x} \lesssim \||\psi_1|^2 - |\psi_2|^2\|_{L^{\frac{6}{5}}_x} \lesssim \|\psi_1 - \psi_2\|_{L^2_x}(\|\psi_1\|_{L^3_x} + \|\psi_2\|_{L^3_x}),$$

from which we find that

$$\|S(\psi_1) - S(\psi_2)\|_{L^{\frac{4}{3}}_t L^{\frac{3}{2}}_x} \lesssim M^2 T^{\frac{3}{4}} \|\psi_1 - \psi_2\|_{L^\infty_t L^2_x} \le M^2 T^{\frac{3}{4}} d(\psi_1 - \psi_2). \tag{4.8}$$

From (4.7) and (4.8), it follows that

$$d\big(\mathcal{H}(\psi_1), \mathcal{H}(\psi_2)\big) \le \frac{1}{2} d(\psi_1 - \psi_2),$$

provided that T is sufficiently small.

Since $\mathcal{H}$ is a contraction mapping, there exists a unique fixed point ψ, that is, ψ satisfies (4.2). By the Strichartz estimate, we can also find that $\mathcal{H}(\psi) \in C([0,T], H^1(\mathbb{R}^3))$ (see [11, Theorem 2.3.3]) and hence $\psi \in C([0,T], H^1(\mathbb{R}^3))$. Thus ψ is a unique solution on (4.1) in X. Once we could obtain the local well-posedness in H^1, a standard argument shows that the energy conservation law and the charge conservation law hold. □

Finally in this section, we prove the following global well-posedness result in the L^2-subcritical case, which is a direct consequence of the conservation laws.

Proposition 4.4. *Suppose that $1 < p < \frac{7}{3}$. Then the unique solution ψ obtained in Proposition 4.1 exists globally in $t > 0$.*

Proof. It is sufficient to show that there exists $C > 0$ independent on t such that $\|\nabla\psi(t)\|_{L^2_x} \le C$ for all t in the existence interval.

Now by the definition of the energy $\mathcal{E}$, the Gagliardo-Nirenberg inequality and Lemma 2.1, one has

$$\|\nabla\psi\|^2_{L^2_x} = 2\mathcal{E}(\psi) + \frac{1}{p+1}\|\psi\|^{p+1}_{L^{p+1}_x} - e^2 A_1(\psi) - 2e^2 A_2(\psi) - e^2 A_0$$
$$\lesssim 2\mathcal{E}(\psi) + \|\psi\|^{\frac{5-p}{2}}_{L^2_x} \|\nabla\psi\|^{\frac{3(p-1)}{2}}_{L^2_x} + \|\psi\|^3_{L^2_x}\|\nabla\psi\|_{L^2_x} + \|\rho\|_{\frac{6}{5}}\|\psi\|^{\frac{3}{2}}_{L^2_x}\|\nabla\psi\|^{\frac{1}{2}}_{L^2_x} + \|\rho\|^2_{\frac{6}{5}}.$$

Moreover using the Young inequality and the two conservations laws, for any $\varepsilon' \in (0,1)$, we deduce that

$$\|\nabla\psi\|^2_{L^2_x} \lesssim 2\mathcal{E}(\psi_0) + \varepsilon'\|\nabla\psi\|^2_{L^2_x} + \|\psi_0\|_2^{\frac{2(5-p)}{7-3p}} + \|\psi_0\|_2^6 + \|\rho\|_{\frac{6}{5}}\|\psi_0\|_2 + \|\rho\|^2_{\frac{6}{5}},$$

from which we conclude. □

5. Stability of standing waves

In this section, we prove the orbital stability of standing waves associated with minimizers for $C(\mu)$, which is a direct consequence of Lemma 3.8.

