A velocity alignment collision model for spatially homogeneous kinetic collective dynamics

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Abstract

The aim of this paper is to present a Boltzmann type model that describes a collective behavior of a large group of individuals. The model considers a mechanism where as two individuals collide, they adopt after the collision the same post-collisional velocity according to a distribution centered at the mid pre-collisional velocity. We show in this paper that the solutions of a spatially homogeneous model on $\mathbb{R}^d$ converge exponentially towards the equilibrium state for the Wasserstein metric. The convergence of the solutions in the strong-norm L^1 is also proved for initial conditions satisfying a stronger regularity property. In a last part, these results are illustrated numerically.

1 Introduction

We consider a group of individuals subject to a social interaction. For this, we consider a Boltzmann type model introduced by Bertin, Droz, Gregoire [2], [3]. In the BDG model, each individual (bird, fish, rod,...) moves independently from the others outside the collisions and are indistinguishable. At the time of the collision if two individuals are close enough, then they will line up in velocity. For each $t \geq 0$, the evolution of the collective behavior is represented by a probability distribution $f_t = f(t, x, v)$ where x denotes the position, and v denotes the velocity of the individuals. The two post-collisional velocities v, $v_\star$ adopted by the two individuals are equal after the collision $v = v_\star$, and randomly distributed according to a probability $K(\,\cdot\,, v', v'_\star)$ centered at the mid pre-collisional velocity $\frac{v'+v'_\star}{2}$. In general, the collision rate is represented by a function $\beta(v' - v'_\star)$ taking its values close to 1 if the two individuals are almost aligned before the collision and taking its values close to 0 in the case of grazing collisions. The model may also take into account a velocity confinment as in [5]. In [12] Raoul studies a similar model. The population of the individuals is structured by a continuous one-dimensional trait. At the time two individuals meet, they interact sexually and the trait of the offspring is distributed according to a Gaussian measure centered at the mid trait of the parents.

The present paper deals with a simplified version of the BDG model: the density f is independent of the position of the individuals, the velocity is d-dimensional, and the collision rate β is constant equal to 1, the so-called Maxwellian case. In [2], [3], [4], the dimension d of the space of velocities is equal to 1, in [8], the space of velocities may be a manifold of any dimension $d \geq 1$, but the probability $K(dv, v', v'_\star) = \delta_{(v'+v'_\star)/2}(dv)$ must be a Dirac mass at the mid velocity. Our results are more general in the sense that they are valid in any dimension and for any distribution $K(dv, v', v'_\star)$. However the

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technique of proofs we use assume that the space of velocities is euclidean. A model where the velocity is constrained to be of norm 1 as in [8] is out of reached by our methods.

For general collision rate, the unknown probability distribution f satisfies the following Boltzmann like equation in the sense of distributions:

$$\frac{\partial f}{\partial t} = Q(f,f) = Q_+(f,f) - Q_-(f,f), \tag{1.1}$$

where $Q(f,f)$ is the collision operator which is decomposed into a gain term $Q_+(f,f)$ and a loss term $Q_-(f,f)$. For any test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^d)$ with compact support,

$$\langle Q_+(f,f),\varphi\rangle := \int_0^{+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(t,v)K(dv,v',v'_\star)\beta(v'-v'_\star)f(t,dv')f(t,dv'_\star)dt \tag{1.2}$$

and

$$\langle Q_-(f,f),\varphi\rangle := \int_0^{+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(t,v)\beta(v-v')f(t,dv)f(t,dv')dt. \tag{1.3}$$

In ref. [8], the model (1.1) is studied when $\beta = 1$ and $K(\,\cdot\,,v',v'_\star) = \delta_{(v'+v'_\star)/2}$. This model is called the discrete midpoint model. In ref. [2], [3], [4], the dimension of the velocity is $d = 1$, the direction taken after the collision is chosen according to a density probability distribution centered at the mean $(v'+v'_\star)/2$. This model is called the continuous midpoint model. In a probabilistic framework, in both cases, the velocity after the collision is written under the form $v = (v'+v'_\star)/2 + X$ where X is a random variable of law g, considered discrete or continuous.

