Undular bores in channels and estuaries: from Favre's experiments to a new geometrical Green-Naghdi system

M. Ricchiuto

Inria at University of Bordeaux

THANKS TO

credit to them for the good stuff, blame me for the rest

P. Bonneton (CNRS)**,**

R. Chassagne (U. Grenoble)**,**

A.G. Filippini (BRGM)**,**

S. Gavrilyuk (U. Aix-Marseille),

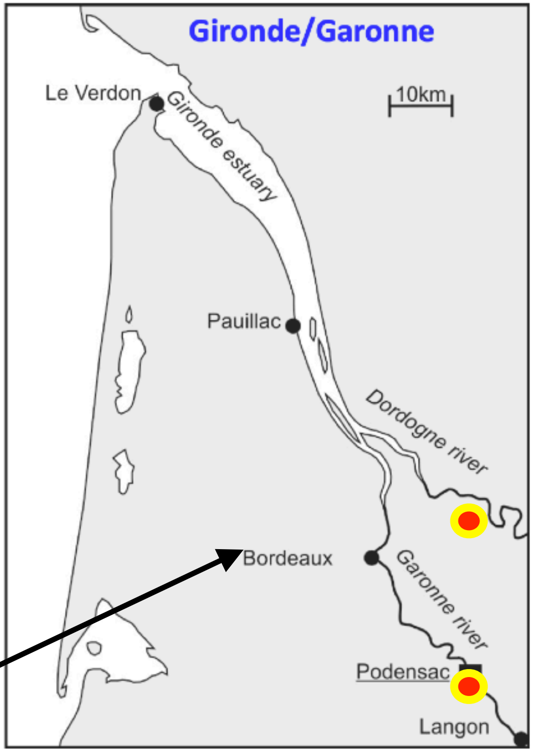
B. Jouy (EDF)**,**

M. Kazolea (Inria)**,**

M. Le (LHSV)**,**

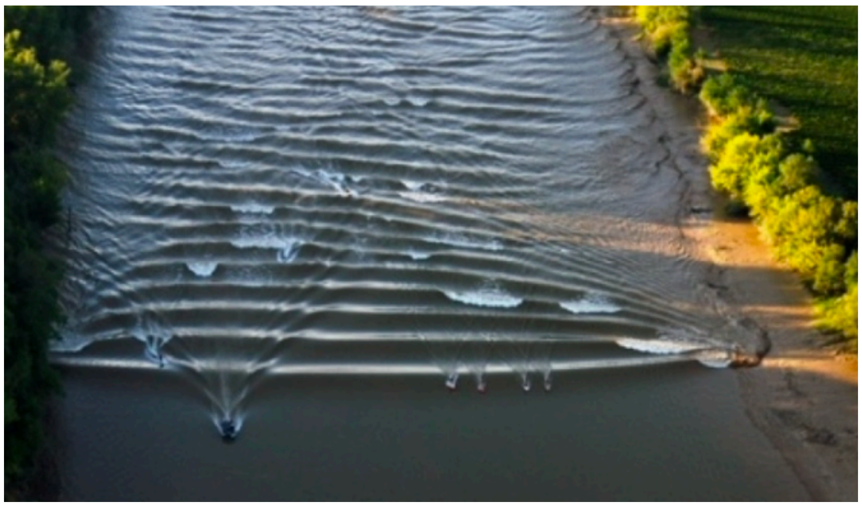
D. Violeau (EDF)

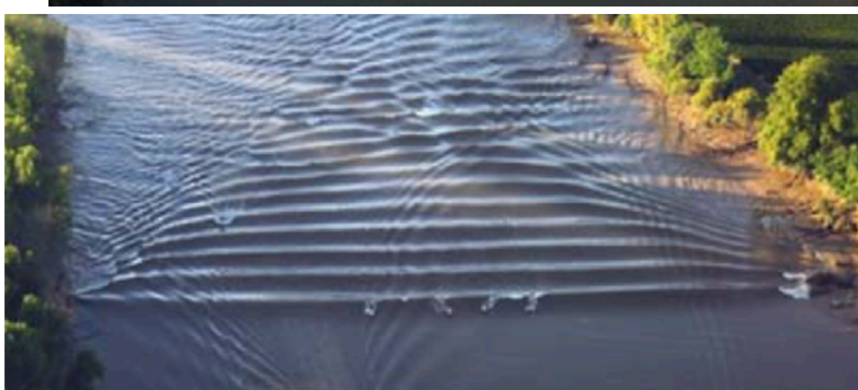

Belgium
Paris
France
Switzerla
Geneva
GIRONDE
Monaco
Marseille

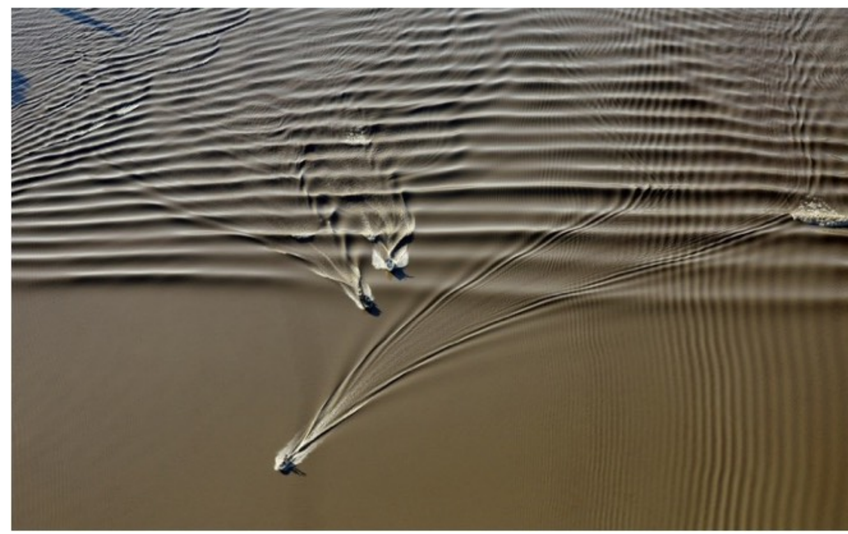

Gironde/Garonne
Le Verdon
Gironde estuary
10km
Pauillac
Dordogne river
Bordeaux
Garonne river
Podensac
Langon


La Rochelle
Limoge
Parc naturel régional Périgord-Limousin
Parc naturel régional Médoc
GIRONDE
A63
A62
Parc naturel régional des Landes de Gascogne
A65

Dordogne river in 2021

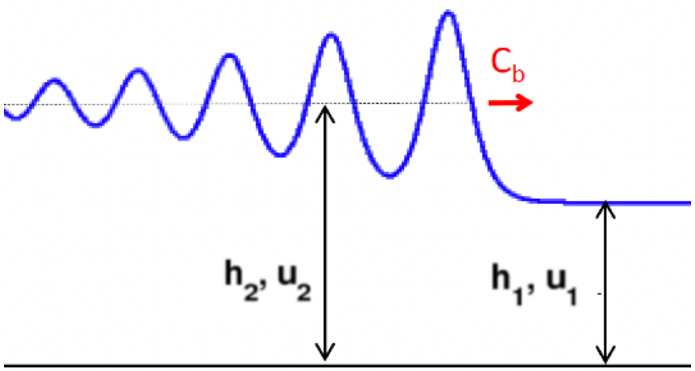






Garonne river in 2010

These undulating processes are known to be modelled by some dispersive PDE as e.g.



$$\partial_t u - \alpha \partial_{txx} u + \partial_x f(u) - \beta \partial_{xxx} u = 0$$

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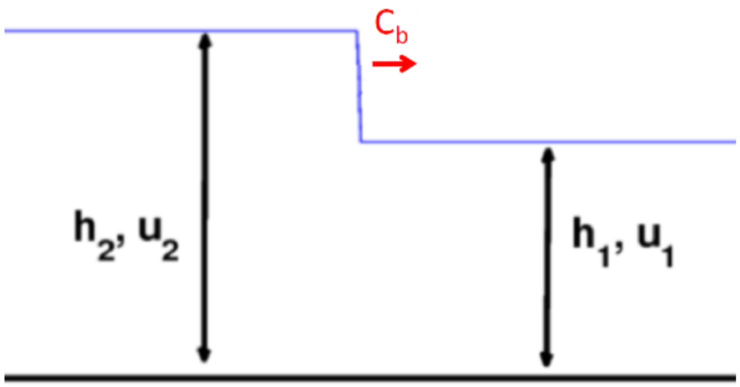

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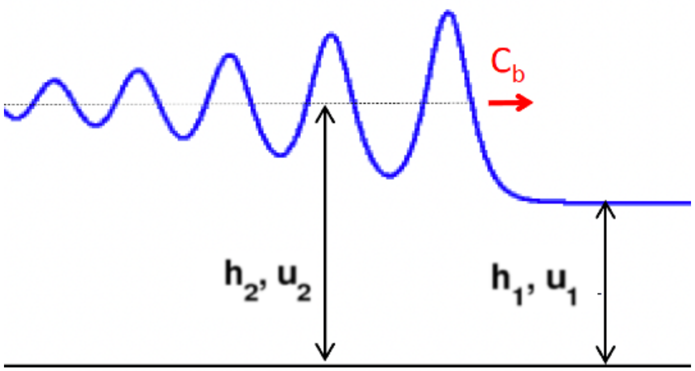


In this talk we show that

1. hyperbolic models as the shallow water equations may exhibit dispersive propagation instead of shock formation

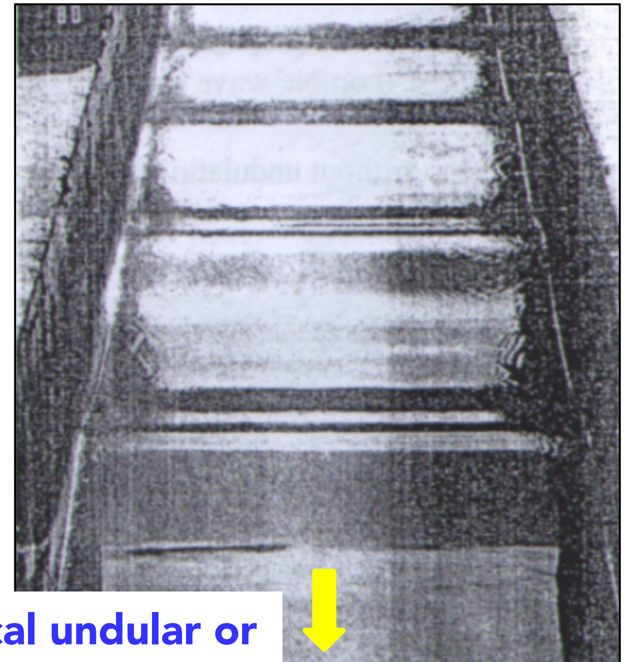
2. these *dispersive-like* undular bores waves exist, they have been measured in lab and in nature, and seem to occur systematically for Froude numbers below ~ 1.15-1.17

Undular bores:

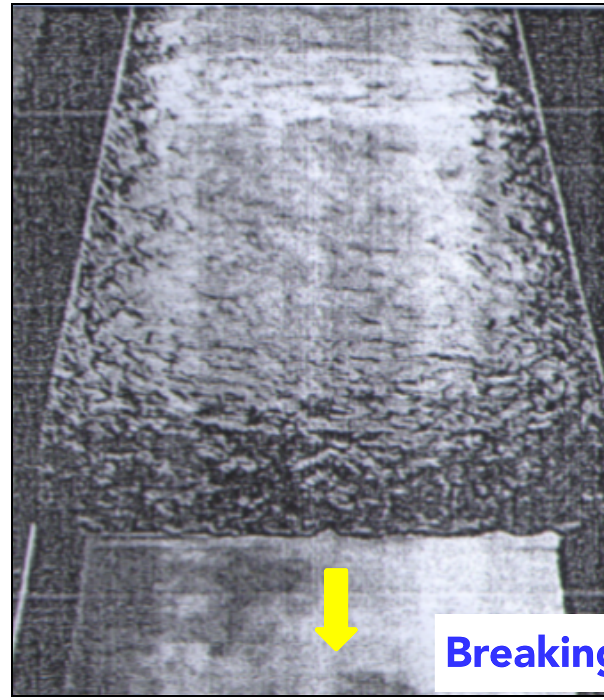
straight walled channels, trapezoidal channels

and natural environnements

Favre/Treske experiments in rectangular channels (vertical walls)



classical undular or "dispersive bore" or "Favre wave"



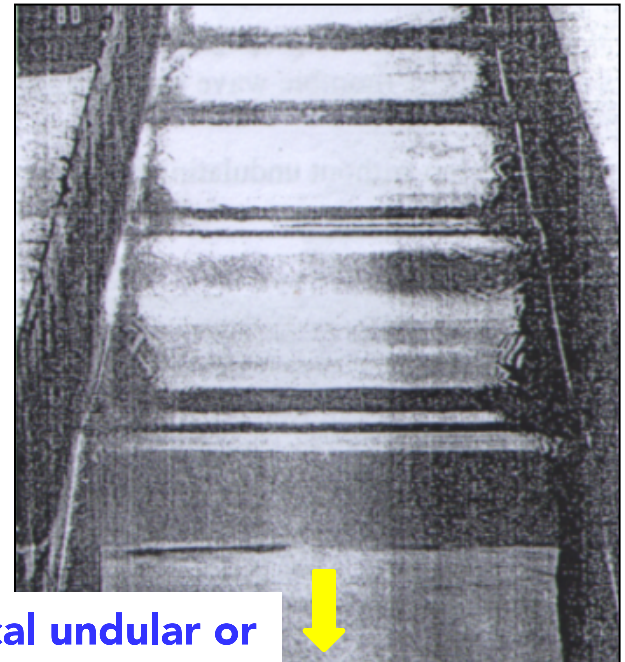
Breaking bore

$F_{t2} > Fr > 1$ — F_{t2} — $Fr > F_{t2}$ → Fr

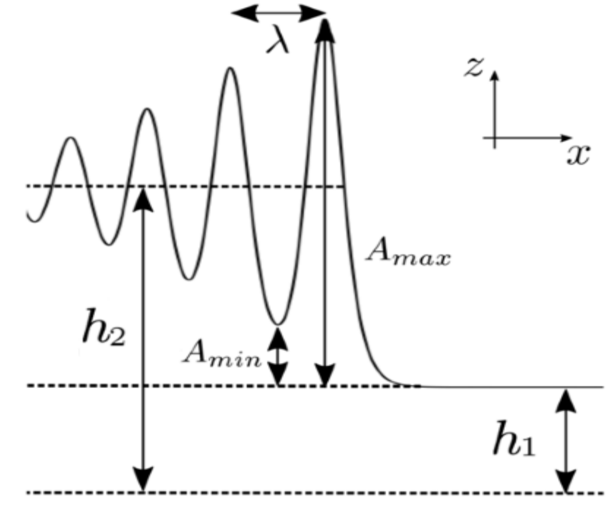
Favre, *Dunod, 1935*

Treske, *J. Hydraulic Research, 1994*

Experiments in rectangular channels (no banks)



classical undular or "dispersive bore" or "Favre wave"



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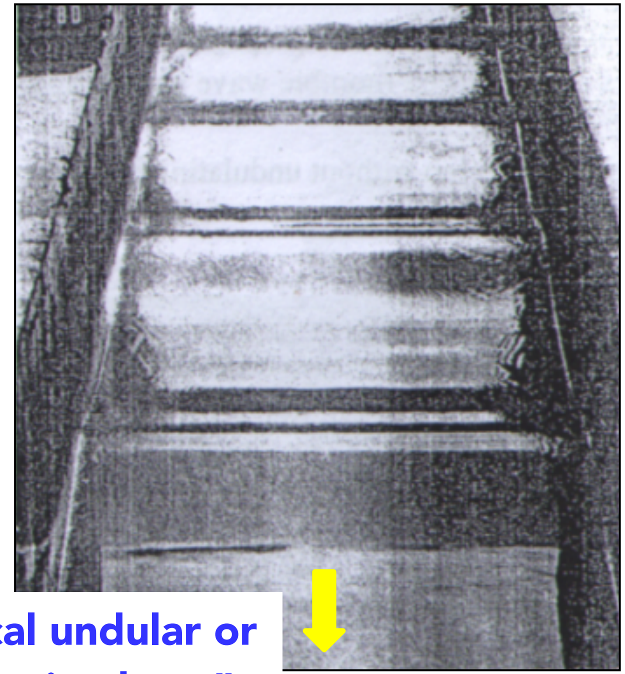

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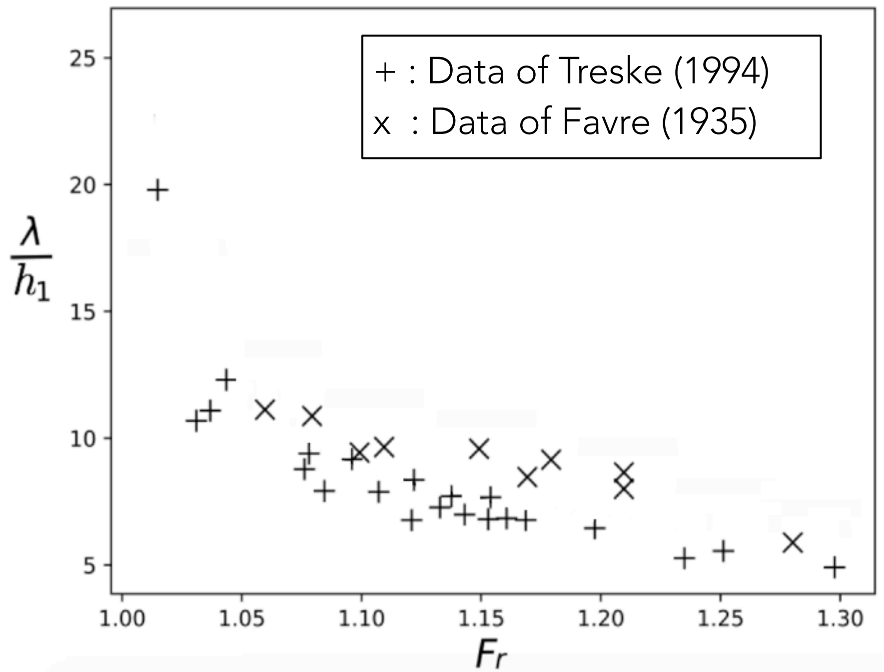
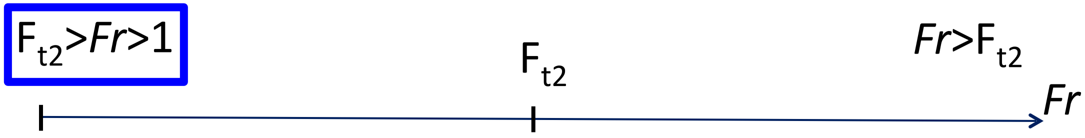
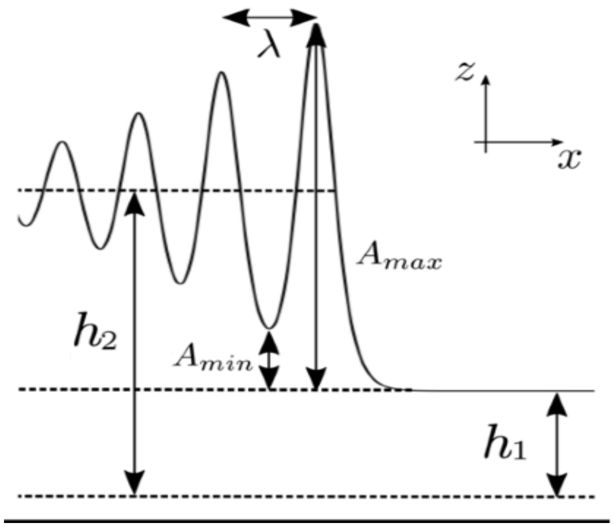




Favre, *Dunod, 1935*

Treske, *J. Hydraulic Research, 1994*



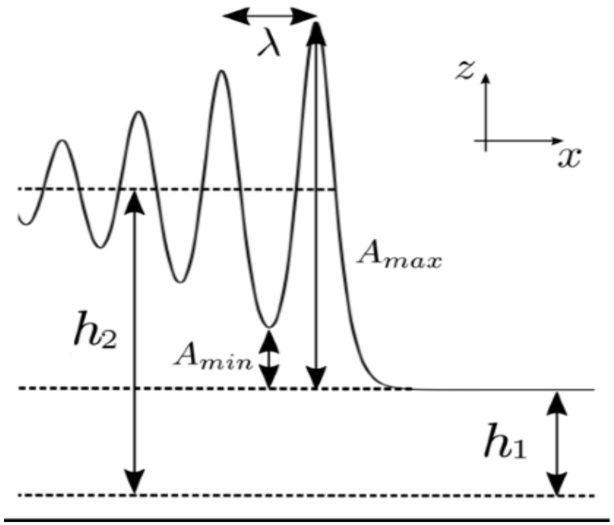
Lemoine analogy

Lemoine, La Houille Blanche, 1948

1. Secondary waves conserve mass/momentum

2. The undular front moves at the speed of the bore: $C_\lambda + U_2 = C_b$

3. No energy dissipation, energy goes into the secondary waves



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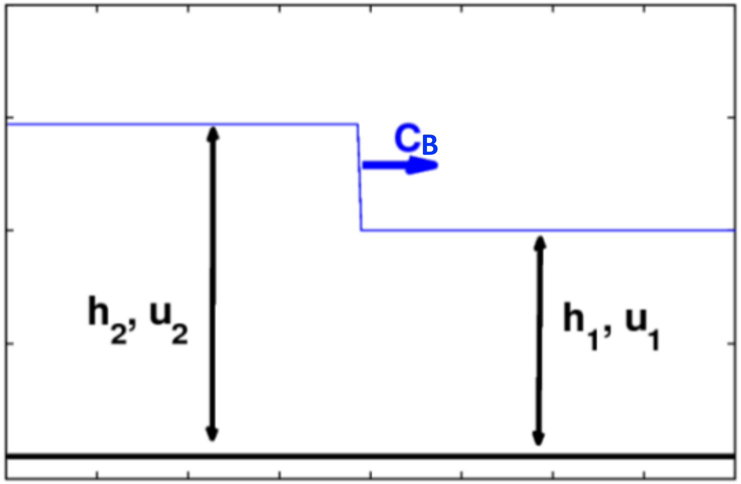
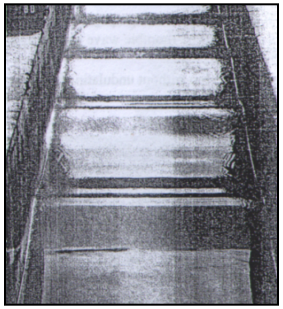
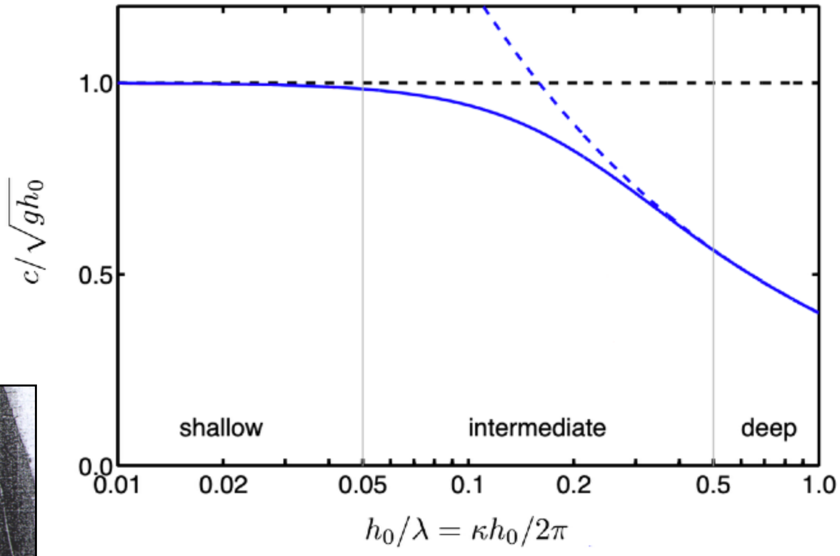
Bores (straight walled channels) Lemoine analogy







Bore:

Using the shallow water Rankine-Hugoniot relation (no dispersion !!):

$$C_b - U_2 = \sqrt{\frac{h_1}{h_2} g\bar{h}}$$

Water waves:

exact dispersion of Euler equations (Airy theory)

$$C_\lambda = \sqrt{g\frac{\lambda}{2\pi}\tanh(\frac{2\pi}{\lambda}h)}$$

$$C_b = U_2 + C_\lambda \Longrightarrow \lambda(Fr)$$

Lemoine, La Houille Blanche, 1948

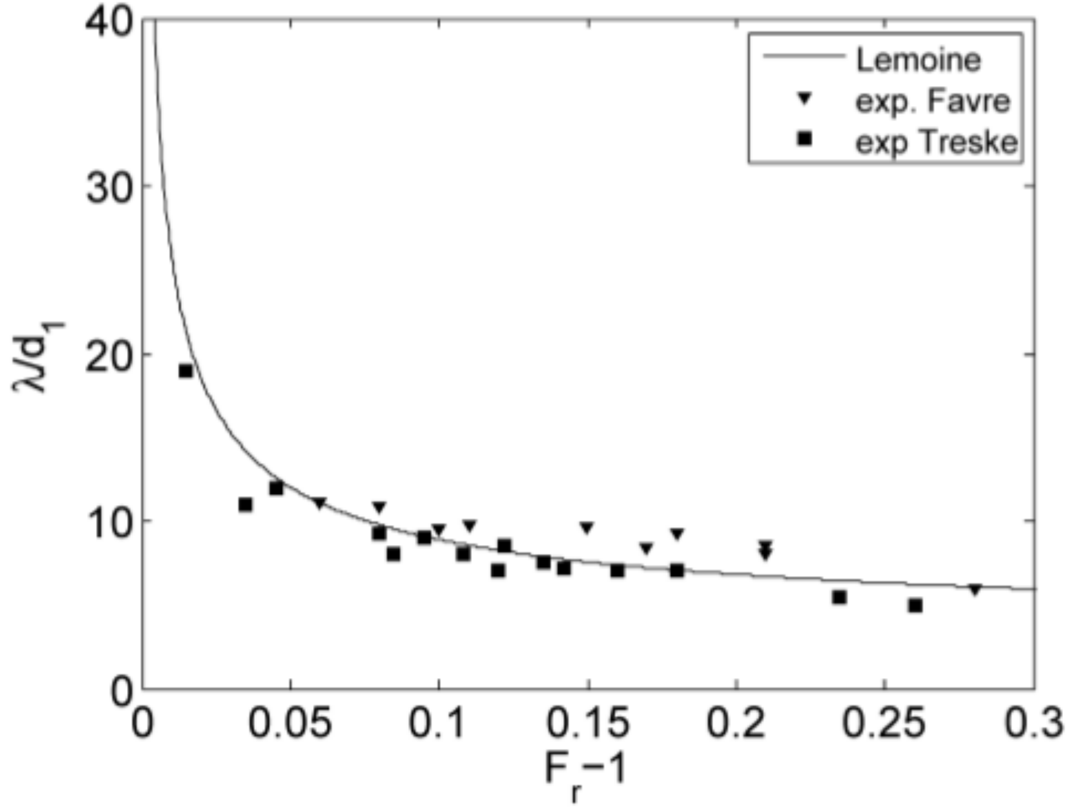


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Fig. 8. Undular bore at Froude ~ 1.04.