Proof of Theorem 1.2. The proof follows the argument of [12]. First we observe, since $\phi(t,\cdot) = \frac{e}{2}(-\Delta)^{-1}|\psi(t,\cdot)|^2$, that if

$$\sup_{t>0}\Big\{\inf_{u\in M(\mu)} \big\|\psi(t,\cdot) - u(\cdot)\big\|_{H^1}\Big\} < \varepsilon,$$

one also has

$$\sup_{t>0}\inf_{u\in M(\mu)}\left\|\phi(t,\cdot)-\frac{e}{2}(-\Delta)^{-1}|u(\cdot)|^2\right\|_{D^{1,2}}<C\varepsilon$$

for some $C>0$ independent of ε. Thus it is enough to prove that for every $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for any initial data $\psi_{(0)}$ satisfying

$$\inf_{u\in M(\mu)}\|\psi_{(0)}-u\|_{H^1}<\delta,$$

the corresponding solution ψ verifies

$$\sup_{t>0}\inf_{u\in M(\mu)}\|\psi(t,\cdot)-u(\cdot)\|_{H^1}<\varepsilon.$$

For that purpose, we assume by contradiction that there exist $\varepsilon_0>0$,

$$\left(\psi_{(0)j}\right)_{j\in\mathbb{N}}\subset H^1(\mathbb{R}^3,\mathbb{C})$$

and $\{t_j\}\subset\mathbb{R}$ such that

$$\inf_{u\in M(\mu)}\|\psi_{(0)j}-u\|_{H^1}\to 0 \text{ as } j\to\infty, \tag{5.1}$$

but the corresponding solution (ψ_j) satisfies

$$\inf_{u\in M(\mu)}\|\psi(t_j,\cdot)-u(\cdot)\|_{H^1}\geq\varepsilon_0. \tag{5.2}$$

For simplicity, we write $u_j=\psi_j(t_j,\cdot)$. Then by the charge conservation law and from (5.1), there exists $u_\mu\in B(\mu)$ such that

$$\|u_j\|_2^2=\|\psi_{(0)j}\|_2^2\to\|u_\mu\|_2^2=\mu. \tag{5.3}$$

By the energy conservation law, we also have

$$\begin{aligned}\mathcal{E}(u_j)&=\mathcal{E}(\psi_{(0)j})\\&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\psi_{(0)j}|^2\,dx+e^2A_1(\psi_{(0)j})+2e^2A_2(\psi_{(0)j})+e^2A_0-\frac{1}{p+1}\int_{\mathbb{R}^3}|\psi_{(0)j}|^{p+1}\,dx.\end{aligned}$$

From (5.1), one gets

$$\mathcal{E}(u_j)\to\mathcal{E}(u_\mu)=\mathcal{C}(\mu). \tag{5.4}$$

Then from (5.3), (5.4) and by Lemma 3.8, there exists $\{y_j\}\subset\mathbb{R}^3$ such that $u_j(\cdot)-u_\mu(\cdot+y_j)\to 0$ in $H^1(\mathbb{R}^3)$, in contradiction with (5.2). This ends the proof of Theorem 1.2. $\square$

6. The case ρ is a characteristic function

In this section, we consider the case where the doping profile ρ is a characteristic function, which appears frequently in physical literatures [21, 25, 27]. More precisely, let $\{\Omega_i\}_{i=1}^m\subset\mathbb{R}^3$ be disjoint bounded open sets with smooth boundary. For $\alpha_i\in\mathbb{R}$ $(i=1,\cdots,m)$, we assume that the doping profile ρ has the form:

$$\rho(x)=\sum_{i=1}^m\alpha_i\chi_{\Omega_i}(x),\quad \chi_{\Omega_i}(x)=\begin{cases}1 & (x\in\Omega_i),\\0 & (x\notin\Omega_i).\end{cases} \tag{6.1}$$

In this case, ρ cannot be weakly differentiable so that the assumption $\|x\cdot\nabla\rho\|_{\frac{6}{5}}\leq\rho_0$ does not make sense. Even so, we are able to obtain the existence of stable standing waves by imposing some smallness condition related with Ω_i.