We choose from now on any probability $g(dv)$ with zero mean on $\mathbb{R}^d$, for example $g = \delta_0$ in ref. [8], or $g(v)dv$, a density with respect to the Lebesgue measure as in ref. [4]. The model considered in this paper is given by the following kernel K and collision rate β,

$$K(dv,v',v'_\star) := \left(\tau\left[\frac{v'+v'_\star}{2}\right]\#g\right)(dv) \qquad \text{and} \qquad \beta(v) = 1.$$

We have denoted by $\tau[u] : v \longmapsto v+u$ the translation by u, by $\tau[u]\#g$ the push forward of the measure g by $\tau[u]$, that is, for any bounded mesurable function φ,

$$\int \varphi(v)(\tau[u]\#g)(dv) := \int \varphi\circ\tau[u](v)g(dv).$$

For every $\alpha > 0$, let $\mathcal{P}_\alpha(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$ admitting a finite moment of order α. We recall the notion of a moment of order $\alpha > 0$ of a measure $\mu \in \mathcal{P}_\alpha(\mathbb{R}^d)$

$$M_\alpha(\mu) := \int_{\mathbb{R}^d}|v|^\alpha d\mu(v).$$

For $\alpha > 1$ and $m \in \mathbb{R}^d$, let $\mathcal{P}_\alpha^m(\mathbb{R}^d)$ be the set of measures $\mu \in \mathcal{P}_\alpha(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} v d\mu(v) = m.$$

Let $g \in \mathcal{P}_2^0(\mathbb{R}^d)$ and $f_0 \in \mathcal{P}_2^m(\mathbb{R}^d)$. The evolution equation (1.1) becomes

$$\begin{cases} \dfrac{\partial f}{\partial t} = \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\tau\left[\frac{v'+v'_\star}{2}\right]\#g\right)f(t,dv')f(t,dv'_\star) - f(t,\,\cdot\,)\int_{\mathbb{R}^d}f(t,dv') \\ f(0,\,\cdot\,) = f_0. \end{cases} \tag{1.4}$$

For μ, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, we define the Wasserstein metric W_2 by

$$W_2(\mu,\nu) := \sqrt{\inf_{\pi\in\Pi} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\pi(x,y)}$$

where Π denotes the set of couplings of (μ,ν). Recall that a sequence of measures $(\mu_n)_n$ in $\mathcal{P}_2(\mathbb{R}^d)$ converges to a measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ for the distance W_2 if and only if

$$(\mu_n)_n \overset{\text{weak}*}{\to} \mu \quad \text{and} \quad \lim_{n\to+\infty} \int_{\mathbb{R}^d} |x|^2 d\mu_n(x) = \int_{\mathbb{R}^d} |x|^2 d\mu(x).$$

Therefore, W_2 metrizes the weak topology on $\mathcal{P}_2(\mathbb{R}^d)$ and makes that space complete (Definition 6.8, Theorems 6.9 and 6.18 in ref. [15]). This will allow us to etablish the existence of solutions for the equation (1.4). We define a solution in the sense of distributions as follows

Definition 1.1. *Let $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}^d))$ and $f_0 \in \mathcal{P}_2(\mathbb{R}^d)$. For any test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^d)$ with compact support, we define*

$$\langle f, \varphi\rangle := \int_0^{+\infty} \int_{\mathbb{R}^d} \varphi(t,v) f(t,dv) dt.$$

A solution of the equation (1.4) in the sense of distributions is a measured-valued function $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}^d))$ satisfying for every test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^d)$ with compact support

$$-\left\langle f, \frac{\partial\varphi}{\partial t}\right\rangle = \int_{\mathbb{R}^d} \varphi(0,v) f_0(dv) + \langle Q_+(f,f), \varphi\rangle - \langle Q_-(f,f),\varphi\rangle. \tag{1.5}$$

We define a second notion of mild solution as follows.