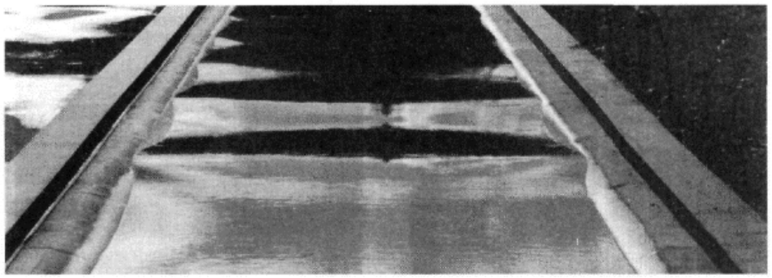


Fig. 9. Undular bore at Froude ~ 1.06.

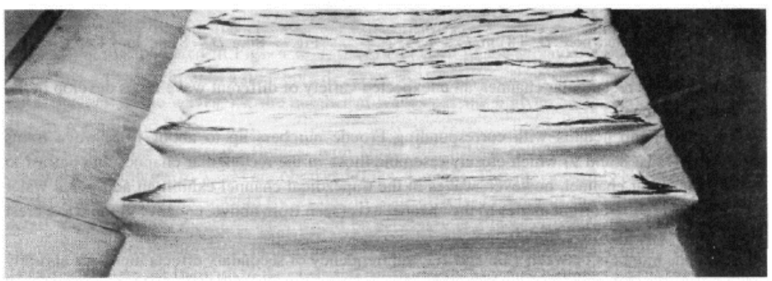


Fig. 10. Undular bore at Froude ~ 1.10.

Fr

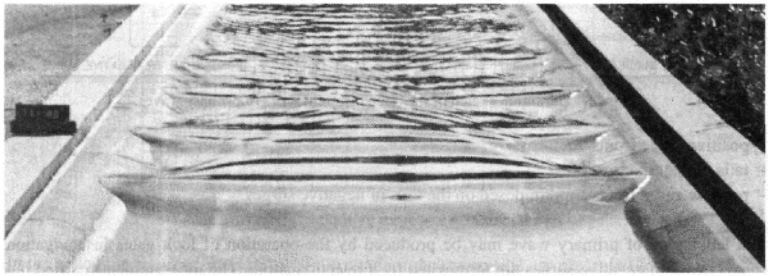


Fig. 11. Undular bore at Froude ~ 1.12.

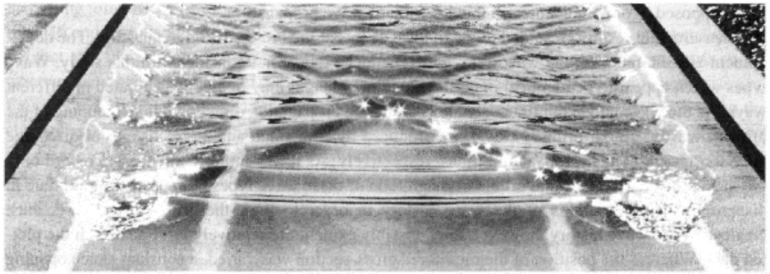


Fig. 12. Undular bore at Froude ~ 1.24.

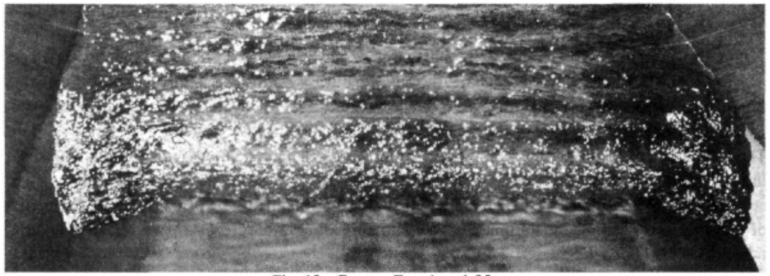
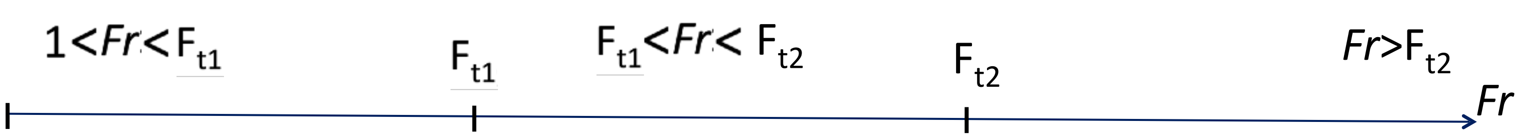


Fig. 13. Bore at Froude ~ 1.35.



Treske, *J. Hydraulic Research, 1994*

Treske's experiments

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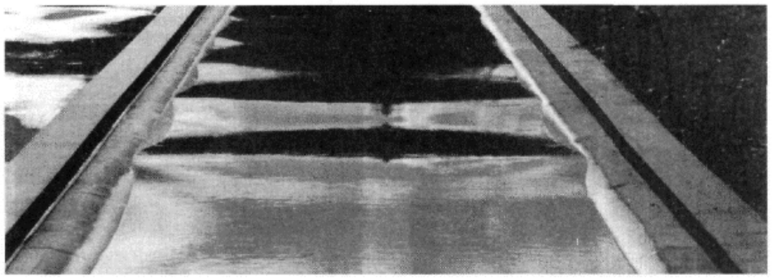

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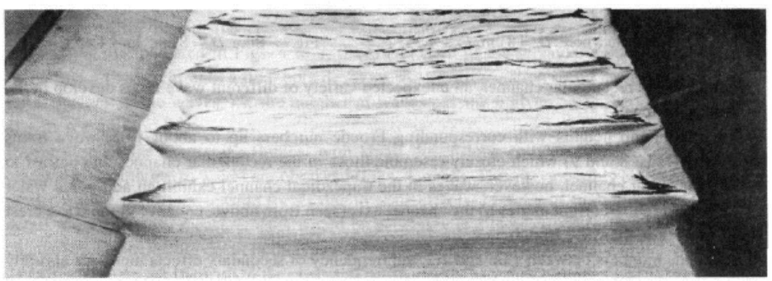

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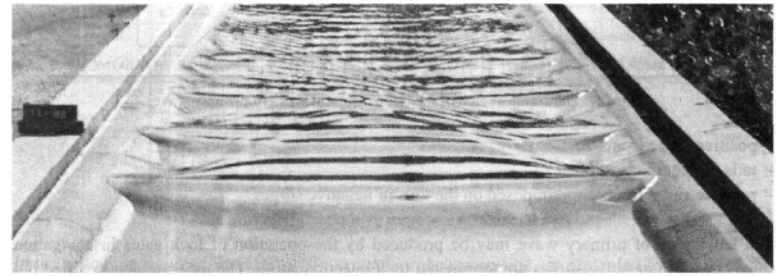
Fr



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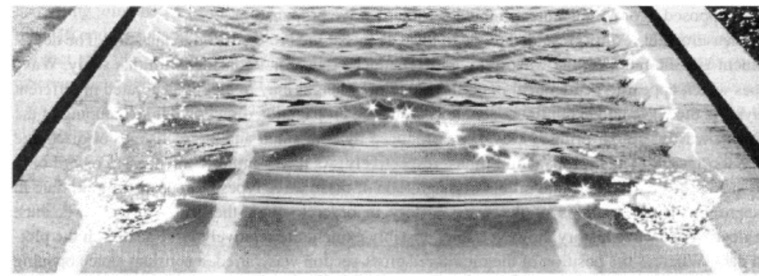

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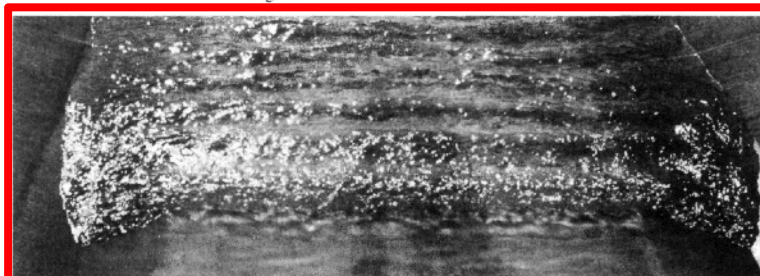

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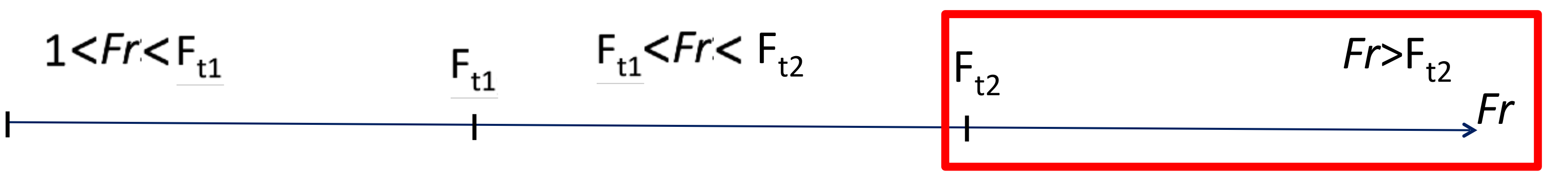

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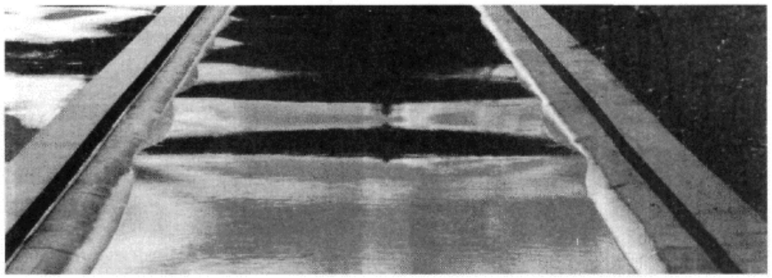

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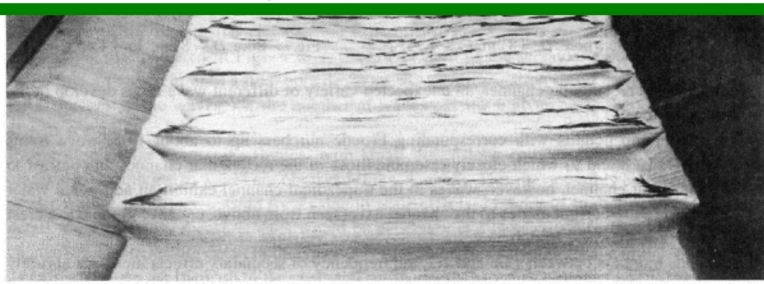

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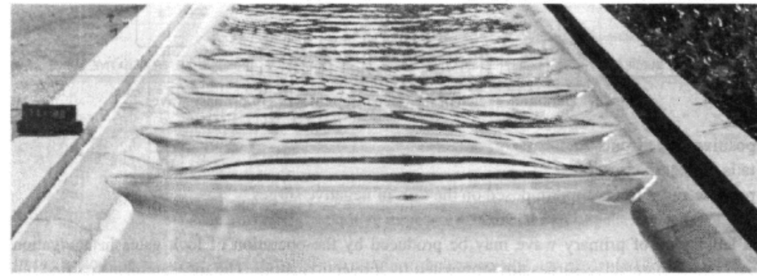
Fr



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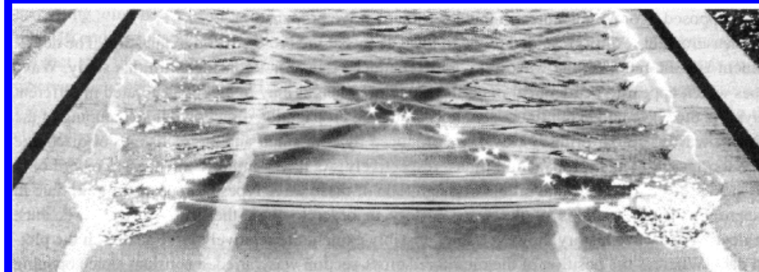

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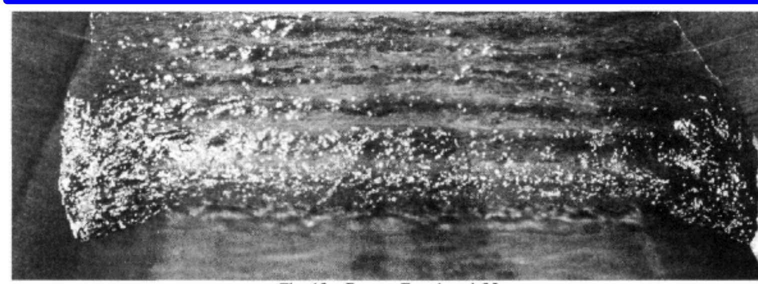

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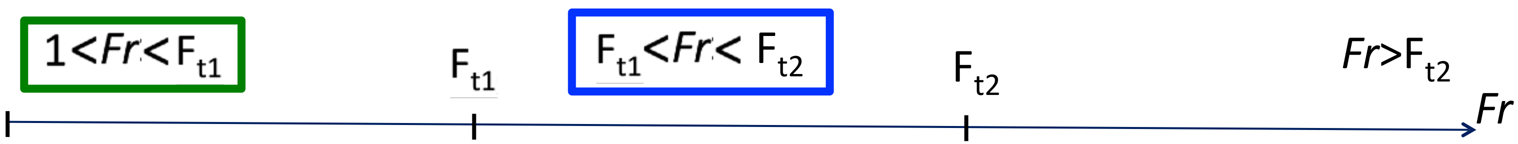

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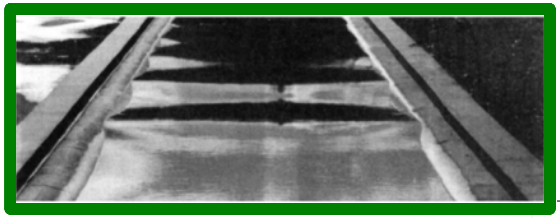

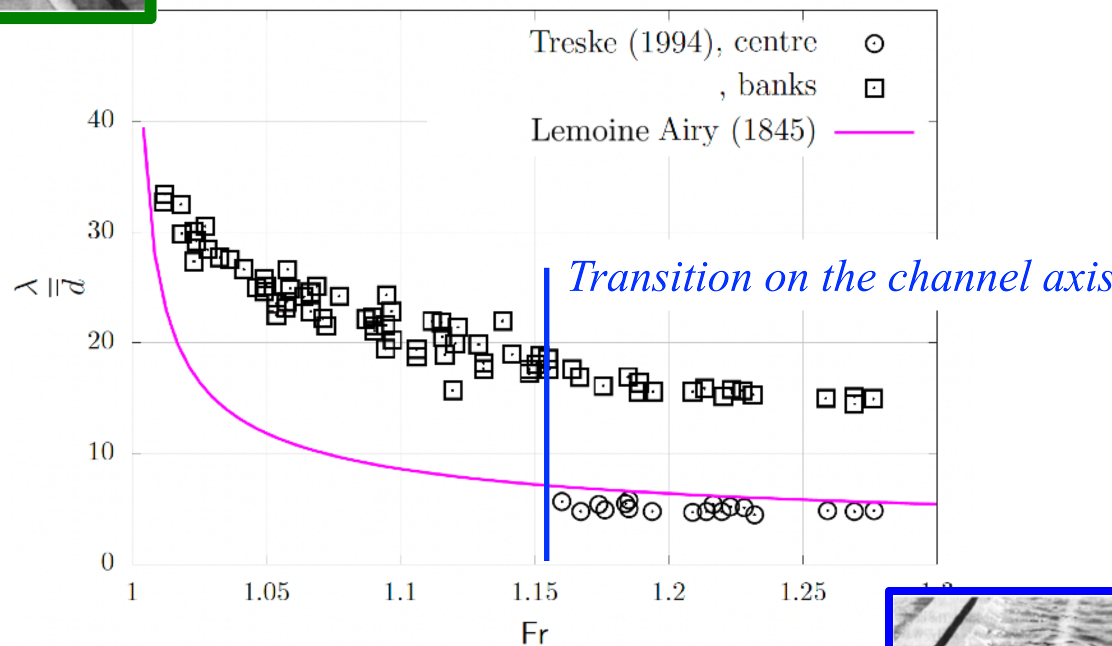

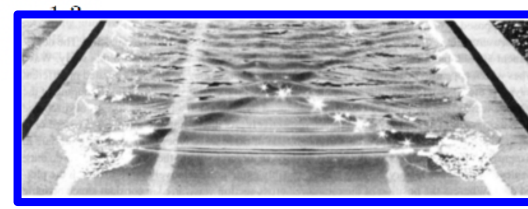

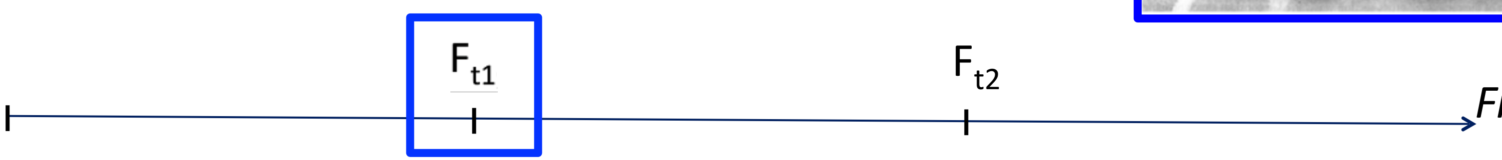

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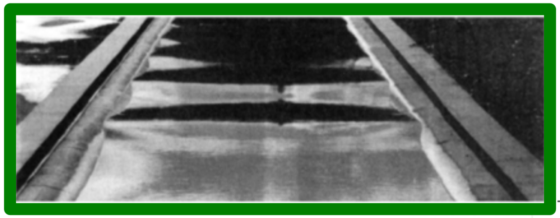
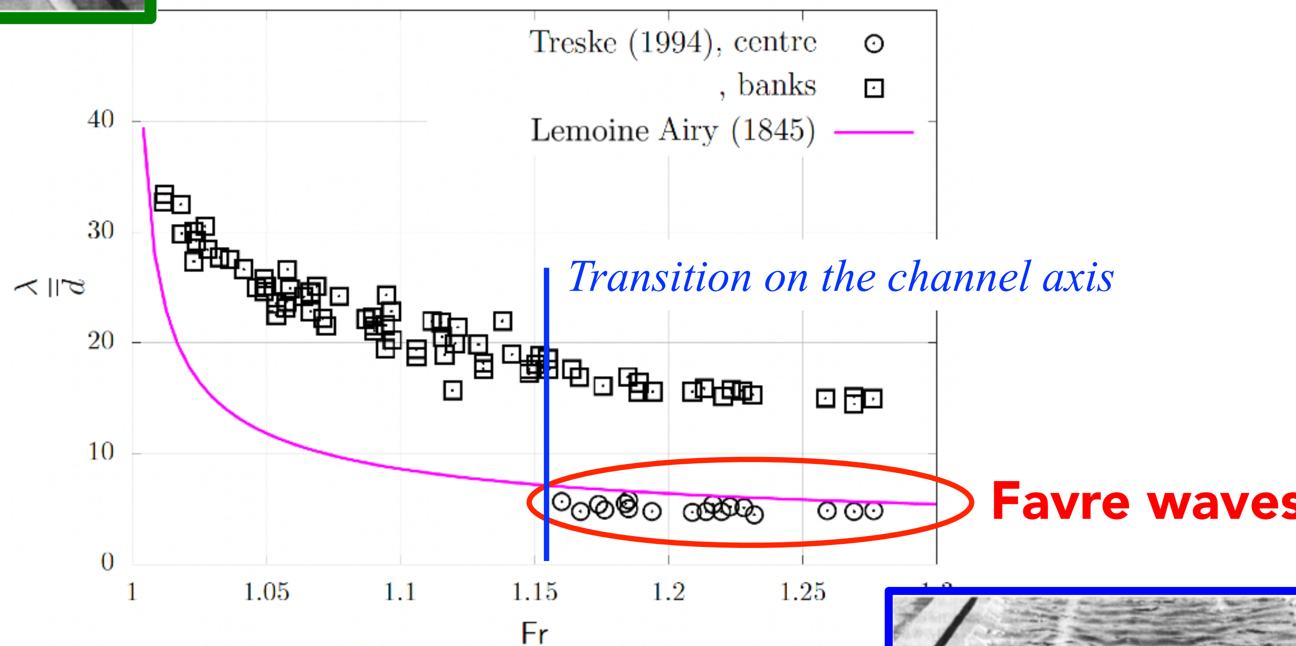
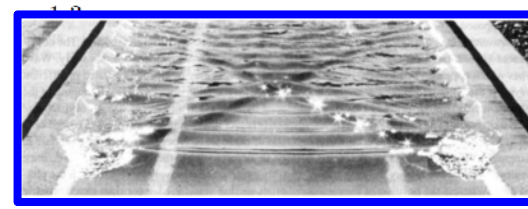
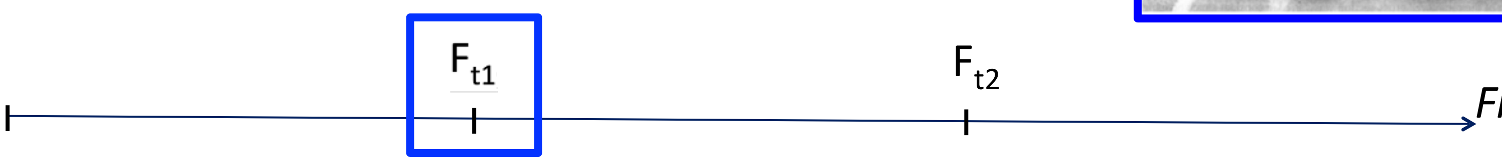