To state our main result for this case, let us put $L:=\sup_{x\in\partial\Omega}|x|<\infty$. A key is the following *sharp boundary trace inequality* due to [1, Theorem 6.1], which we present here according to the form used in this paper.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace operator. Then it holds that*

$$\int_{\partial\Omega} |\gamma(u)|^2 \, dS \le \kappa_1(\Omega) \int_\Omega |u|^2 \, dx + \kappa_2(\Omega) \left(\int_\Omega |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \quad \textit{for any } u \in H^1(\Omega),$$

where $\kappa_1(\Omega) = \frac{|\partial\Omega|}{|\Omega|}$, $\kappa_2(\Omega) = \big\| |\nabla w| \big\|_{L^\infty(\partial\Omega)}$ and w is a unique solution of the torsion problem:

$$\Delta w = \kappa_1(\Omega) \textit{ in } \Omega, \quad \frac{\partial w}{\partial n} = 1 \textit{ on } \partial\Omega.$$

In relation to the size of ρ, we define

$$D(\Omega) := L|\Omega|^{\frac{1}{6}} |\partial\Omega|^{\frac{1}{2}} \Big(\kappa_1(\Omega)|\Omega|^{\frac{1}{3}} + \kappa_2(\Omega) \Big)^{\frac{1}{2}}.$$

Remark 6.2. *It is known that $\kappa_2(\Omega) \ge 1$; see [1]. Then by the isoperimetric inequality in $\mathbb{R}^3$:*

$$|\partial\Omega| \ge 3|\Omega|^{\frac{2}{3}} |B_1|^{\frac{1}{3}},$$

and the fact $|\Omega| \le |B_L(0)| = L^3|B_1|$, we find that

$$D(\Omega) \ge \left(\frac{|\Omega|}{|B_1|} \right)^{\frac{1}{3}} |\Omega|^{\frac{1}{6}} \cdot \sqrt{3} |\Omega|^{\frac{1}{3}} |B_1|^{\frac{1}{6}} \Big(3|B_1|^{\frac{1}{3}} + 1 \Big)^{\frac{1}{2}} = C|\Omega|^{\frac{5}{6}} = C\|\chi_\Omega\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}, \tag{6.2}$$

where C is a positive constant independent of Ω.

Under these preparations, we have the following result.

Theorem 6.3. *Under the assumption (6.1), we have the followings.*

(i) *Suppose that $2 < p < \frac{7}{3}$ and choose $\mu > 0$ so that $c_\infty(\mu) < 0$. Then there exists $\rho_0 = \rho_0(e, \mu) > 0$ such that if $\sum_{i=1}^m |\alpha_i| D(\Omega_i) \le \rho_0$, the minimization problem (1.5) admits a minimizer u_μ.*

Moreover the associated Lagrange multiplier $\omega = \omega(\mu)$ is positive.

(ii) *Suppose that $1 < p \le 2$. Then there exists $e_0 = e_0(\mu, \rho) > 0$ such that if $0 < e \le e_0$, the minimization problem (1.5) admits a minimizer u_μ.*

Similarly to Theorem 1.2, the orbital stability of $e^{i\omega t} u_\mu(x)$ also holds true.

We mention that the first part $x \cdot \nabla\rho(x)$ appeared was the definition of $A_3(u)$ and A_4 in Subsection 2.3. Under the assumption (6.1), we replace them by

$$A_3(u) := -\frac{1}{2} \sum_{i=1}^m \alpha_i \int_{\partial\Omega_i} S_1(u) x \cdot n_i \, dS_i,$$

$$A_4 := -\frac{1}{2} \sum_{i=1}^m \alpha_i \int_{\partial\Omega_i} S_2 x \cdot n_i \, dS_i,$$

where n_i is the unit outward normal on $\partial\Omega_i$. Indeed we have the following.