Definition 1.2. *A mild solution of the equation (1.4) is a function $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}^d))$ taking values in the space of probability measures equipped with the Wasserstein metric W_2 satisfying for all $t \geq 0$*

$$f(t,\cdot) = e^{-t} f_0 + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-(t-s)} \left(\tau\left[\frac{v'+v'_\star}{2}\right] \# g\right) f(s,dv') f(s,dv'_\star) ds. \tag{1.6}$$

The notion of mild solution is stronger than the notion of solution in the sense of distribution. We will prove the existence as well as the uniqueness of mild solutions by using a fixed point type argument.

We consider next the equilibrium states of the collision operator Q corresponding to probability measures f satisfying $Q(f,f) = 0$. We will mainly focus on the convergence to the unique equilibrium state of the equation (1.4) that is defined as follows.

Definition 1.3. *An equilibrium state of the equation (1.4) is a probability distribution $f \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying the fixed point equation*

$$f = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(\tau\left[\frac{v'+v'_\star}{2}\right] \# g\right) f(dv') f(dv'_\star). \tag{1.7}$$

We will show the existence and uniqueness of the equilibrium state by using a fixed point type argument. The main result of this paper concerns the exponential convergence of the solution of (1.4) towards the equilibrium state for the Wasserstein metric W_2 and for the strong-norm L^1. We will also make the link with the convergence result of the discrete midpoint model in ref. [8].

Theorem 1.1. *Let $m \in \mathbb{R}^d$, $f_0 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and $g \in \mathcal{P}_2^0(\mathbb{R}^d)$.*

(1) There exists a unique mild solution $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}^d))$ *to the equation* (1.4) *with* $f(0,\,\cdot\,) = f_0$*. Moreover, we have for all* $t \geq 0$

$$\int_{\mathbb{R}^d} v f(t, dv) = \int_{\mathbb{R}^d} v f_0(dv) := m.$$

(2) There exists a unique equilibrium state $f_m^\infty \in \mathcal{P}_2^m(\mathbb{R}^d)$*, that is a probability measure* f_m^∞ *satisfying*

$$Q(f_m^\infty, f_m^\infty) = 0.$$

(3) For all $t \geq 0$

$$W_2(f(t,\,\cdot\,), f_m^\infty) \leq e^{-t/4} W_2(f_0, f_m^\infty). \tag{1.8}$$

(4) If $f_0 \in H^s(\mathbb{R}^d) \cap \mathcal{P}_2^m(\mathbb{R}^d)$ *and* $g \in H^s(\mathbb{R}^d) \cap \mathcal{P}_2^0(\mathbb{R}^d)$ *are densities with* $s > 2 + d/2$*, then there exists a constant* $C > 0$ *explicitly computable such that for all* $t \geq 0$

$$\|f(t,\,\cdot\,) - f_m^\infty\|_{L^1(\mathbb{R}^d)} \leq C e^{-t/(d+4)}. \tag{1.9}$$

Results close to item (2) are present in ref. [4] and in Lemma 2.2 in ref. [12] (the model is different and g is Gaussian). Item (3) is similar to step two of the proof of Lemma 2.1 in ref. [12]. The author shows a perturbation result about the a Gaussian solution; we show an exponential convergence to an equilibrium state which may be non Gaussian. Our result is also valid in any dimension.

To prove (4) of theorem 1.1, we control the strong-norm L^1 by the Fourier-Toscani-based distance d_2 introduced in Carrillo, Toscani [7] and Toscani, Villani [13]. Then we show that the solution converges exponentially towards the equilibrium state of $Q(f, f)$ defined in (1.1, 1.2, 1.3) for the distance d_2. To bound the L^1 norm by the distance d_2, an estimate on the Sobolev norm $\|\cdot\|_{H^s(\mathbb{R}^d)}$ is needed for $s \geq 0$.