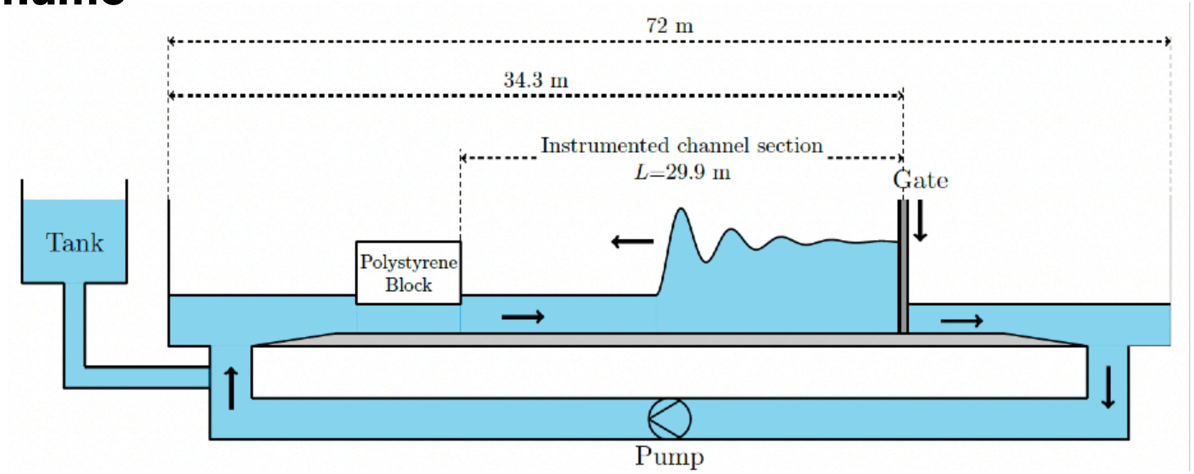






Treske, *J. Hydraulic Research, 1994*

EDF Chatou's flume



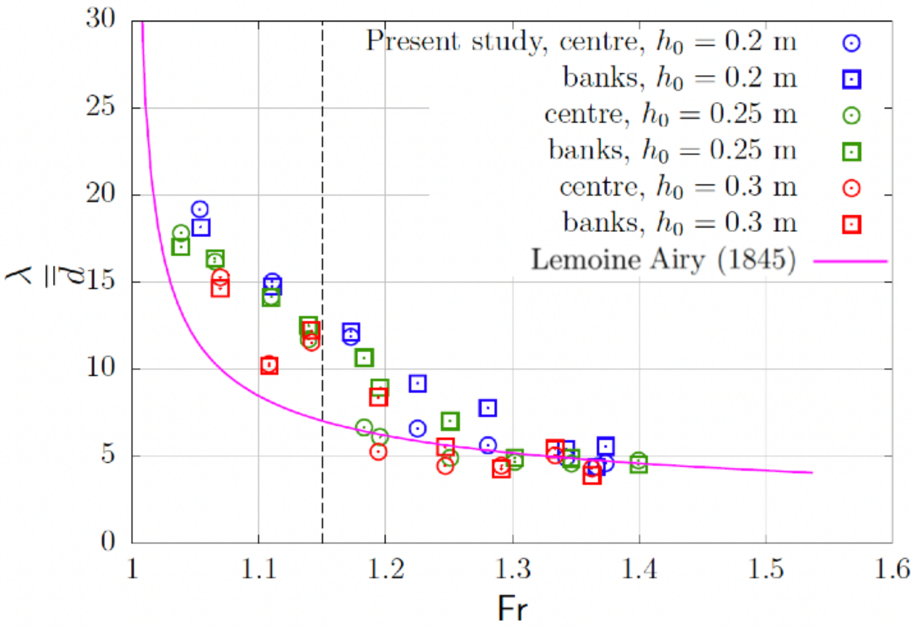
Fr = 1.13

Fr = 1.18

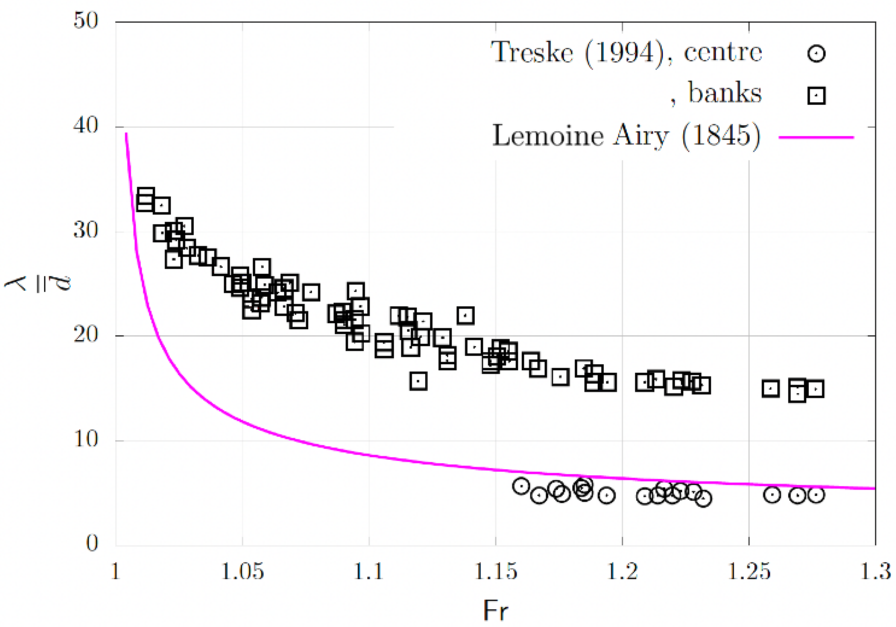
Jouy et al, IAHR-Int.Symp.Env.Hyd. 2024

Experiments in trapezoidal channels

Jouy's data



Treske data



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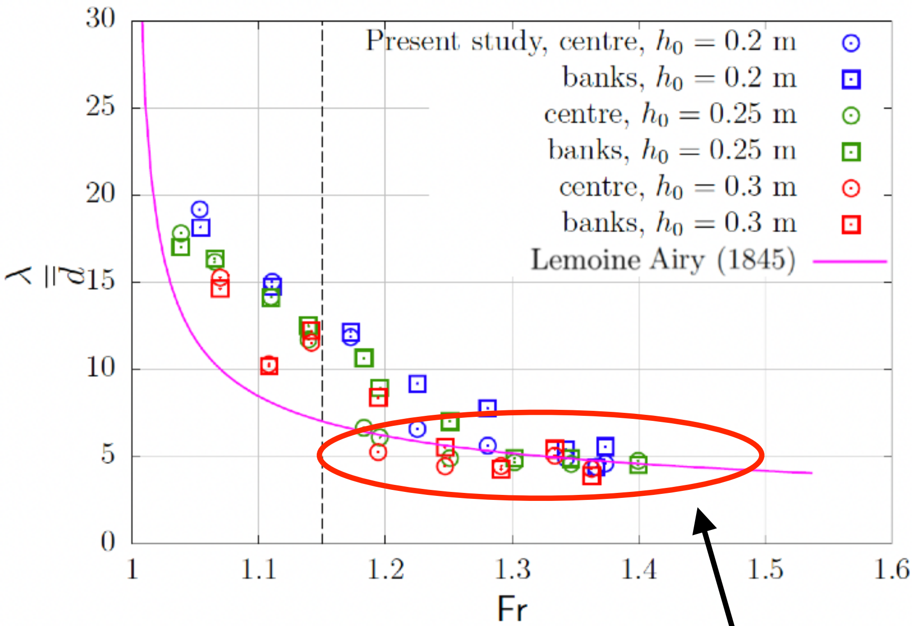

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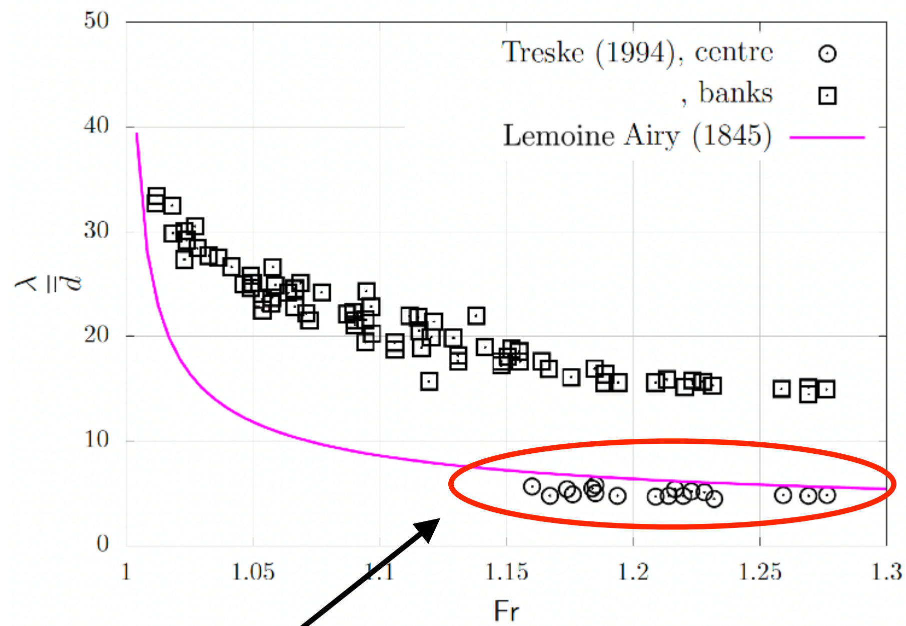


Favre waves

Low Fr transition in Seine and Gironde: the unseen Mascaret

3 field campaigns :
a unique long-term high-frequency database



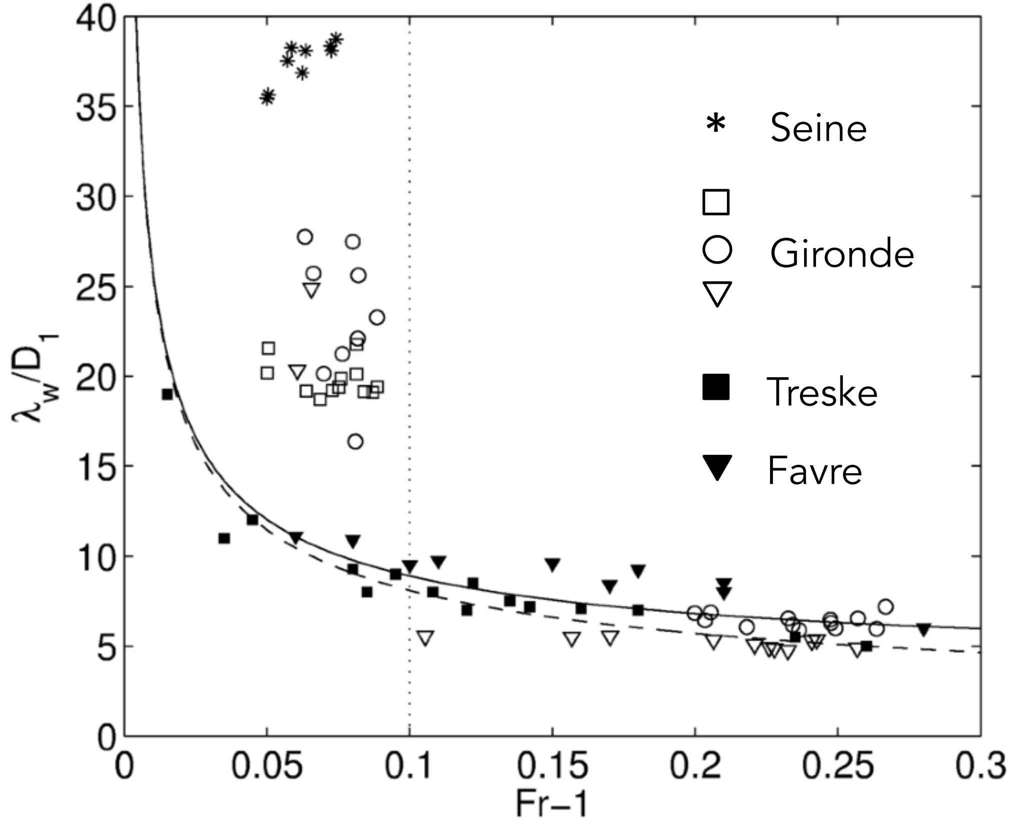


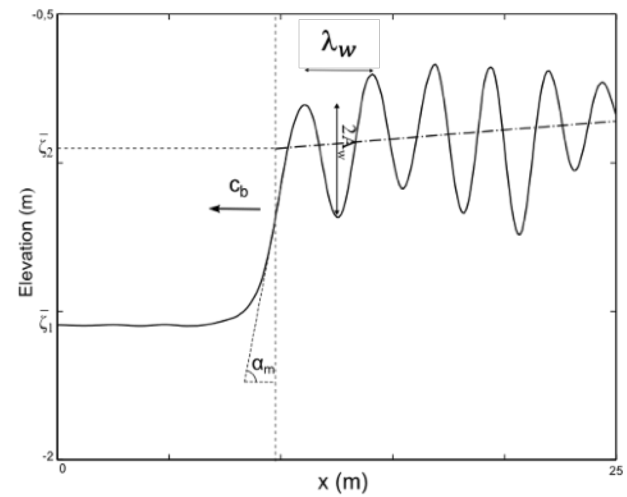


Bonneton et al, *Comptes Rendus Geoscience*, 2012

Bonneton et al, *J. Geophysical Research - Oceans*, 2015

Baptised *Ressaut de marée* in **Bonneton et al**, *Comptes Rendus Geoscience*, 2012
- not visible naked eyes
- mechanism not known





Common undular tidal bore (mascaret): Favre wave

Transition for Fr=F_T (F_T≈1.1)

Baptised *Ressaut de marée* in **Bonneton et al**, *Comptes Rendus Geoscience*, 2012
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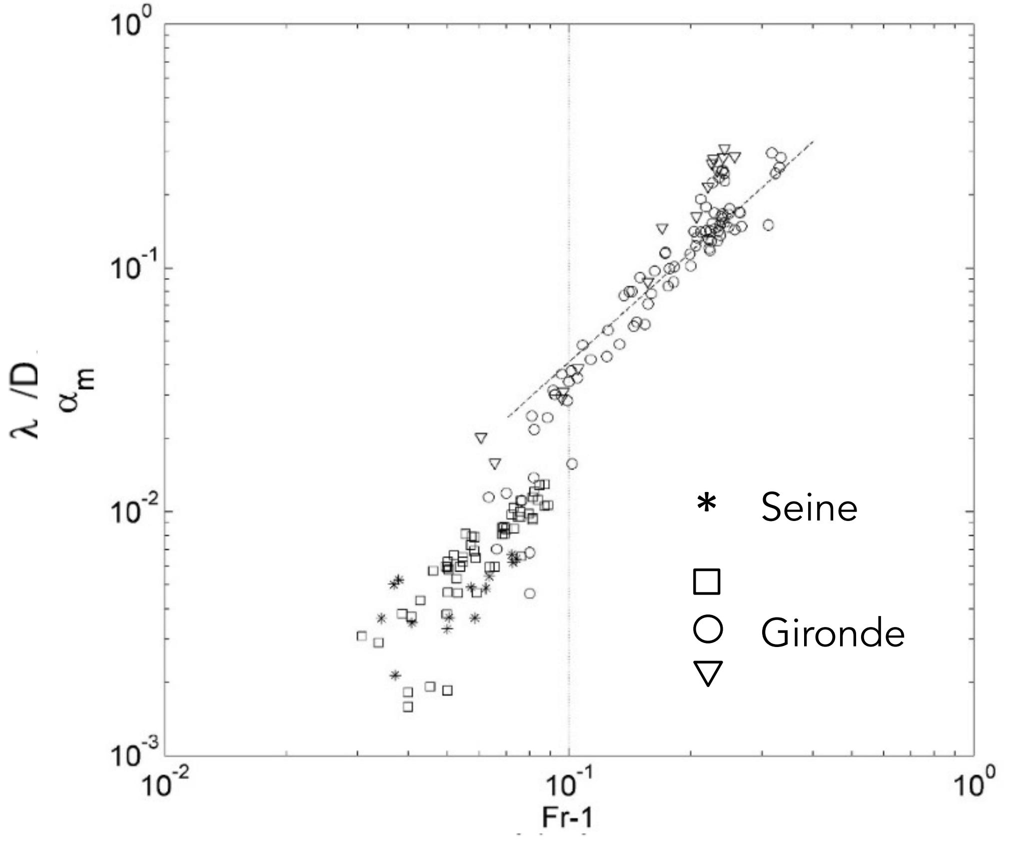

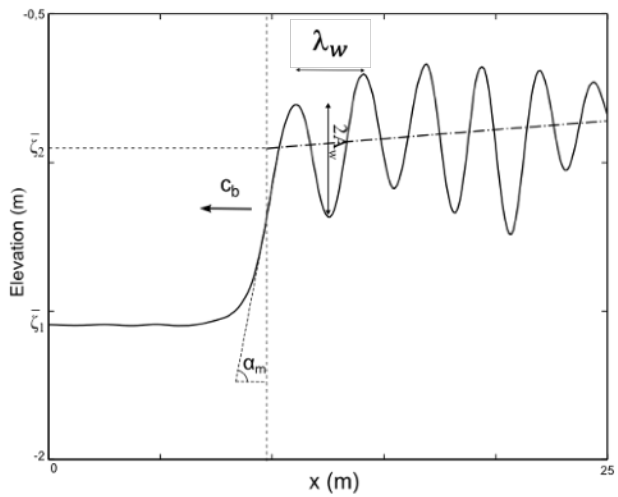

Low steepness secondary wave regime
→ **not visually observable**

high steepness secondary wave regime
→ **mascaret**

Common undular tidal bore (mascaret): Favre wave

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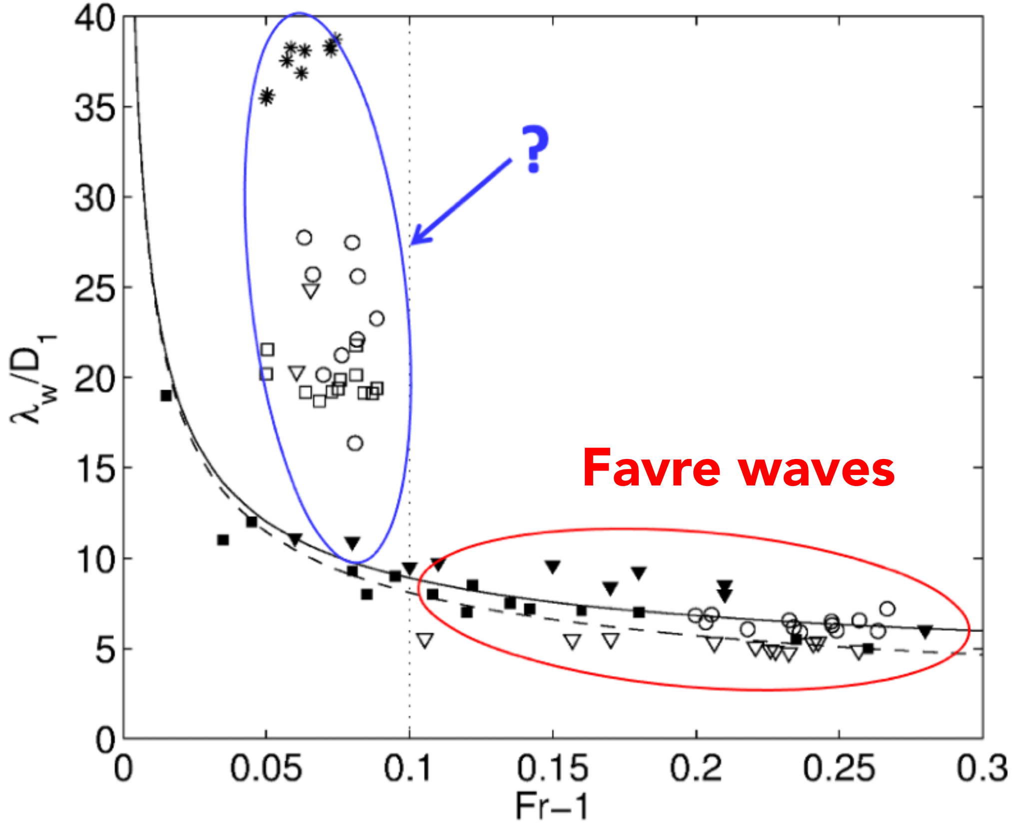
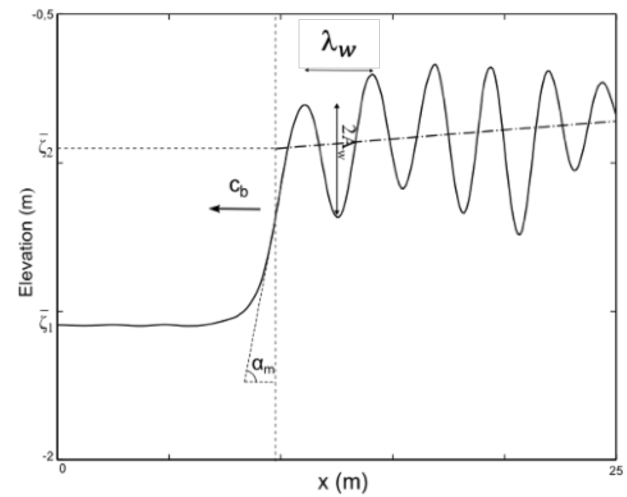
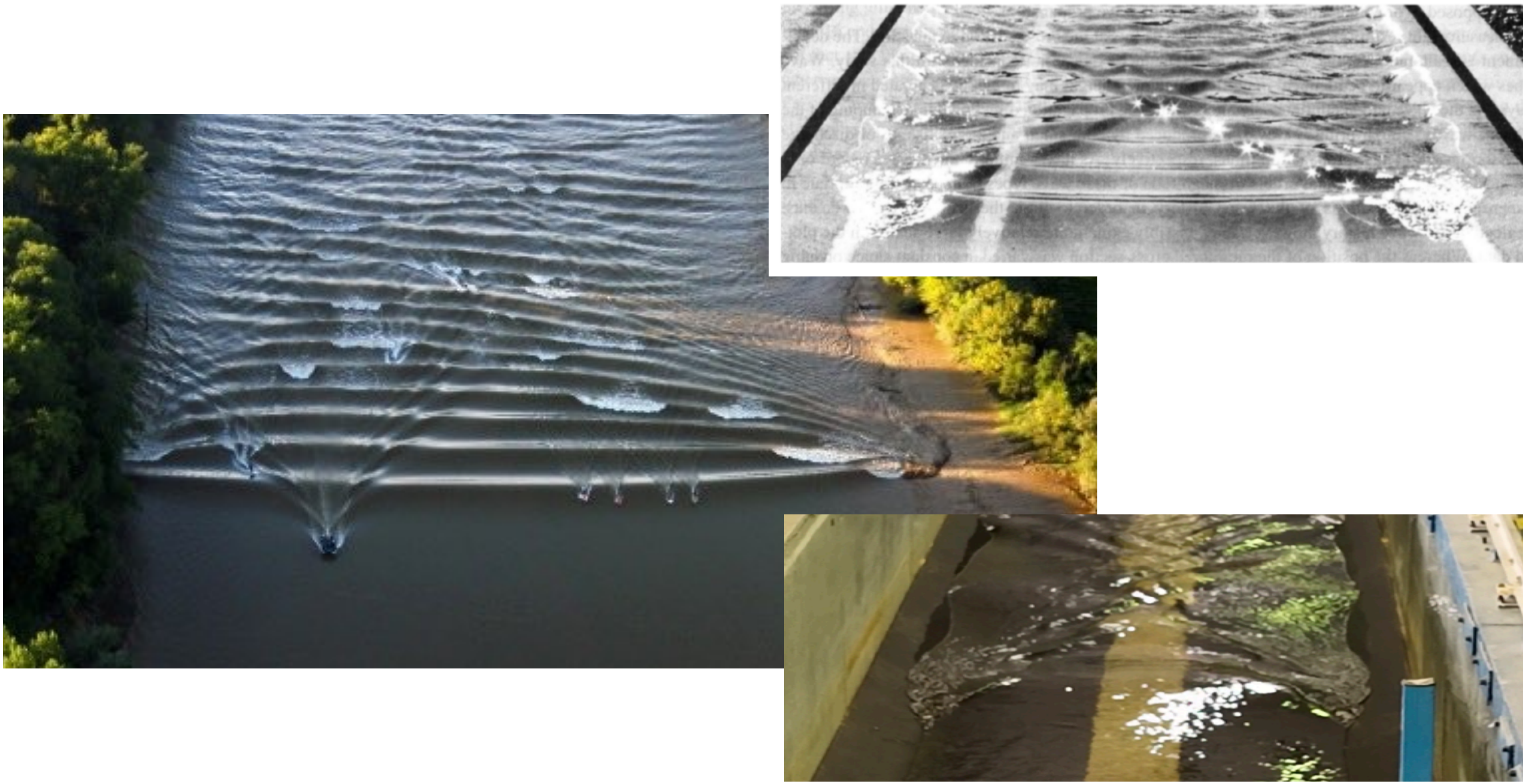




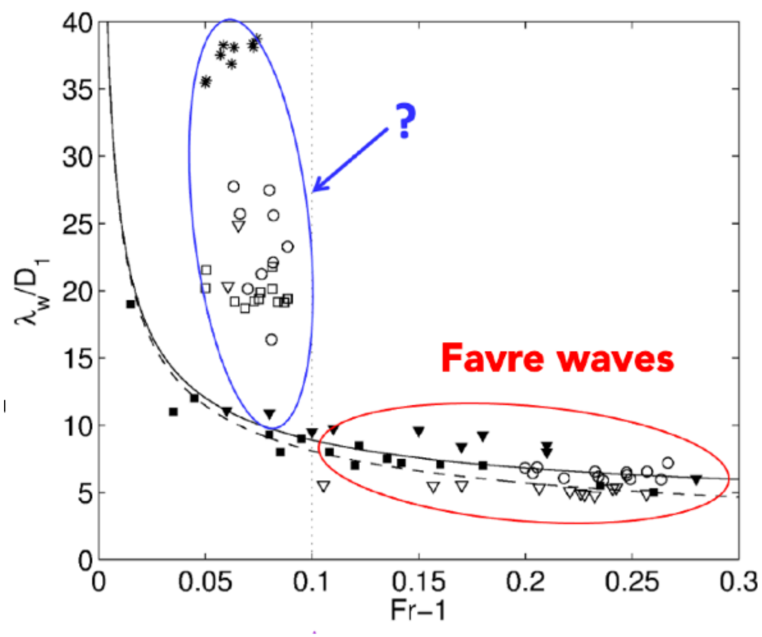
Common undular tidal bore (mascaret): Favre wave