Lemma 6.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Then it holds that*

$$\frac{d}{d\tau} \int_{\mathbb{R}^3} S_1(u)(x) \chi_\Omega \left(\frac{x}{\tau} \right) dx \bigg|_{\tau=1} = \int_{\partial\Omega} S_1(u) x \cdot n \, dS.$$

Proof. The proof is based on the domain deformation as in [30]. In fact, one has

$$\begin{aligned}\frac{d}{d\tau}\int_{\mathbb{R}^3} S_1(u)(x)\chi_\Omega\left(\frac{x}{\tau}\right)\,dx\bigg|_{\tau=1} &= \frac{d}{d\tau}\int_{\tau\Omega} S_1(u)\,dx\bigg|_{\tau=1}\\ &= \lim_{h\to 0}\frac{1}{h}\left(\int_{(1+h)\Omega} S_1(u)\,dx - \int_\Omega S_1(u)\,dx\right)\\ &= \lim_{h\to 0}\frac{1}{h}\int_{(1+h)\Omega\setminus\Omega} S_1(u)\,dx\\ &= \lim_{h\to 0}\frac{1}{h}\int_1^{1+h}\left(\int_{\partial(t\Omega)} S_1(u)x\cdot n_{\partial(t\Omega)}\,dS\right)\,dt\\ &= \int_{\partial\Omega} S_1(u)x\cdot n\,dS,\end{aligned}$$

from which we conclude. □

By Lemma 6.4, the Pohozev identity can be reformulated as follows.

Lemma 6.5. *Under the assumption* (6.1)*, any nontrivial solution u of* (1.1) *satisfies the following identity.*

$$0 = \frac{1}{2}\|\nabla u\|_2^2 + \frac{3\omega}{2}\|u\|_2^2 - \frac{3}{p+1}\|u\|_{p+1}^{p+1} + 5e^2 A(u) + \frac{e^2}{2}\sum_{i=1}^m \alpha_i\int_{\partial\Omega_i} S(u)x\cdot n_i\,dS_i,$$

$$\begin{aligned}(5p-7)\mathcal{E}(u) &= 2(p-2)\|\nabla u\|_2^2 - \frac{(3p-5)\omega}{2}\|u\|_2^2\\ &\quad - e^2\sum_{i=1}^m \alpha_i\left(2\int_{\Omega_i} S(u)\,dx - \frac{3-p}{2}\int_{\partial\Omega_i} S(u)x\cdot n_i\,dS_i\right).\end{aligned}$$

Proof. As we have seen in Subsection 2.5, the Pohozaev identity can be obtained by considering $\frac{d}{d\lambda}I(u_\lambda)|_{\lambda=1}$ with $u_\lambda(x) = u\left(\frac{x}{\lambda}\right)$. Applying Lemma 6.4 with $\tau = \lambda^{-1}$, we then obtain

$$\begin{aligned}0 &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{3\omega}{2}\|u\|_2^2 - \frac{3}{p+1}\|u\|_{p+1}^{p+1} + 5e^2 A(u) + \frac{e^2}{2}\sum_{i=1}^m \alpha_i\int_{\partial\Omega_i} S(u)x\cdot n_i\,dS_i\\ &\quad - 5e^2 A_0 - \frac{e^2}{2}\sum_{i=1}^m \alpha_i\int_{\partial\Omega_i} S_2 x\cdot n_i\,dS_i.\end{aligned}$$

Now recalling that

$$A_0 = -\frac{1}{4}\int_{\mathbb{R}^3} S_2\rho(x)\,dx = -\frac{1}{4}\sum_{i=1}^m \alpha_i\int_{\Omega_i} S_2\,dx,$$

$$S_2(x) = -\frac{1}{8\pi}\int_{\mathbb{R}^3}\frac{\rho(y)}{|x-y|}\,dy = -\frac{1}{8\pi}\sum_{i=1}^m \alpha_i\int_{\Omega_i}\frac{1}{|x-y|}\,dy,$$