This model is new and interesting because it is located at the interface between collective dynamics and kinetic theory. The transport equation has no forcing or diffusion term in velocity, the change of velocity is computed as in Boltzmann framework. As the collisions are not micro-reversible, it is not obvious to find an entropy functional. In the Boltzmann equations, micro-reversibility is a crucial element for obtaining the H Theorem. Consequently, the classical tools for dealing with the problems of returns to equilibrium, such as for example the Csiszàr-Kullback-Pinsker inequality [9], are inoperative. In our case, we have instead a phenomenon of contraction in the collision process which does not take place for the Boltzmann operator but drives the density towards an equilibrium state.

The plan of this paper is the following. We start by establishing the existence of a mild solution of the equation (1.4). We show item (1) of Theorem 1.1 in section 2. We show the existence of the equilibrium state, item (2), and the proof of convergence, item (3) of Theorem 1.1 in section 3. We also make the link with the midpoint model in this same section. Then, we show the exponential convergence of the solution towards the equilibrium state for the distance d_2 in section 4, which will imply the convergence in L^1, item (4) of Theorem 1.1 in section 5. The last section is devoted to numerical simulations in dimension 1.

2 Existence

For all $t \geq 0$, we denote by $\rho(t)$, $u(t)$ and $\Sigma_f(t)$ the mass, bulk velocity and covariance matrix at the instant t of the solution f. Note that the equation (1.4) can be written equivently as

$$\frac{\partial f}{\partial t} = g * (U\#f(t,\,\cdot\,)) * (U\#f(t,\,\cdot\,)) - \rho(t) f(t,\,\cdot\,) \tag{2.1}$$

with

$$U : v \in \mathbb{R}^d \longmapsto \frac{v}{2} \in \mathbb{R}^d. \tag{2.2}$$

Indeed by definition of the convolution product we have for any test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^d)$

$$\begin{aligned}
\langle g * (U\#f) * (U\#f), \varphi\rangle &= \int_0^{+\infty} \int_{\mathbb{R}^d} \varphi(t,v) g * (U\#f(t,\cdot\,)) * (U\#f(t,\cdot\,))(dv)dt \\
&= \int_0^{+\infty} \iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(t, v + v' + v'_\star) g(dv)(U\#f(t,\cdot\,))(dv')(U\#f(t,\cdot\,))(dv'_\star)dt \\
&= \int_0^{+\infty} \iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi\left(t, v + \frac{v' + v'_\star}{2}\right) g(dv) f(t, dv') f(t, dv'_\star)dt \\
&= \int_0^{+\infty} \iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(t,v)(\tau_{(v'+v'_\star)/2}\#g)(dv) f(t, dv') f(t, dv'_\star)dt \\
&= \langle Q_+(f,f), \varphi\rangle.
\end{aligned}$$

By direct computation, we have (with the notations $v\,{}^t v = (v_i v_j)_{1\le i,j\le d}$)

$$\int_{\mathbb{R}^d} \begin{bmatrix} 1 \\ v \\ v\,{}^t v \end{bmatrix} g * (U\#f(t,\cdot\,)) * (U\#f(t,\cdot\,))(dv) = \begin{bmatrix} \rho(t)^2 \\ \rho(t)u(t) \\ \rho(t)^2(\Sigma_g + u(t)\,{}^t u(t)) + \Sigma_f(t)/(2\rho(t)). \end{bmatrix}. \tag{2.3}$$

The previous computation shows that the mass and the mean velocity are preserved, but not the energy. Note that the gain term $Q_+(f,f)$ is a density if g is a density (even if f is a probability measure).

We start by proving item (1) of Theorem 1.1. Some properties on W_2 are needed first.