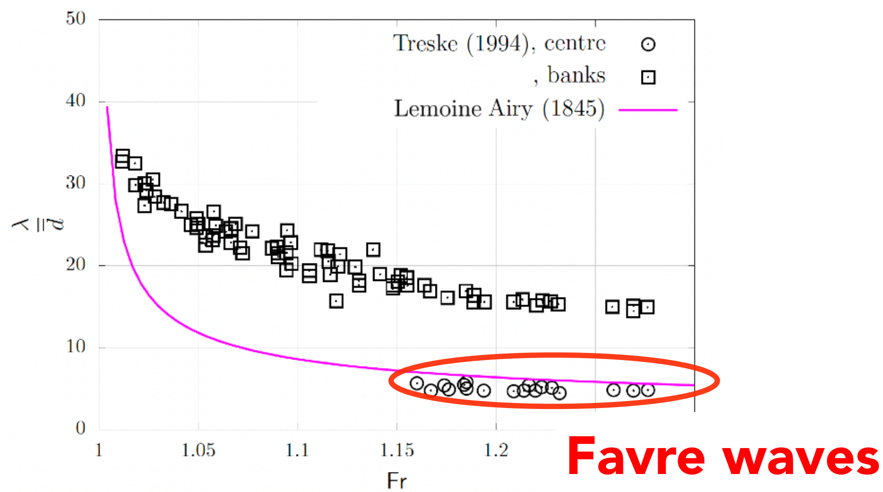
Transition for **Fr=F$_T$** ($F_T \approx 1.1$)

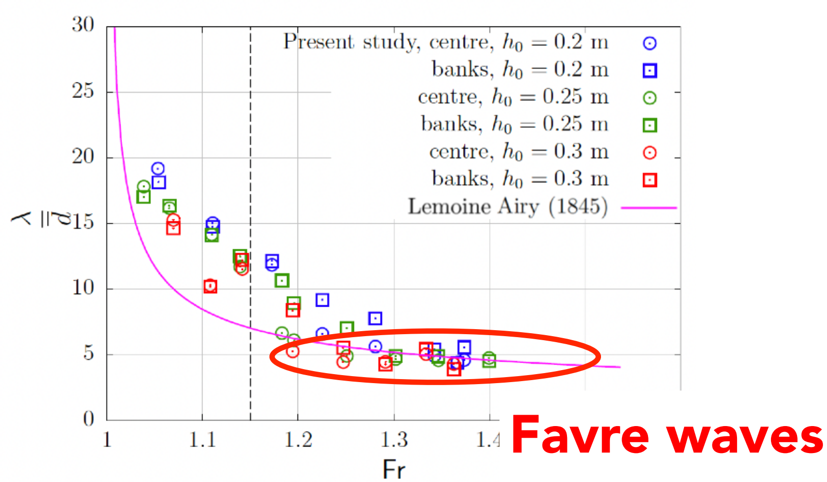
Similarities between the low Fr transition in natural and artificial environnements



Similarities between the low Fr transition in natural and artificial environnements



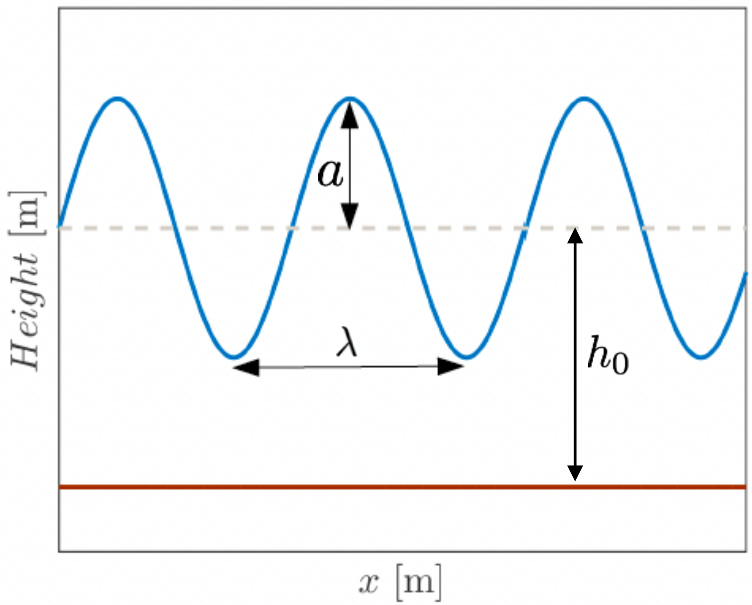




Numerical modelling

using

Serre-Green-Nagdhi and Shallow Water (!)



Dimensionless parameters

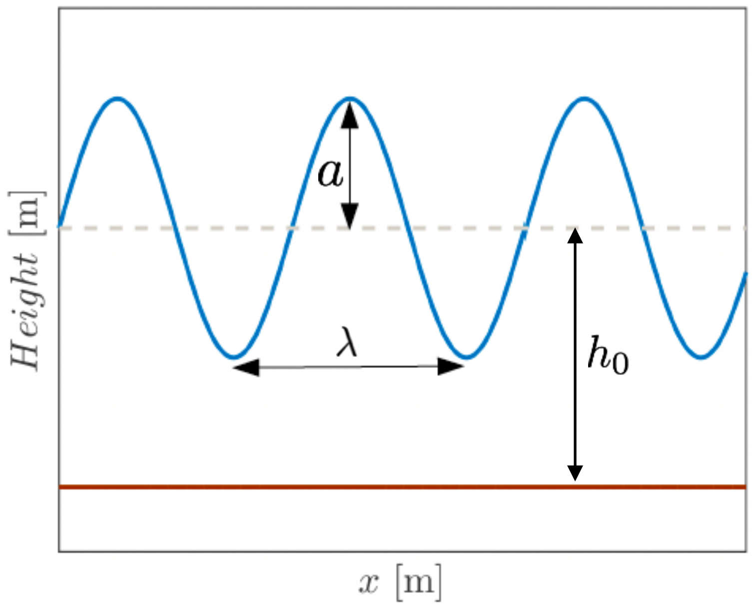
- dispersion: $\mu = \dfrac{h_0}{\lambda} = \dfrac{\kappa h_0}{2\pi}$
- non-linearity: $\epsilon = \dfrac{a}{h_0}$

Physical hypotheses

Long waves : small μ

Weakly dispersive waves : $\mu^2 \ll 1\,,\quad \mu^4$ negligible

Weak/full non-linearity : $\epsilon = \mathcal{O}(\mu^2)$ and $\epsilon = \mathcal{O}(1)$ respectively



Dimensionless parameters

- dispersion: $\mu = \frac{h_0}{\lambda} = \frac{\kappa h_0}{2\pi}$
- non-linearity: $\epsilon = \frac{a}{h_0}$

This was
β in Jerry's talk
σ in Angel's talk

Physical hypotheses

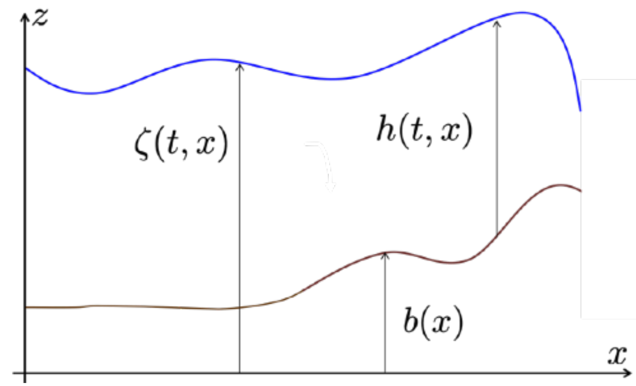
This is the $S = 1$
of Jerry's talk

Long waves : small μ

Weakly dispersive waves : $\mu^2 \ll 1$, μ^4 negligible

Weak/full non-linearity : $\epsilon = \mathcal{O}(\mu^2)$ and $\epsilon = \mathcal{O}(1)$ respectively

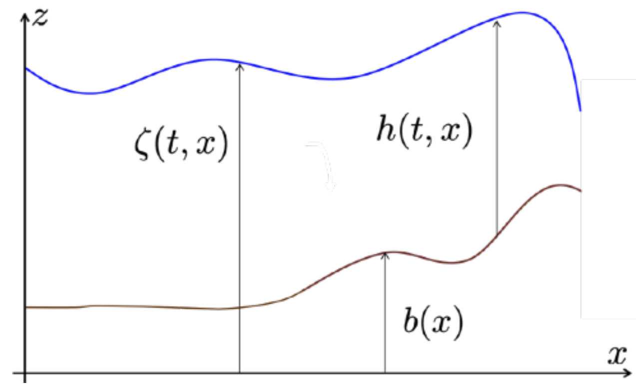
Shallow water equations in 1D



$$\partial_t h + \partial_x(h\mathrm{u}) = 0$$
$$\partial_t(h\mathrm{u}) + \partial_x(h\mathrm{u}^2 + gh^2/2) + gh\partial_x b = 0$$

$\mu = 0$ limit, or equivalently zero-th order term in the μ expansion

Enhanced Serre-Green-Naghdi equations in 1D



$$\partial_t h + \partial_x(h\mathrm{u}) = 0$$
$$\partial_t(h\mathrm{u}) + \partial_x(h\mathrm{u}^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = \alpha\partial_x(h^2\partial_x\dot{\mathrm{u}}) + (\alpha - 1)\partial_x(h^2\partial_{xx}\zeta) + \mathcal{Q}(\mathrm{u}, b)$$

μ^2 correction

here $\epsilon = \mathcal{O}(1)$

$$\mathcal{Q}(\mathrm{u}, b) = h\partial_x h^2(\partial_x\mathrm{u})^2 + \frac{2}{3}h^3\partial_x(\partial_x\mathrm{u})^2$$
$$+ h^2\partial_x b(\partial_x\mathrm{u})^2 + \frac{h}{2}\partial_{xx}b\partial_x\mathrm{u}^2 + \left(\partial_x(h^2\partial_{xx}b) + \partial_x(\partial_x b)^2\right)\frac{\mathrm{u}^2}{2}$$

Green & Naghdi, J.Fluid Mech, 1976

Chazel et al, J.Sci.Comp. 2011

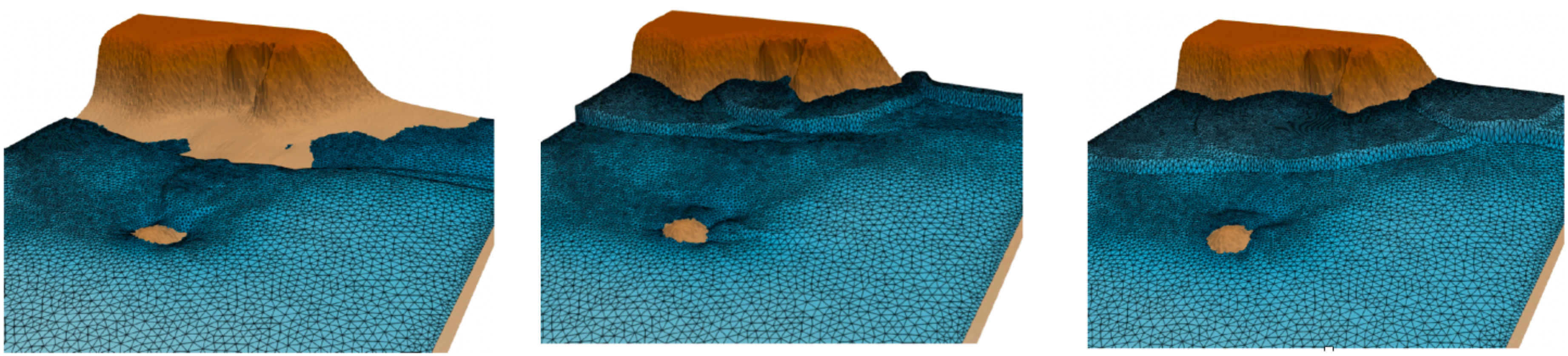
Lannes, Nonlinearity 2020

Equations in 2D

Kazolea et al, Ocean Mod., 2023

$$\partial_t h + \nabla \cdot (h\mathrm{u}) = 0$$
$$\partial_t (h\mathrm{u}) + \nabla \cdot (h\mathrm{u} \otimes \mathrm{u}) + gh\nabla\zeta = \mathcal{D}$$

Shallow water eq.s for $\mathcal{D} = 0$



Equations in 2D

Kazolea et al, Ocean Mod., 2023

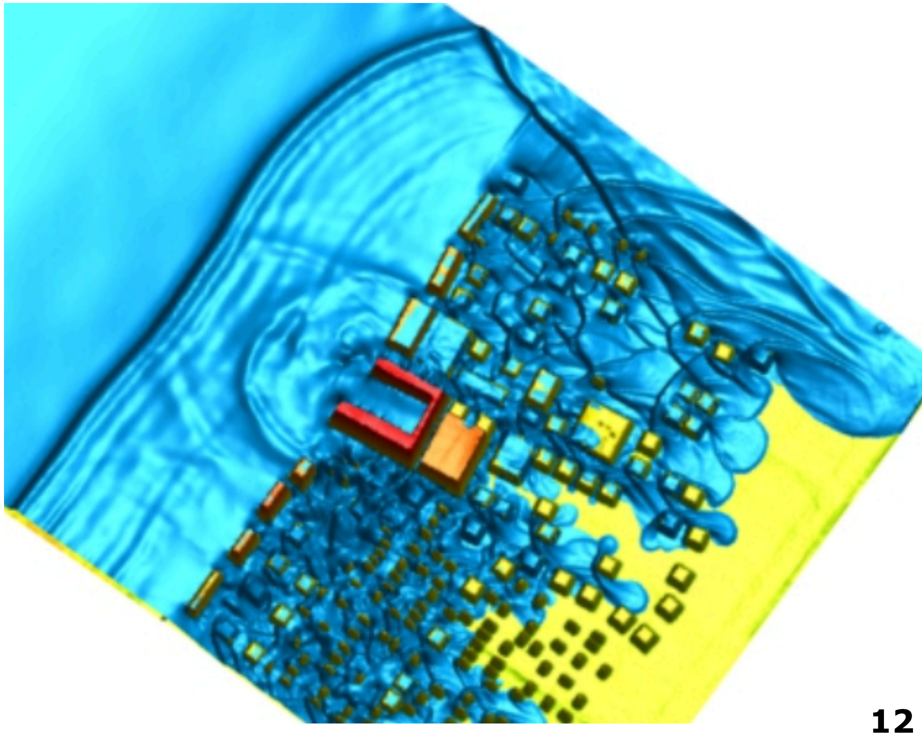
$$\partial_t h + \nabla \cdot (h\mathsf{u}) = 0$$
$$\partial_t (h\mathsf{u}) + \nabla \cdot (h\mathsf{u} \otimes \mathsf{u}) + gh\nabla\zeta = \mathcal{D}$$

Optimized SGN if $\mathcal{D}$ satisfies
(constant bathymetry case)

$$\mathcal{D} + \alpha \mathrm{T}_h(\mathcal{D}) = \mathcal{R}(h, \mathsf{u}, b)$$

with

$$\mathrm{T}_h(\mathsf{v}) = -\nabla(h\nabla \cdot \mathsf{v}) + \nabla(\mathsf{v} \cdot \nabla h)$$
$$\mathcal{R}(h, \mathsf{u}, b) = \mathrm{T}_h(gh\nabla\zeta) + \mathcal{Q}(\mathsf{u})$$



$$\partial_t A(h) + \partial_x (A(h)U) = 0$$

$$\partial_t (A(h)U) + \partial_x (A(h)U^2 + K(h)) = 0$$

$$\partial K = gh\partial A$$

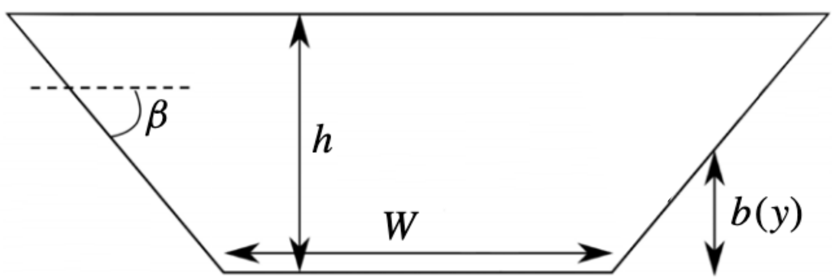
Smoothed initial discontinuous state from Rankine-Hugoniot condition of section averaged shallow water system

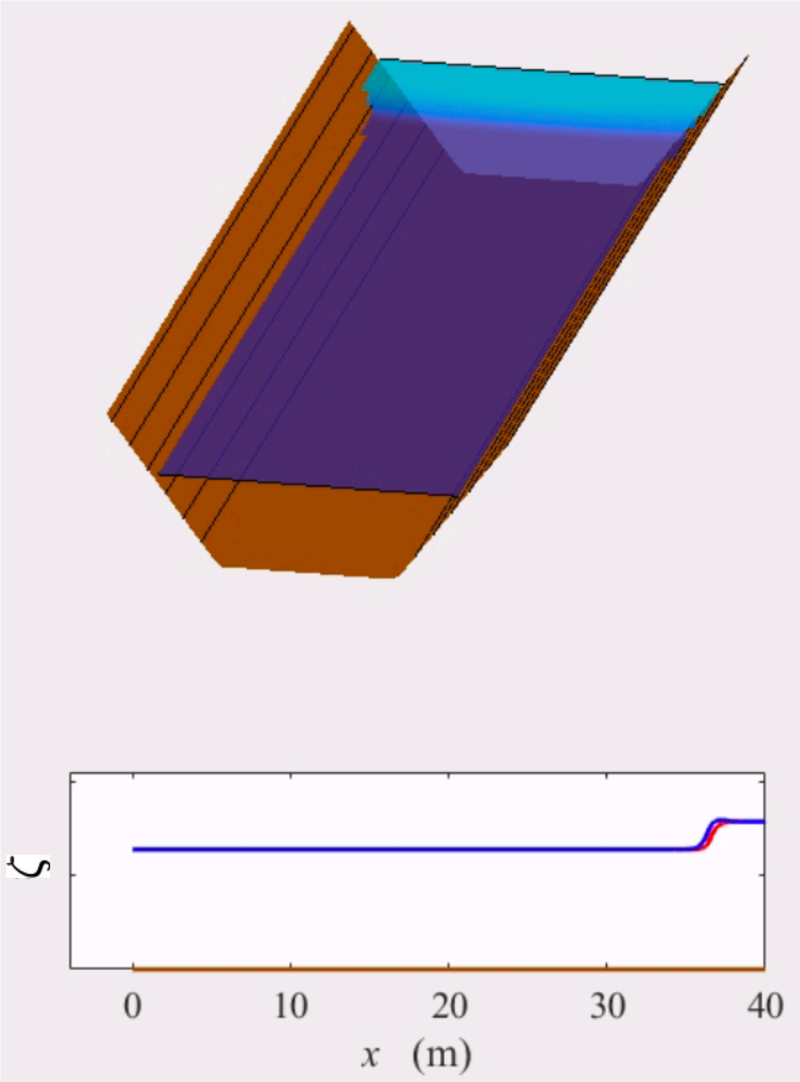
Chanson, Elsevier, 2024

for a trapezium

$$A(h) = Wh + \frac{h^2}{\tan\beta}$$

$$K(h) = Wg\frac{h^2}{2} + \frac{g}{\tan\beta}\frac{h^3}{3}$$





Smoothed initial discontinuous state from Rankine-Hugoniot condition of section averaged shallow water system

$$h = h_2 + \frac{h_2 - h_1}{2}\tanh\left(\frac{x - x_0}{\ell}\right) - b(y)$$

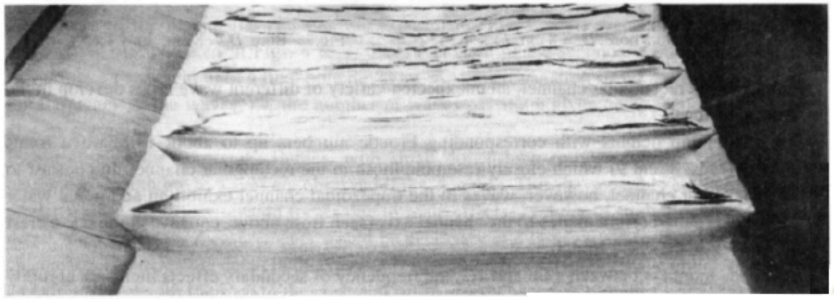
$$u = u_2 + \frac{u_2 - u_1}{2}\tanh\left(\frac{x - x_0}{\ell}\right)$$

for different Froude numbers

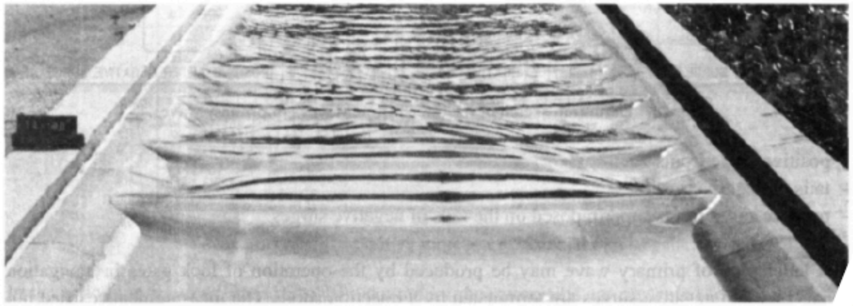
$$\mathrm{Fr} := \frac{u_1 - C_b}{\sqrt{g\bar{h}_1}}, \quad \bar{h}_1 = \frac{A_1}{\partial_h A(h_1)}$$

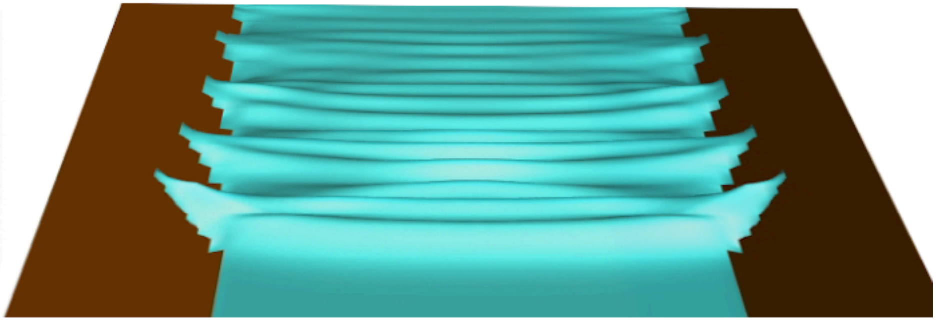
2D simulations (trapezoidal) | Treske's experiments

Fr = 1.10



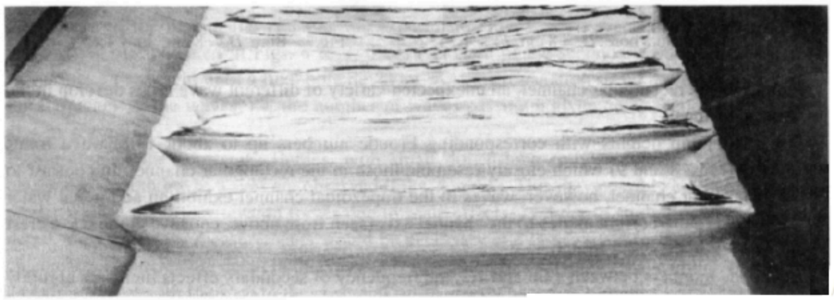
Fr = 1.17



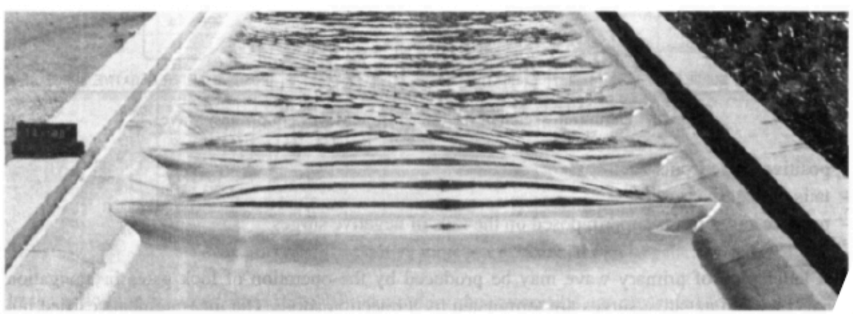


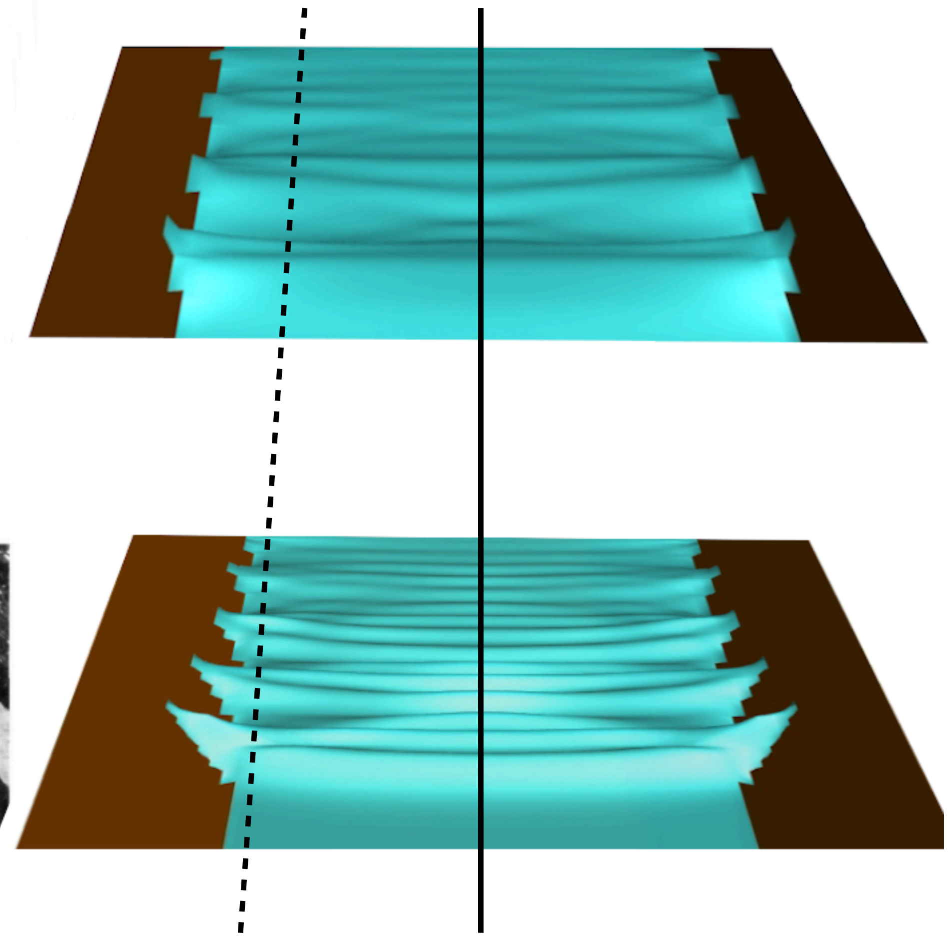
2D simulations (trapezoidal) | Treske's experiments

