

GROUND STATE SOLUTIONS FOR SCHRÖDINGER-POISSON SYSTEM WITH A DOPING PROFILE

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ABSTRACT. This paper is devoted to the study of the nonlinear Schrödinger-Poisson system with a doping profile. We are interested in the existence of ground state solutions by considering the minimization problem on a Nehari-Pohozaev set. The presence of a doping profile causes several difficulties, especially in the proof of the uniqueness of a maximum point of a fibering map. A key ingredient is to establish the energy inequality inspired by [30]. We also establish the relation between ground state solutions and L^2 -constraint minimizers obtained in [16]. When the doping profile is a characteristic function supported on a bounded smooth domain, some geometric quantities related to the domain, such as the mean curvature, are responsible for the existence of ground state solutions.

1. INTRODUCTION

In this paper, we are concerned with the following nonlinear Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + \omega u + e\phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = \frac{e}{2}(|u|^2 - \rho(x)) \end{cases} \quad (1.1) \quad \boxed{\text{eq: 1.1}}$$

where $\omega > 0$, $e > 0$ and $1 < p < 5$. Equation (1.1) appears as a stationary problem for the time-dependent nonlinear Schrödinger-Poisson system:

$$\begin{cases} i\psi_t + \Delta \psi - e\phi \psi + |\psi|^{p-1}\psi = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ -\Delta \phi = \frac{e}{2}(|\psi|^2 - \rho(x)) & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \psi(0, x) = \psi_0. \end{cases} \quad (1.2) \quad \boxed{\text{eq: 1.2}}$$

Indeed when we look for a standing wave of the form: $\psi(t, x) = e^{i\omega t}u(x)$, we are led to the elliptic problem (1.1). In this paper, we are interested in the existence of ground state solutions of (1.1) and their relation with L^2 -constraint minimizers obtained in [16].

The Schrödinger-Poisson system appears in various fields of physics, such as quantum mechanics, black holes in gravitation and plasma physics. Especially, the Schrödinger-Poisson system plays an important role in the study of semi-conductor theory; see [22, 26, 28], and then the function $\rho(x)$ is referred as *impurities* or a *doping profile*. The doping profile comes from the difference of the number densities of positively charged donor ions and negatively charged acceptor ions, and the most typical examples are characteristic functions, step functions or Gaussian functions. Equation (1.1) also appears as a stationary problem for the Maxwell-Schrödinger system. We refer to [6, 13, 14] for the physical background and the stability result of standing waves for the Maxwell-Schrödinger system. In this case, the constant e describes the strength of the interaction between a particle and an external electromagnetic field.

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The nonlinear Schrödinger-Poisson system with $\rho \equiv 0$:

$$\begin{cases} -\Delta u + \omega u + e\phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3 \\ -\Delta \phi = \frac{e}{2}|u|^2 \end{cases} \quad (1.3) \quad \boxed{\text{eq:1.3}}$$

has been studied widely in the last two decades. Especially, the existence of non-trivial solutions and ground state solutions of (1.3) has been considered in detail. Furthermore, the existence of associated L^2 -constraint minimizers depending on p and the size of the mass and their stability have been investigated as well. We refer to e.g. [3, 4, 5, 11, 13, 21, 23, 25, 27, 31, 32, 35] and references therein. On the other hand, the nonlinear Schrödinger-Poisson system with a doping profile is less studied. In [17, 18], the corresponding 1D problem has been considered. Moreover, the linear Schrödinger-Poisson system (that is, the problem (1.1) without $|u|^{p-1}u$) with a doping profile in \mathbb{R}^3 has been studied in [7, 8]. In [16], the authors have investigated the existence of stable standing waves for (1.2) by considering the corresponding L^2 -minimization problem. As far as we know, there is no literature concerning with the existence of ground state solutions of (1.1), which is exactly the purpose of this paper.

To state our main results, let us give some notation. For $u \in H^1(\mathbb{R}^3, \mathbb{C})$, the energy functional associated with (1.1) is given by

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^3} |u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} dx + e^2 \mathcal{A}(u). \quad (1.4) \quad \boxed{\text{eq:1.4}}$$

Here we denote the nonlocal term by $S(u) = S_0(u) + S_1$ with

$$\begin{aligned} S_0(u)(x) &:= (-\Delta)^{-1} \left(\frac{|u(x)|^2}{2} \right) = \frac{1}{8\pi|x|} * |u(x)|^2, \\ S_1(x) &:= (-\Delta)^{-1} \left(\frac{-\rho(x)}{2} \right) = -\frac{1}{8\pi|x|} * \rho(x), \end{aligned}$$

and the functional corresponding to the nonlocal term by

$$\mathcal{A}(u) := \frac{1}{4} \int_{\mathbb{R}^3} S(u)(|u|^2 - \rho(x)) dx = \frac{1}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(|u(x)|^2 - \rho(x))(|u(y)|^2 - \rho(y))}{|x-y|} dx dy.$$

A function u_0 is said to be a *ground state solution* (GSS) of (1.1) if u_0 has a least energy among all nontrivial solutions of (1.1), namely u_0 satisfies

$$\mathcal{I}(u_0) = \inf\{\mathcal{I}(u) \mid u \in H^1(\mathbb{R}^3, \mathbb{C}), \mathcal{I}'(u) = 0\}.$$

For the doping profile ρ , we assume that

$$\rho(x) \in L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^q_{loc}(\mathbb{R}^3) \text{ for some } q > 3, \quad x \cdot \nabla \rho(x) \in L^{\frac{6}{5}}(\mathbb{R}^3), \quad x \cdot (D^2 \rho(x)x) \in L^{\frac{6}{5}}(\mathbb{R}^3), \quad (1.5) \quad \boxed{\text{eq:1.5}}$$

where $D^2 \rho$ is the Hessian matrix of ρ , and

$$\rho(x) \geq 0, \neq 0 \quad \text{for } x \in \mathbb{R}^3. \quad (1.6) \quad \boxed{\text{eq:1.6}}$$

Typical examples are the Gaussian function $\rho(x) = \varepsilon e^{-\alpha|x|^2}$ and $\rho(x) = \frac{\varepsilon}{1+\alpha|x|^r}$ for $r > \frac{5}{2}$.

In this setting, our first main result can be expressed as follows.

thm:1.1

Theorem 1.1. *Suppose that $2 < p < 5$ and assume (1.5)-(1.6). There exists ρ_0 independent of e, ρ such that if*

$$e^2 \left(\|\rho\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|x \cdot \nabla \rho\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|x \cdot (D^2 \rho x)\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \right) \leq \rho_0,$$

then (1.1) has a ground state solution u_0 . Moreover any ground state solution of (1.1) is real-valued up to phase shift.

When $1 < p < 2$, we are able to obtain the existence of a *radial* ground state solution of (1.1), which is a weak result; see Section 6 below.

Our second purpose of this paper is to investigate the relation between the ground state solution of (1.1) obtained in Theorem 1.1 and the L^2 -constraint minimizer in [16]. There has been significant progress in this relationship in recent years; see [15, 19, 21] for this direction. Based on the terminology in [15, 19], we call u_0 obtained in Theorem 1.1 an *action ground state solution* of (1.1).

To state our second main result, we define the energy functional $\mathcal{E} : H^1(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}$ by

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} dx + e^2 \mathcal{A}(u).$$

For $\mu > 0$, let us consider the minimization problem:

$$\mathcal{C}(\mu) = \inf_{u \in B(\mu)} \mathcal{E}(u), \quad (1.7) \quad \boxed{\text{eq:1.7}}$$

where $B(\mu) = \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \mid \|u\|_{L^2(\mathbb{R}^3)}^2 = \mu\}$. In this setting, the constant ω in (1.1) appears as a Lagrange multiplier. We also define the energy associated with (1.3):

$$E_\infty(u) := \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} dx + \frac{e^2}{4} \int_{\mathbb{R}^3} S_0(u) |u|^2 dx.$$

Indeed if we assume $\rho(x) \rightarrow 0$ as $|x| \rightarrow \infty$, (1.3) can be seen as a problem at infinity. We define the minimum energy associated with (1.3) by

$$c_{e,\infty}(\mu) = c_\infty(\mu) := \inf_{u \in B(\mu)} E_\infty(u). \quad (1.8) \quad \boxed{\text{eq:1.8}}$$

The existence of minimizers for $c_{e,\infty}(\mu)$ has been studied widely; we refer to [4, 5, 13, 11] and references therein. Especially in the case $2 < p < \frac{7}{3}$, $c_{e,\infty}(\mu)$ is attained if and only if $c_{e,\infty}(\mu) < 0$. Moreover $c_{e,\infty}(\mu) < 0$ when μ is large for fixed e or e is small for fixed μ .

In [16], it was shown that if $2 < p < \frac{7}{3}$ and $c_\infty(\mu) < 0$, there exists $\rho_0 = \rho_0(e, \mu) > 0$ such that if $\|\rho\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|x \cdot \nabla \rho\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \leq \rho_0$, the minimization problem (1.7) admits a minimizer u_μ and the associated Lagrange multiplier $\omega = \omega(\mu)$ is positive.

As in [15, 19], we call u_μ an *energy ground state solution* of (1.1). Our second main result of this paper is the following.

thm:1.2

Theorem 1.2. *Let $\mu > 0$ be given so that $c_\infty(\mu) < 0$ and suppose that $2 < p < \frac{7}{3}$. Under the assumptions in Theorem 1.1, the following properties hold.*

- (i) *The energy ground state solution u_μ is an action ground state solution of (1.1) with $\omega = \omega_\mu$.*
- (ii) *Let $\Omega(\mu)$ be the set of Lagrange multipliers associated with energy ground state solutions for B_μ , namely*

$$\Omega(\mu) := \{ \omega_\mu > 0 \mid \omega_\mu \text{ is the Lagrange multiplier associated with an energy ground state of (1.1) under the constraint } B_\mu \},$$

and w_μ be an action ground state solution of (1.1) with $\omega = \omega_\mu$. Then w_μ is an energy ground state solution of (1.1) under the constraint B_μ .

We emphasize that up to authors' knowledge, Theorem 1.2 is new even for the case $\rho \equiv 0$. (See also Remark 5.3 below.)

The assumption (1.5) rules out the case ρ is a characteristic function supported on a bounded smooth domain. Even in this case, we are still able to obtain the existence of ground state solutions and their relation under a smallness condition on some geometric quantities related to the domain; See Section 7.

Here we briefly explain our strategy and its difficulty. For the existence of a ground state solution of (1.1), we adapt a strategy in [3, 27], that is, we aim to obtain a ground state solution as a minimizer of the *Nehari-Pohozaev* constraint $0 = J(u) = N(u) - 2P(u)$, where $N(u) = 0$ is the Nehari identity and $P(u) = 0$ is the Pohozaev identity for (1.1). It is standard to show that for any $u \in H^1(\mathbb{R}^3, \mathbb{C})$, there exists $t = t_u > 0$ such that $J(u_t) = 0$ provided that $u_t(x) = t^2 u(tx)$. A crucial point is then to show that t_u is unique. In [27] for the case $\rho \equiv 0$, this was carried out by considering up to third derivatives of the fibering map, which requires an assumption involving third derivative of ρ in our case. To avoid this difficulty, we follow the idea in [30] to establish the following *energy inequality*:

$$I(u) - I(u_t) \geq \frac{1-t^3}{3} J(u) + \frac{(1-t)^2 \omega}{3} \|u\|_2^2 + \frac{\alpha}{6(p+1)} (1-t)^2 \|u\|_{p+1}^{p+1} - \beta(1-t)^2 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) \quad (1.9) \quad \boxed{\text{eq: 1.9}}$$

for any $u \in H^1(\mathbb{R}^3, \mathbb{C})$, $0 \leq t \leq T$ and some $T, \alpha, \beta > 0$. This key inequality is enough to prove the uniqueness of t_u for $u \in H^1(\mathbb{R}^3, \mathbb{C})$ with $J(u) \leq 0$. Moreover we can apply the energy inequality to show the relation between two types of ground state solutions of (1.1).

As we can easily imagine, if a doping profile ρ is considered, scaling arguments do not work straightforwardly because of the loss of spatial homogeneity. Furthermore the presence of the doping profile ρ satisfying (1.5) and (1.6) causes additional difficulties. Firstly we cannot expect that a remainder term in (1.9) is zero. Secondly we are not able to use the concentration function as in [3]. See Remark 3.8 and Remark 4.3 for details.

Compared to [16], we impose two additional assumptions on ρ . One is (1.6) which guarantees that the least energy for (1.1) is strictly less than that of (1.3), that is,

$$m < m_\infty.$$

This property appears naturally when we apply the concentration compactness principle. Second additional assumption is the $L^{\frac{6}{5}}$ -integrability of $x \cdot (D^2 \rho x)$. Unlike the existence of minimizers, we need detailed geometric properties of the functional \mathcal{I} . Especially we apply the second-order Taylor expansion to obtain (1.9), causing that some estimate for $x \cdot (D^2 \rho x)$ is required.

When ρ is a characteristic function, further consideration is required because ρ cannot be weakly differentiable. In this case, a key of the proof is the *sharp boundary trace inequality* which was developed in [2], and a variation of domain related with the *calculus of moving surfaces* due to Hadamard [20]. Then by imposing a smallness condition of some geometric quantities related to the support of ρ , we are able to obtain the existence of ground state solutions.

This paper is organized as follows. In Section 2, we introduce several properties of the energy functional and some lemmas which will be used later on. We investigate basic properties of the Nehari-Pohozaev set in Section 3. In Section 4, we prove the existence of a ground state solution of (1.1) by applying the concentration compactness principle and completes the proof of Theorem 1.1. Section 5 is devoted to the relation between action ground state solutions and energy ground state solutions. We also study the case $1 < p < 2$ in Section 6. In Section 7, we finish this paper by considering the case ρ is a characteristic function and present the existence of ground state solutions and their relation for this case.

Hereafter in this paper, unless otherwise specified, we write $\|u\|_{L^p(\mathbb{R}^3)} = \|u\|_p$. We also set $\|u\|^2 := \|\nabla u\|_2^2 + \|u\|_2^2$.

2. VARIATIONAL FORMULATION AND PRELIMINARIES

The aim of this section is to prepare several properties of the energy functional and present intermediate lemmas which will be used later on.

2.1. Decomposition of the energy.

In this subsection, we rewrite the energy functional \mathcal{I} in a more convenient way. We put

$$A(u) = \|\nabla u\|_2^2, \quad B(u) = \|u\|_2^2, \quad C(u) = \|u\|_{p+1}^{p+1},$$

and decompose \mathcal{I} in the following way:

$$\begin{aligned} \mathcal{I}(u) &= \frac{1}{2}A(u) + \frac{\omega}{2}B(u) - \frac{1}{p+1}C(u) \\ &\quad + \frac{e^2}{4} \int_{\mathbb{R}^3} S_0(u)|u|^2 dx + \frac{e^2}{4} \int_{\mathbb{R}^3} S_1|u|^2 dx - \frac{e^2}{4} \int_{\mathbb{R}^3} S_0(u)\rho(x) dx - \frac{e^2}{4} \int_{\mathbb{R}^3} S_1\rho(x) dx. \end{aligned}$$

Next we define three nonlocal terms:

$$D(u) = \frac{1}{4} \int_{\mathbb{R}^3} S_0(u)|u|^2 dx, \tag{2.1} \quad \boxed{\text{eq:2.1}}$$

$$E_1(u) = -\frac{1}{4} \int_{\mathbb{R}^3} S_0(u)\rho(x) dx = -\frac{1}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(y)|^2 \rho(x)}{|x-y|} dx dy = \frac{1}{4} \int_{\mathbb{R}^3} S_1|u|^2 dx, \tag{2.2} \quad \boxed{\text{eq:2.2}}$$

$$F = -\frac{1}{4} \int_{\mathbb{R}^3} S_1\rho(x) dx.$$

Note that F is independent of $u \in H^1(\mathbb{R}^3, \mathbb{C})$. Then we are able to write \mathcal{I} defined in (1.4) in the following form:

$$\mathcal{I}(u) = \frac{1}{2}A(u) + \frac{\omega}{2}B(u) - \frac{1}{p+1}C(u) + e^2D(u) + 2e^2E_1(u) + e^2F.$$

Now it is convenient to define $I(u) := \mathcal{I}(u) - e^2F$, which yields that

$$I(u) = \frac{1}{2}A(u) + \frac{\omega}{2}B(u) - \frac{1}{p+1}C(u) + e^2D(u) + 2e^2E_1(u). \tag{2.3} \quad \boxed{\text{eq:2.3}}$$

Since F is independent of u , we have only to consider the existence of nontrivial critical point of I . Recalling that

$$S_0(u)(x) = (-\Delta)^{-1} \left(\frac{|u(x)|^2}{2} \right) \geq 0,$$

we find that

$$A(u), B(u), C(u), D(u) \geq 0 \quad \text{for all } u \in H^1(\mathbb{R}^3, \mathbb{C}). \tag{2.4} \quad \boxed{\text{eq:2.4}}$$

For later use, let us also define

$$E_2(u) := \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)x \cdot \nabla \rho(x) dx = \frac{1}{2} \int_{\mathbb{R}^3} S_2|u|^2 dx, \tag{2.5} \quad \boxed{\text{eq:2.5}}$$

$$E_3(u) := \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)x \cdot (D^2 \rho(x)x) dx = \frac{1}{2} \int_{\mathbb{R}^3} S_3|u|^2 dx,$$

$$S_2(x) = (-\Delta)^{-1} \left(\frac{x \cdot \nabla \rho(x)}{2} \right) = \frac{1}{8\pi|x|} * (x \cdot \nabla \rho(x)),$$

$$S_3(x) = (-\Delta)^{-1} \left(\frac{x \cdot (D^2 \rho(x)x)}{2} \right) = \frac{1}{8\pi|x|} * (x \cdot (D^2 \rho(x)x)),$$

which is well-defined for $u \in H^1(\mathbb{R}^3, \mathbb{C})$ by (1.5).