one finds that

$$\begin{aligned}\int_{\Omega_i} x\cdot\nabla S_2\,dx &= \frac{1}{8\pi}\sum_{i=1}^m \alpha_i\int_{\Omega_i}\int_{\Omega_i}\frac{x\cdot(x-y)}{|x-y|^3}\,dy\,dx\\ &= \frac{1}{8\pi}\sum_{i=1}^m \alpha_i\int_{\Omega_i}\int_{\Omega_i}\left(\frac{1}{|x-y|} + \frac{y\cdot(x-y)}{|x-y|^3}\right)\,dy\,dx\\ &= -\int_{\Omega_i} S_2\,dx - \int_{\Omega_i} x\cdot\nabla S_2\,dx,\end{aligned}$$

and hence

$$\int_{\Omega_i} x\cdot\nabla S_2\,dx = -\frac{1}{2}\int_{\Omega_i} S_2\,dx.$$

Here we used the Fubini theorem. Thus by the divergence theorem, we get

$$\begin{aligned}-5e^2A_0-\frac{e^2}{2}\sum_{i=1}^m\alpha_i\int_{\partial\Omega_i}S_2x\cdot n_i\,dS_i &= \frac{5e^2}{4}\sum_{i=1}^m\alpha_i\int_{\Omega_i}S_2\,dx-\frac{e^2}{2}\sum_{i=1}^m\alpha_i\int_{\Omega_i}\mathrm{div}(xS_2)\,dx\\ &= -\frac{e^2}{4}\sum_{i=1}^m\alpha_i\int_{\Omega_i}S_2\,dx-\frac{e^2}{2}\sum_{i=1}^m\alpha_i\int_{\Omega_i}x\cdot\nabla S_2\,dx=0.\end{aligned}$$

This completes the proof of the Pohozaev identity. Then similarly to Lemma 2.3, we can show the second identity. □

Next we establish estimates for A_0, A_2, A_3 and A_4.

Lemma 6.6. *For any $u\in H^1(\mathbb{R}^3,\mathbb{C})$, A_2 and A_3 satisfy the estimates:*

$$|A_2(u)|\le C\sum_{i=1}^m|\alpha_i||\Omega_i|^{\frac{5}{6}}\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}},$$

$$|A_3(u)|\le C\sum_{i=1}^m|\alpha_i|D(\Omega_i)\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}},$$

where $C>0$ is a constant independent of Ω_i.

Moreover the constants A_0 and A_4 can be estimated as follows:

$$|A_0|\le C\left(\sum_{i=1}^m|\alpha_i||\Omega_i|^{\frac{5}{6}}\right)^2,$$

$$|A_4|\le C\left(\sum_{i=1}^m|\alpha_i|D(\Omega_i)\right)^2.$$

Proof. First we observe that

$$|A_2(u)|\le\frac{1}{4}\sum_{i=1}^m|\alpha_i|\int_{\Omega_i}|S_1(u)|\,dx,$$

from which the estimate for A_2 can be obtained by the Hölder inequality and Lemma 2.1. Next by Proposition 6.1, the Hölder inequality and the Sobolev inequality, one has