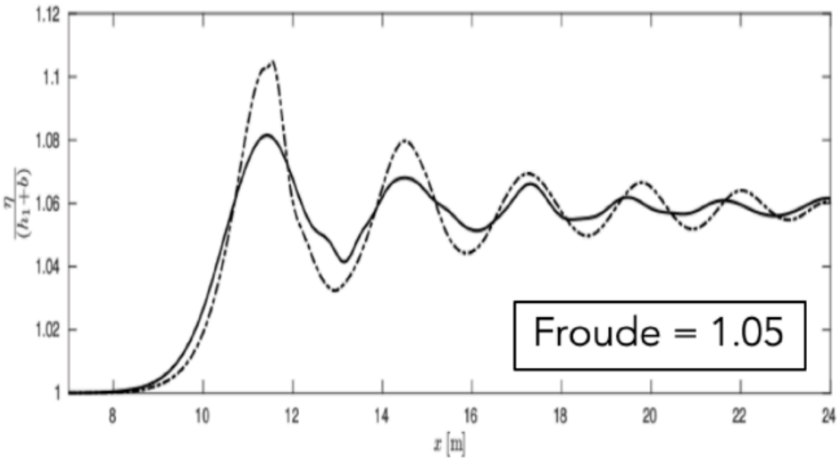
$Fr = 1.10$

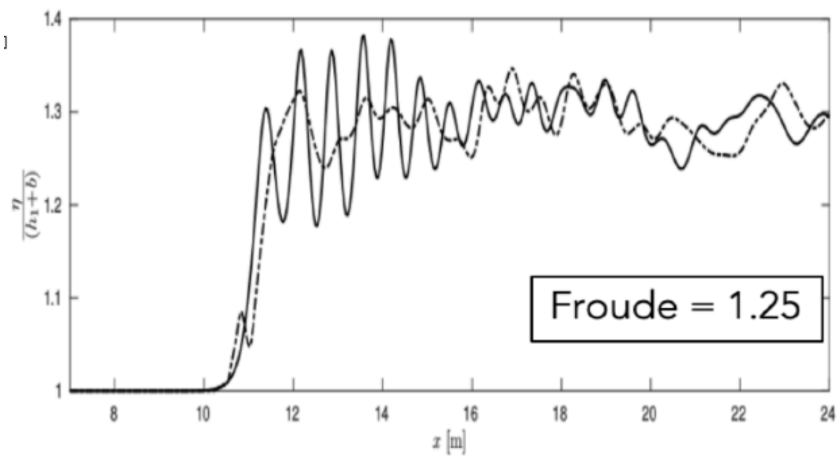


$Fr = 1.17$







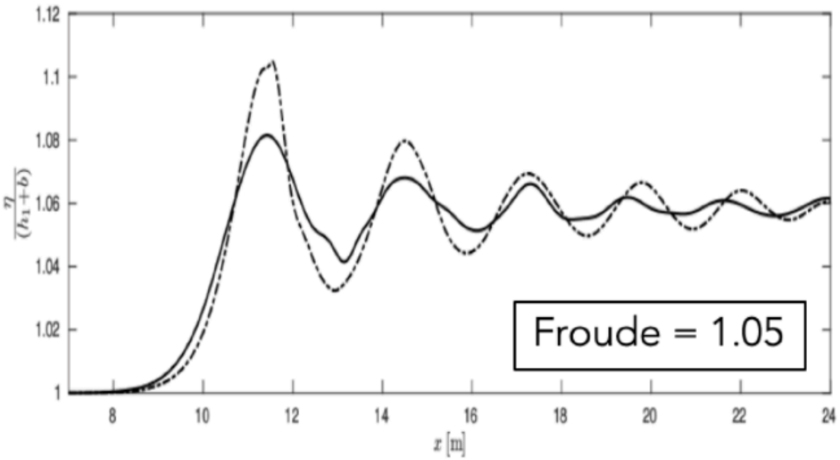


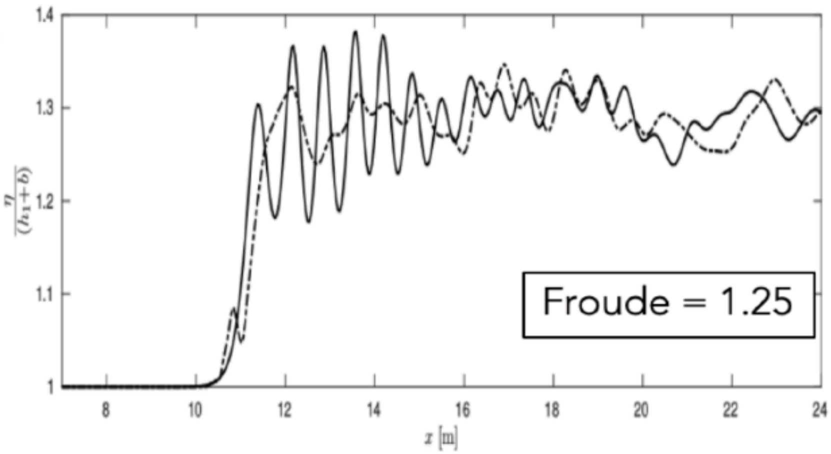
- - - - - - - - - - - banks

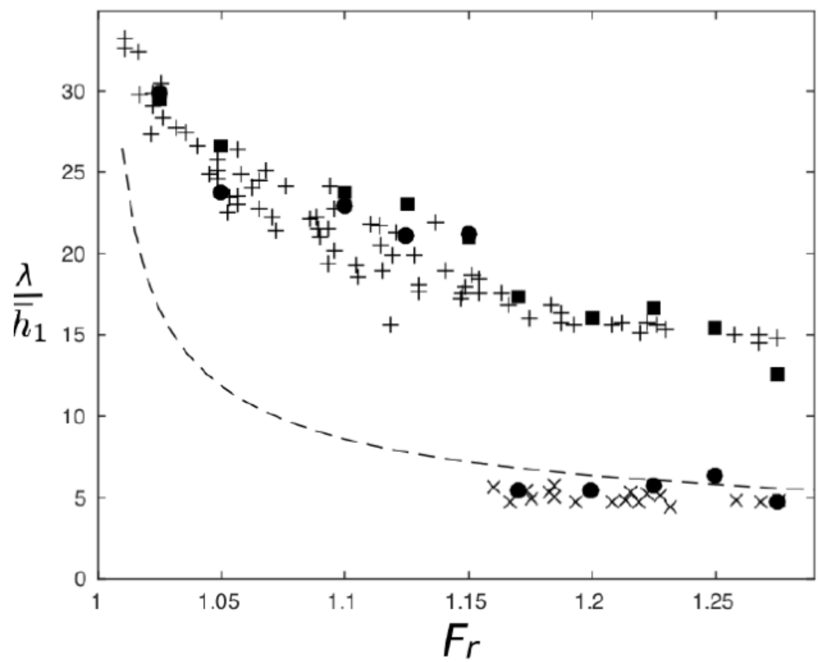
——————— axis

2D simulations (trapezoidal)

Treske's experiments







- - - - - - banks

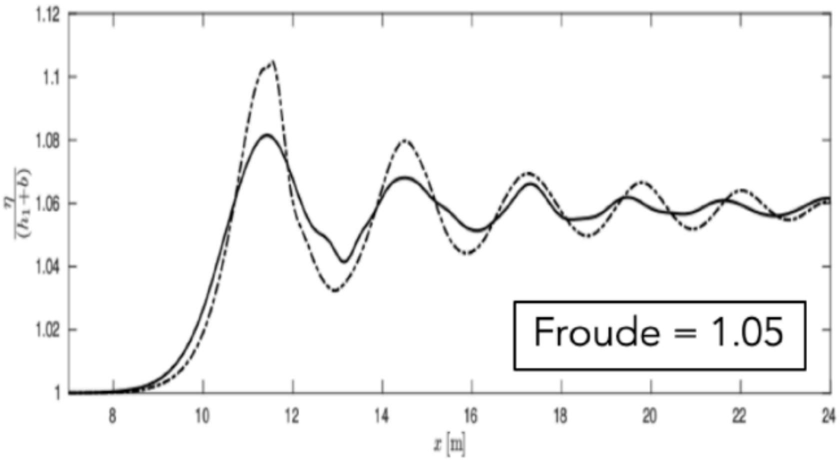
——— axis

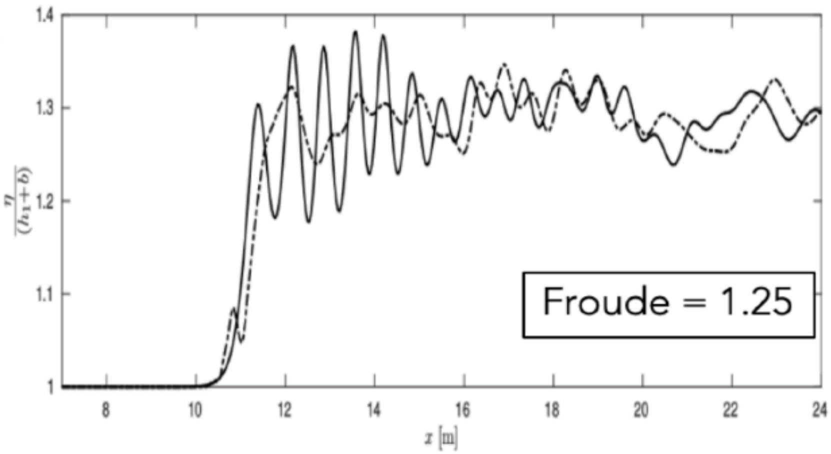
■ SGN banks + Treske banks

● SGN axis x Treske axis

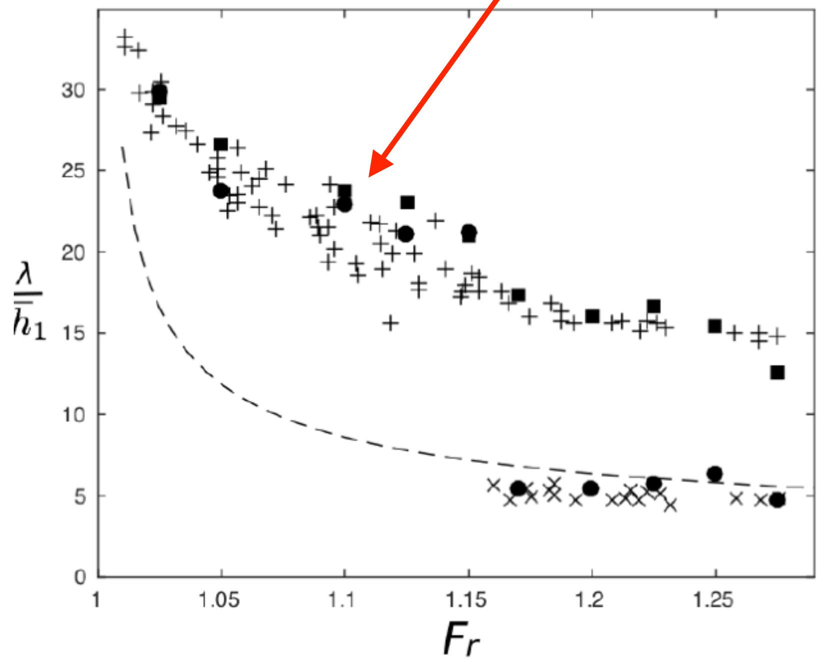
2D simulations (trapezoidal)

Treske's experiments





So, what are these ?



- - - - - - - - - - banks

———— axis

■ SGN banks + Treske banks

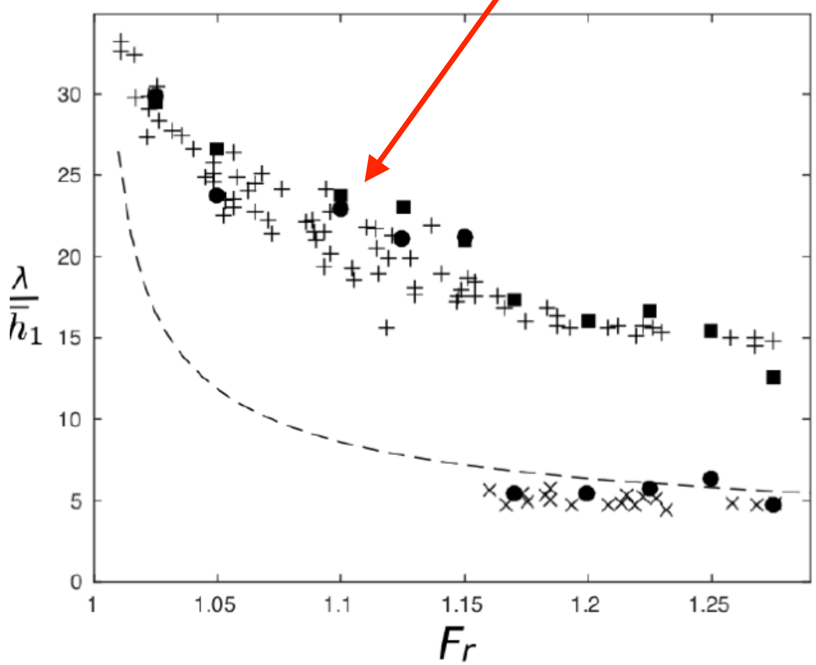
● SGN axis x Treske axis

Several elements hint that it may be an hydrostatic process:

- It is predominant on the banks (very shallow limit)
- It involves long(er) waves
- Previous work on dispersion in wave propagation in heterogenous media:

Ketcheson & Quessada de Luna, SIAM Multiscale Mod. Simul., 2015

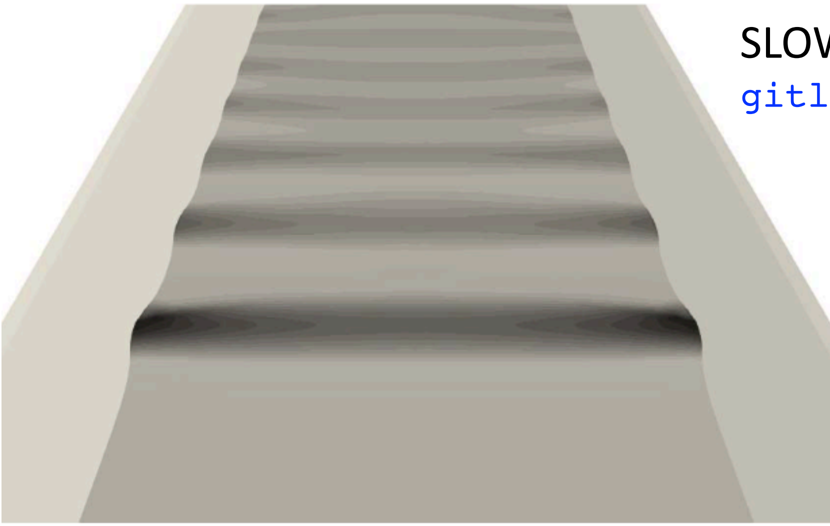
So, what are these ?



■ SGN banks
+ Treske banks
● SGN axis
x Treske axis

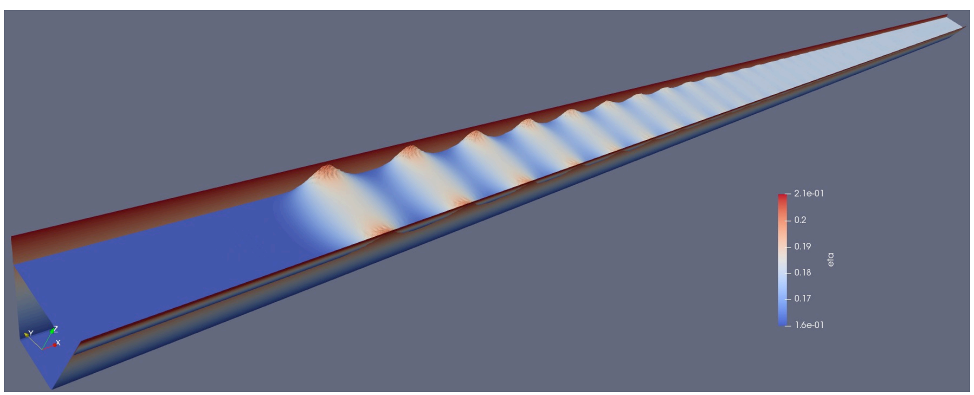
Shallow water simulations with different codes for Fr = 1.05

Shallow water simulations with different codes for Fr = 1.05

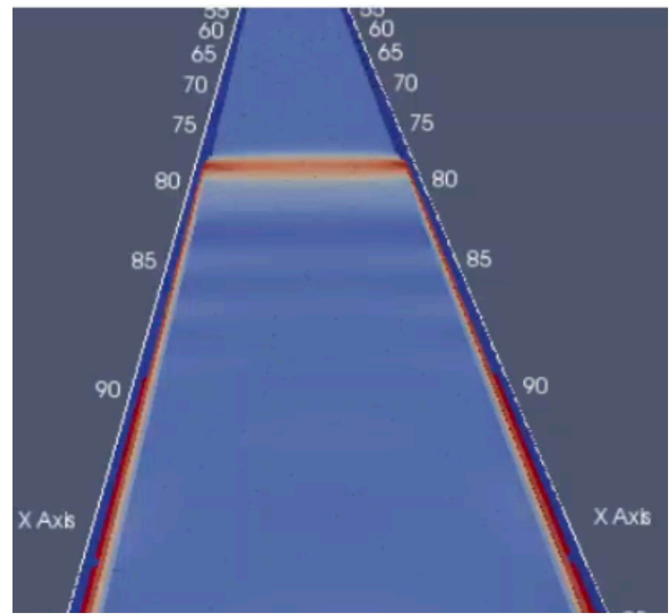


$Fr = 1.05$

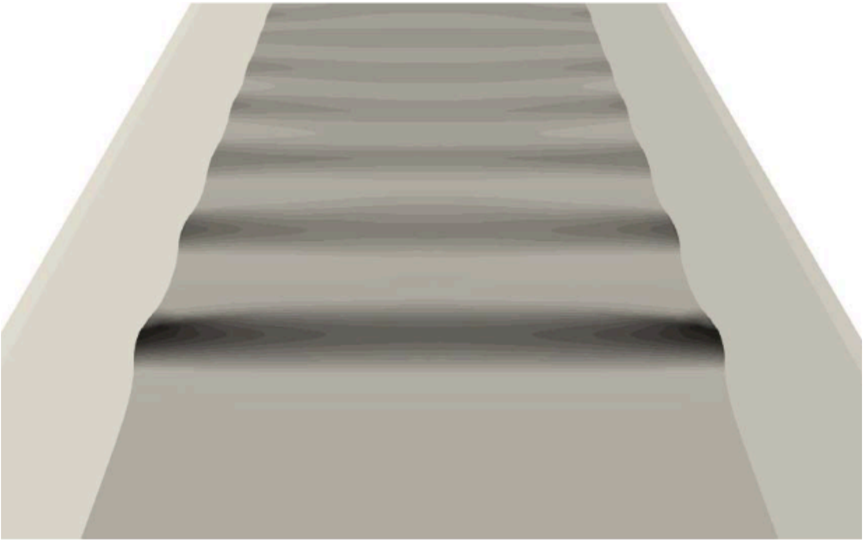
SLOWS, developed by inria
gitlab.inria.fr/slows-public-group/slows_public



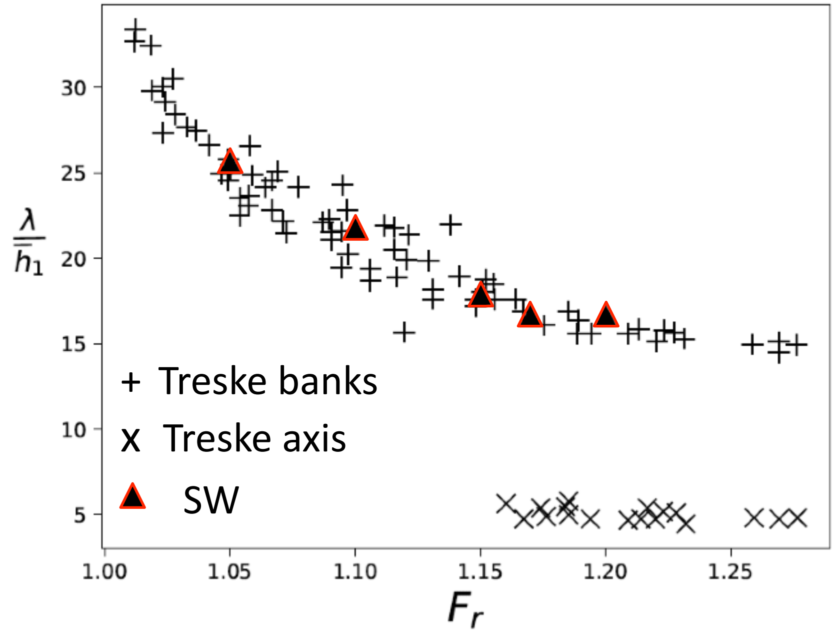
UHAINA, by the French Geophysical survey
www.brgm.fr



Eole-SW, developed by PRINCIPIA
www.principia-group.com

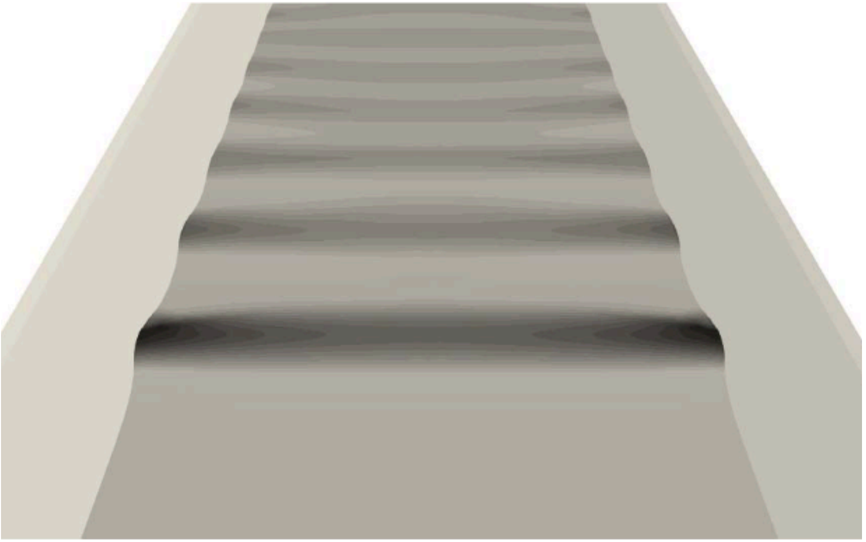


$Fr = 1.05$

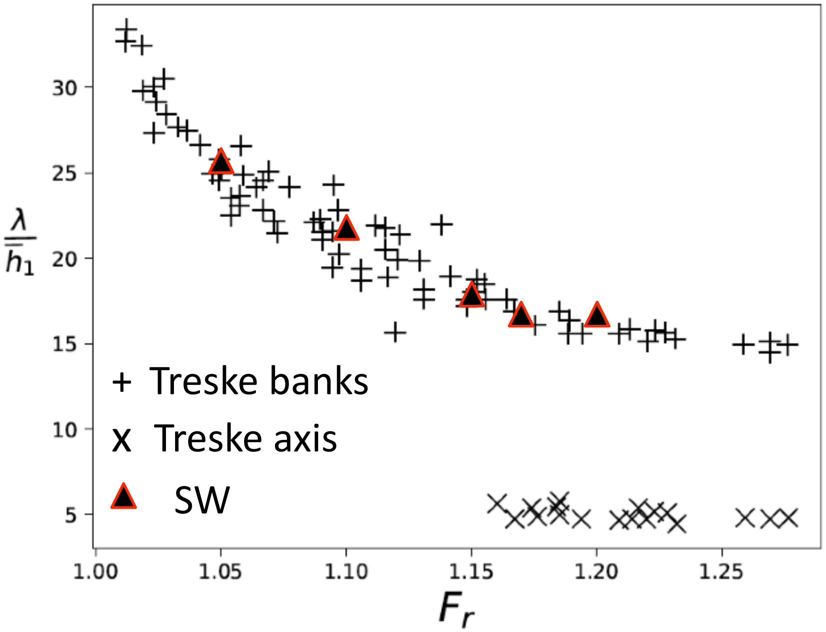


Dispersive waves described by the hyperbolic shallow water eq.s !