2.2. Derivatives of nonlocal terms.

We next investigate Frechet derivatives of D , E_1 and E_2 which will be needed later.

lem:2.1

Lemma 2.1. (i) For $\varphi \in H^1(\mathbb{R}^3, \mathbb{C})$, it holds that

$$\begin{aligned} D'(u)\varphi &= \int_{\mathbb{R}^3} S_0(u)u\bar{\varphi} dx, \\ E_1'(u)\varphi &= \frac{1}{2} \int_{\mathbb{R}^3} S_1 u \bar{\varphi} dx, \\ E_2'(u)\varphi &= \int_{\mathbb{R}^3} S_2 u \bar{\varphi} dx. \end{aligned}$$

Especially we have

$$\begin{aligned} D'(u)u &= \int_{\mathbb{R}^3} S_0(u)|u|^2 dx = 4D(u), \\ E_1'(u)u &= \frac{1}{2} \int_{\mathbb{R}^3} S_1 |u|^2 dx = -\frac{1}{2} \int_{\mathbb{R}^3} S_0(u)\rho(x) dx = 2E_1(u), \\ E_2'(u)u &= \int_{\mathbb{R}^3} S_2 |u|^2 dx = \int_{\mathbb{R}^3} S_0(u)x \cdot \nabla \rho(x) dx = 2E_2(u). \end{aligned}$$

(ii) It follows that

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{B_R(0)} S_0(u)ux \cdot \nabla \bar{u} dx &= -\frac{5}{4} \int_{\mathbb{R}^3} S_0(u)|u|^2 dx = -5D(u), \\ \lim_{R \rightarrow \infty} \int_{B_R(0)} S_1(u)ux \cdot \nabla \bar{u} dx &= \frac{5}{2} \int_{\mathbb{R}^3} S_0(u)\rho(x) dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)x \cdot \nabla \rho(x) dx \\ &= -10E_1(u) + E_2(u), \\ \lim_{R \rightarrow \infty} \int_{B_R(0)} S_2(u)ux \cdot \nabla \bar{u} dx &= -3 \int_{\mathbb{R}^3} S_0(u)x \cdot \nabla \rho(x) dx - \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)x \cdot (D^2 \rho(x)x) dx \\ &= -6E_2(u) - E_3(u). \end{aligned}$$

Proof. (i) We observe from (2.1) that

$$\begin{aligned} D'(u)\varphi &= \frac{1}{4} \int_{\mathbb{R}^3} S_0'(u)\varphi |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)u\bar{\varphi} dx \\ &= \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(y)\overline{\varphi(y)}|u(x)|^2}{|x-y|} dy dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)u\bar{\varphi} dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)u\bar{\varphi} dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)u\bar{\varphi} dx = \int_{\mathbb{R}^3} S_0(u)u\bar{\varphi} dx. \end{aligned}$$

The derivatives of E_1 and E_2 can be derived readily by (2.2) and (2.5).

(ii) By the divergence theorem and the fact $S_0(u)|u|^2 \in L^1(\mathbb{R}^3)$, arguing as in [9, 12], one has

$$\begin{aligned} &\lim_{R \rightarrow \infty} \int_{B_R(0)} S_0(u)ux \cdot \nabla \bar{u} dx \\ &= \lim_{R \rightarrow \infty} \left\{ \frac{1}{2} \int_{\partial B_R(0)} S_0(u)|u|^2 x \cdot n dS - \frac{3}{2} \int_{B_R(0)} S_0(u)|u|^2 dx - \frac{1}{2} \int_{B_R(0)} |u|^2 x \cdot \nabla S_0(u) dx \right\} \\ &= -\frac{3}{2} \int_{\mathbb{R}^3} S_0(u)|u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_0(u) dx. \end{aligned}$$

Using

$$-\Delta S_0(u) = \frac{1}{2}|u|^2 \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla S_0(u)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)|u|^2 dx,$$

we find that

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_0(u) dx &= \int_{\mathbb{R}^3} \nabla S_0(u) \cdot \nabla (x \cdot \nabla S_0(u)) dx \\
&= \int_{\mathbb{R}^3} |\nabla S_0(u)|^2 dx + \int_{\mathbb{R}^3} \nabla S_0(u) \cdot (D^2 S_0(u)x) dx \\
&= \int_{\mathbb{R}^3} |\nabla S_0(u)|^2 dx - \int_{\mathbb{R}^3} S_0(u) \Delta S_0(u) dx - \int_{\mathbb{R}^3} S_0(u)x \cdot \nabla (\Delta S_0(u)) dx \\
&= \int_{\mathbb{R}^3} |\nabla S_0(u)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u) |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u)x \cdot \nabla |u|^2 dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^3} S_0(u) |u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_0(u) dx.
\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_0(u) dx = -\frac{1}{2} \int_{\mathbb{R}^3} S_0(u) |u|^2 dx$$

and hence

$$\lim_{R \rightarrow \infty} \int_{B_R(0)} S_0(u) u x \cdot \nabla \bar{u} dx = -\frac{5}{4} \int_{\mathbb{R}^3} S_0(u) |u|^2 dx.$$

Next since $\Delta S_1 = \frac{\rho}{2}$, it holds that

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_1 dx &= \int_{\mathbb{R}^3} \nabla S_0(u) \cdot \nabla (x \cdot \nabla S_1) dx \\
&= - \int_{\mathbb{R}^3} S_0(u) \rho(x) dx - \frac{1}{2} \int_{\mathbb{R}^3} S_0(u) x \cdot \nabla \rho(x) dx,
\end{aligned}$$

yielding that

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_{B_R(0)} S_1(u) u x \cdot \nabla \bar{u} dx &= -\frac{3}{2} \int_{\mathbb{R}^3} S_1 |u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_1 dx \\
&= \frac{5}{2} \int_{\mathbb{R}^3} S_0(u) \rho(x) dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u) x \cdot \nabla \rho(x) dx.
\end{aligned}$$

Similarly from $\Delta S_2 = -\frac{x \cdot \nabla \rho}{2}$, we obtain

$$\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_2 dx = \frac{3}{2} \int_{\mathbb{R}^3} S_0(u) x \cdot \nabla \rho(x) dx + \frac{1}{2} \int_{\mathbb{R}^3} S_0(u) x \cdot (D^2 \rho(x)x) dx,$$

and hence

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_{B_R(0)} S_2(u) u x \cdot \nabla \bar{u} dx &= \int_{\mathbb{R}^3} S_2 u x \cdot \nabla \bar{u} dx = -\frac{3}{2} \int_{\mathbb{R}^3} S_2 |u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 x \cdot \nabla S_2 dx \\
&= -3 \int_{\mathbb{R}^3} S_0(u) x \cdot \nabla \rho(x) dx - \frac{1}{2} \int_{\mathbb{R}^3} S_0(u) x \cdot (D^2 \rho(x)x) dx.
\end{aligned}$$

This completes the proof. \square

2.3. Estimates of nonlocal terms.

This subsection is devoted to present estimates for the nonlocal terms.

lem:2.2

Lemma 2.2. For any $u \in H^1(\mathbb{R}^3, \mathbb{C})$, S_0 , D , E_1 , E_2 and E_3 satisfy the estimates:

$$\begin{aligned} \|S_0(u)\|_6 &\leq C \|\nabla S_0(u)\|_2 \leq C \|u\|_{\frac{12}{5}}^2 \leq C \|u\|^2, \\ \|\nabla S_0(u)\|_2 &\leq C \|u\|_2^{\frac{5p-7}{3(p-1)}} \|u\|_{p+1}^{\frac{p+1}{3(p-1)}} \leq C (\|u\|_2^2 + \|u\|_{p+1}^2) \quad \text{if } 2 < p < 5, \\ D(u) &\leq C \|S_0(u)\|_6 \|u\|_{\frac{12}{5}}^2 \leq C \|u\|^4, \\ |E_1(u)| &\leq \frac{1}{4} \|S_0(u)\|_6 \|\rho\|_{\frac{6}{5}} \leq C \|\rho\|_{\frac{6}{5}} \|u\|^2, \\ |E_2(u)| &\leq \frac{1}{2} \|S_0(u)\|_6 \|x \cdot \nabla \rho\|_{\frac{6}{5}} \leq C \|x \cdot \nabla \rho\|_{\frac{6}{5}} \|u\|^2, \\ |E_3(u)| &\leq \frac{1}{2} \|S_0(u)\|_6 \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}} \leq C \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}} \|u\|^2. \end{aligned}$$

For the proof of the inequality on $S_0(u)$, we refer to [27]. The other estimates can be obtained by the Hölder inequality and the Sobolev inequality.

2.4. Convergence properties of nonlocal terms.

Next we introduce the Brezis-Lieb type convergence result for $D(u)$.

lem:2.3

Lemma 2.3. Assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Then it follows that

$$\lim_{n \rightarrow \infty} \{D(u_n - u) - D(u_n) + D(u)\} = 0.$$

Moreover if $u_n \rightarrow u$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$, we also have

$$S_0(u_n) \rightarrow S_0(u) \quad \text{in } L^6(\mathbb{R}^3) \quad \text{and} \quad \lim_{n \rightarrow \infty} D(u_n) = D(u).$$

Proof. The proof can be found in [35, Lemma 2.2]. \square

As for E_1 and E_2 , we have the following compactness property, which follows from the integrability of ρ and $x \cdot \nabla \rho$.

lem:2.4

Lemma 2.4. Assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Then it follows that

$$\lim_{n \rightarrow \infty} E_1(u_n) = E_1(u) \quad \text{and} \quad \lim_{n \rightarrow \infty} E_2(u_n) = E_2(u).$$

Proof. First we observe that since u_n converges weakly in $H^1(\mathbb{R}^3, \mathbb{C})$, there exists $C > 0$ such that $\|u_n\| \leq C$. Moreover passing to a subsequence, we may assume that $u_n \rightarrow u$ in $L_{loc}^r(\mathbb{R}^3)$ for $2 \leq r < 6$. Then similarly as Lemma 2.3, one can see that $S_0(u_n) \rightarrow S_0(u)$ in $L_{loc}^6(\mathbb{R}^3)$. Furthermore since $\rho \in L^{\frac{6}{5}}(\mathbb{R}^3)$, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\left(\int_{|x| \geq R_\varepsilon} |\rho(x)|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} < \varepsilon.$$

Now from (2.2) and the Hölder inequality, it follows that

$$\begin{aligned} |E_1(u_n) - E_1(u)| &\leq \frac{1}{4} \int_{\mathbb{R}^3} |S_0(u_n) - S_0(u)| |\rho| dx \\ &= \frac{1}{4} \int_{|x| \leq R_\varepsilon} |S_0(u_n) - S_0(u)| |\rho| dx + \frac{1}{4} \int_{|x| \geq R_\varepsilon} |S_0(u_n) - S_0(u)| |\rho| dx \\ &\leq \frac{1}{4} \left(\int_{|x| \leq R_\varepsilon} |S_0(u_n) - S_0(u)|^6 dx \right)^{\frac{1}{6}} \|\rho\|_{\frac{6}{5}} \\ &\quad + \frac{1}{4} (\|S_0(u_n)\|_6 + \|S_0(u)\|_6) \left(\int_{|x| \geq R_\varepsilon} |\rho(x)|^{\frac{6}{5}} dx \right)^{\frac{5}{6}}. \end{aligned}$$

Thus by Lemma 2.2 and Lemma 2.3, we obtain

$$\limsup_{n \rightarrow \infty} |E_1(u_n) - E_1(u)| \leq \frac{1}{4} (C + \|S_0(u)\|_6) \varepsilon.$$

Since ε is arbitrary, we deduce that $E_1(u_n) \rightarrow E_1(u)$ as $n \rightarrow \infty$. $E_2(u)$ can be treated in a similar manner. \square

2.5. Scaling properties.

In this subsection, we collect scaling properties of the nonlocal terms D and E_1 . For $a, b \in \mathbb{R}$ and $t > 0$, let us adapt the scaling $u_t(x) := t^a u(t^b x)$. Then we have

$$\begin{aligned} S_0(u_t)(x) &= \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{|u_t(y)|^2}{|x-y|} dy = \frac{t^{2a}}{8\pi} \int_{\mathbb{R}^3} \frac{|u(t^b y)|^2}{|x-y|} dy = \frac{t^{2a+b}}{8\pi} \int_{\mathbb{R}^3} \frac{|u(t^b y)|^2}{|t^b x - t^b y|} dy \\ &\stackrel{y=t^{-b}z}{=} \frac{t^{2a-2b}}{8\pi} \int_{\mathbb{R}^3} \frac{|u(z)|^2}{|t^b x - z|} dz. \end{aligned}$$

Thus one finds that

$$\begin{aligned} S_0(u_t)(x) &= t^{2a-2b} S_0(u)(t^b x), \\ D(u_t) &= t^{4a-5b} D(u), \end{aligned} \tag{2.6} \quad \boxed{\text{eq:2.6}}$$

$$\begin{aligned} E_1(u_t) &= -\frac{1}{4} \int_{\mathbb{R}^3} S_0(u_t)(x) \rho(x) dx = -\frac{t^{2a-2b}}{4} \int_{\mathbb{R}^3} S_0(u)(t^b x) \rho(x) dx \\ &= -\frac{t^{2a-5b}}{4} \int_{\mathbb{R}^3} S_0(u)(x) \rho(t^{-b} x) dx. \end{aligned} \tag{2.7} \quad \boxed{\text{eq:2.7}}$$

By the Hölder inequality, it follows that

$$|E_1(u_t)| \leq \frac{t^{2a-5b}}{4} \|S_0(u)\|_6 \|\rho(t^{-b} \cdot)\|_{\frac{6}{5}} \leq C t^{2a-\frac{5}{2}b} \|\rho\|_{\frac{6}{5}} \|u\|^2. \tag{2.8} \quad \boxed{\text{eq:2.8}}$$

2.6. Nehari and Pohozaev identities.

This subsection is devoted to establish the Nehari identity and the Pohozaev identity associated with (1.1).

lem:2.5

Lemma 2.5. *Let $u \in H^1(\mathbb{R}^3, \mathbb{C})$ be a weak solution of (1.1). Then u satisfies the Nehari identity $N(u) = 0$ and the Pohozaev identity $P(u) = 0$, where*

$$N(u) = A(u) + \omega B(u) - C(u) + 4e^2 D(u) + 4e^2 E_1(u), \tag{2.9} \quad \boxed{\text{eq:2.9}}$$

$$P(u) = \frac{1}{2} A(u) + \frac{3\omega}{2} B(u) - \frac{3}{p+1} C(u) + 5e^2 D(u) + 10e^2 E_1(u) - e^2 E_2(u). \tag{2.10} \quad \boxed{\text{eq:2.10}}$$

Proof. First by Lemma 2.1 (i), one has

$$\begin{aligned} 0 &= I'(u)u = \|\nabla u\|_2^2 + \omega \|u\|_2^2 - \|u\|_{p+1}^{p+1} + e^2 D'(u)u + 2e^2 E_1'(u)u \\ &= A(u) + \omega B(u) - C(u) + 4e^2 D(u) + 4e^2 E_1(u). \end{aligned}$$

Next by Lemma 2.1 (ii), formally it holds that

$$\begin{aligned} 0 &= I'(u)x \cdot \nabla u = -\frac{1}{2} \|\nabla u\|_2^2 - \frac{3\omega}{2} \|u\|_2^2 + \frac{3}{p+1} \|u\|_{p+1}^{p+1} + e^2 D'(u)x \cdot \nabla u + 2e^2 E_1'(u)x \cdot \nabla u \\ &= -\frac{1}{2} A(u) - \frac{3\omega}{2} B(u) + \frac{3}{p+1} C(u) - 5e^2 D(u) - 10e^2 E_1(u) + e^2 E_2(u). \end{aligned}$$

A rigorous proof can be done by establishing the $C_{loc}^{1,\alpha}$ -regularity of any weak solution of (1.1) for some $\alpha \in (0, 1)$. Note that since $\rho \in L_{loc}^q(\mathbb{R}^3)$ for some $q > 3$, it follows by the elliptic regularity theory that $S_1 \in W_{loc}^{2,q}(\mathbb{R}^3) \hookrightarrow C_{loc}^{1,\alpha}(\mathbb{R}^3)$. The smoothness of u can be shown similarly by applying the elliptic regularity theory. Then multiplying $x \cdot \nabla \bar{u}$ and $ex \cdot \nabla S(u)$ by (1.1)

respectively, integrating them over $B_R(0)$ and passing to a limit $R \rightarrow \infty$, we are able to prove (2.10) as in [9, 12]. \square

3. PROPERTIES OF NEHARI-POHOZAEV SET

In this section, we investigate fundamental properties of a Nehari-Pohozaev set, which we will use to obtain the existence of a ground state solution of (1.1).