$$\begin{aligned}|A_3(u)| &\le \frac{1}{2}\sum_{i=1}^m|\alpha_i|\int_{\partial\Omega_i}|S_1(u)||x|\,dS_i\\ &\le \frac{1}{2}\sum_{i=1}^m|\alpha_i|\left(\int_{\partial\Omega_i}|S_1(u)|^2\,dS_i\right)^{\frac{1}{2}}\left(\int_{\partial\Omega_i}|x|^2\,dS_i\right)^{\frac{1}{2}}\\ &\le \frac{1}{2}\sum_{i=1}^m|\alpha_i|L_i|\partial\Omega_i|^{\frac{1}{2}}\left(\kappa_1(\Omega_i)\|S_1(u)\|^2_{L^2(\Omega_i)}+\kappa_2(\Omega_i)\|S_1(u)\|_{L^2(\Omega_i)}\|\nabla S_1(u)\|_{L^2(\Omega_i)}\right)^{\frac{1}{2}}\\ &\le \frac{1}{2}\sum_{i=1}^m|\alpha_i|L_i|\partial\Omega_i|^{\frac{1}{2}}\left(\kappa_1(\Omega_i)|\Omega_i|^{\frac{2}{3}}\|S_1(u)\|^2_{L^6(\mathbb{R}^3)}+\kappa_2(\Omega_i)|\Omega_i|^{\frac{1}{3}}\|S_1(u)\|_{L^6(\mathbb{R}^3)}\|\nabla S_1(u)\|_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}\\ &\le C\sum_{i=1}^m|\alpha_i|L_i|\Omega_i|^{\frac{1}{6}}|\partial\Omega_i|^{\frac{1}{2}}\left(\kappa_1(\Omega_i)|\Omega_i|^{\frac{1}{3}}+\kappa_2(\Omega_i)\right)^{\frac{1}{2}}\|\nabla S_1(u)\|_{L^2(\mathbb{R}^3)}\\ &\le C\sum_{i=1}^m|\alpha_i|D(\Omega_i)\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}}.\end{aligned}$$

The estimates for A_0 and A_4 can be shown similarly. $\square$

Now we are ready to prove Theorem 6.3.

Proof of Theorem 6.3. We establish the existence of a minimizer for $2 < p < \frac{7}{3}$. For this purpose, it suffices to modify the proof of Lemma 3.4 only, because the other part of the existence proof does not rely on $x \cdot \nabla \rho(x)$. Under the same notation as the proof of Lemma 3.4 and applying Lemma 6.4, we arrive at

$$f'(1) = \frac{p-1+(3p-7)b}{2}E(u) - \frac{p-1+(3p-7)b}{4}\|\nabla u\|_2^2 + \frac{1}{2}\big(3-p+(5-3p)b\big)e^2A_1(u)$$
$$- \big(p-1+(3p+1)b\big)e^2A_2(u) + be^2A_3(u).$$

Furthermore we choose $b = 1$ and use (6.2). By Lemma 6.6, it follows that

$$\begin{aligned} f'(1) &\le 2(p-2)E(u) - (p-2)\|\nabla u\|_2^2 \\ &\quad + Ce^2\sum_{i=1}^m |\alpha_i||\Omega_i|^{\frac{5}{6}}\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}} + Ce^2\sum_{i=1}^m |\alpha_i|D(\Omega_i)\|u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}} \\ &\le 2(p-2)E(u) + Ce^{\frac{8}{3}}\mu\left(\sum_{i=1}^m |\alpha_i|D(\Omega_i)\right)^{\frac{4}{3}}. \end{aligned}$$

Then similarly to Lemma 3.4, there exists $\rho_0 > 0$ such that

$$\sum_{i=1}^m |\alpha_i|D(\Omega_i) \le \rho_0 \quad \Rightarrow \quad f'(1) < 0,$$

from which we can conclude.

We can also show the other parts of Theorem 6.3 by modifying the proof of Lemma 3.7 and Lemma 3.10 in a similar way. $\square$

7. Concluding remark and an open problem

In this paper, the nonlinear Schrödinger-Poisson system with a doping profile has been investigated. By establishing the existence of L^2-constraint minimizers when

$$c_\infty(\mu) < 0 \text{ and } \|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} \ll 1 \text{ for } 2 < p < \frac{7}{3}, \quad \text{or} \quad e \ll 1 \text{ for } 1 < p \le 2,$$

we are able to obtain stable standing waves. The presence of a doping profile ρ causes a difficulty of proving the strict sub-additivity which is the key of the existence and the stability of standing waves. This paper concludes by providing one open problem.

Problem: Non-existence of minimizers for large ρ ?