With a discontinuous initial state !



$Fr = 1.05$

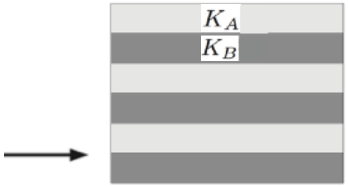


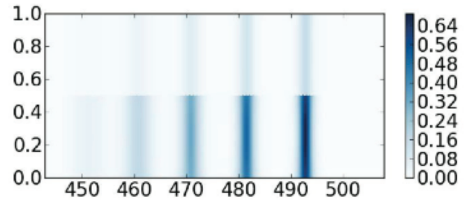
Dispersive waves described by the hyperbolic shallow water eq.s !

What is the origin?

- Dispersion in (hyperbolic) wave propagation in heterogenous media

Ketcheson & Quessada de Luna, SIAM Multiscale Mod. Simul., 2015



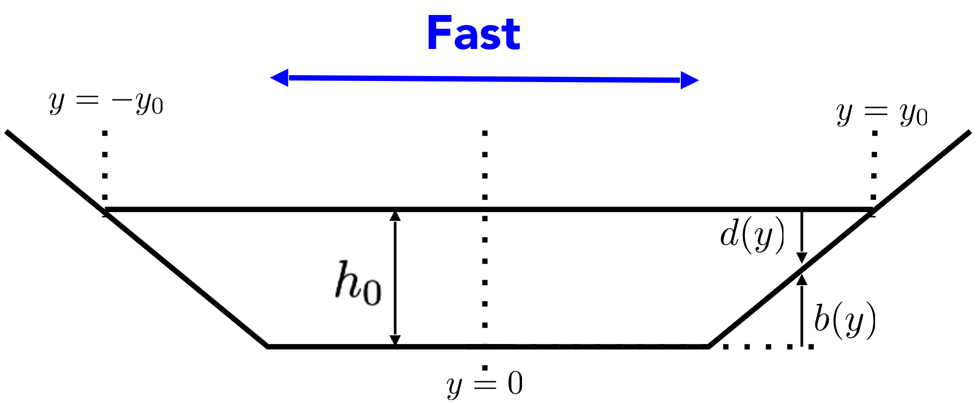


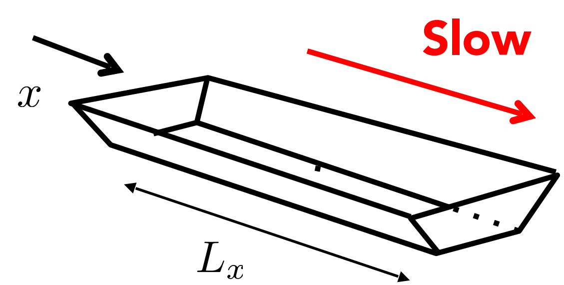
Linear asymptotic modelling of (weakly) dispersive-like waves in channels

Asymptotic analysis

Ansatz

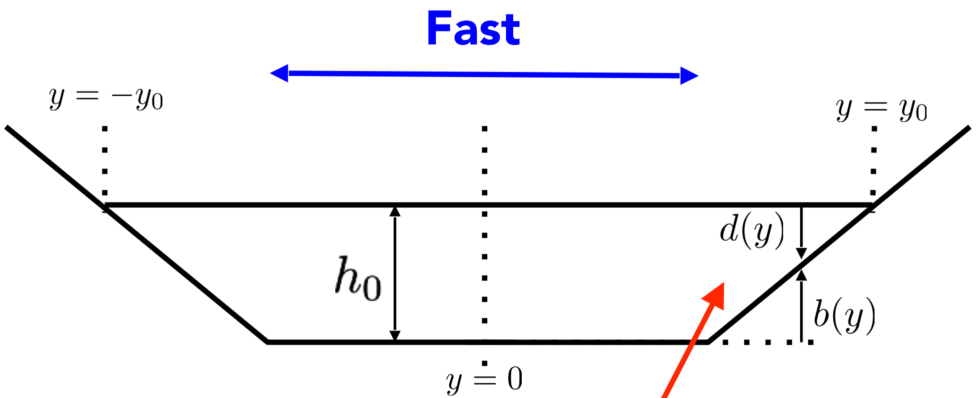
- Shallow water waves (hydrostatic, no dispersion terms)
- Linear waves
- Scale separation between transverse (fast) and longitudinal (slow) waves



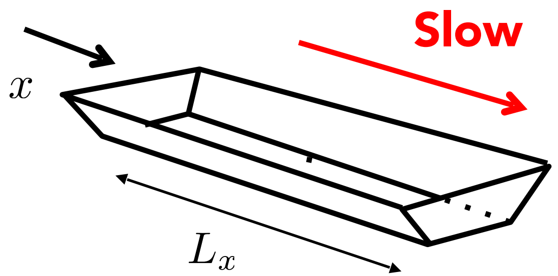


$$\tau_y \ll \tau_x$$

$$L_y = \tau_y \sqrt{g h_0} \ll L_x \quad \longrightarrow \quad \delta = \frac{L_y}{L_x} \ll 1$$



$$d = d(y)$$



Dimensionless variables (the *-rred ones are now the dimensional ones)

$$b^* = bh_0\ , \quad \zeta^* = \epsilon\zeta A\ , \quad d^* = h_0 d$$

$$x^* = xL_x\ , \quad y^* = yL_y\ , \quad t^* = \frac{\sqrt{gh_0}}{L_x} \qquad \epsilon = A/h_0$$

$$u^* = \epsilon\sqrt{gh_0}\ , \quad v^* = \epsilon\sqrt{gh_0}$$

Scaling used when deriving Boussinesq equations (cf. Jerry's and Angel's talks) plus the new ansatz. Same unit for the x- and y-velocity components.

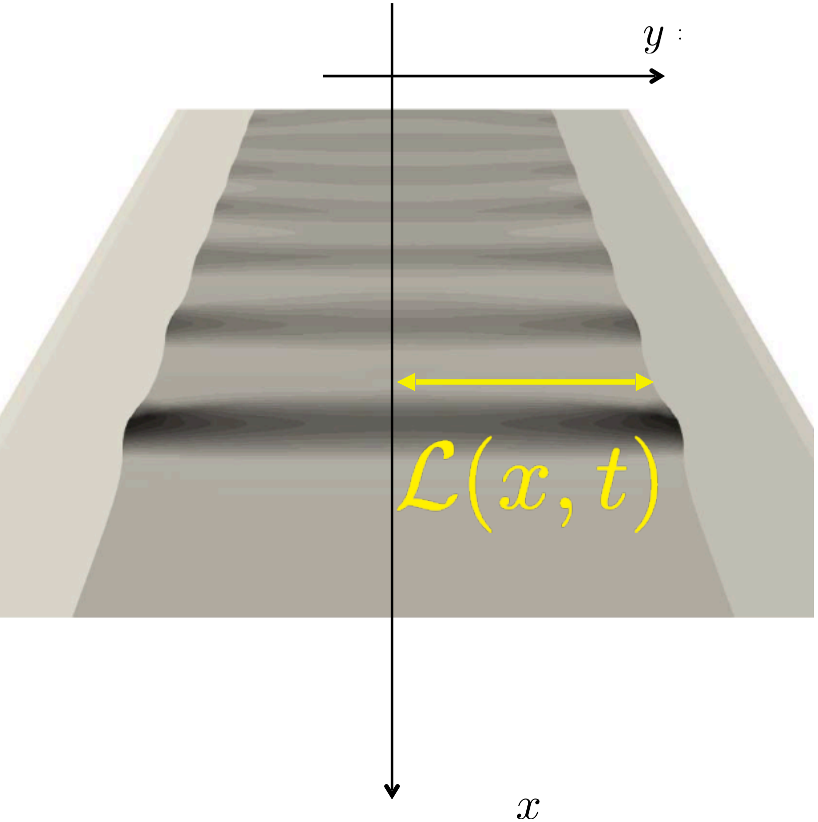
2D NLSW in dimensionless form

$$\partial_t \zeta + \partial_x((d+\epsilon\zeta)u) + \frac{1}{\delta}\partial_y((d+\epsilon\zeta)v) = 0$$

$$\partial_t u + \epsilon u \partial_x u + \frac{\epsilon}{\delta} v \partial_y u + \partial_x \zeta = 0$$

$$\partial_t v + \epsilon u \partial_x v + \frac{\epsilon}{\delta} v \partial_y v + \frac{1}{\delta}\partial_y \zeta = 0$$

NOTA BENE: compared to Jerry's and Angel's talks all the μ (or β, or σ) terms are absent since we start from the shallow water equation which is the zero-th order approximation in that scaling (no "vertical" kinematics).



$$\mathcal{L}(t,x) = y_0 + \frac{\epsilon}{\delta}\int_0^t v(s,x,y_{bank}(s,x))ds$$

$$\mathcal{L} = y_0$$ for linear waves

$$hv(t,x,y=\mathcal{L}) = hv(t,x,y=-\mathcal{L})$$

for banks, straight walls, and periodic

$$\overline{(\cdot)} := \frac{1}{2\mathcal{L}}\int_{-\mathcal{L}(t,x)}^{\mathcal{L}(t,x)} (\cdot)(t,x,y)dy$$ transverse averaging

Linearized problem

$$\delta(\partial_t \zeta + d\partial_x u) + \partial_y(dv) = 0$$

$$\partial_t u + \partial_x \zeta = 0$$

$$\delta \partial_t v + \partial_y \zeta = 0$$

$$\overline{\zeta}_{tt} - \overline{(d\zeta)}_{xx} = 0$$

This is exact

$$d = d(y)$$

$$dv\big|_{y=\mathcal{L}} = dv\big|_{y=-\mathcal{L}}$$

$$\zeta = \sum_{j\geq 0} \delta^j \zeta_j\,, \quad u = \sum_{j\geq 0} \delta^j u_j\,, \quad v = \sum_{j\geq 0} \delta^j v^j$$

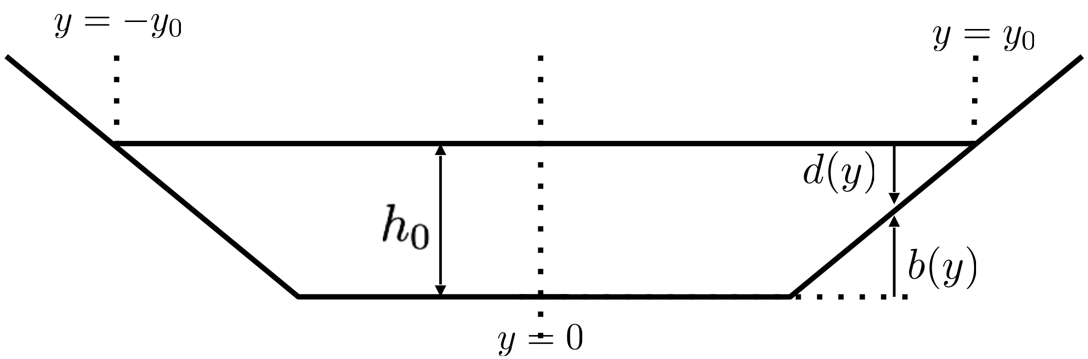
$$\delta(\partial_t \zeta + d\partial_x u) + \partial_y(dv) = 0 \longrightarrow dv_{n+1} = \int_{-1}^{y} (\partial_t \zeta_n + \partial_x u_n) ds$$

$$\partial_t u + \partial_x \zeta = 0 \longrightarrow \partial_t u_{n+1} + \partial_x \zeta_{n+1} = 0$$

$$\delta \partial_t v + \partial_y \zeta = 0 \longrightarrow \zeta_{n+1} - \bar{\zeta}_{n+1} = Z(\partial_t v_n)$$

With I.C.

$$\zeta_0 = \bar{\zeta}\,, \quad \partial_t u_0 = -\partial_x \zeta_0\,, \quad v_0 = 0$$



$$\zeta(x,y,t) = \overline{\zeta}(x,t) + \delta^2 (K(y) - \overline{K})\overline{\zeta}_{xx} + \mathcal{O}(\delta^4)$$

$$K(y) := \int_{-y_0}^{y} \frac{h_0(y_0+s) - D(s)}{d(s)} ds$$

$$\mathcal{D}(y) := \int_{-y_0}^{y} d(s) ds$$

$$\overline{\zeta}_{tt} - \overline{(d\zeta)}_{xx} = 0$$

$$\overline{\zeta}_{tt} - c_0^2\overline{\zeta}_{xx} - \boxed{\chi c_0^2 \overline{\zeta}_{xxxx}} = 0$$

$$\chi := \overline{d(y)(K(y) - \overline{K})}$$

Geometrical dispersion due to diffraction within each section

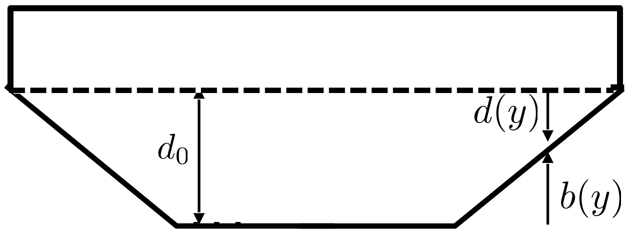
$$\boxed{\omega^2 = \kappa^2 c_0^2 (1 - \chi(\kappa y_0)^2)}$$

$$\omega^2 = \kappa^2 c_0^2 (1 - \chi(\kappa y_0)^2)$$

For most cases of physical relevance (mild sloped channels) we have $\chi \geq 0$

So this is a bad Boussinesq-like model. It may be possible to correct this, however ATM we don't really care and focus on the implication of this model for long waves.

A better approach is discussed in the last part of the talk.

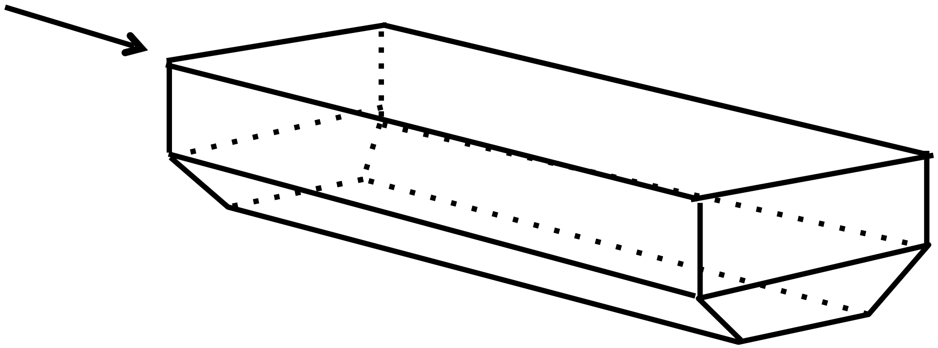


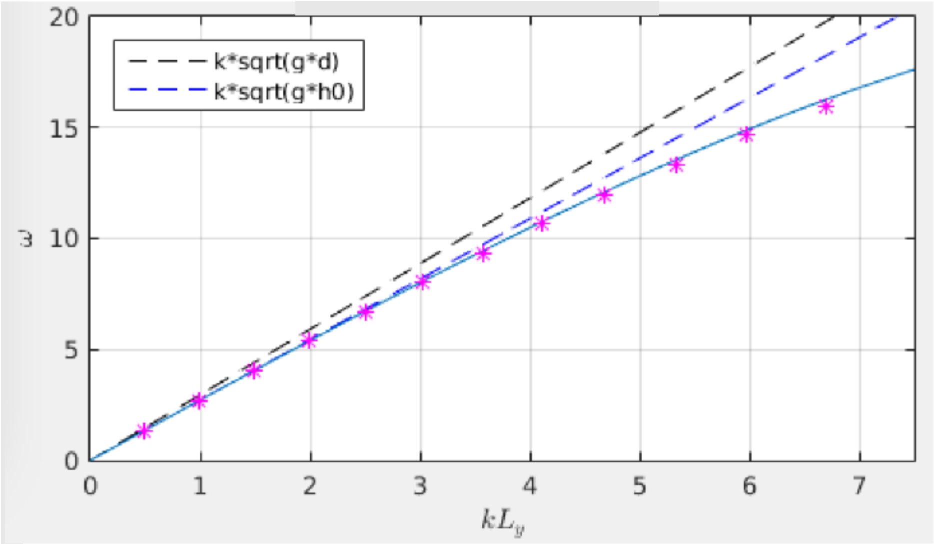
Shallow water simulations with walls:

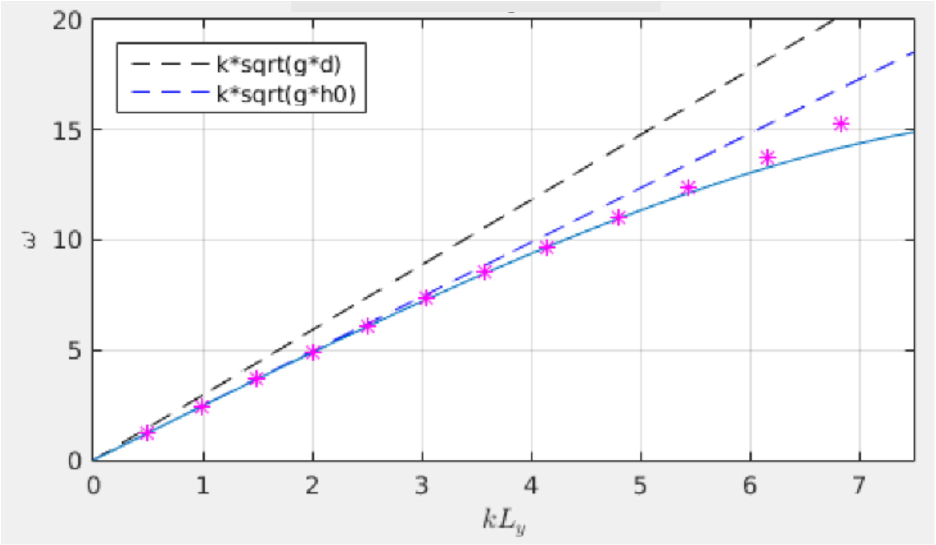
1- Linear periodic signal imposed at the inlet

2- The whole signal is (section-)averaged

3- Period and wavelength are measured downstream




20
15
10
5
0
ω
0 1 2 3 4 5 6 7
kL_y
k*sqrt(g*d)
k*sqrt(g*h0)


20
15
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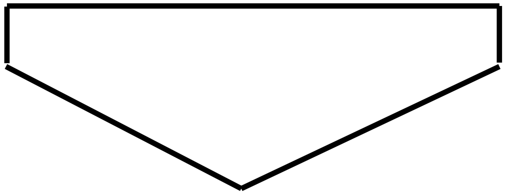
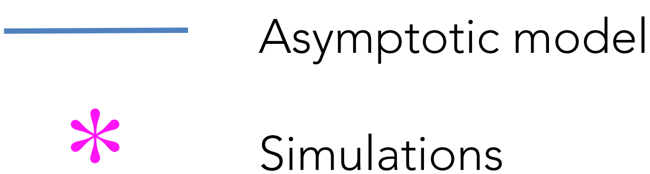
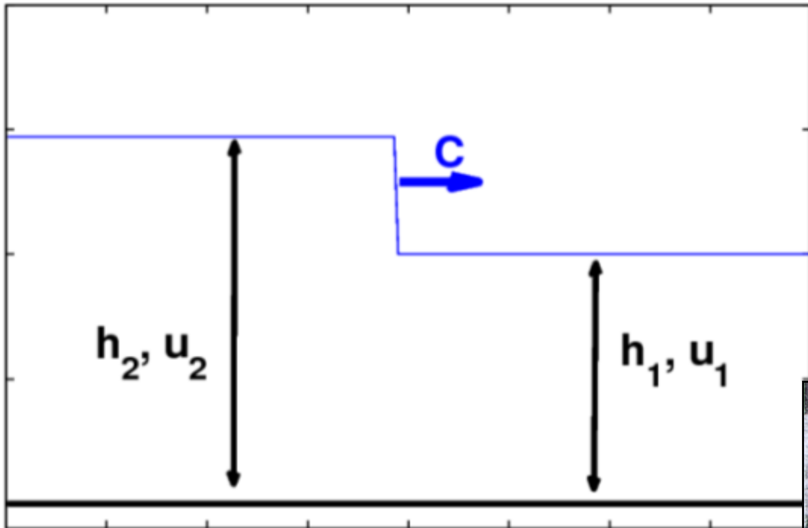
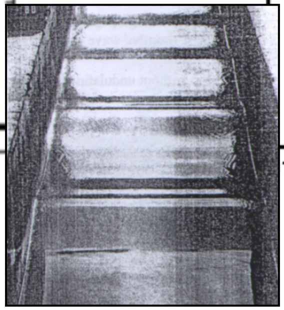
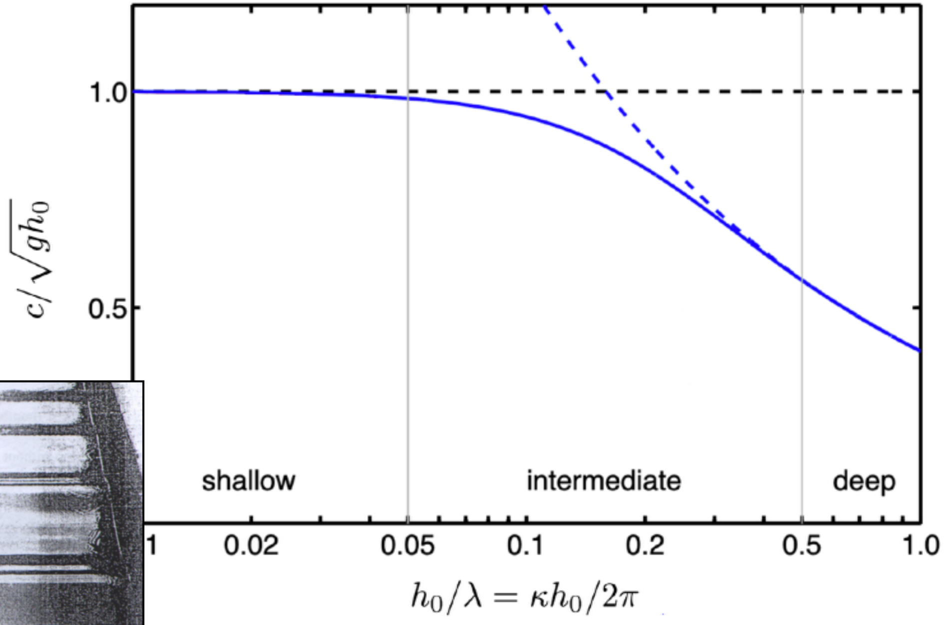
5
0
ω
0 1 2 3 4 5 6 7
kL_y
k*sqrt(g*d)
k*sqrt(g*h0)




Asymptotic model
Simulations







Bore:

Using the shallow water Rankine-Hugoniot relation (no dispersion !!):

$$C_b - U_2 = \sqrt{\frac{A(h_1)}{A(h_2)} g \frac{K(h_2) - K(h_1)}{A(h_2) - A(h_1)}}$$

Dispersive like-waves:

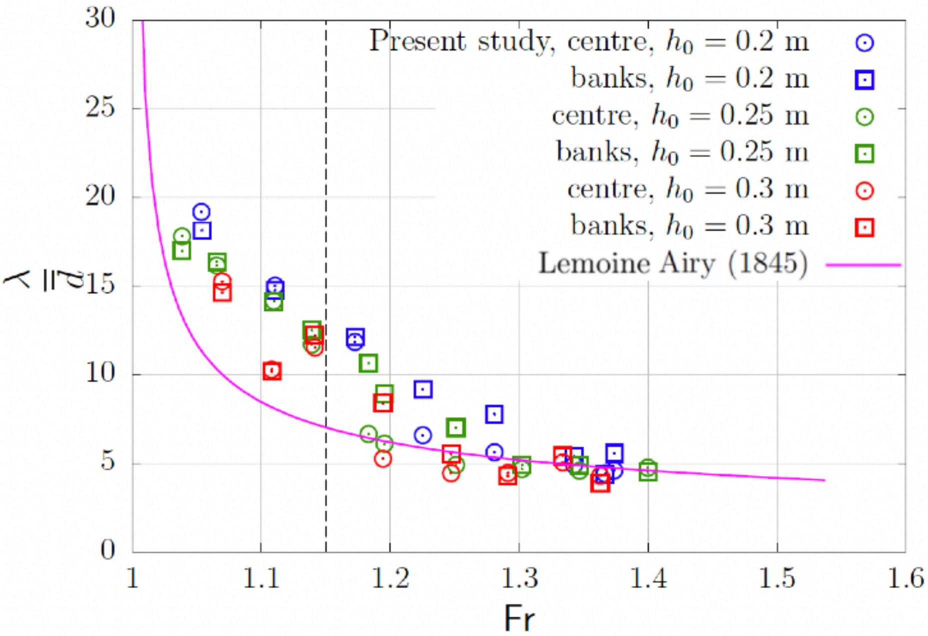
dispersion relation of approximate transverse average PDE

$$C_\lambda^2 = g\bar{d}\sqrt{1 - \chi(\kappa y_0)^2}$$

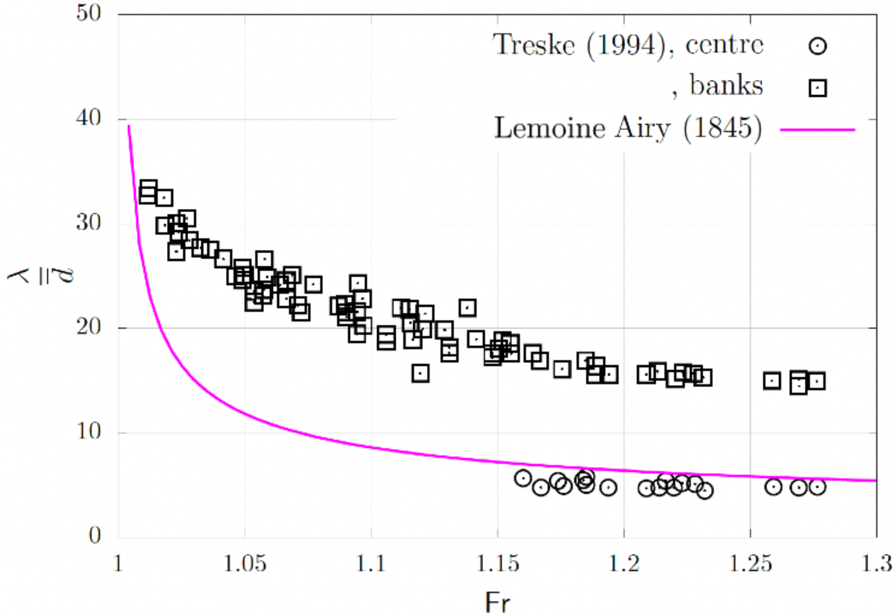
$$C_b = U_2 + C_\lambda \Longrightarrow \lambda(Fr)$$