Now let us define

$$J(u) := 2N(u) - P(u).$$

From (2.9) and (2.10), it holds that

$$J(u) = \frac{3}{2}A(u) + \frac{\omega}{2}B(u) - \frac{2p-1}{p+1}C(u) + 3e^2D(u) - 2e^2E_1(u) + e^2E_2(u). \quad (3.1) \quad \boxed{\text{eq:3.1}}$$

We also denote by \mathcal{M} the Nehari-Pohozaev set:

$$\mathcal{M} = \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\} \mid J(u) = 0\}.$$

By Lemma 2.5, one knows that any weak solution of (1.1) belongs to \mathcal{M} . We will show later that a minimizer $I|_{\mathcal{M}}$ is actually a ground state solution of (1.1).

For this purpose, we begin with the following lemma. Hereafter we let

$$u_t(x) := t^2u(tx) \quad \text{for } u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\} \text{ and } t > 0.$$

lem:3.1

Lemma 3.1. *Suppose that $2 < p < 5$. For any $u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$, there exists $t_u > 0$ such that $u_{t_u} \in \mathcal{M}$. Especially the set \mathcal{M} is non-empty.*

Proof. Taking $a = 2$ and $b = 1$, we have from (2.3), (2.6) and (2.7) that

$$f(t) := I(u_t) = \frac{t^3}{2}A(u) + \frac{\omega t}{2}B(u) - \frac{t^{2p-1}}{p+1}C(u) + e^2t^3D(u) - \frac{e^2t^{-1}}{2} \int_{\mathbb{R}^3} S_0(u)\rho(t^{-1}x) dx. \quad (3.2) \quad \boxed{\text{eq:3.2}}$$

By (2.4) and (2.8), it follows that

$$f(t) \geq \frac{\omega t}{2}B(u) - \frac{t^{2p-1}}{p+1}C(u) - Ce^2t^{\frac{3}{2}}\|\rho\|_{\frac{6}{5}}\|u\|^2,$$

from which we deduce that $f(t) > 0$ for small $t > 0$. On the other hand since $2p - 1 > 3$, one also finds that $f(t) \rightarrow -\infty$ as $t \rightarrow \infty$. This implies that there exists a maximum point $t = t_u > 0$ so that $f'(t_u) = 0$.

By a direct computation, $0 = t_u f'(t_u)$ shows that

$$\begin{aligned} 0 &= \frac{3t_u^3}{2}A(u) + \frac{\omega t_u}{2}B(u) - \frac{2p-1}{p+1}t_u^{2p-1}C(u) + 3e^2t_u^3D(u) \\ &\quad + \frac{e^2t_u^{-1}}{2} \int_{\mathbb{R}^3} S_0(u)\rho(t_u^{-1}x) dx + \frac{e^2t_u^{-1}}{2} \int_{\mathbb{R}^3} S_0(u)t_u^{-1}x \cdot \nabla \rho(t_u^{-1}x) dx. \end{aligned}$$

Using (2.6) and (2.7) again, we find that $0 = J(u_{t_u})$, which ends the proof. \square

lem:3.2

Lemma 3.2. *Suppose that $2 < p < 5$. There exist ρ_0, δ_0 and $\alpha_0 > 0$ independent of e, ρ such that if*

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \leq \rho_0,$$

then it holds that

$$J(u) \geq \alpha_0 \|u\|^2 \quad \text{for any } u \in H^1(\mathbb{R}^3, \mathbb{C}) \text{ with } 0 < \|u\| < \delta_0.$$

Furthermore if $J(u) \leq 0$, there exists $\delta_1 > 0$ independent of e, ρ such that

$$\|u\|_{p+1} \geq \delta_1. \quad (3.3) \quad \boxed{\text{eq:3.3}}$$

Proof. By the Sobolev inequality, Lemma 2.2 and from (3.1), one has

$$J(u) \geq \frac{\min\{3, \omega\}}{2} \|u\|^2 - C_1 \|u\|^{p+1} - C_2 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \|u\|^2,$$

where $C_1, C_2 > 0$ are independent of e and ρ . Thus if

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \leq \frac{1}{4C_2} \min\{3, \omega\},$$

it follows that

$$J(u) \geq \frac{\min\{3, \omega\}}{4} \|u\|^2 - C_1 \|u\|^{p+1}.$$

Putting $\delta_0 = \left(\frac{\min\{3, \omega\}}{8C_1} \right)^{\frac{1}{p-1}}$, we obtain

$$J(u) \geq \frac{\min\{3, \omega\}}{8} \|u\|^2 \quad \text{for } 0 < \|u\| < \delta_0.$$

Next suppose that $J(u) \leq 0$. Then by Lemma 2.2, one finds that

$$\begin{aligned} 0 \geq J(u) &\geq \frac{3}{2} \|\nabla u\|_2^2 + \frac{\omega}{2} \|u\|_2^2 - \frac{2p-1}{p+1} \|u\|_{p+1}^{p+1} - C_3 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) \\ &\geq \frac{\min\{3, \omega\}}{2} \|u\|^2 - \frac{2p-1}{p+1} \|u\|_{p+1}^{p+1} - C_3 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) (\|u\|^2 + \|u\|_{p+1}^2). \end{aligned}$$

Choosing

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \leq \frac{1}{4C_3} \min\{3, \omega\},$$

and using the Sobolev inequality, we obtain

$$\begin{aligned} 0 \geq J(u) &\geq \frac{\min\{3, \omega\}}{4} \|u\|^2 - \frac{2p-1}{p+1} \|u\|_{p+1}^{p+1} - C_3 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \|u\|_{p+1}^2 \\ &\geq \left\{ \frac{C_4 \min\{3, \omega\}}{4} - C_3 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \right\} \|u\|_{p+1}^2 - \frac{2p-1}{p+1} \|u\|_{p+1}^{p+1}. \end{aligned}$$

Thus if

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \leq \frac{C_4}{8C_3} \min\{3, \omega\},$$

it holds that

$$0 \geq J(u) \geq \left(\frac{C_4 \min\{3, \omega\}}{8} - \frac{2p-1}{p+1} \|u\|_{p+1}^{p-1} \right) \|u\|_{p+1}^2,$$

which implies that

$$\|u\|_{p+1} \geq \left(\frac{(p+1)C_4 \min\{3, \omega\}}{8(2p-1)} \right)^{\frac{1}{p-1}}.$$

This completes the proof. \square

Now by Lemma 3.2, we can define

$$\sigma := \inf_{u \in \mathcal{M}} I(u). \tag{3.4} \quad \boxed{\text{eq:3.4}}$$

lem:3.3

Lemma 3.3. *Suppose that $2 < p < 5$. There exist $\rho_0, \alpha_1 > 0$ independent of e, ρ such that if*

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \leq \rho_0,$$

then it holds that

$$I(u) \geq \alpha_1 \|u\|^2 \quad \text{for any } u \in \mathcal{M}.$$

Especially σ is positive.

Proof. From (2.3) and (3.1), one has

$$\begin{aligned} I(u) &= \frac{1}{2}A(u) + \frac{\omega}{2}B(u) - \frac{1}{p+1}C(u) + e^2D(u) + 2e^2E_1(u), \\ 0 &= \frac{3}{2}A(u) + \frac{\omega}{2}B(u) - \frac{2p-1}{p+1}C(u) + 3e^2D(u) - 2e^2E_1(u) + e^2E_2(u), \end{aligned}$$

from which we deduce that

$$(2p-1)I(u) = (p-2)A(u) + (p-1)\omega B(u) + 2(p-2)e^2D(u) + 4pe^2E_1(u) - e^2E_2(u).$$

By Lemma 2.2 and from (2.4), it follows that

$$(2p-1)I(u) \geq \min\{p-2, (p-1)\omega\}\|u\|^2 - C_1e^2\left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}}\right)\|u\|^2,$$

where $C_1 > 0$ is independent of e and ρ . Thus if

$$e^2\left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}}\right) \leq \frac{1}{2C_1}\min\{p-2, (p-1)\omega\},$$

then we have

$$(2p-1)I(u) \geq \frac{\min\{p-2, (p-1)\omega\}}{2C_1}\|u\|^2,$$

from which we conclude. \square

lem:3.4

Lemma 3.4. *Suppose that $2 < p < 5$. There exists $\rho_0 > 0$ such that if*

$$e^2\left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}}\right) \leq \rho_0,$$

then \mathcal{M} is a co-dimension one manifold.

Proof. By Lemma 3.2, it suffices to show that $J'(u) \neq 0$ if $u \in \mathcal{M}$. Suppose by contradiction that $u \in \mathcal{M}$ satisfies $J'(u) = 0$. Then by Lemma 2.1 (i), one finds that u is a weak solution of the problem:

$$-3\Delta u + \omega u - (2p-1)|u|^{p-1}u + 3e^2S_0(u)u - e^2S_1u + e^2S_2u = 0. \quad (3.5) \quad \text{eq:3.5}$$

Especially we have

$$0 = J'(u)u = 3A(u) + \omega B(u) - (2p-1)C(u) + 12e^2D(u) - 4e^2E_1(u) + 2e^2E_2(u). \quad (3.6) \quad \text{eq:3.6}$$

Furthermore multiplying $x \cdot \nabla u$ by (3.5), using Lemma 2.1 (ii) and arguing as in Lemma 2.5, one also finds that

$$\begin{aligned} 0 = Q(u) &:= \frac{3}{2}A(u) + \frac{3\omega}{2}B(u) - \frac{3(2p-1)}{p+1}C(u) + 15e^2D(u) \\ &\quad - 10e^2E_1(u) + 7e^2E_2(u) + e^2E_3(u). \end{aligned} \quad (3.7) \quad \text{eq:3.7}$$

Now from (2.3), (3.1), (3.6) and (3.7), we obtain the following system of equations:

$$\begin{pmatrix} I - 2e^2E_1 \\ 2e^2E_1 - e^2E_2 \\ 4e^2E_1 - 2e^2E_2 \\ 10e^2E_1 - 7e^2E_2 - e^2E_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\omega}{2} & -\frac{1}{p+1} & e^2 \\ \frac{3}{2} & \frac{\omega}{2} & -\frac{2p-1}{p+1} & 3e^2 \\ 3 & \omega & -(2p-1) & 12e^2 \\ \frac{3}{2} & \frac{3\omega}{2} & -\frac{3(2p-1)}{p+1} & 15e^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}. \quad (3.8) \quad \text{eq:3.8}$$

Solving (3.8) with the aid of Mathematica [34], it follows that

$$D(u) = \frac{2p-1}{24(p-2)}\{16e^2E_1(u) - 7e^2E_2(u) - e^2E_3(u) - 3I(u)\}.$$

By Lemma 2.2 and Lemma 3.3, we find that

$$0 \leq D(u) \leq -\frac{2p-1}{8(p-2)}\alpha_1\|u\|^2 + Ce^2\left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}}\right)\|u\|^2,$$

from which we arrive at a contradiction provided that $e^2(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}})$ is sufficiently small. \square

By Lemma 3.4, we are able to apply the method of Lagrange multiplier, which yields that if u is a nontrivial critical point of $I|_{\mathcal{M}}$, there exists $\mu \in \mathbb{R}$ such that

$$I'(u) - \mu J'(u) = 0. \quad (3.9) \quad \text{eq:3.9}$$

lem:3.5 **Lemma 3.5.** *Suppose that $2 < p < 5$. There exists $\rho_0 > 0$ such that if*

$$e^2(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}}) \leq \rho_0,$$

then it holds that $\mu = 0$, that is, the set \mathcal{M} is a natural constraint for the functional I .

Proof. First we have from (2.9), (3.6) and (3.9) that

$$\begin{aligned} 0 &= I'(u)u - \mu J'(u)u = N(u) - \mu J'(u)u \\ &= (1 - 3\mu)A(u) + (1 - \mu)\omega B(u) - (1 - (2p - 1)\mu)C(u) + (4 - 12\mu)e^2 D(u) \\ &\quad + (4 + 4\mu)e^2 E_1(u) - 2\mu e^2 E_2(u). \end{aligned} \quad (3.10) \quad \text{eq:3.10}$$

Furthermore from (2.10) and (3.7) and arguing as in Lemma 2.5, one also finds that

$$\begin{aligned} 0 &= P(u) - \mu Q(u) \\ &= \frac{1 - 3\mu}{2}A(u) + \frac{3(1 - \mu)\omega}{2}B(u) - \frac{3 - 3\mu(2p - 1)}{p + 1}C(u) + (5 - 15\mu)e^2 D(u) \\ &\quad + (10 + 10\mu)e^2 E_1(u) - (1 + 7\mu)e^2 E_2(u) - \mu e^2 E_3(u). \end{aligned} \quad (3.11) \quad \text{eq:3.11}$$

Combining (2.3), (3.1), (3.10) and (3.11), we arrive at the following system of equations:

$$\begin{pmatrix} I - 2e^2 E_1 \\ 2e^2 E_1 - e^2 E_2 \\ (4 + 4\mu)e^2 E_1 - 2\mu e^2 E_2 \\ (10 + 10\mu)e^2 E_1 - (1 + 7\mu)e^2 E_2 - e^2 \mu E_3 \end{pmatrix} = \Lambda \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}, \quad (3.12) \quad \text{eq:3.12}$$

where

$$\Lambda = \begin{pmatrix} \frac{1}{2} & \frac{\omega}{2} & -\frac{1}{p+1} & e^2 \\ \frac{3}{2} & \frac{\omega}{2} & -\frac{2p-1}{p+1} & 3e^2 \\ 3\mu - 1 & (\mu - 1)\omega & 1 - (2p - 1)\mu & (12\mu - 4)e^2 \\ \frac{3\mu-1}{2} & \frac{3(\mu-1)\omega}{2} & -\frac{3(2p-1)\mu-3}{\beta+1} & (15\mu - 5)e^2 \end{pmatrix}.$$

By a direct calculation with the aid of Mathematica, it follows that

$$\det \Lambda = \frac{4(p-2)(p-1)e^2 \omega}{p+1} \mu(3\mu - 1).$$

If $\mu = \frac{1}{3}$, we can perform row operations on the augmented matrix to obtain

$$\begin{aligned} &\left(\Lambda \left| \begin{array}{l} I - 2e^2 E_1 \\ 2e^2 E_1 - e^2 E_2 \\ (4 + 4\mu)e^2 E_1 - 2\mu e^2 E_2 \\ (10 + 10\mu)e^2 E_1 - (1 + 7\mu)e^2 E_2 - e^2 \mu E_3 \end{array} \right. \right) \\ &\rightarrow \left(\begin{array}{cccc|cccc} \frac{1}{2} & \frac{\omega}{2} & -\frac{1}{p+1} & e^2 & & & & \\ 0 & -\omega & \frac{4-2p}{p+1} & 0 & & & & \\ 0 & -\omega & 2-p & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \end{array} \left| \begin{array}{l} I - 2e^2 E_1 \\ -3I + 8e^2 E_1 - e^2 E_2 \\ 8e^2 E_1 - e^2 E_2 \\ 3I + \frac{16}{3}e^2 E_1 - \frac{7}{3}e^2 E_2 - \frac{1}{3}e^2 E_3 \end{array} \right. \right). \end{aligned}$$

However by Lemma 2.2 and Lemma 3.3, it holds that

$$3I(u) + \frac{16}{3}e^2E_1(u) - \frac{7}{3}e^2E_2(u) - \frac{1}{3}e^2E_3(u) > 0 \quad \text{for any } u \in \mathcal{M}$$

if $e^2(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho(x)x)\|_{\frac{6}{5}}) \ll 1$, which is a contradiction.