We have shown the existence of minimizers when ρ is small, but don't know what happens if ρ is large. In 1D case, it was shown in [16] that no minimizer exists in the case $\mu < \|\rho\|_{L^1(\mathbb{R})}$, which was referred to the *supercritical case*. (Note that [16] deals with the Schrödinger-Poisson system with $\Delta\phi = \frac{1}{2}(|u|^2 - \rho(x))$ so that the sign in the front of $A(u)$ in (1.4) is opposite.) Hence a natural question is whether a similar result holds for the 3D problem.

To explain the idea in [16], let us consider the problem in $\mathbb{R}^N$ and denote by $G(x)$ the fundamental solution of $-\Delta$ on $\mathbb{R}^N$. Under the assumption $\rho \ge 0$ and $\rho \in L^1(\mathbb{R}^N)$, the nonlocal term $A(u)$ can be expressed as follows.

$$\begin{aligned}
A(u) &= \frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(x-y)\big(|u(x)|^2-\rho(x)\big)\big(|u(y)|^2-\rho(y)\big)\,dx\,dy \\
&= \frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(x-y)|u(x)|^2(u(y)|^2\,dx\,dy - \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(x-y)|u(x)|^2\rho(y)\,dx\,dy + A_0 \\
&= \frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \big(G(x-y)-G(x)-G(y)\big)|u(x)|^2|u(y)|^2\,dx\,dy \\
&\quad + \frac{1}{4}\big(\|u\|^2_{L^2(\mathbb{R}^N)} - \|\rho\|_{L^1(\mathbb{R}^N)}\big)\int_{\mathbb{R}^3} G(x)|u(x)|^2\,dx \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big(G(x-y)-G(x)\big)|u(x)|^2\rho(y)\,dx\,dy + A_0.
\end{aligned}$$

Here we have used Fubini's theorem and wrote $A_0 = \frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(x-y)\rho(x)\rho(y)\,dx\,dy$. Thus one has $c(\mu) = -\infty$ if we could show that there is a family $\{u_\lambda\} \subset H^1(\mathbb{R}^N)$ satisfying

$$\|u_\lambda\|^2_{L^2(\mathbb{R}^N)} = \mu, \quad \|\nabla u_\lambda\|_{L^2(\mathbb{R}^N)} \le C, \tag{7.1}$$

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big(G(x-y)-G(x)-G(y)\big)|u_\lambda(x)|^2|u_\lambda(y)|^2\,dx\,dy\right| \le C,$$

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big(G(x-y)-G(x)\big)|u_\lambda(x)|^2\rho(y)\,dx\,dy\right| \le C,$$

$$\text{but} \int_{\mathbb{R}^3} G(x)|u_\lambda(x)|^2\,dx \to \infty \quad \text{as } \lambda \to \infty, \tag{7.2}$$

where $C > 0$ is a constant independent of λ.

When $N = 1$, it follows that $G(x) = \frac{1}{2}|x|$ and $\{u_\lambda\}$ can be constructed by considering a function whose mass is supported near the origin and infinity. (See [16, Example 4.1 and Remark 4.6].) However in the 3D case, which yields $G(x) = \frac{1}{4\pi|x|}$, it cannot happen that both (7.1) and (7.2) are fulfilled. Indeed by the Hardy inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} G(x)|u(x)|^2\,dx = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|u(x)|^2}{|x|}\,dx &\le \frac{1}{4\pi}\left(\int_{\mathbb{R}^3}\frac{|u(x)|^2}{|x|^2}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|u(x)|^2\,dx\right)^{\frac{1}{2}} \\
&\le C\|\nabla u\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Moreover as we have shown in Lemma 3.1, the energy functional on $\mathbb{R}^3$ is always bounded from below, regardless of the size of ρ. Therefore the only possibility for the non-existence is that the strict sub-additivity does not hold when $\mu < \|\rho\|_{L^1(\mathbb{R}^N)}$. Moreover as we have observed in this paper, in the 3D problem, it is rather natural to measure the size of ρ by $L^{\frac{6}{5}}$-norm, which makes us to conjecture that the non-existence result may be obtained if $\|\rho\|_{\frac{6}{5}}$ is large.

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