Lemoine, La Houille Blanche, 1948

Jouy's data



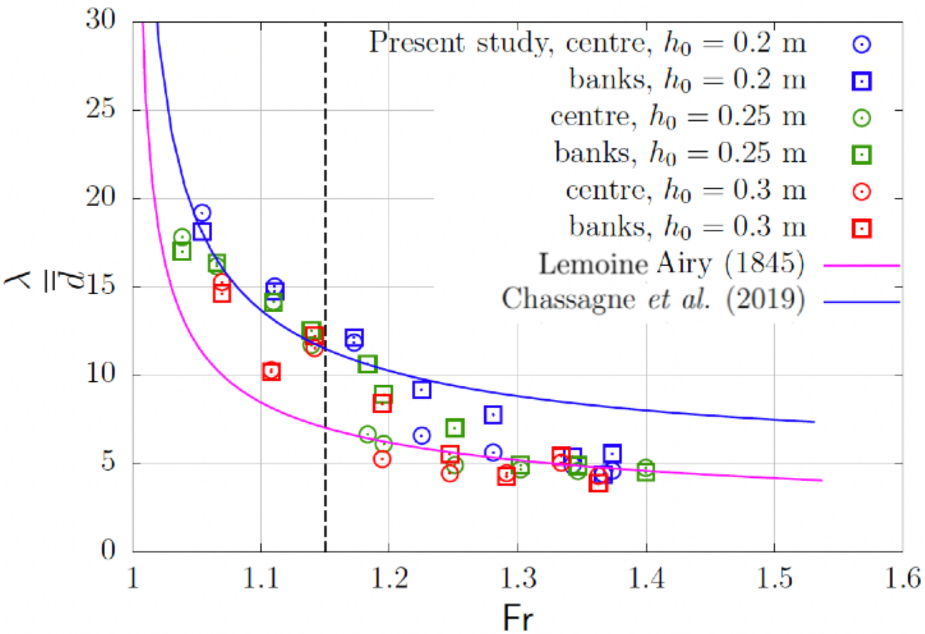
Treske data



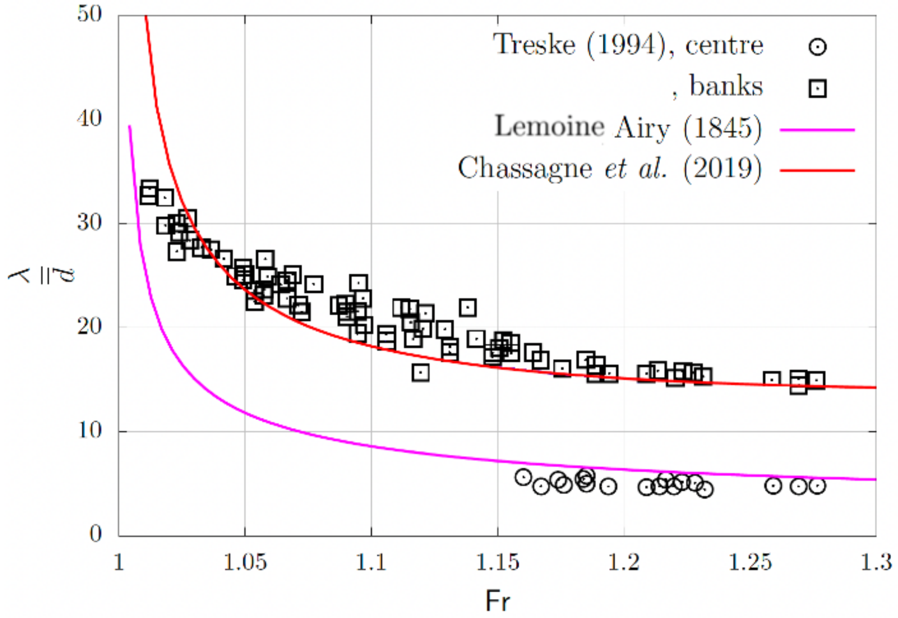
$$C_b = U_2 + C_\lambda \Longrightarrow \lambda(Fr)$$

Asymptotic analysis	Lemoine analogy

Jouy's data



Treske data



$$C_b = U_2 + C_\lambda \Longrightarrow \lambda(Fr)$$

Fully-nonlinear modelling of

(weakly) dispersive-like waves in channels

The waves generated by transverse diffraction have been baptised in **Chassagne et al**, JFM 2019 "dispersive-like" to underline that a process different from the usual "vertical" dispersion is at play.

We have seen that the model derived predicts within quite some quantitative accuracy the wavelengths for the low Froude waves. However

1- it is not Galilean invariant
2- it does not conserve energy
3- it is linear
4- it is a bad Boussinesq model

not well suited for actual computations. Let's write a better one

$$\partial_t h + \partial_x(hu) + \partial_y(hv) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + \partial_y(huv) = 0$$

$$\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2 + gh^2/2) = -gh\partial_y b(y)$$

$$b^* = bh_0 \; , \quad \zeta^* = \epsilon\zeta A \; , \quad d^* = h_0 d$$

$$x^* = xL_x \; , \quad y^* = yL_y \; , \quad t^* = \frac{\sqrt{gh_0}}{L_x}$$

$$\epsilon = A/h_0$$

$$\delta = L_y/L_x$$

$$u^* = \epsilon\sqrt{gh_0} \; , \quad v^* = \frac{L_y}{L_x}\epsilon\sqrt{gh_0}$$

$$\epsilon = 1 \; , \quad \delta \ll 1$$

Differently from **Chassagne et al**, JFM 2019, we account for the anisotropy is also accounted for in the velocity scaling.

$$b^* = bh_0 \ , \quad \zeta^* = \zeta h_0 \ , \quad d^* = h_0 d$$

$$x^* = xL_x \ , \quad y^* = yL_y \ , \quad t^* = \frac{\sqrt{gh_0}}{L_x}$$

$$u^* = \sqrt{gh_0} \ , \quad v^* = \varepsilon\sqrt{gh_0}$$

$$\varepsilon = L_y/L_x \ll 1$$

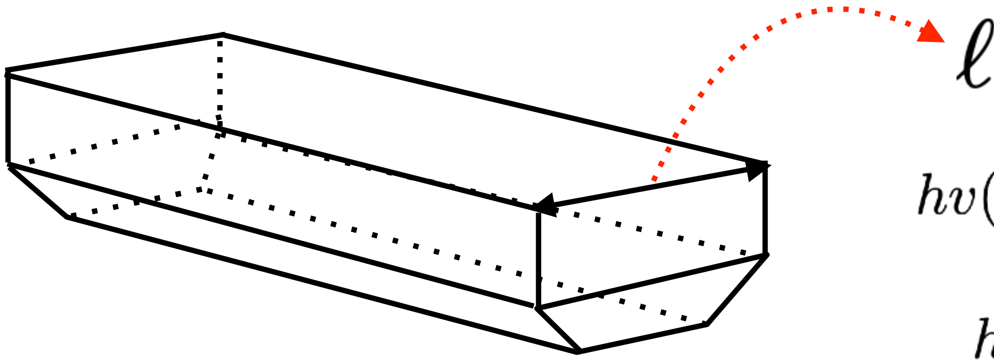
Only one small parameter, from now on change of notation…

$$\partial_t h + \partial_x(hu) + \partial_y(hv) = 0$$

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x(h+b) = 0$$

$$\varepsilon^2(\partial_t v + u\partial_x v + v\partial_y v) + \partial_y(h+b) = 0$$

Differently from **Chassagne et al**, JFM 2019, we account for the anisotropy is also accounted for in the velocity scaling: full mass conservation.



$$hv(t,x,y=\ell) = hv(t,x,y=-\ell)$$

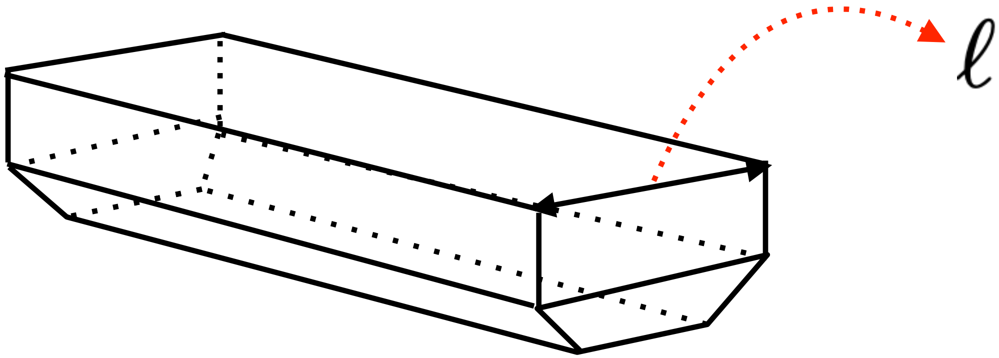
$h>0$ no banks for the moment

straight walls, and periodic topography

In other words ℓ is fixed…

$$\overline{(\cdot)} = \frac{1}{\ell}\int_0^{\ell} (\cdot)(t,x,y)dy$$ transverse averaging

$$\langle\cdot\rangle = \frac{1}{\ell\,\overline{h}}\int_0^{\ell} h(t,x,y)(\cdot)(t,x,y)dy$$ Favre transverse averaging

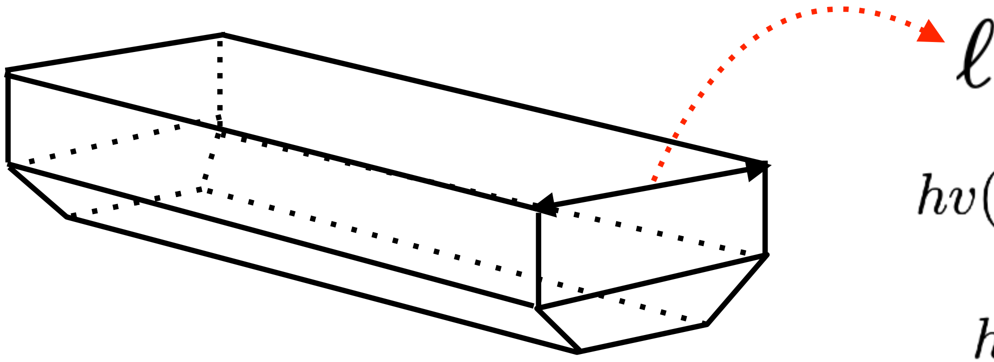


$$u(t,x,y) = \langle u \rangle(t,x) + \varepsilon^2 u_1(t,x,y) + \mathcal{O}(\varepsilon^3)$$

$$v = v_0(t,x,y) + \mathcal{O}(\varepsilon),$$

$\overline{hu_1} = 0$ shear (in y) is negligible $\Longrightarrow$ $\overline{hu^2} = \overline{h}\langle u \rangle^2 + \mathcal{O}(\varepsilon^3)$

We inject this in the system and average …



$$hv(t,x,y=\ell) = hv(t,x,y=-\ell)$$

$h>0$ no banks for the moment

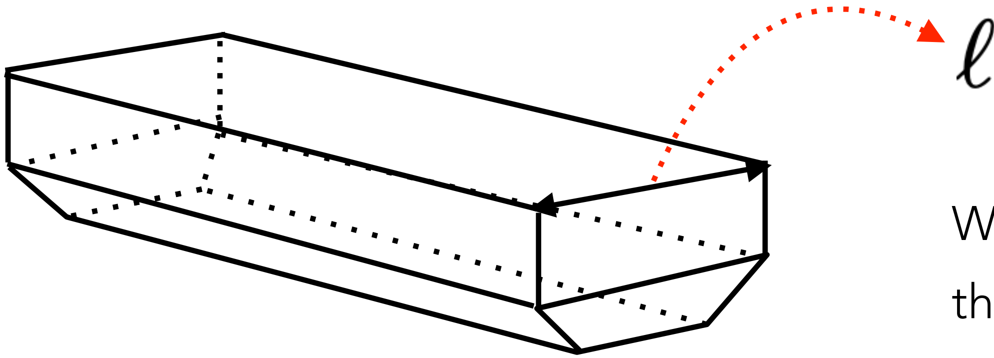
straight walls, and periodic topography

$$\partial_t \bar{h} + \partial_x(\bar{h}\langle u\rangle) = 0$$

$$\partial_t(\bar{h}\langle u\rangle) + \partial_x(\bar{h}\langle u\rangle^2 + \overline{h^2/2}) = 0$$

$$h(t,x,y) + b(y) = \bar{h} + \bar{b} + \mathcal{O}(\varepsilon^2)$$

We seek the transversal profiles of depth and x velocity to close the problem



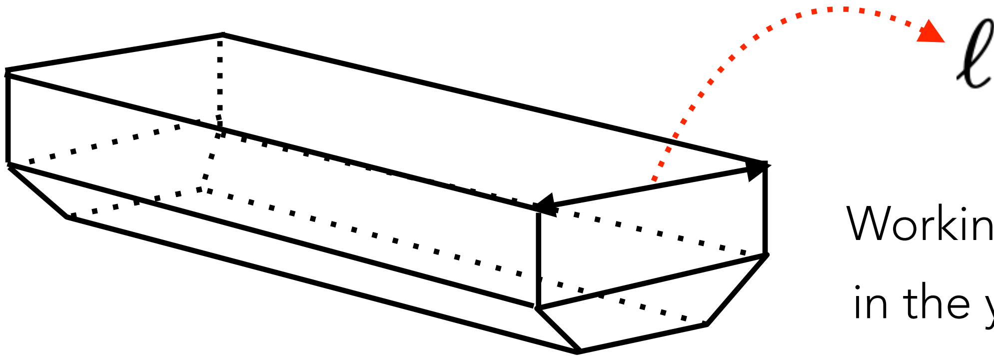
We now combine the 2D and the averaged 1D mass equation and get

$$v_0(t,x,y) \approx \frac{(\overline{S} - S(y))}{\overline{h} + S_y} \langle u \rangle_x = (\text{use } \bar{h} \text{ equation}) = \sigma(y)\dot{\tau}$$

where

$$\tau = 1/\bar{h}\ , \qquad S(y) = \int_0^y (\bar{b} - b(y'))dy'\ , \quad \sigma(y) = \frac{\bar{S} - S(y)}{1 + S'(y)/\bar{h}}$$

We seek the transversal profiles of depth and x velocity to close the problem

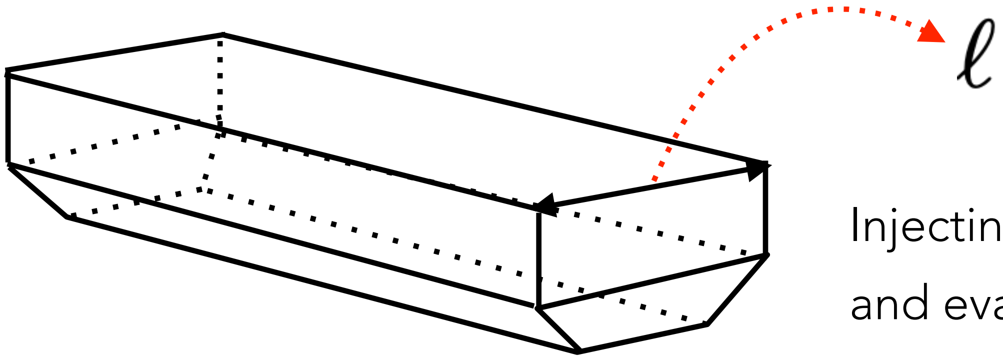


Working out the averages of the terms in the y-momentum we get

$$h + b \approx \overline{h} + \overline{b} - \varepsilon^2 \left(\left(D - \overline{D} \right) \ddot{\tau} + \frac{1}{2} \dot{\tau}^2 \left(\sigma^2 - \overline{\sigma^2} \right) \right)$$

where

$$\tau = 1/\bar{h} \qquad D = \int_0^y \sigma(y') dy' \; , \qquad \sigma(y) = \frac{\bar{S} - S(y)}{1 + S'(y)/\bar{h}}$$



Injecting in the averaged x-momentum
and evaluating the remaining integrals
we get the final system

$$\overline{h}_t + (\overline{h}\langle u\rangle)_x = 0,$$
$$(\overline{h}\langle u\rangle)_t + (\overline{h}\langle u\rangle^2 + \frac{\overline{h}^2}{2} + p)_x = 0,$$

$$p = -\varepsilon^2 \chi \ddot{\tau}, \quad \tau = \frac{1}{\overline{h}} \qquad \chi = \overline{S^2} - \overline{S}^2 > 0\,, \quad S(y) = \int_0^y (\overline{b} - b(y'))dy'.$$

we baptised this a "geometrical" Green-Naghdi system

The new model

- is Galilean invariant
- has a Lagrangian formulation and the energy conservation law

$$\left(\overline{h}\left(\frac{\langle u\rangle^2}{2}+\frac{\overline{h}}{2}+\frac{\varepsilon^2\chi\dot{\tau}^2}{2}\right)\right)_t+$$
$$+\left(\overline{h}\langle u\rangle\left(\frac{\langle u\rangle^2}{2}+\frac{\overline{h}}{2}+\frac{\varepsilon^2\chi\dot{\tau}^2}{2}\right)+\left(\frac{\overline{h}^2}{2}+p\right)\langle u\rangle\right)_x=0.$$

- exhibits travelling wave solutions
- has dispersion properties very close to the linear model

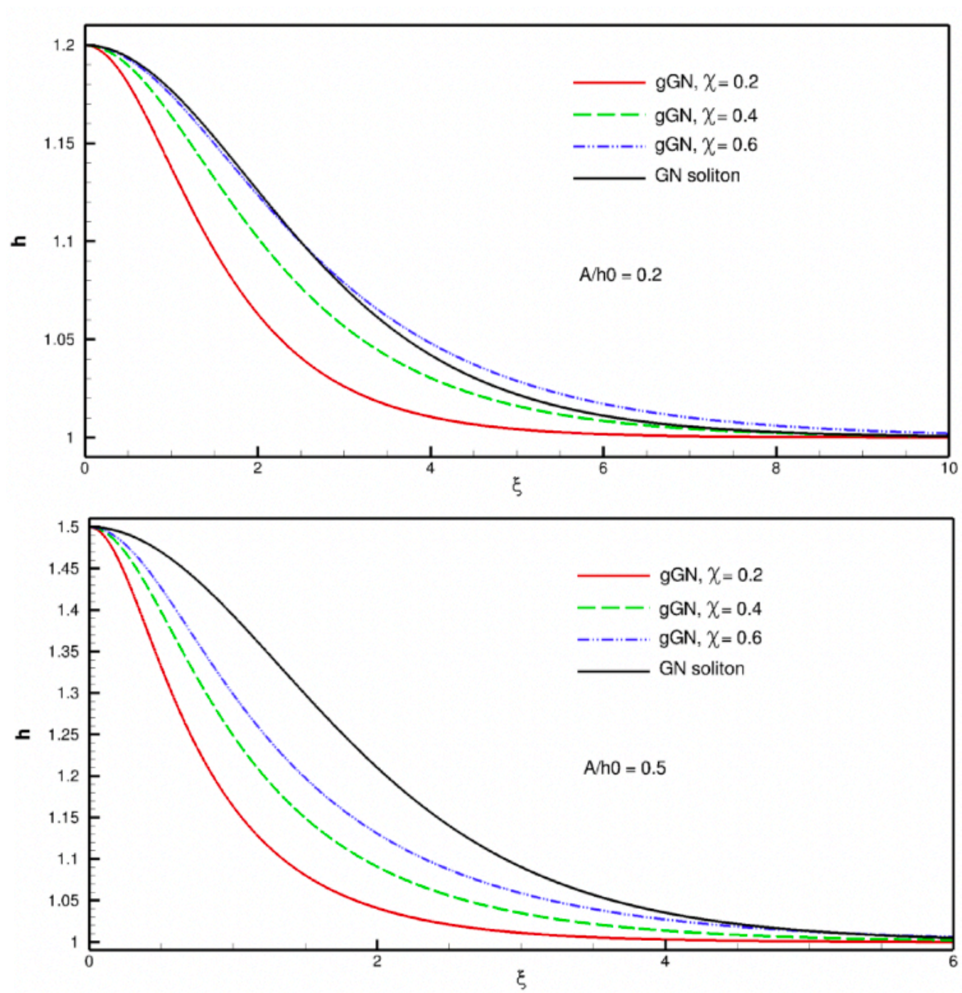
by **Chassagne et al**, JFM 2019

GGN model

solitary waves

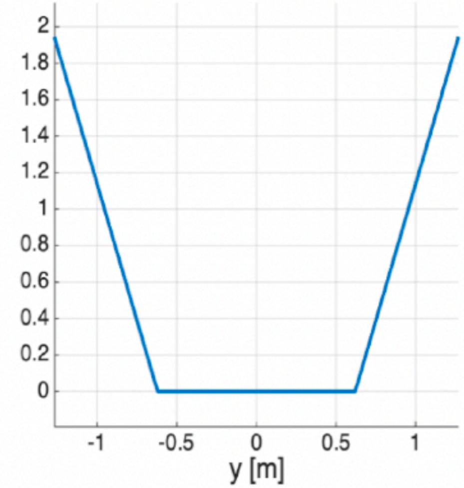
Solitary ODE:

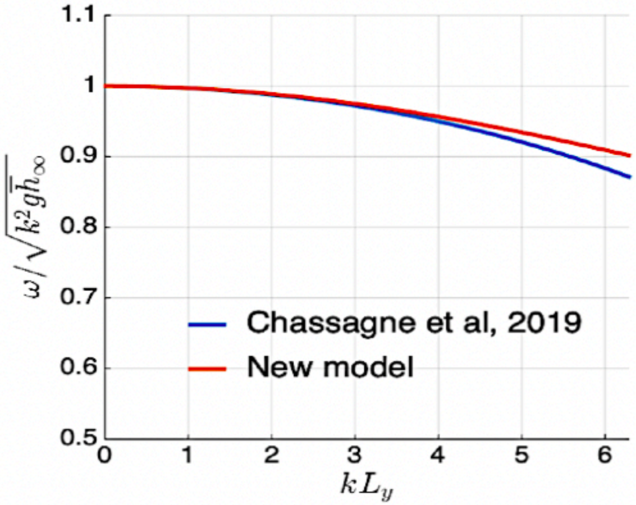
$$\bar{h}' = -h^2\left(\frac{\bar{h}}{\bar{h}_0} - 1\right)\sqrt{\frac{1}{\chi}\left(1 - \frac{\bar{h}}{\bar{h}_0 + A}\right)}$$