On the other hand when $\det \Lambda \neq 0$, one can solve (3.12) to obtain

$$0 \leq C(u) = \frac{p+1}{4(p-1)(p-2)} \{16e^2E_1(u) - 7e^2E_2(u) - e^2E_3(u) - 3I(u)\}.$$

This leads a contradiction provided that $e^2(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}})$ is sufficiently small, from which we conclude that $\mu = 0$. \square

Now let us define the ground state energy level for (1.1) by

$$m := \inf_{u \in \mathcal{S}} I(u), \quad \mathcal{S} = \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\} \mid I'(u) = 0\}.$$

By Lemma 3.5, we are able to prove the following.

prop:3.6

Proposition 3.6. *Suppose that $2 < p < 5$ and assume that*

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right) \leq \rho_0$$

for sufficiently small $\rho_0 > 0$.

If σ defined in (3.4) is attained by some $u_0 \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$, then u_0 is a ground state solution of (1.1), namely u_0 satisfies

$$\sigma = I(u_0) = m.$$

Proof. Let $\tilde{u} \in H^1(\mathbb{R}^3, \mathbb{C})$ be a nontrivial critical point of I . Then by Lemma 2.5, it follows that $\tilde{u} \in \mathcal{M}$ and hence

$$\sigma = \inf_{u \in \mathcal{M}} I(u) \leq I(\tilde{u}).$$

Taking infimum over \mathcal{S} , we find that $\sigma \leq m$.

On the other hand by Lemma 3.5, if σ is achieved by $u_0 \in \mathcal{M}$, one has $I'(u_0) = 0$. This implies that $u_0 \in \mathcal{S}$ and

$$m \leq \inf_{u \in \mathcal{S}} I(u) \leq I(u_0) = \sigma,$$

from which we conclude that $m = \sigma$. \square

By Proposition 3.6, it suffices to investigate the attainability of σ . To this aim, we next establish the following energy estimate, which is a key tool in this paper and will be also used to prove the equivalence between two types of ground state solutions in Section 5.

lem:3.7

Lemma 3.7. *Suppose that $2 < p < 5$ and take $T \geq 4$ so that $T^{2p-4} \geq 3$. There exist $\alpha = \alpha(T) > 0$ and $\beta = \beta(T) > 0$ independent of e, ρ, t such that the following estimates hold: For any $u \in H^1(\mathbb{R}^3, \mathbb{C})$,*

$$\begin{aligned} I(u) - I(u_t) - \frac{1-t^3}{3}J(u) & \tag{3.13} \tag{3.7-1} \\ & \geq \frac{(1-t)^2\omega}{3}\|u\|_2^2 + \frac{\alpha}{6(p+1)}(1-t)^2\|u\|_{p+1}^{p+1} \\ & \quad - \beta(1-t)^2e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) \quad \text{for } 0 \leq t \leq T, \end{aligned}$$

$$\begin{aligned}
I(u) - I(u_t) - \frac{1-t^3}{3}J(u) & \quad (3.14) \quad \boxed{3.7-2} \\
& \geq \frac{(1-t)^2\omega}{6}\|u\|_2^2 + \frac{t^3\omega}{3T}\|u\|_2^2 + \frac{\alpha}{12(p+1)}(1-t)^2\|u\|_{p+1}^{p+1} + \frac{\alpha}{6(p+1)}t^3\|u\|_{p+1}^{p+1} \\
& \quad - \beta t^3 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) \quad \text{for } t \geq T.
\end{aligned}$$

Assume further that $\|u\|_{p+1} \geq \delta$ for some $\delta > 0$ independent of e, ρ . There exists $\rho_0 > 0$ independent of e, ρ, t such that if

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}} \right) \leq \rho_0,$$

then the following estimate holds:

$$I(u) - I(u_t) - \frac{1-t^3}{3}J(u) \geq \frac{(1-t)^2\omega}{6}\|u\|_2^2 + \frac{\alpha\delta^{p-1}}{12(p+1)}(1-t)^2\|u\|_{p+1}^2 \quad \text{for all } t > 0. \quad (3.15) \quad \boxed{\text{eq:3.13}}$$

Proof. The proof consists of four steps.

Step 1 (Transformation of $I(u) - I(u_t)$): First we observe from (3.2) that

$$\begin{aligned}
I(u) - I(u_t) &= \frac{1-t^3}{2}A(u) + \frac{(1-t)\omega}{2}B(u) - \frac{1-t^{2p-1}}{p+1}C(u) + (1-t^3)e^2D(u) \\
& \quad + 2e^2E_1(u) + \frac{e^2t^{-1}}{2} \int_{\mathbb{R}^3} S_0(u)\rho(t^{-1}x) dx.
\end{aligned}$$

Transforming (3.1), one also has

$$\frac{1}{2}A(u) + e^2D(u) = \frac{1}{3}J(u) - \frac{\omega}{6}B(u) + \frac{2p-1}{3(p+1)}C(u) + \frac{2}{3}e^2E_1(u) - \frac{1}{3}e^2E_2(u),$$

from which we deduce that

$$\begin{aligned}
I(u) - I(u_t) &= \frac{1-t^3}{3}J(u) + \frac{(1-t)^2(t+2)\omega}{6}B(u) \\
& \quad + \frac{1}{3(p+1)} (3t^{2p-1} - (2p-1)t^3 + 2p-4) C(u) \\
& \quad + \frac{8-2t^3}{3}e^2E_1(u) - \frac{1-t^3}{3}e^2E_2(u) + \frac{e^2t^{-1}}{2} \int_{\mathbb{R}^3} S_0(u)\rho(t^{-1}x) dx.
\end{aligned}$$

Moreover putting

$$\begin{aligned}
R(t, u) &:= \frac{8-2t^3}{3}e^2E_1(u) - \frac{1-t^3}{3}e^2E_2(u) + \frac{e^2t^{-1}}{2} \int_{\mathbb{R}^3} S_0(u)\rho(t^{-1}x) dx \\
&= e^2 \int_{\mathbb{R}^3} S_0(u)M(t, x) dx, \\
M(t, x) &:= \frac{t^3-1}{6}(\rho(x) + x \cdot \nabla \rho(x)) - \frac{\rho(x)}{2} + \frac{\rho(t^{-1}x)}{2t},
\end{aligned}$$

we arrive at

$$\begin{aligned}
I(u) - I(u_t) &= \frac{1-t^3}{3}J(u) + \frac{(1-t)^2(t+2)\omega}{6}B(u) \\
& \quad + \frac{1}{3(p+1)} (3t^{2p-1} - (2p-1)t^3 + 2p-4) C(u) + R(t, u).
\end{aligned} \quad (3.16) \quad \boxed{\text{eq:3.14}}$$

Step 2 (Evaluation of coefficients): Now for $T \geq 4$, one has

$$\begin{aligned} \frac{(1-t)^2(t+2)}{6} &= \frac{(1-t)^2}{3} + \frac{t^3}{6} - \frac{t^2}{3} + \frac{t}{6} \\ &\geq \frac{(1-t)^2}{3} + \frac{2t^3}{3T} - \frac{t^3}{3T} = \frac{(1-t)^2}{3} + \frac{t^3}{3T} \quad \text{for } t \geq T, \end{aligned}$$

from which we get

$$\frac{(1-t)^2(t+2)}{6} \geq \begin{cases} \frac{(1-t)^2}{3} & \text{for } 0 \leq t \leq T, \\ \frac{(1-t)^2}{6} + \frac{t^3}{3T} & \text{for } t \geq T. \end{cases} \quad (3.17) \quad \boxed{3.7-3}$$

Next we claim that

$$\frac{1}{3(p+1)} (3t^{2p-1} - (2p-1)t^3 + 2p-4) \geq \begin{cases} \frac{\alpha}{6(p+1)}(1-t)^2 & \text{for } 0 \leq t \leq T, \\ \frac{\alpha}{12(p+1)}(1-t)^2 + \frac{\alpha}{6(p+1)}t^3 & \text{for } t \geq T. \end{cases} \quad (3.18) \quad \boxed{3.7-4}$$

For this purpose, we put $g(t) := 3t^{2p-1} - (2p-1)t^3 + 2p-4$. For $t \geq T$, one has

$$g(t) \geq t^3 (3t^{2p-4} - (2p-1)) \geq t^3 (3T^{2p-4} - (2p-1)).$$

Since $2 < p < 5$ and $T^{2p-4} \geq 3$, it holds $\frac{2p-1}{3} < 3 \leq T^{2p-4}$. Taking $C_1 = 3T^{2p-4} - (2p-1) > 0$, we get

$$g(t) \geq C_1 t^3 \quad \text{for } t \geq T. \quad (3.19) \quad \boxed{3.7-5}$$

Now we take $0 < \tau < 1$ so that $(1-\tau)^{2p-4} = \frac{p}{2(p-1)}$, which is possible because $p > 2$. For $0 \leq t \leq 1-\tau$, it holds that $g'(t) = 3(2p-1)t^2(t^{2p-4} - 1) < 0$ and thus

$$g(t) \geq g(1-\tau) \geq g(1-\tau)(1-t)^2 \quad \text{for } 0 \leq t \leq 1-\tau. \quad (3.20) \quad \boxed{3.7-6}$$

Next we show that

$$g(t) \geq C_2(1-t)^2 \quad \text{for } t \geq 1+\tau \text{ and some } C_2 > 0. \quad (3.21) \quad \boxed{3.7-7}$$

Letting $h(t) := g(t) - C_2(1-t)^2$, one finds that

$$\begin{aligned} h'(t) &= 3(2p-1)t^2(t^{2p-4} - 1) - 2C_2t + 2C_2 \geq t^2(3(2p-1)(t^{2p-4} - 1) - 2C_2) \\ &\geq t^2(3(2p-1)((1+\tau)^{2p-4} - 1) - 2C_2) \quad \text{for } t \geq 1+\tau. \end{aligned}$$

If $0 < C_2 \leq \frac{3}{2}(2p-1)((1+\tau)^{2p-4} - 1)$, then $h'(t) \geq 0$ on $[1+\tau, \infty)$. Thus taking

$$C_2 = \min \left\{ \frac{g(1+\tau)}{\tau^2}, \frac{3}{2}(2p-1)((1+\tau)^{2p-4} - 1) \right\},$$

we deduce that $h'(t) \geq 0$ and $h(t) \geq h(1+\tau) \geq 0$, which shows (3.21).

Finally since $g(1) = g'(1) = 0$ and $g''(t) = 6(2p-1)((p-1)t^{2p-4} - 1)$, we have by the Taylor theorem that

$$g(t) = \frac{1}{2}g''(\xi)(1-t)^2 \quad \text{for some } \xi \text{ between } t \text{ and } 1.$$

When $1-\tau \leq t \leq 1+\tau$, it follows that $t^{2p-4} \geq (1-\tau)^{2p-4} = \frac{p}{2(p-1)}$, from which we obtain $g''(t) \geq 6(2p-1)(\frac{p}{2} - 1)$ and

$$g(t) \geq \frac{3}{2}(2p-1)(p-2)(1-t)^2 \quad \text{for } 1-\tau \leq t \leq 1+\tau. \quad (3.22) \quad \boxed{3.7-8}$$

Combining (3.19)-(3.22) and putting

$$\alpha := \min \left\{ \frac{3}{2}(p-1)(p-2), g(1-\tau), C_1, C_2 \right\},$$

we arrive at

$$g(t) \geq \begin{cases} \alpha(1-t)^2 & \text{for } t \geq 0, \\ \alpha t^3 & \text{for } t \geq T. \end{cases}$$

Moreover since

$$\frac{1}{3(p+1)} (3t^{2p-1} - (2p-1)t^3 + 2p-4) = \frac{1}{3(p+1)}g(t) = \frac{1}{6(p+1)}g(t) + \frac{1}{6(p+1)}g(t),$$

we obtain (3.18).

Step 3 (Estimate for $R(t, u)$): First when $t \geq T$, one find from the assumption (1.6) that

$$\begin{aligned} M(t, x) &\geq -\frac{t^3-1}{6} (|\rho(x)| + |x \cdot \nabla \rho(x)|) - \frac{1}{2}|\rho(x)| \geq -\frac{t^3}{6} (|\rho(x)| + |x \cdot \nabla \rho(x)|) - \frac{1}{2}|\rho(x)| \\ &\geq -t^3 \left(\frac{2}{3}|\rho(x)| + \frac{1}{6}|x \cdot \nabla \rho(x)| \right). \end{aligned}$$

Thus by Lemma 2.2, we get

$$\begin{aligned} R(t, u) &\geq -t^3 e^2 \int_{\mathbb{R}^3} |S_0(u)| \left(\frac{2}{3}|\rho(x)| + \frac{1}{6}|x \cdot \nabla \rho(x)| \right) dx \\ &\geq -t^3 e^2 \left(\frac{2}{3}\|\rho\|_{\frac{6}{5}} + \frac{1}{6}\|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \|S_0(u)\|_6 \\ &\geq -C_3 t^3 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) \quad \text{for } t \geq T, \end{aligned} \quad (3.23) \quad \boxed{3.7-9}$$

where $C_3 > 0$ is independent of e, ρ, t . Next for $0 \leq t \leq \frac{1}{2}$, we have

$$M(t, x) \geq \frac{t^3-1}{6} (|\rho(x)| + |x \cdot \nabla \rho(x)|) - \frac{1}{2}|\rho(x)| \geq -\frac{2}{3}|\rho(x)| - \frac{1}{6}|x \cdot \nabla \rho(x)|,$$

from which one concludes that

$$\begin{aligned} R(t, u) &\geq -e^2 \left(\frac{2}{3}\|\rho\|_{\frac{6}{5}} + \frac{1}{6}\|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \|S_0(u)\|_6 \\ &\geq -4(1-t)^2 e^2 \left(\frac{2}{3}\|\rho\|_{\frac{6}{5}} + \frac{1}{6}\|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \|S_0(u)\|_6 \\ &\geq -C_4(1-t)^2 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) \quad \text{for } 0 \leq t \leq \frac{1}{2} \end{aligned} \quad (3.24) \quad \boxed{3.7-10}$$

for some $C_4 > 0$ independent of e, ρ, t . For $\frac{1}{2} \leq t \leq T$, we first observe that

$$\begin{aligned} \frac{\partial M}{\partial t}(t, x) &= \frac{t^2}{2}(\rho(x) + x \cdot \nabla \rho(x)) - \frac{\rho(t^{-1}x)}{2t^2} - \frac{x \cdot \nabla \rho(t^{-1}x)}{2t^3}, \\ \frac{\partial^2 M}{\partial t^2}(t, x) &= t(\rho(x) + x \cdot \nabla \rho(x)) + \frac{\rho(t^{-1}x)}{t^3} + \frac{2x \cdot \nabla \rho(t^{-1}x)}{t^4} + \frac{x \cdot (D^2 \rho(t^{-1}x)x)}{2t^5}. \end{aligned}$$

Then for fixed $x \in \mathbb{R}^3$, one finds that $M(1, x) = \frac{\partial M}{\partial t}(1, x) = 0$ and

$$\begin{aligned} \frac{\partial^2 M}{\partial t^2}(t, x) &\geq -t(|\rho(x)| + |x \cdot \nabla \rho(x)|) \\ &\quad - \frac{1}{t^3} \left(|\rho(t^{-1}x)| + 2|(t^{-1}x) \cdot \nabla \rho(t^{-1}x)| + \frac{1}{2}|(t^{-1}x) \cdot D^2 \rho(t^{-1}x)(t^{-1}x)| \right) \\ &=: -N(t, x). \end{aligned}$$

By the Taylor theorem, there exists $\xi = \xi(t, x) \in (\frac{1}{2}, T)$ such that

$$M(t, x) = \frac{1}{2} \frac{\partial^2 M}{\partial t^2}(\xi, x)(1-t)^2 \geq -\frac{1}{2}N(\xi, x)(1-t)^2.$$

Using the Hölder inequality, we deduce that

$$R(t, u) \geq -\frac{1}{2}(1-t)^2 e^2 \|N(\xi, x)\|_{\frac{6}{5}} \|S_0(u)\|_6 \quad \text{for } \frac{1}{2} \leq t \leq T.$$

Moreover since $\frac{1}{2} < \xi < T$, we have

$$\begin{aligned} \|N(\xi, x)\|_{\frac{6}{5}} &\leq \|\xi\rho\|_{\frac{6}{5}} + \|\xi x \cdot \nabla\rho\|_{\frac{6}{5}} + \|\xi^{-\frac{1}{2}}\rho\|_{\frac{6}{5}} + 2\|\xi^{-\frac{1}{2}}x \cdot \nabla\rho\|_{\frac{6}{5}} + \frac{1}{2}\|\xi^{-\frac{1}{2}}x \cdot (D^2\rho x)\|_{\frac{6}{5}} \\ &\leq T \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} \right) + \sqrt{2} \left(\|\rho\|_{\frac{6}{5}} + 2\|x \cdot \nabla\rho\|_{\frac{6}{5}} + \frac{1}{2}\|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right). \end{aligned}$$

Then by Lemma 2.2, it follows that

$$R(t, u) \geq -C_5(1-t)^2 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) \quad \text{for } \frac{1}{2} \leq t \leq T, \quad (3.25) \quad \boxed{3.7-11}$$

where $C_5 > 0$ is independent of e, ρ, t . From (3.23)-(3.25), letting $\beta = \min\{C_3, C_4, C_5\} > 0$, we find that

$$\begin{aligned} R(t, u) & \quad (3.26) \quad \boxed{3.7-12} \\ &\geq \begin{cases} -\beta(1-t)^2 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) & \text{for } 0 \leq t \leq T, \\ -\beta t^3 e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right) (\|u\|_2^2 + \|u\|_{p+1}^2) & \text{for } t \geq T. \end{cases} \end{aligned}$$

Step 4 (Conclusion): Now from (3.16), (3.17), (3.18) and (3.26), we can see that (3.13) and (3.14) hold.

Finally suppose that $\|u\|_{p+1} \geq \delta$. If

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla\rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right) \leq \frac{1}{2\beta} \min \left\{ \frac{2\omega}{3T}, \frac{\alpha\delta^{p-1}}{6(p+1)} \right\},$$

using (3.13) and (3.14), we obtain

$$I(u) - I(u_t) - \frac{1-t^3}{3} J(u) \geq \frac{(1-t)^2 \omega}{6} \|u\|_2^2 + \frac{\alpha\delta^{p-1}}{12(p+1)} (1-t)^2 \|u\|_{p+1}^2 \quad \text{for all } t > 0.$$

This finishes the proof. \square

rem:3.8

Remark 3.8. *Letting*

$$F(t, x) := \frac{1}{t^4} (\rho(t^{-1}x) + (t^{-1}x) \cdot \nabla\rho(t^{-1}x)),$$

we find that $M(t, x)$ can be written as

$$\begin{aligned} M(t, x) &= \frac{t^3 - 1}{6} (\rho(x) + x \cdot \nabla\rho(x)) - \frac{\rho(x)}{2} + \frac{\rho(t^{-1}x)}{2t} \\ &= \int_1^t \frac{s^2}{2} (\rho(x) + x \cdot \nabla\rho(x)) ds + \frac{1}{2} \int_1^t \frac{d}{ds} (s^{-1}\rho(s^{-1}x)) ds \\ &= \frac{1}{2} \int_1^t s^2 (F(1, x) - F(s, x)) ds. \end{aligned}$$

Thus if $F(t, x)$ is non-increasing with respect to t for every $x \in \mathbb{R}^3$, it holds that $M(t, x) \geq 0$ and hence $R(t, u) \geq 0$ for all $t > 0$, which is a same situation to [30]. However we cannot expect that $F(t, x)$ is non-increasing in t for doping profiles.

Indeed by a direct calculation, $F(t, x)$ is non-increasing in t if ρ satisfies

$$4\rho(x) + 6x \cdot \nabla\rho(x) + x \cdot (D^2\rho(x)x) \geq 0 \quad \text{for all } x \in \mathbb{R}^3. \quad (3.27) \quad \boxed{\text{eq:3.18}}$$

When ρ is a Gaussian function $e^{-\alpha|x|^2}$ for $\alpha > 0$, one finds that

$$2\rho(x) + 3x \cdot \nabla \rho(x) + \frac{1}{2}x \cdot (D^2\rho(x)x) = (2\alpha^2|x|^4 - 7\alpha|x|^2 + 2)e^{-\alpha|x|^2}.$$

Hence no matter how we choose α , (3.27) fails to hold near the inflection point. Moreover if we consider $\rho(x) = \frac{1}{1+|x|^r}$, we see that

$$2\rho(x) + 3x \cdot \nabla \rho(x) + \frac{1}{2}x \cdot (D^2\rho(x)x) = \frac{1}{2(1+|x|^r)^3} \left\{ (r-1)(r-4)|x|^{2r} - (r^2 + 5r - 8)|x|^r + 4 \right\}.$$

Then it follows that

$$\begin{aligned} r > 4 &\Rightarrow (3.27) \text{ fails to hold near the inflection point,} \\ 1 \leq r \leq 4 &\Rightarrow (3.27) \text{ fails to hold for large } |x|, \\ 0 < r < 1 &\Rightarrow \rho \notin L^{\frac{6}{5}}(\mathbb{R}^3). \end{aligned}$$

In this sense, the assumption (1.5) and (3.27) seem to be inconsistent.

Now using 3.7, we can show the following.

lem:3.9 **Lemma 3.9.** Suppose that $2 < p < 5$ and assume that

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right) \leq \rho_0$$

for sufficiently small $\rho_0 > 0$.

Then for any $u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $u_{t_u} \in \mathcal{M}$. Especially for $u \in \mathcal{M}$,

$$u_t \in \mathcal{M} \quad \text{if and only if} \quad t_u = 1.$$

Proof. By Lemma 3.1, we know that there exists $t_u > 0$, which is a maximum point of $f(t)$ defined in (3.2), such that $u_{t_u} \in \mathcal{M}$. Thus it suffices to show that $f(t)$ has a unique critical point for $t > 0$.

Now suppose by contradiction that there exist $0 < t_1 < t_2$ such that $f'(t_1) = f'(t_2) = 0$. As we have observed in the proof of Lemma 3.1, it holds that $J(u_t) = tf'(t)$, which yields that

$$J(u_{t_1}) = J(u_{t_2}) = 0. \tag{3.28}$$

eq:3.19

Since $u_{t_1}, u_{t_2} \in \mathcal{M}$, we can use (3.3) to show that $\|u_{t_1}\|_{p+1} \geq \delta_1$ and $\|u_{t_2}\|_{p+1} \geq \delta_1$. Then we are able to apply (3.15) if $e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2\rho x)\|_{\frac{6}{5}} \right)$ is sufficiently small.

Applying (3.15) with $u = u_{t_1}$ and $t = \frac{t_2}{t_1}$, we have

$$I(u_{t_1}) \geq I(u_{t_2}) + \frac{t_1^3 - t_2^3}{3t_1^3} J(u_{t_1}) + \frac{(t_1 - t_2)^2 \omega}{6t_1^2} \|u_{t_1}\|_2^2 + \frac{\alpha \delta_1^{p-1} (t_1 - t_2)^2}{12(p+1)t_1^2} \|u_{t_1}\|_{p+1}^2.$$

Similarly one gets

$$I(u_{t_2}) \geq I(u_{t_1}) + \frac{t_2^3 - t_1^3}{3t_2^3} J(u_{t_2}) + \frac{(t_2 - t_1)^2 \omega}{6t_2^2} \|u_{t_2}\|_2^2 + \frac{\alpha \delta_1^{p-1} (t_2 - t_1)^2}{12(p+1)t_2^2} \|u_{t_2}\|_{p+1}^2.$$

Thus from (3.28), it follows that

$$0 \geq \frac{\omega}{6} (t_1 - t_2)^2 \left(\frac{1}{t_1^2} \|u_{t_1}\|_2^2 + \frac{1}{t_2^2} \|u_{t_2}\|_2^2 \right) + \frac{\alpha \delta_1^{p-1}}{12(p+1)} (t_1 - t_2)^2 \left(\frac{1}{t_1^2} \|u_{t_1}\|_{p+1}^2 + \frac{1}{t_2^2} \|u_{t_2}\|_{p+1}^2 \right) > 0,$$

from which arrive at a contradiction and conclude that $f(t)$ has a unique critical point. \square

rem:3.10

Remark 3.10. By Proposition 3.6 and Lemma 3.9, we can obtain the following minimax characterization of m :

$$m = \inf_{u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}} \max_{t > 0} I(u_t).$$

4. EXISTENCE OF A GROUND STATE SOLUTION FOR $2 < p < 5$

In this section, we establish the existence of a ground state solution of (1.1). For this purpose, we define the energy functional I_∞ associated with (1.3) by

$$I_\infty(u) := \frac{1}{2}A(u) + \frac{\omega}{2}B(u) - \frac{1}{p+1}C(u) + e^2D(u) \quad \text{for } u \in H^1(\mathbb{R}^3, \mathbb{C}).$$

Indeed if $\rho(x) \rightarrow 0$ as $|x| \rightarrow \infty$, I_∞ can be seen as a functional at infinity. We define the ground state energy corresponding to (1.3) by

$$m_\infty := \inf_{u \in \mathcal{S}_\infty} I_\infty(u), \quad \mathcal{S}_\infty := \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \mid I'_\infty(u) = 0\}.$$

Let us denote by J_∞ the Nehari-Pohozaev functional for I_∞ , that is,

$$J_\infty(u) := \frac{3}{2}A(u) + \frac{\omega}{2}B(u) - \frac{2p-1}{p+1}C(u) + 3e^2D(u).$$

We also set

$$\sigma_\infty := \inf_{u \in \mathcal{M}_\infty} I_\infty(u), \quad \mathcal{M}_\infty := \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\} \mid J_\infty(u) = 0\}.$$

Then by the result in [3] with minor modifications considering $u \in H^1(\mathbb{R}^3, \mathbb{C})$, it follows that $\sigma_\infty = m_\infty$ and σ_∞ is achieved by some $u_\infty \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ for $2 < p < 5$ and any $e > 0$. Moreover arguing as Lemma 3.7 (see also [30, Corollary 3.3]), we have

$$I_\infty(u) \geq I_\infty(u_t) + \frac{1-t^3}{3}J_\infty(u) \quad \text{for all } u \in H^1(\mathbb{R}^3, \mathbb{C}) \text{ and } t > 0, \quad (4.1) \quad \boxed{\text{eq:4.1}}$$

where $u_t(x) = t^2u(tx)$.

To prove the attainability of σ , we need the following lemma. Note that this is the only part where we require (1.6).

lem:4.1

Lemma 4.1. *Suppose that $2 < p < 5$. Assume further (1.5) and (1.6). Then it follows that*

$$\sigma < \sigma_\infty \quad \text{and} \quad m < m_\infty.$$

Proof. First we observe from (1.6) that

$$I(u) < I_\infty(u) \quad \text{for all } u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$$

because $E_1(u) < 0$ by (2.2).

Now let u_∞ be a ground state solution for (1.3). By Lemma 3.1, there exists $t_\infty > 0$ such that $\tilde{u}_\infty(x) := t_\infty^2 u_\infty(t_\infty x) \in \mathcal{M}$. Then from (4.1) and $J_\infty(u_\infty) = 0$, we obtain

$$\sigma \leq I(\tilde{u}_\infty) < I_\infty(\tilde{u}_\infty) \leq I_\infty(u_\infty) = \sigma_\infty,$$

which ends the proof. \square

Now we are ready to prove the following result.

prop:4.2

Proposition 4.2. *Suppose that $2 < p < 5$. Assume further (1.5), (1.6) and*

$$e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} + \|x \cdot (D^2 \rho x)\|_{\frac{6}{5}} \right) \leq \rho_0$$

for sufficiently small $\rho_0 > 0$.

Then there exists $u_0 \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ such that

$$I(u_0) = \sigma \quad \text{and} \quad J(u_0) = 0.$$

Proof. Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence for σ , that is,

$$I(u_n) = \sigma + o(1) \quad \text{and} \quad J(u_n) = 0. \quad (4.2) \quad \boxed{\text{eq:4.2}}$$

Then by Lemma 3.3, it follows that

$$\sigma + o(1) = I(u_n) \geq \alpha_1 \|u\|^2,$$

which implies that $\|u_n\|$ is bounded. Thus passing to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^3, \mathbb{C})$ for some $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$. We divide the proof into two steps.

Step 1: We claim that $u_0 \not\equiv 0$.

Suppose by contradiction that $u_0 \equiv 0$ so that $u_n \rightarrow 0$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Then by Lemma 2.4, it follows that

$$E_1(u_n) \rightarrow 0 \quad \text{and} \quad E_2(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since we can write

$$I(u) = I_\infty(u) + 2e^2 E_1(u) \quad \text{and} \quad J(u) = J_\infty(u) - 2e^2 E_1(u) + e^2 E_2(u),$$

we have from (4.2) that

$$I_\infty(u_n) \rightarrow \sigma \quad \text{and} \quad J_\infty(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.3) \quad \boxed{\text{eq:4.3}}$$

Arguing as in Lemma 3.2, we also deduce that $\|u_n\|_{p+1} \geq \delta$ for some $\delta > 0$ independent of e, ρ and $n \in \mathbb{N}$.

Now we apply the concentration compactness principle [24, 33] to show that there exist $\tilde{\delta} > 0$ and $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n(x)|^{p+1} dx \geq \tilde{\delta}.$$

Letting $\hat{u}_n(x) := u_n(x + y_n)$, we have

$$\|\hat{u}_n\| = \|u_n\| \quad \text{and} \quad \int_{B_1(0)} |\hat{u}_n(x)|^{p+1} dx \geq \tilde{\delta}.$$

Especially, there exists $\hat{u} \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ such that $\hat{u}_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Moreover from (4.3), one also finds that

$$I_\infty(\hat{u}_n) = I_\infty(u_n) = \sigma + o(1) \quad \text{and} \quad J_\infty(\hat{u}_n) = J_\infty(u_n) = o(1). \quad (4.4) \quad \boxed{\text{eq:4.4}}$$

Let us put $v_n := \hat{u}_n - \hat{u}$. Then by the Brezis-Lieb lemma [10] and Lemma 2.3, it follows that

$$I_\infty(\hat{u}_n) = I_\infty(\hat{u}) + I_\infty(v_n) + o(1) \quad \text{and} \quad J_\infty(\hat{u}_n) = J_\infty(\hat{u}) + J_\infty(v_n) + o(1).$$

We also define

$$K_\infty(u) := I_\infty(u) - \frac{1}{3} J_\infty(u) = \frac{\omega}{3} B(u) + \frac{2(p-2)}{3(p-1)} C(u).$$

Then from (4.4), one finds that

$$K_\infty(v_n) = \sigma - K_\infty(\hat{u}) + o(1) \quad \text{and} \quad J_\infty(v_n) = -J_\infty(\hat{u}) + o(1). \quad (4.5) \quad \boxed{\text{eq:4.5}}$$

If there exists a subsequence $\{v_{n_j}\} \subset \{v_n\}$ such that $v_{n_j} = 0$, then passing to a limit along this subsequence, it holds from (4.5) that $K_\infty(\hat{u}) = \sigma$ and $J_\infty(\hat{u}) = 0$. This implies that

$$\sigma_\infty \leq I_\infty(\hat{u}) = K_\infty(\hat{u}) + \frac{1}{3} J_\infty(\hat{u}) = \sigma,$$

which is absurd by Lemma 4.1. Thus we may assume that $v_n \neq 0$. Arguing as Lemma 3.9 (see also [27, Lemma 3.3], [30, Lemma 2.4]), there exists a unique $\hat{t}_n > 0$ such that $\hat{\phi}_n(x) := (\hat{t}_n)^2 v_n(\hat{t}_n x) \in \mathcal{M}_\infty$.