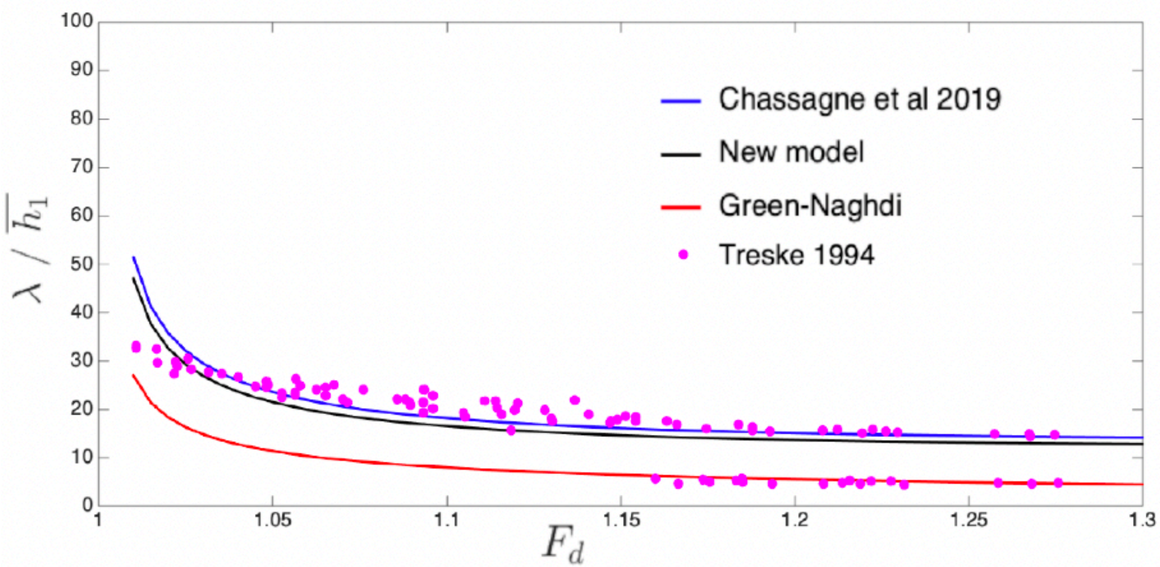


Dispersion relation:

$$\omega^2 = k^2 \frac{g\overline{h}_0}{1 + \frac{\chi}{\overline{h}_0^2} k^2}$$

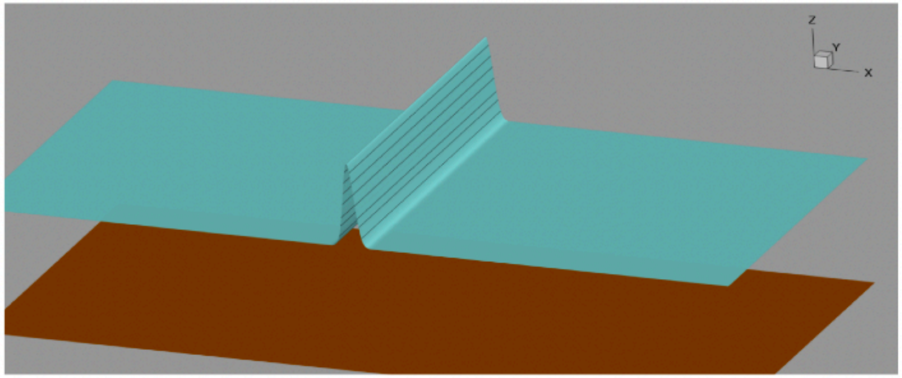


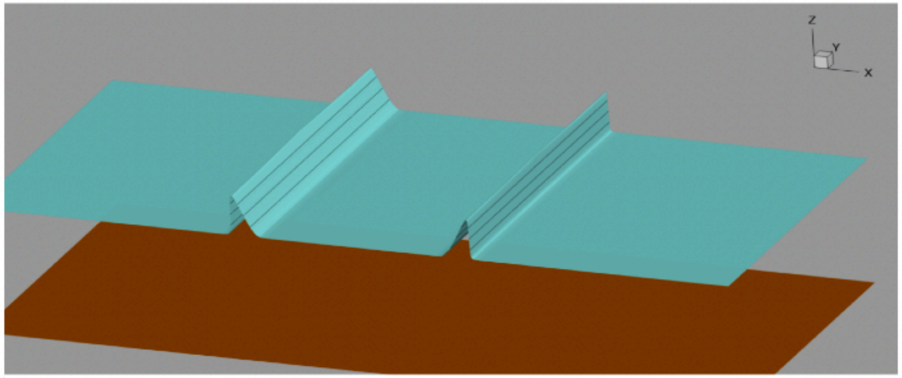


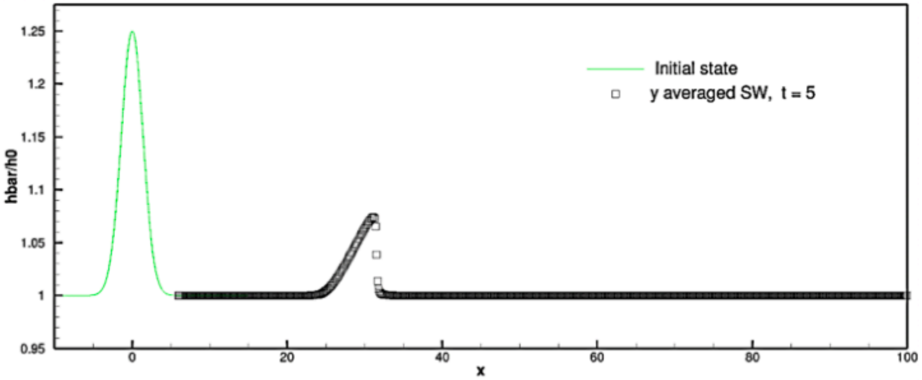


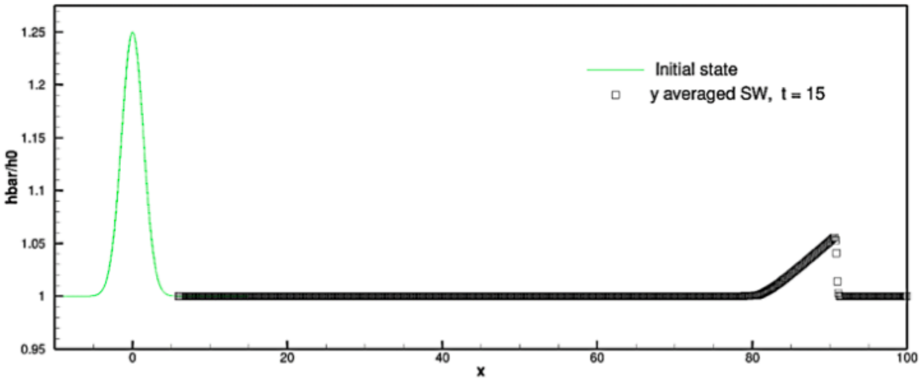
Lemoine analogy

Gaussian bell breakdown:

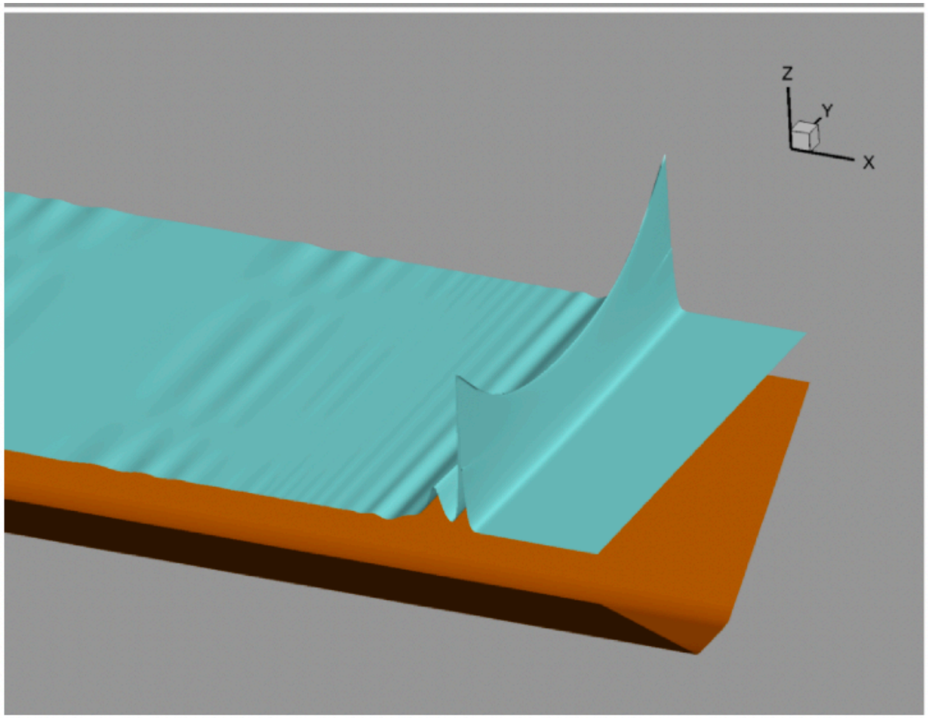


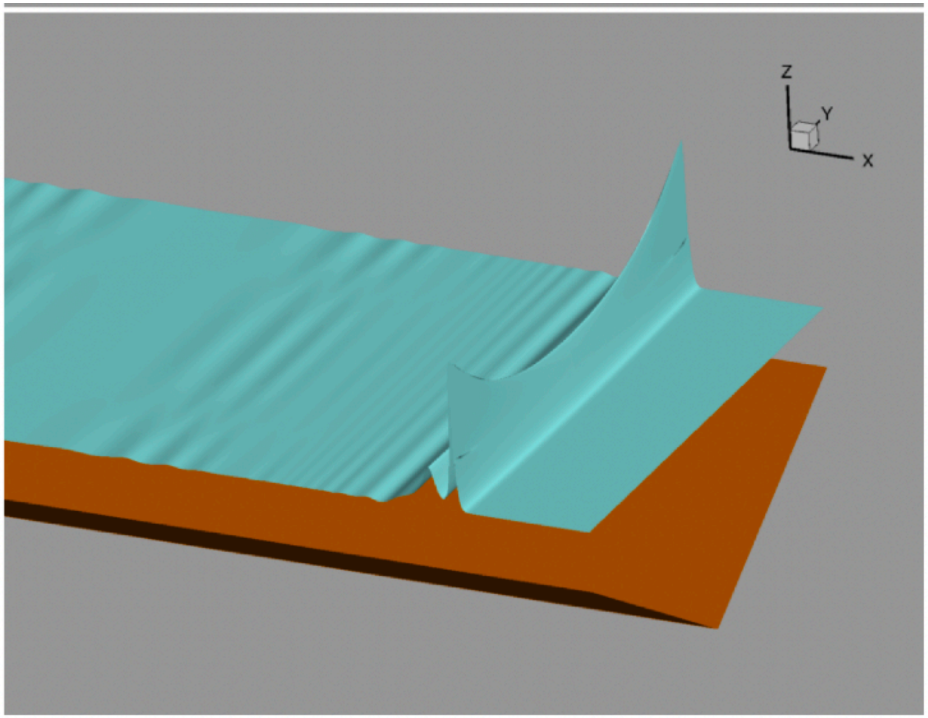




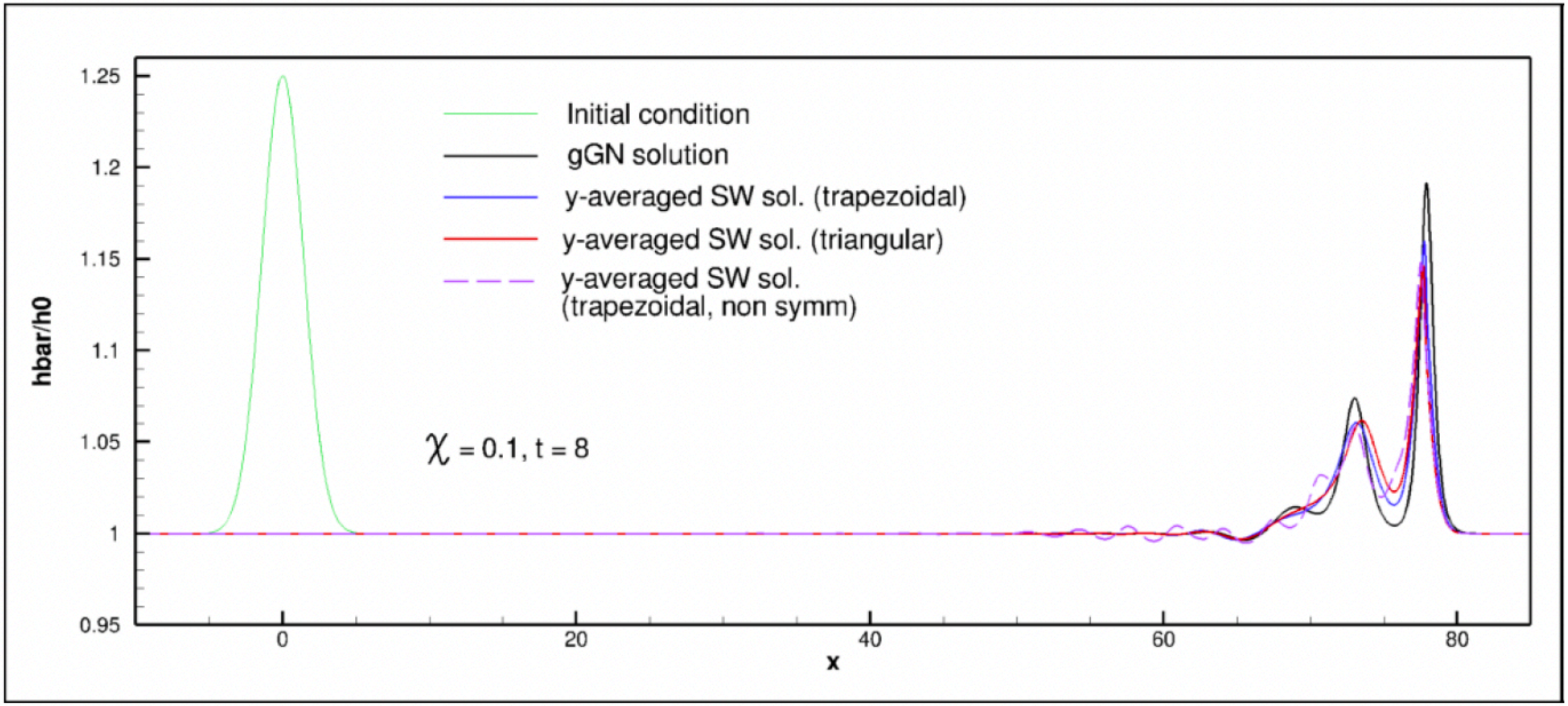


Gaussian bell breakdown:

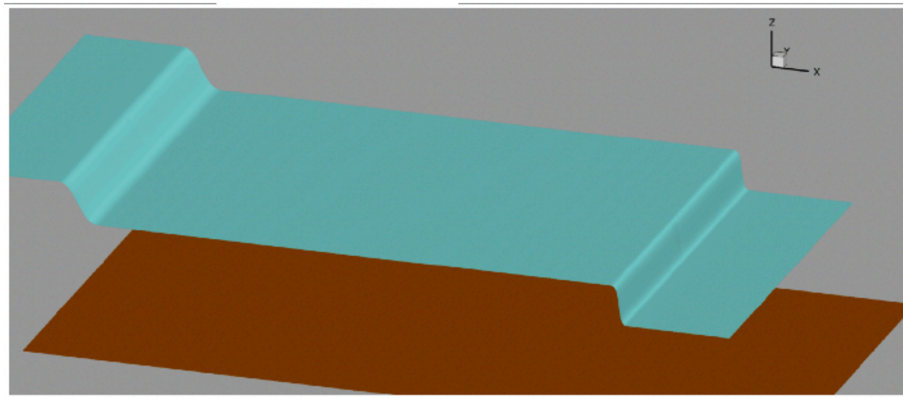


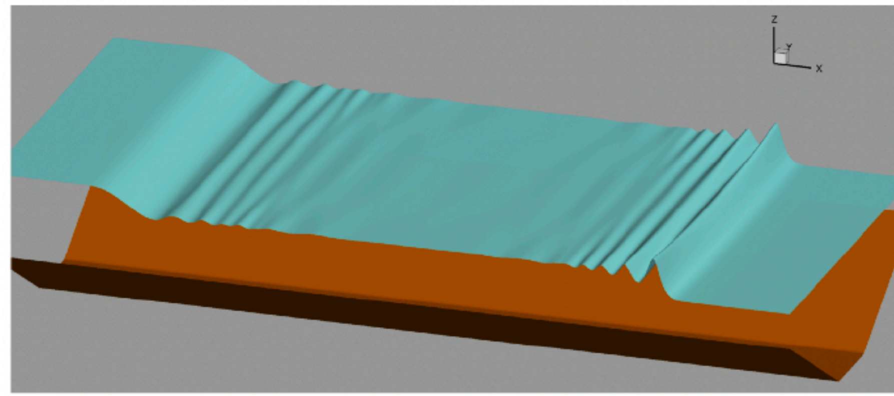


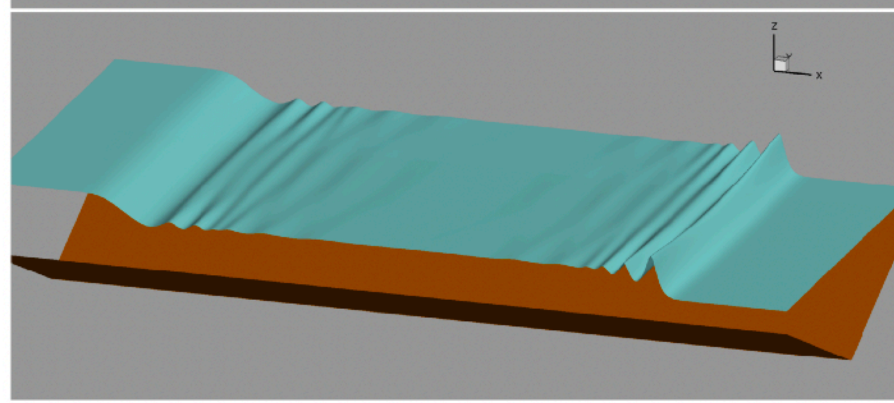
Gaussian bell breakdown:



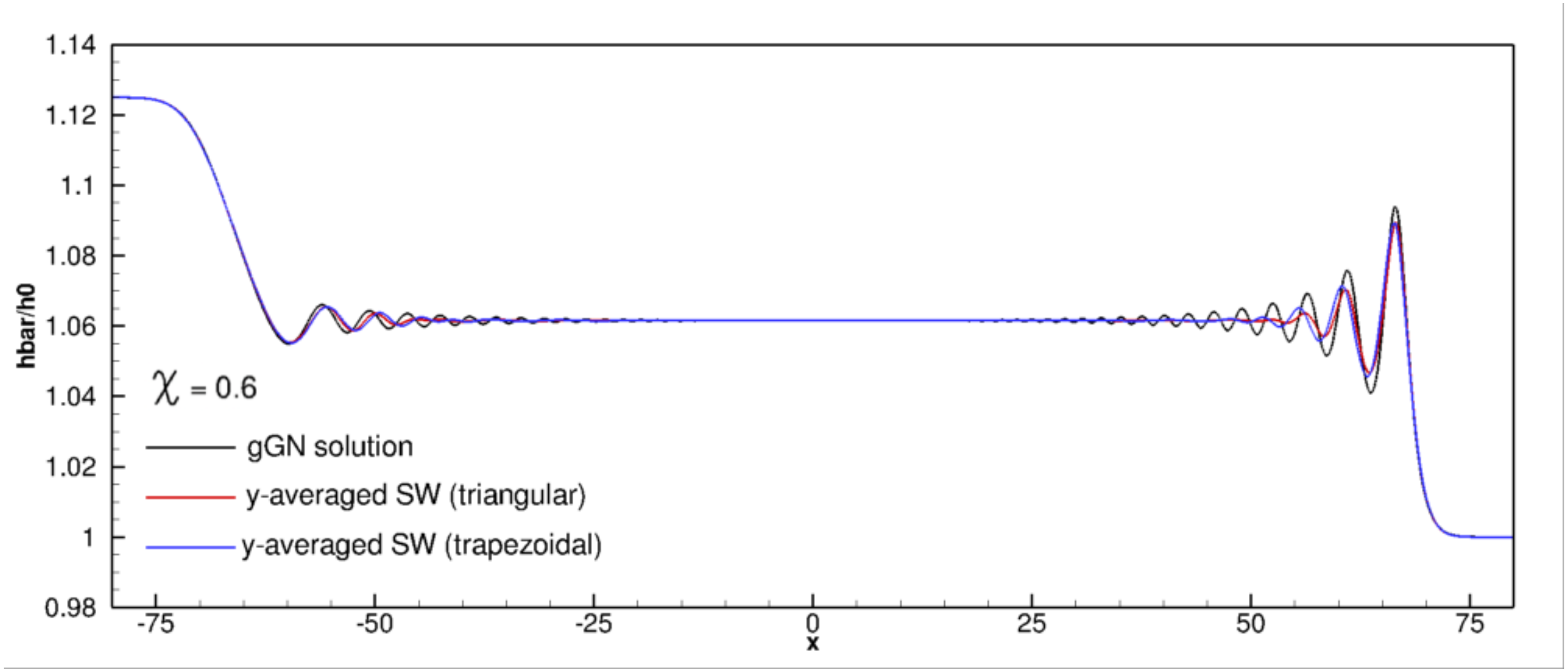
(smoothed) Riemann problem



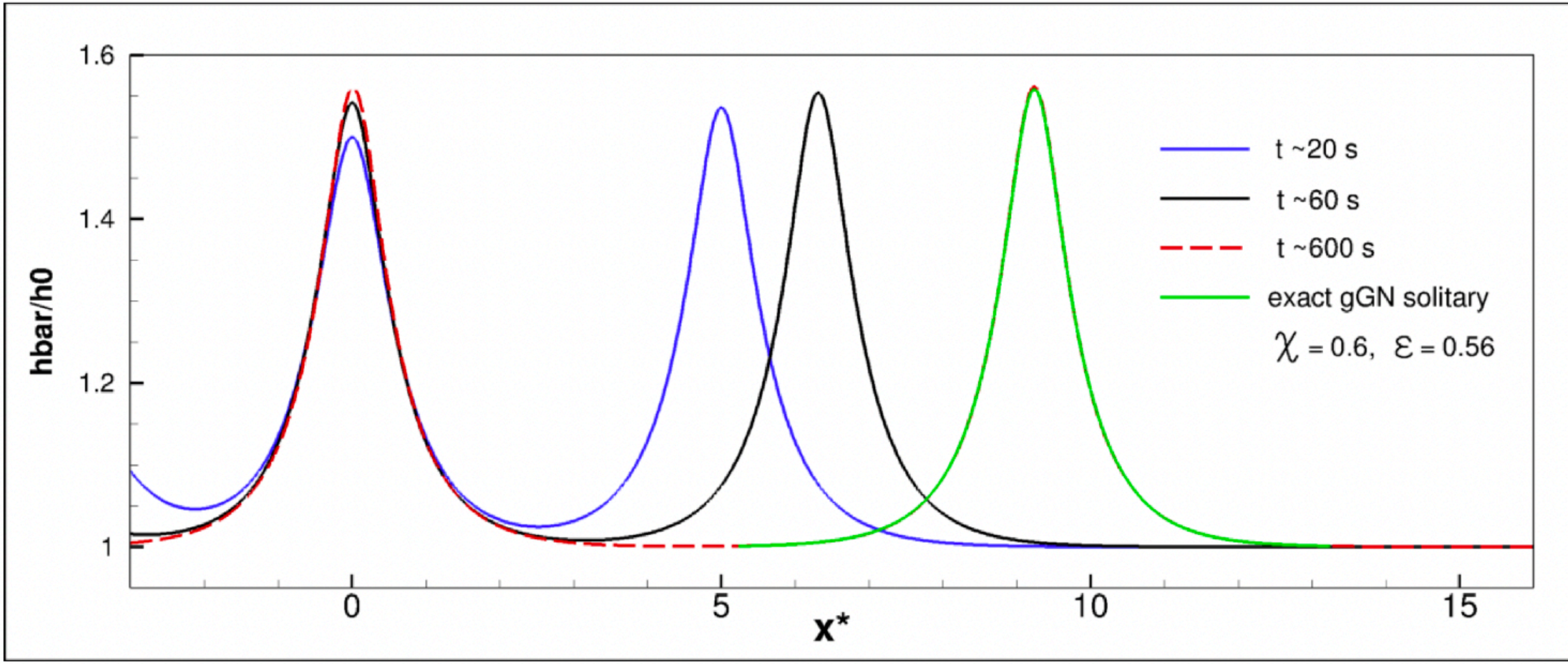




(smoothed) Riemann problem



(smoothed) Riemann problem: long time limit - soliton fission



The second peak of all solution is translated onto the origin.

A few take away messages and remarks

- The study of Favre waves has revealed importance of “transverse" dispersion
- “Transverse" dispersion is mostly hydrostatic, or anyways can be well approximated already using non-dispersive models (in multi-D)
- “Transverse" dispersion is related to the geometry of the bathymetry
- The question is: which dispersive process dominates and when, what is the transition mechanism from “geometrical" Favre waves to the usual ones
- How to model both using transverse and depth averaged (1D) equations
- How to model the transition ?

Some bibliography

oooops

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Linear waves in anisotropic periodic media exhibit dispersion (asymptotic PDE derived)

R. Chassagne, A.G. Filippini, **M. Ricchiuto** and P. Bonneton, JFM 2019
low Fr data from Treske can be reproduced with 2D shallow water models,
1D dispersive asymptotic (linear) section averaged model derived starting from 2D shallow water eq.s

D. Ketcheson and M. Quezada de Luna, JFM 2021
Same developments as Chassagne et al 2019, and show the existence of solitary waves
in hydrostatic (shallow water) propagation

B. Jouy, D. Violeau, **M. Ricchiuto** and M. Le, AMM 2024
Simulation using a 1D section averaged Boussinsq model to reproduce Treske's experiment

D. Ketcheson, L. Loczi, and G. Russo, https://arxiv.org/abs/2311.02603 2023
1D shallow water with high frequency variations of bathymetry exhibit dispersion

D. Ketcheson and G. Russo, https://arxiv.org/abs/2409.00076 2024
2D shallow water equations with transverse periodic variations in bathymetry exhibit dispersion,
weakly nonlinear model derived

S. Gavrilyuk and **M. Ricchiuto** https://arxiv.org/abs/2408.08625 2024- submitted to JFM
Bores in trapezoidal channels: geometrical Green-Naghfi model characterising dispersive waves originating in hydrostatic propagation in channels (fully nonlinear generalisation of the model by Chassagne et al 2019)