Next we prove that $J_\infty(\hat{u}) \leq 0$. Suppose by contradiction that $J_\infty(\hat{u}) > 0$. Then from (4.5), it follows that $J_\infty(v_n) \leq 0$ and hence $\|v_n\|_{p+1} \geq \delta_1$ by Lemma 3.2. Using (3.15) with $\rho \equiv 0$ and (4.5), one deduces that

$$\begin{aligned} \sigma - K_\infty(\hat{u}) + o(1) &= K_\infty(v_n) = I_\infty(v_n) - \frac{1}{3}J_\infty(v_n) \\ &\geq I_\infty(\hat{\phi}_n) - \frac{(\hat{t}_n)^3}{3}J_\infty(v_n) \\ &\geq \sigma_\infty - \frac{(\hat{t}_n)^3}{3}J_\infty(v_n) \geq \sigma_\infty, \end{aligned}$$

which yields that

$$\sigma_\infty \leq \sigma - K_\infty(\hat{u}) < \sigma_\infty - K_\infty(\hat{u}) \quad \text{and thus} \quad K_\infty(\hat{u}) < 0.$$

This is a contradiction to the fact $K_\infty(u) > 0$ for any $u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ and hence $J_\infty(\hat{u}) \leq 0$.

Now by Lemma 3.9, there exists a unique $\hat{t} > 0$ such that $\hat{\phi}(x) := (\hat{t})^2 \hat{u}(\hat{t}x) \in \mathcal{M}_\infty$. Then from (3.15) with $\rho \equiv 0$, (4.4) and by the Fatou lemma, we obtain

$$\begin{aligned} \sigma &= \lim_{n \rightarrow \infty} \left\{ I_\infty(\hat{u}_n) - \frac{1}{3}J_\infty(\hat{u}_n) \right\} = \lim_{n \rightarrow \infty} K_\infty(\hat{u}_n) \geq K_\infty(\hat{u}) \\ &= I_\infty(\hat{u}) - \frac{1}{3}J_\infty(\hat{u}) \\ &\geq I_\infty(\hat{\phi}) - \frac{(\hat{t})^3}{3}J_\infty(\hat{u}) \\ &\geq \sigma_\infty - \frac{(\hat{t})^3}{3}J_\infty(\hat{u}) \geq \sigma_\infty. \end{aligned}$$

This is a contradiction to Lemma 4.1 and hence we conclude that $u_0 \neq 0$.

Step 2: We prove that $I(u_0) = \sigma$ and $J(u_0) = 0$.

Let us define $w_n := u_n - u_0$. By Lemma 2.3, Lemma 2.4 and the Brezis-Lieb lemma, we have

$$I(u_n) = I(u_0) + I(w_n) + o(1) \quad \text{and} \quad J(u_n) = J(u_0) + J(w_n) + o(1).$$

We also put

$$\begin{aligned} K(u) &:= I(u) - \frac{1}{3}J(u) \\ &= \frac{\omega}{3}B(u) + \frac{2(p-2)}{3(p+1)}C(u) + \frac{8}{3}e^2E_1(u) - \frac{1}{3}e^2E_2(u) \quad \text{for } u \in H^1(\mathbb{R}^3, \mathbb{C}). \end{aligned}$$

Then from (4.2), it holds that

$$K(w_n) = \sigma - K(u_0) + o(1) \quad \text{and} \quad J(w_n) = -J(u_0) + o(1). \tag{4.6} \quad \boxed{\text{eq:4.6}}$$

If there exists a subsequence $\{w_{n_j}\} \subset \{w_n\}$ such that $w_{n_j} = 0$, then passing to a limit along this subsequence, we arrive at $K(u_0) = \sigma$ and $J(u_0) = 0$. This implies that $I(u_0) = \sigma$ and hence we conclude. Thus we may assume that $w_n \neq 0$.

Next we show that $J(u_0) \leq 0$. Indeed if $J(u_0) > 0$, it follows from (4.6) that $J(w_n) \leq 0$ and hence $\|w_n\|_{p+1} \geq \delta_1$ by Lemma 3.2. Then by Lemma 3.9, there exists a unique $t_n > 0$ such that $\phi_n(x) := t_n^2 w_n(t_n x) \in \mathcal{M}$. Using (3.15) and (4.6), we deduce that

$$\begin{aligned} \sigma - K(u_0) + o(1) &= K(w_n) = I(w_n) - \frac{1}{3}J(w_n) \\ &\geq I(\phi_n) - \frac{t_n^3}{3}J(w_n) \\ &\geq \sigma - \frac{t_n^3}{3}J(w_n) \geq \sigma, \end{aligned}$$

yielding that $K(u_0) \leq 0$. However this is a contradiction because $K(u) > 0$ for any $u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ by Lemma 2.2 provided that $e^2(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}})$ is sufficiently small, from which we conclude that $J(u_0) \leq 0$.

Now using Lemma 3.9 again, there exists a unique $t_0 > 0$ such that $\phi_0(x) := t_0^2 u_0(t_0 x) \in \mathcal{M}$. Moreover since $J(u_0) \leq 0$, we are able to use (3.15) by Lemma 3.2. Then from (4.2), Lemma 2.4 and by the Fatou lemma, one finds that

$$\begin{aligned} \sigma &= \lim_{n \rightarrow \infty} \left\{ I(u_n) - \frac{1}{3} J(u_n) \right\} = \lim_{n \rightarrow \infty} K(u_n) \geq K(u_0) \\ &= I(u_0) - \frac{1}{3} J(u_0) \\ &\geq I(\phi_0) - \frac{t_0^3}{3} J(u_0) \\ &\geq \sigma - \frac{t_0^3}{3} J(u_0) \geq \sigma, \end{aligned} \tag{4.7} \quad \boxed{\text{eq:4.7}}$$

from which we get $K(u_0) = \sigma$. Moreover if $J(u_0) < 0$, (4.7) leads a contradiction. Thus it follows that $J(u_0) = 0$ and hence $I(u_0) = \sigma$. This completes the proof. \square

rem:4.3

Remark 4.3. *As we have observed in the proof of Proposition 4.2,*

$$\begin{aligned} K(u) &= \frac{\omega}{3} B(u) + \frac{2(p-2)}{3(p+1)} C(u) + \frac{8}{3} e^2 E_1(u) - \frac{1}{3} e^2 E_2(u) \\ &= \int_{\mathbb{R}^3} \left\{ \frac{\omega}{3} |u(x)|^2 + \frac{2(p-2)}{3(p+1)} |u(x)|^{p+1} - \frac{e^2}{6} S_0(u)(x) (\rho(x) + x \cdot \nabla \rho(x)) \right\} dx \end{aligned}$$

plays an important role. Especially it is natural to apply the concentration compactness principle to the function

$$\rho_n(x) := \frac{\omega}{3} |u_n(x)|^2 + \frac{2(p-2)}{3(p+1)} |u_n(x)|^{p+1} - \frac{e^2}{6} S_0(u_n)(x) (\rho(x) + x \cdot \nabla \rho(x))$$

for a minimizing sequence $\{u_n\} \subset \mathcal{M}$. However we don't know whether $\rho_n(x) \geq 0$ for all $x \in \mathbb{R}^3$, although one knows that $K(u_n) \geq 0$ if $e^2(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}})$ is small by Lemma 2.2.

If ρ satisfies

$$\rho(x) + x \cdot \nabla \rho(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^3, \tag{4.8} \quad \boxed{\text{eq:4.8}}$$

it follows that $\rho_n \geq 0$ on \mathbb{R}^3 . However the condition (4.8) seems to be inconsistent with (1.5) and (1.6). For example, if we consider the Gaussian function $\rho(x) = e^{-\alpha|x|^2}$ for $\alpha > 0$, we have

$$\rho(x) + x \cdot \nabla \rho(x) = (1 - 2\alpha|x|^2) e^{-\alpha|x|^2}.$$

Thus no matter how we choose α , (4.8) fails to hold near the origin.

Proof of Theorem 1.1. By Proposition 3.6 and Proposition 4.2, there exists $u_0 \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ such that

$$I(u_0) = \sigma = m \quad \text{and} \quad J(u_0) = 0, \tag{4.9} \quad \boxed{\text{eq:4.9}}$$

namely, u_0 is a ground state solution of (1.1). We claim that u_0 can be assumed to be real-valued up to phase shift.

For this purpose, we argue as in [1] and show that

$$|\nabla |u_0|| = |\nabla u_0| \quad \text{a.e. in } \mathbb{R}^3. \tag{4.10} \quad \boxed{\text{eq:4.10}}$$

Indeed suppose by contradiction that $\mathcal{L}(\{x \in \mathbb{R}^3 \mid |\nabla |u_0(x)|| < |\nabla u_0(x)|\}) > 0$, where $\mathcal{L}(A)$ is the Lebesgue measure for the set $A \subset \mathbb{R}^3$. Then it follows that

$$\sigma = I(u_0) > I(|u_0|), \quad 0 = J(u_0) > J(|u_0|) \quad \text{and} \quad K(u_0) > K(|u_0|). \tag{4.11} \quad \boxed{\text{eq:4.11}}$$

Moreover by Lemma 3.9, there exists a unique $t_0 > 0$ such that $v_0(x) := t_0^2 |u_0|(t_0 x) \in \mathcal{M}$. Then from (3.15), (4.9) and (4.11), we find that

$$\begin{aligned} \sigma &= I(u_0) - \frac{1}{3}J(u_0) = K(u_0) > K(|u_0|) \\ &= I(|u_0|) - \frac{1}{3}J(|u_0|) \\ &\geq I(v_0) - \frac{t_0^3}{3}J(|u_0|) > I(v_0) \geq \sigma, \end{aligned}$$

which is a contradiction and hence (4.10) holds.

Now using the polar form, we can write $u_0(x) = |u_0(x)|e^{i\eta(x)}$. Then a direct calculation shows that

$$|\nabla u_0|^2 = |\nabla |u_0||^2 + |u_0|^2 |\nabla \eta|^2,$$

yielding that $\nabla \eta \equiv 0$ a.e. in \mathbb{R}^3 from (4.10). This implies that $u_0(x) = e^{i\theta} |u_0(x)|$ for some $\theta \in \mathbb{R}$ and hence u_0 can be assumed to be real-valued up to phase shift. \square

5. RELATION BETWEEN ACTION GSS AND ENERGY GSS

In this section, we investigate the relation between action ground state solutions and energy ground state solutions of (1.1).

For given $\mu > 0$, let $u_\mu \in H^1(\mathbb{R}^3, \mathbb{C})$ be an energy ground state solution of (1.1), that is,

$$\begin{aligned} E(u_\mu) &= \inf_{B(\mu)} E(u), \quad B_\mu = \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \mid \|u\|_2^2 = \mu\}, \\ E(u) &:= \frac{1}{2}A(u) - \frac{1}{p+1}C(u) + e^2D(u) + 2e^2E_1(u). \end{aligned}$$

By the result in [16], if $2 < p < \frac{7}{3}$, $c_\infty(\mu) < 0$ and $e^2 \left(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}} \right) \leq \rho_0$ for sufficiently small $\rho_0 > 0$, u_μ exists and the corresponding Lagrange multiplier $\omega = \omega_\mu$ is positive. Here $c_\infty(\mu)$ is the minimum energy defined in (1.8). Especially u_μ is a nontrivial solution of (1.1) with $\omega = \omega_\mu$.

To clarify the dependence with respect to ω , we write $I = I_\omega$, $J = J_\omega$, $\mathcal{S} = \mathcal{S}_\omega$ and $m = m_\omega$. Under these preparations, we have the following result.

prop:5.1

Proposition 5.1. *Let $\mu > 0$ be given so that $c_\infty(\mu) < 0$ and suppose that $2 < p < \frac{7}{3}$. Under the assumptions in Theorem 1.1, the energy ground state solution u_μ is an action ground state solution of (1.1) with $\omega = \omega_\mu$.*

Proof. Since u_μ is a nontrivial critical point of I_{ω_μ} , we have

$$u_\mu \in \mathcal{S}_{\omega_\mu} \quad \text{and} \quad m_{\omega_\mu} \leq I_{\omega_\mu}(u_\mu). \tag{5.1} \quad \text{eq:5.1}$$

Thus it suffices to show that $I_{\omega_\mu}(u_\mu) \leq m_{\omega_\mu}$.

Now let $w_\mu \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ be an action ground state solution of (1.1) with $\omega = \omega_\mu$, which exists by Theorem 1.1. Then by (3.15) and $J_{\omega_\mu}(w_\mu) = 0$, it follows that

$$I_{\omega_\mu}(w_\mu) - I_{\omega_\mu}((w_\mu)_t) \geq \frac{1-t^3}{2}J_{\omega_\mu}(w_\mu) = 0 \quad \text{for all } t > 0, \tag{5.2} \quad \text{eq:5.2}$$

where $(w_\mu)_t = t^2 w_\mu(tx)$. Taking $t_\mu := \frac{\|u_\mu\|_2^2}{\|w_\mu\|_2^2}$, one has

$$\|(w_\mu)_{t_\mu}\|_2^2 = \|u_\mu\|_2^2 \quad \text{and hence} \quad E(u_\mu) \leq E((w_\mu)_{t_\mu}).$$

Thus from (5.1) and (5.2), we obtain

$$\begin{aligned} m_{\omega_\mu} &\leq I_{\omega_\mu}(u_\mu) = E(u_\mu) + \frac{\omega_\mu}{2} \|u_\mu\|_2^2 \\ &\leq E((w_\mu)_{t_\mu}) + \frac{\omega_\mu}{2} \|(w_\mu)_{t_\mu}\|_2^2 \\ &= I_{\omega_\mu}((w_\mu)_{t_\mu}) \leq I_{\omega_\mu}(w_\mu) = m_{\omega_\mu}, \end{aligned}$$

from which we conclude. \square

In order to consider the converse, let $\Omega(\mu)$ be the set of Lagrange multipliers associated with energy ground state solutions for B_μ , namely

$$\Omega(\mu) := \{ \omega_\mu > 0 \mid \omega_\mu \text{ is the Lagrange multiplier associated with an energy ground state of (1.1) under the constraint } B_\mu \}.$$

By the result in [16], we know that $\Omega(\mu) \neq \emptyset$ for every $\mu > 0$, when $2 < p < \frac{7}{3}$ and $e^2(\|\rho\|_{\frac{6}{5}} + \|x \cdot \nabla \rho\|_{\frac{6}{5}})$ is small. Moreover for any $\omega_\mu \in \Omega(\mu)$, there exists an action ground state solution $w_\mu \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\}$ of (1.1) with $\omega = \omega_\mu$ by Theorem 1.1.

In this setting, the following result holds.

prop:5.2

Proposition 5.2. *Let $\mu > 0$ be given so that $c_\infty(\mu) < 0$ and suppose that $2 < p < \frac{7}{3}$. Under the assumptions of Theorem 1.1, w_μ is an energy ground state solution of (1.1) under the constraint B_μ .*

Proof. Since w_μ is an action ground state solution of (1.1) with $\omega = \omega_\mu$, by using (3.15), we have $J_{\omega_\mu}(w_{\omega_\mu}) = 0$ and

$$I_{\omega_\mu}(w_\mu) - I_{\omega_\mu}((w_\mu)_t) \geq \frac{(1-t)^2 \omega_\mu}{6} \|w_\mu\|_2^2 \geq 0 \quad \text{for all } t > 0. \quad (5.3) \quad \text{eq:5.3}$$

Now let \tilde{u}_μ be an energy ground state solution of (1.1) under the constraint B_μ whose Lagrange multiplier coincides with ω_μ . Then it follows that

$$m_{\omega_\mu} \leq I_{\omega_\mu}(\tilde{u}_\mu) \quad \text{and} \quad E(\tilde{u}_\mu) \leq E(u) \quad \text{for any } u \in H^1(\mathbb{R}^3, \mathbb{C}) \text{ with } \|u\|_2^2 = \mu.$$

Especially choosing $\tilde{t}_\mu = \frac{\mu}{\|w_\mu\|_2^2}$, we have

$$\|(w_\mu)_{\tilde{t}_\mu}\|_2^2 = \mu = \|\tilde{u}_\mu\|_2^2 \quad \text{and} \quad E(\tilde{w}_\mu) \leq E((w_\mu)_{\tilde{t}_\mu}). \quad (5.4) \quad \text{eq:5.4}$$

Then from (5.3) and (5.4), we deduce that

$$\begin{aligned} m_{\omega_\mu} &\leq I_{\omega_\mu}(\tilde{u}_\mu) = E(\tilde{u}_\mu) + \frac{\omega_\mu}{2} \|\tilde{u}_\mu\|_2^2 \\ &\leq E((w_\mu)_{\tilde{t}_\mu}) + \frac{\omega_\mu}{2} \|(w_\mu)_{\tilde{t}_\mu}\|_2^2 \\ &= I_{\omega_\mu}((w_\mu)_{\tilde{t}_\mu}) \leq I_{\omega_\mu}(w_\mu) = m_{\omega_\mu}, \end{aligned}$$

which yields that $I_{\omega_\mu}((w_\mu)_{\tilde{t}_\mu}) = I_{\omega_\mu}(w_\mu)$. Going back to (5.3), one finds that

$$0 \geq \frac{(1-\tilde{t}_\mu)^2 \omega_\mu}{6} \|w_\mu\|_2^2 \geq 0,$$

from which we conclude that $\tilde{t}_\mu = 1$ and hence

$$E(w_\mu) = E(\tilde{u}_\mu) = \inf_{B_\mu} E(u).$$

This completes the proof. \square

rem:5.3

Remark 5.3. (i) By Proposition 5.2, if $\omega_\mu \in \Omega(\mu)$, every action ground state solution of (1.1) with $\omega = \omega_\mu$ share the same L^2 -norm.

(ii) Although $\Omega(\mu) \neq \emptyset$ for every $\mu > 0$, we don't know whether $\Omega(\mu)$ is an interval. Moreover we don't know for given $\omega_\mu \in \Omega(\mu)$, there exists a unique $\mu > 0$ such that the corresponding Lagrange multiplier coincides with ω_μ . In other words, it is not known if

$$\mu \mapsto \omega_\mu : (0, \infty) \rightarrow (0, \ell) \text{ is one to one mapping for some } \ell > 0. \quad (5.5)$$

eq:5.5

(5.5) is important in investigating further properties of ground state solutions, such as uniqueness and stability. However (5.5) is known only in limited situations; see [13, 16, 19]. In particular, it is not known whether (5.5) holds true even when $\rho \equiv 0$.

Proof of Theorem 1.2. The claim follows by Propositions 5.1 and 5.2. \square

6. EXISTENCE OF A RADIAL GROUND STATE SOLUTION FOR $1 < p < 2$

In this section, we consider the case $1 < p < 2$, for which arguments in sections 3-5 do not work well. Especially we cannot handle the energy inequality (3.15) in Lemma 3.7 because the coefficient in front of $\|u\|_{p+1}^{p+1}$ becomes negative when $1 < p < 2$.

In the case $1 < p < 2$, we only have the following weak result.

thm:6.1

Theorem 6.1. Suppose that $1 < p < 2$. Assume $\rho \in L^{\frac{6}{5}}(\mathbb{R}^3)$ and $\rho(x) = \rho(|x|)$. There exists $e_0 > 0$ such that if $0 < e \leq e_0$, (1.1) has a radial ground state solution u_0 , that is, u_0 satisfies

$$I(u_0) = \inf_{u \in \mathcal{S}_{rad}} I(u), \quad \mathcal{S}_{rad} = \{u \in H_{rad}^1(\mathbb{R}^3) \mid I'(u) = 0\},$$

where $H_{rad}^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3, \mathbb{R}) \mid u(x) = u(|x|)\}$.

To this aim, let us recall the functional I_∞ for $\rho \equiv 0$, which was defined by

$$I_\infty(u) := \frac{1}{2}A(u) + \frac{\omega}{2}B(u) - \frac{1}{p+1}C(u) + e^2D(u) \quad \text{for } u \in H^1(\mathbb{R}^3, \mathbb{C}).$$

When $1 < p < 2$, the following properties are known; see [27, Theorem 4.3].

prop:6.2

Proposition 6.2. Suppose that $1 < p < 2$ and $e > 0$. Then it holds:

- (i) $\inf_{H_{rad}^1(\mathbb{R}^3)} I_\infty(u) > -\infty$,
- (ii) I_∞ satisfies the Palais-Smale condition on $H_{rad}^1(\mathbb{R}^3)$.

Now by Lemma 2.2, one finds that

$$I(u) = I_\infty(u) + e^2E_1(u) \geq I_\infty(u) - Ce^2\|\rho\|_{\frac{6}{5}}\|u\|^2 \quad \text{for any } u \in H^1(\mathbb{R}^3, \mathbb{C}),$$

from which we conclude that $\inf_{H_{rad}^1(\mathbb{R}^3)} I(u) > -\infty$. Moreover since E_1 is a compact operator by Lemma 2.4, I also satisfies the Palais-Smale condition on $H_{rad}^1(\mathbb{R}^3)$.

Proof of Theorem 6.1. Let us define the energy functional I_0 for $e = 0$:

$$I_0(u) = \frac{1}{2}A(u) + \frac{\omega}{2}B(u) - \frac{1}{p+1}C(u) \quad \text{for } u \in H^1(\mathbb{R}^3, \mathbb{C}).$$

It is standard to show that there exists $\tilde{u} \in H_{rad}^1(\mathbb{R}^3)$ such that $I_0(\tilde{u}) < 0$. Then by Lemma 2.2, one has

$$\begin{aligned} I(\tilde{u}) &= I_0(\tilde{u}) + e^2D(\tilde{u}) + e^2E_1(\tilde{u}) \\ &\leq I_0(\tilde{u}) + Ce^2\|\tilde{u}\|^4 + Ce^2\|\rho\|_{\frac{6}{5}}\|\tilde{u}\|^2. \end{aligned}$$

Choosing a small $e_0 > 0$, we have $I(\tilde{u}) < 0$ for $0 < e \leq e_0$ and hence $\inf_{u \in H_{rad}^1(\mathbb{R}^3)} I(u) < 0$.

By Proposition 6.2, we are able to apply Ekeland's variational principle. Then there exists $u_0 \in H_{rad}^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$I(u_0) = \min_{u \in H_{rad}^1(\mathbb{R}^3)} I(u) < 0.$$

Particularly, u_0 is a nontrivial solution of (1.1). Moreover since u_0 is a global minimizer on $H_{rad}^1(\mathbb{R}^3)$, it follows that

$$I(u_0) \geq \inf_{u \in \mathcal{S}_{rad}} I(u) \geq \inf_{u \in H_{rad}^1(\mathbb{R}^3)} I(u) = I(u_0),$$

yielding that u_0 is a radial ground state solution of (1.1). \square

rem:6.3

Remark 6.3. (i) In [16], we have shown that (1.1) has an energy ground state solution if $1 < p < 2$ and e is sufficiently small. Unfortunately unlike the case $2 < p < \frac{7}{3}$, we cannot say anything about the relation between the radial ground state solution obtained in Theorem 6.1 and the energy ground state solution in [16].

(ii) In the case $p = 2$, it remains open whether (1.1) has a nontrivial solution for any $\omega > 0$ and small $e > 0$. Moreover in [13], thanks to the scaling invariance of (1.3) when $p = 2$, it was proved that an energy ground state solution of (1.3) is an action ground state solution and vice versa. However due to the loss of scaling invariance for $\rho \neq 0$, we don't know if an energy ground state solution of (1.1) obtained in [16] is an action ground state of (1.1).

7. THE CASE ρ IS A CHARACTERISTIC FUNCTION

In this section, we consider the case where the doping profile ρ is a characteristic function, which appears frequently in physical literatures [22, 26, 28]. More precisely, let $\{\Omega_i\}_{i=1}^m \subset \mathbb{R}^3$ be disjoint bounded open sets with smooth boundary. For $\alpha_i > 0$ ($i = 1, \dots, m$), we assume that the doping profile ρ has the form:

$$\rho(x) = \sum_{i=1}^m \alpha_i \chi_{\Omega_i}(x), \quad \chi_{\Omega_i}(x) = \begin{cases} 1 & (x \in \Omega_i), \\ 0 & (x \notin \Omega_i). \end{cases} \quad (7.1) \quad \text{eq:7.1}$$

In this case, ρ cannot be weakly differentiable so that the assumption (1.5) does not make sense. Even so, we are able to obtain the existence of ground state solutions by imposing some smallness condition related with α_i and Ω_i .

To state our main result for this case, let us put $L := \sup_{x \in \partial\Omega} |x| < \infty$. A key is the following *sharp boundary trace inequality* due to [2, Theorem 6.1], which we present here according to the form used in this paper.

prop:7.1

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $\gamma : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ be the trace operator. Then it holds that*

$$\int_{\partial\Omega} |\gamma(u)|^2 dS \leq \kappa_1(\Omega) \int_{\Omega} |u|^2 dx + \kappa_2(\Omega) \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \quad \text{for any } u \in H^1(\Omega),$$

where $\kappa_1(\Omega) = \frac{|\partial\Omega|}{|\Omega|}$, $\kappa_2(\Omega) = \|\|\nabla w\|\|_{L^\infty(\partial\Omega)}$ and w is a unique solution of the torsion problem:

$$\Delta w = \kappa_1(\Omega) \text{ in } \Omega, \quad \frac{\partial w}{\partial n} = 1 \text{ on } \partial\Omega.$$

In relation to the size of ρ , we define

$$D(\Omega) := L|\Omega|^{\frac{1}{6}} \left(L\|H\|_{L^2(\partial\Omega)} + |\partial\Omega|^{\frac{1}{2}} \right) \left(\kappa_1(\Omega)|\Omega|^{\frac{1}{3}} + \kappa_2(\Omega) \right)^{\frac{1}{2}},$$

where H is the mean curvature of $\partial\Omega$.

rem:7.2

Remark 7.2. It is known that $\kappa_2(\Omega) \geq 1$; see [2]. Then by the isoperimetric inequality in \mathbb{R}^3 :

$$|\partial\Omega| \geq 3|\Omega|^{\frac{2}{3}}|B_1|^{\frac{1}{3}},$$

and the fact $|\Omega| \leq |B_L(0)| = L^3|B_1|$, we find that

$$D(\Omega) \geq \left(\frac{|\Omega|}{|B_1|}\right)^{\frac{1}{3}} |\Omega|^{\frac{1}{6}} \cdot \sqrt{3}|\Omega|^{\frac{1}{3}}|B_1|^{\frac{1}{6}} \left(3|B_1|^{\frac{1}{3}} + 1\right)^{\frac{1}{2}} = C|\Omega|^{\frac{5}{6}} = C\|\chi_\Omega\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}, \quad (7.2) \quad \text{eq:7.2}$$

where C is a positive constant independent of Ω .

Under these preparations, we have the following result.

thm:7.3

Theorem 7.3. Suppose that $2 < p < 5$ and assume (7.1). There exists $\rho_0 > 0$ such that if

$$e^2 \sum_{i=1}^m \alpha_i D(\Omega_i) \leq \rho_0,$$

then (1.1) has a ground state solution u_0 . Moreover the statement of Theorem 1.2 holds true.

Note that when $1 < p < 2$, we have only assumed that $\rho(x) \in L^{\frac{6}{5}}(\mathbb{R}^3)$ and $\rho(x) = \rho(|x|)$ in Theorem 6.1, which covers the case (7.1).

We mention that the first part $x \cdot \nabla \rho(x)$ and $x \cdot (D^2 \rho(x)x)$ appeared was the definition of $E_2(u)$ and $E_3(u)$ in (2.5). Under the assumption (7.1), we replace them by

$$\begin{aligned} E_1(u) &= -\frac{1}{4} \sum_{i=1}^m \alpha_i \int_{\Omega_i} S_0(u) dx, \\ E_2(u) &:= -\frac{1}{2} \sum_{i=1}^m \alpha_i \int_{\partial\Omega_i} S_0(u) x \cdot n_i dS_i, \\ E_3(u) &:= -\frac{1}{2} \sum_{i=1}^m \alpha_i \int_{\partial\Omega_i} H_i(x) S_0(u) (x \cdot n_i)^2 dS_i, \end{aligned}$$

where n_i is the unit outward normal on $\partial\Omega_i$. Indeed we have the following.

lem:7.4

Lemma 7.4. It holds that

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{B_R(0)} S_0(u) u x \cdot \nabla \bar{u} dx &= -10E_1(u) + E_2(u), \\ \lim_{R \rightarrow \infty} \int_{B_R(0)} S_1(u) u x \cdot \nabla \bar{u} dx &= -6E_2(u) - E_3(u). \end{aligned}$$

Proof. For simplicity, let us consider the case $m = 1$ and $\alpha = 1$. First by the divergence theorem and the fact $S_0(u)|u|^2 \in L^1(\mathbb{R}^3)$, one finds that

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{B_R(0)} S_0(u) u x \cdot \nabla \bar{u} dx &= -\frac{1}{8\pi} \lim_{R \rightarrow \infty} \int_{\Omega} \int_{B_R(0)} \frac{u(y) y \cdot \nabla \overline{u(y)}}{|x-y|} dy dx \\ &= \frac{1}{16\pi} \int_{\Omega} \int_{\mathbb{R}^3} \frac{|u(y)|^2 \operatorname{div}_y y}{|x-y|} dy dx + \frac{1}{16\pi} \int_{\Omega} \int_{\mathbb{R}^3} |u(y)|^2 y \cdot \nabla_y \left(\frac{1}{|x-y|} \right) dy dx \\ &= \frac{3}{16\pi} \int_{\Omega} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy dx + \frac{1}{16\pi} \int_{\Omega} \int_{\mathbb{R}^3} |u(y)|^2 \frac{y \cdot (x-y)}{|x-y|^3} dy dx. \end{aligned}$$

Using the identity $y \cdot (x - y) = -|x - y|^2 + x \cdot (x - y)$, the Fubini theorem and the divergence theorem, we get

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \int_{B_R(0)} S_0(u) u x \cdot \nabla \bar{u} \, dx \\
&= \frac{1}{8\pi} \int_{\Omega} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x - y|} \, dy \, dx - \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\Omega} |u(y)|^2 x \cdot \nabla_x \left(\frac{1}{|x - y|} \right) \, dx \, dy \\
&= \frac{1}{8\pi} \int_{\Omega} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x - y|} \, dy \, dx - \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\Omega} \operatorname{div}_x \left(\frac{|u(y)|^2 x}{|x - y|} \right) \, dx \, dy + \frac{3}{16\pi} \int_{\mathbb{R}^3} \int_{\Omega} \frac{|u(y)|^2}{|x - y|} \, dx \, dy \\
&= \frac{5}{16\pi} \int_{\Omega} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x - y|} \, dy \, dx - \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\Omega} \frac{|u(y)|^2}{|x - y|} x \cdot n \, dS \, dy \\
&= \frac{5}{2} \int_{\Omega} S_0(u) \, dx - \frac{1}{2} \int_{\partial\Omega} S_0(u) x \cdot n \, dS = -10E_1(u) + E_2(u).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \int_{B_R(0)} S_1(u) u x \cdot \nabla \bar{u} \, dx \\
&= -\frac{1}{8\pi} \lim_{R \rightarrow \infty} \int_{\partial\Omega} \int_{B_R(0)} \frac{u(y) y \cdot \nabla \overline{u(y)}}{|x - y|} x \cdot n \, dy \, dS \\
&= \frac{1}{8\pi} \int_{\partial\Omega} \int_{\mathbb{R}^3} \frac{|u(y)|^2 x \cdot n}{|x - y|} \, dy \, dS - \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\partial\Omega} |u(y)|^2 x \cdot \nabla_x \left(\frac{1}{|x - y|} \right) x \cdot n \, dS \, dy \\
&= \frac{1}{8\pi} \int_{\partial\Omega} \int_{\mathbb{R}^3} \frac{|u(y)|^2 x \cdot n}{|x - y|} \, dy \, dS - \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\partial\Omega} \operatorname{div}_x \left(\frac{|u(y)|^2 (x \cdot n) x}{|x - y|} \right) \, dS \, dy \\
&\quad + \frac{3}{16\pi} \int_{\mathbb{R}^3} \int_{\partial\Omega} \frac{|u(y)|^2 x \cdot n}{|x - y|} \, dS \, dy + \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\partial\Omega} \frac{|u(y)|^2 x \cdot \nabla_x (x \cdot n)}{|x - y|} \, dS \, dy \\
&= \frac{5}{2} \int_{\partial\Omega} S_0(u) x \cdot n \, dS - \frac{1}{2} \int_{\partial\Omega} \operatorname{div}_x (S_0(u) (x \cdot n) x) \, dS + \frac{1}{2} \int_{\partial\Omega} S_0(u) x \cdot \nabla_x (x \cdot n) \, dS. \quad (7.3) \quad \boxed{\text{eq:7.3}}
\end{aligned}$$

Applying the surface divergence theorem (see e.g. [29, 7.6]) and noticing that $\partial(\partial\Omega) = \emptyset$, it follows that

$$\int_{\partial\Omega} \operatorname{div}_x (S_0(u) (x \cdot n) x) \, dS = - \int_{\partial\Omega} (S_0(u) (x \cdot n) x)^\perp \cdot \vec{H} \, dS = - \int_{\partial\Omega} H(x) S_0(u) (x \cdot n)^2 \, dS,$$

where x^\perp is the normal component of x and \vec{H} is the mean curvature vector $\vec{H} = Hn$. Finally since $x \cdot \nabla_x (x \cdot n) = x \cdot n$, we deduce from (7.3) that

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \int_{B_R(0)} S_1(u) u x \cdot \nabla \bar{u} \, dx \\
&= 3 \int_{\partial\Omega} S_0(u) x \cdot n \, dS + \frac{1}{2} \int_{\partial\Omega} H(x) S_0(u) (x \cdot n)^2 \, dS = -6E_2(u) - E_3(u),
\end{aligned}$$

which ends the proof. The general case can be shown by summing up the integrals. \square

By Lemma 7.4, the Pohozev identity can be reformulated as follows.

lem:7.5

Lemma 7.5. *Under the assumption (7.1), the functionals $P(u)$ defined in (2.10) and $Q(u)$ in (3.7) have the same form.*

Proof. As we have mentioned in the proof of Lemma 2.5, the Pohozaev identity can be shown by multiplying $x \cdot \nabla \bar{u}$ and $ex \cdot S_0(u)$ by (1.1), integrating them over $B_R(0)$ and passing to a limit $R \rightarrow \infty$. Then we are able to obtain (2.10) and (3.7) by Lemma 7.4. \square

Next we establish estimates for E_1 , E_2 and E_3 .

lem:7.6

Lemma 7.6. *For any $u \in H^1(\mathbb{R}^3, \mathbb{C})$, E_1 , E_2 and E_3 satisfy the estimates:*

$$\begin{aligned} |E_1(u)| &\leq C \sum_{i=1}^m \alpha_i |\Omega_i|^{\frac{5}{6}} \|\nabla S_0(u)\|_2, \\ |E_2(u)| &\leq C \sum_{i=1}^m \alpha_i D(\Omega_i) \|\nabla S_0(u)\|_2, \\ |E_3(u)| &\leq C \sum_{i=1}^m \alpha_i D(\Omega_i) \|\nabla S_0(u)\|_2, \end{aligned}$$

where $C > 0$ is a constant independent of Ω_i .

Proof. First we observe that

$$|E_1(u)| \leq \frac{1}{4} \sum_{i=1}^m \alpha_i \int_{\Omega_i} |S_0(u)| dx \leq \frac{1}{4} \sum_{i=1}^m \alpha_i \left(\int_{\Omega_i} |S_0(u)|^6 dx \right)^{\frac{1}{6}} \left(\int_{\Omega_i} dx \right)^{\frac{5}{6}},$$

from which the estimate for E_1 can be obtained by the Sobolev inequality. Next by Proposition 7.1, the Hölder inequality and the Sobolev inequality, one has

$$\begin{aligned} |E_2(u)| &\leq \frac{1}{2} \sum_{i=1}^m \alpha_i \int_{\partial\Omega_i} |S_0(u)| |x| dS_i \\ &\leq \frac{1}{2} \sum_{i=1}^m \alpha_i \left(\int_{\partial\Omega_i} |S_0(u)|^2 dS_i \right)^{\frac{1}{2}} \left(\int_{\partial\Omega_i} |x|^2 dS_i \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \sum_{i=1}^m \alpha_i L_i |\partial\Omega_i|^{\frac{1}{2}} \left(\kappa_1(\Omega_i) \|S_0(u)\|_{L^2(\Omega_i)}^2 + \kappa_2(\Omega_i) \|S_0(u)\|_{L^2(\Omega_i)} \|\nabla S_0(u)\|_{L^2(\Omega_i)} \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \sum_{i=1}^m \alpha_i L_i |\partial\Omega_i|^{\frac{1}{2}} \left(\kappa_1(\Omega_i) |\Omega_i|^{\frac{2}{3}} \|S_0(u)\|_{L^6(\mathbb{R}^3)}^2 + \kappa_2(\Omega_i) |\Omega_i|^{\frac{1}{3}} \|S_0(u)\|_{L^6(\mathbb{R}^3)} \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)} \right)^{\frac{1}{2}} \\ &\leq C \sum_{i=1}^m \alpha_i L_i |\Omega_i|^{\frac{1}{6}} |\partial\Omega_i|^{\frac{1}{2}} \left(\kappa_1(\Omega_i) |\Omega_i|^{\frac{1}{3}} + \kappa_2(\Omega_i) \right)^{\frac{1}{2}} \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)} \\ &\leq C \sum_{i=1}^m \alpha_i D(\Omega_i) \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} |E_3(u)| &\leq \frac{1}{2} \sum_{i=1}^m \alpha_i \int_{\partial\Omega_i} |H_i| |S_0(u)| |x|^2 dS_i \\ &\leq \frac{1}{2} \sum_{i=1}^m \alpha_i L_i^2 \|H_i\|_{L^2(\partial\Omega_i)} \|S_0(u)\|_{L^2(\partial\Omega_i)} \\ &\leq C \sum_{i=1}^m \alpha_i L_i^2 \|H_i\|_{L^2(\partial\Omega_i)} |\Omega_i|^{\frac{1}{6}} \left(\kappa_1(\Omega_i) |\Omega_i|^{\frac{1}{3}} + \kappa_2(\Omega_i) \right)^{\frac{1}{2}} \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)} \\ &\leq C \sum_{i=1}^m \alpha_i D(\Omega_i) \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

This completes the proof. \square

Our next step is to modify the proof of the energy inequality in Lemma 3.7. For this purpose, we prove the following.

lem:7.7

Lemma 7.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and put*

$$\Omega(t) := \int_{t\Omega} S_0(u)(x) dx = t^3 \int_{\Omega} S_0(u)(ty) dy.$$

Then it holds that

$$\begin{aligned} \Omega'(t) &= t^2 \int_{\partial\Omega} S_0(u)(ty)(y \cdot n) dS, \\ \Omega''(t) &= -2t \int_{\partial\Omega} S_0(u)(ty)(y \cdot n) dS - t \int_{\partial\Omega} H(y) S_0(u)(ty)(y \cdot n)^2 dS. \end{aligned}$$

Proof. First we observe that

$$\begin{aligned} \Omega'(t) &= 3t^2 \int_{\Omega} S_0(u)(ty) dy + t^2 \int_{\Omega} \nabla_y S_0(u)(ty) \cdot y dy \\ &= 3t^2 \int_{\Omega} S_0(u)(ty) dy + t^2 \int_{\Omega} \operatorname{div}_y (S_0(u)(ty)y) dy - t^2 \int_{\Omega} S_0(u)(ty) \operatorname{div}_y y dy \\ &= t^2 \int_{\partial\Omega} S_0(u)(ty)(y \cdot n) dS. \end{aligned}$$

Similarly by the surface divergence theorem, one has

$$\begin{aligned} \Omega''(t) &= 2t \int_{\partial\Omega} S_0(u)(ty)(y \cdot n) dS + t \int_{\partial\Omega} \nabla_y S_0(u)(ty) \cdot y(y \cdot n) dS \\ &= 2t \int_{\partial\Omega} S_0(u)(ty)(y \cdot n) dS \\ &\quad + t \int_{\partial\Omega} \operatorname{div}_y (S_0(u)(ty)(y \cdot n)y) dS - t \int_{\partial\Omega} S_0(u)(ty) \operatorname{div}_y ((y \cdot n)y) dS \\ &= 2t \int_{\partial\Omega} S_0(u)(ty)(y \cdot n) dS - t \int_{\partial\Omega} H(y) S_0(u)(ty)(y \cdot n)^2 dS \\ &\quad - t \int_{\partial\Omega} S_0(u)(ty) \operatorname{div}_y y(y \cdot n) dS - t \int_{\partial\Omega} S_0(u)(ty)y \cdot \nabla_y (y \cdot n) dS \\ &= -2t \int_{\partial\Omega} S_0(u)(ty)(y \cdot n) dS - t \int_{\partial\Omega} H(y) S_0(u)(ty)(y \cdot n)^2 dS. \end{aligned}$$

This completes the proof. \square

rem:7.8

Remark 7.8. *Lemma 7.7 is related with the "calculus of moving surfaces" due to Hadamard; see [20, (38)-(39)].*

Using Lemma 7.6 and Lemma 7.7, we can establish the energy identity as follows.

lem:7.9

Lemma 7.9. *Suppose that $2 < p < 5$ and let $u_t(x) = t^2 u(tx)$ for $t > 0$. Under the assumption (7.1), there exist $\alpha > 0$, $\beta > 0$, $\rho_0 > 0$ independent of e , ρ , t such that if $\|u\|_{p+1} \geq \delta$ for some $\delta > 0$ independent of e , ρ and $e^2 \sum_{i=1}^m \alpha_i D(\Omega_i) \leq \rho_0$, then the following estimate holds.*

$$I(u) - I(u_t) - \frac{1-t^3}{3} J(u) \geq \frac{(1-t)^2 \omega}{6} \|u\|_2^2 + \frac{\alpha \delta^{p-1}}{12(p+1)} (1-t)^2 \|u\|_{p+1}^2 \quad \text{for all } t > 0.$$

Proof. Under the notation of Lemma 7.7, we can write the remainder term $R(t)$ as follows:

$$\begin{aligned} R(t) &= \frac{8-2t^3}{3} e^2 E_1(u) - \frac{1-t^3}{3} e^2 E_2(u) + \frac{e^2 t^{-1}}{2} \int_{\mathbb{R}^3} S_0(u) \rho(t^{-1}x) dx \\ &= e^2 \sum_{i=1}^m \alpha_i \left\{ -\frac{4-t^3}{6} \int_{\Omega_i} S_0(u) dx + \frac{1-t^3}{6} \int_{\partial\Omega_i} S_0(u) x \cdot n dS_i + \frac{t^{-1}}{2} \int_{t\Omega_i} S_0(u) dx \right\} \\ &= e^2 \sum_{i=1}^m \alpha_i \left\{ \frac{t^3-1}{6} (\Omega_i(1) - \Omega'_i(1)) - \frac{\Omega_i(1)}{2} + \frac{\Omega_i(t)}{2t} \right\}. \end{aligned}$$

Let $T \geq 4$ be chosen so that $T^{2p-4} \geq 3$ and put

$$G(t) := \frac{t^3-1}{6} (\Omega_i(1) - \Omega'_i(1)) - \frac{\Omega_i(1)}{2} + \frac{\Omega_i(t)}{2t}.$$

For $t \geq T$, it follows that

$$G(t) \geq -\frac{t^3-1}{6} (|\Omega_i(1)| + |\Omega'_i(1)|) - \frac{1}{2} |\Omega_i(1)| \geq -t^3 \left(\frac{2}{3} |\Omega_i(1)| + \frac{1}{6} |\Omega'_i(1)| \right).$$

Similarly one has

$$G(t) \geq -\left(\frac{2}{3} |\Omega_i(1)| + \frac{1}{6} |\Omega'_i(1)| \right) \geq -4(1-t)^2 \left(\frac{2}{3} |\Omega_i(1)| + \frac{1}{6} |\Omega'_i(1)| \right) \quad \text{for } 0 \leq t \leq \frac{1}{2}.$$

When $\frac{1}{2} \leq t \leq T$, we see that $G(1) = G'(1) = 0$ and

$$\begin{aligned} G''(t) &= t (\Omega_i(1) - \Omega'_i(1)) + \frac{1}{t^3} \left(\Omega'_i(t) - t\Omega'_i(t) + \frac{t^2}{2} \Omega''_i(t) \right) \\ &\geq -t (|\Omega_i(1)| + |\Omega'_i(1)|) - \frac{1}{t^3} \left(|\Omega_i(t)| + t|\Omega'_i(t)| + \frac{t^2}{2} |\Omega''_i(t)| \right) =: -\tilde{N}(t). \end{aligned}$$

Then by the Taylor theorem, there exists $\xi = \xi(t) \in (\frac{1}{2}, T)$ such that

$$G(t) \geq -\frac{1}{2} \tilde{N}(\xi) (1-t)^2 \quad \text{for } \frac{1}{2} \leq t \leq T.$$

Now by Proposition 7.1 and Lemma 7.7, arguing similarly as Lemma 7.6, one finds that

$$\begin{aligned} \frac{|\Omega_i(t)|}{t^3} &\leq \int_{\Omega_i} |S_0(u)(ty)| dy \leq |\Omega_i|^{\frac{5}{6}} \|S_0(u)(ty)\|_{L^6(\mathbb{R}^3)} \leq Ct^{-\frac{1}{2}} D(\Omega_i) \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)}, \\ \frac{|\Omega'_i(t)|}{t^2} &\leq \int_{\partial\Omega_i} |S_0(u)(ty)| |y| dS \leq Ct^{-\frac{1}{2}} D(\Omega_i) \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)}, \\ \frac{|\Omega''_i(t)|}{t} &\leq 2 \int_{\partial\Omega_i} |S_0(u)(ty)| |y| dS + \int_{\partial\Omega_i} |H_i(y)| |S_0(u)(ty)| |y|^2 dS \leq Ct^{-\frac{1}{2}} D(\Omega_i) \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Then by Lemma 2.2, we have

$$\tilde{N}(\xi) \leq T (|\Omega_i(1)| + |\Omega'_i(1)|) + \sqrt{2} (CD(\Omega_i) \|\nabla S_0(u)\|_{L^2(\mathbb{R}^3)}) \leq CD(\Omega_i) (\|u\|_2^2 + \|u\|_{p+1}^2)$$

and hence

$$R(t) \geq \begin{cases} -\beta(1-t)^2 e^2 \sum_{i=1}^m \alpha_i D(\Omega_i) (\|u\|_2^2 + \|u\|_{p+1}^2) & \text{for } 0 \leq t \leq T, \\ -\beta t^3 e^2 \sum_{i=1}^m \alpha_i D(\Omega_i) (\|u\|_2^2 + \|u\|_{p+1}^2) & \text{for } t \geq T, \end{cases}$$

for some $\beta > 0$ independent of e, ρ, t . The remaining parts can be shown in the same way as Lemma 3.7. \square

Proof of Theorem 7.3. By Lemma 7.5, Lemma 7.6 and Lemma 7.9, we are able to modify the proofs in Sections 3-5. \square

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