Proposition 2.1 (Properties of W_2)**.**

(1) ***Convexity****. Given μ_1, μ_2, ν_1 and ν_2 in $\mathcal{P}_2(\mathbb{R}^d)$ and $\alpha \in [0,1]$, then*

$$W_2(\alpha\mu_1 + (1-\alpha)\mu_2, \alpha\nu_1 + (1-\alpha)\nu_2)^2 \le \alpha W_2(\mu_1,\nu_1)^2 + (1-\alpha)W_2(\mu_2,\nu_2)^2. \tag{2.4}$$

(2) ***Convexity with respect to transition kernel****. Let $P_1 : v \in \mathbb{R}^d \longmapsto P_1(v,\cdot\,) \in \mathcal{P}_2(\mathbb{R}^d)$ and $P_2 : v \in \mathbb{R}^d \longmapsto P_2(v,\cdot\,) \in \mathcal{P}_2(\mathbb{R}^d)$ be two transition kernels (that is Borel maps in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$) and μ be a probability measure, then*

$$W_2\left(\int_{\mathbb{R}^d} P_1(v,\cdot\,)d\mu(v), \int_{\mathbb{R}^d} P_2(v,\cdot\,)d\mu(v)\right)^2 \le \int_{\mathbb{R}^d} W_2(P_1(v,\cdot\,), P_2(v,\cdot\,))^2 d\mu(v). \tag{2.5}$$

(3) ***Sub-additivity with respect to convolution****. Given μ_1, $\nu_1 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and μ_2, $\nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$, then*

$$W_2(\mu_1 * \mu_2, \nu_1 * \nu_2)^2 \le W_2(\mu_1,\nu_1)^2 + W_2(\mu_2,\nu_2)^2. \tag{2.6}$$

(Notice that μ_1 and ν_1 must have the same mean value.)

(4) ***Transfert****. Given μ, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ a Borel map, then for all coupling π of (μ,ν),*

$$W_2(f\#\mu, f\#\nu)^2 \le \iint_{\mathbb{R}^d\times\mathbb{R}^d} |f(x) - f(y)|^2 d\pi(x,y). \tag{2.7}$$

For the sake of completeness we give the proof of the previous proposition in Appendix A.

Remark 2.1. *If $\varphi \in \mathcal{C}^0([0,T])$ is a function such that $\int_0^T \varphi(t)dt = 1$ and if $f_1,\ f_2 \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}^d))$, then we obtain by* (2.5)

$$W_2\left(\int_0^T \varphi(t) f_1(t,\,\cdot\,)dt, \int_0^T \varphi(t) f_2(t,\,\cdot\,)dt\right)^2 \le \int_0^T \varphi(t) W_2(f_1(t,\,\cdot\,), f_2(t,\,\cdot\,))^2 dt. \tag{2.8}$$

Remark 2.2. *By taking $\mu_1 = \nu_1$ in* (2.6)*, we have*

$$W_2(\mu_1 * \mu_2, \mu_1 * \nu_2)^2 \le W_2(\mu_1, \mu_1)^2 + W_2(\mu_2, \nu_2)^2 = W_2(\mu_2, \nu_2)^2.$$

And so we obtain for $\mu_1,\ \mu_2,\ \nu \in \mathcal{P}_2(\mathbb{R}^d)$

$$W_2(\nu * \mu_1, \nu * \mu_2) \le W_2(\mu_1, \mu_2). \tag{2.9}$$

Note that it is not necessary that $\mu_1,\ \mu_2$ and ν have the same mean.

Remark 2.3. *Inequality* (2.7) *gives with U defined in equation (2.2),*

$$W_2(U\#\mu, U\#\nu)^2 \le \frac{1}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y)$$

for all coupling π of (μ, ν). Therefore by taking the infimum over π, we obtain

$$W_2(U\#\mu, U\#\nu) \le \frac{1}{2} W_2(\mu, \nu). \tag{2.10}$$

The following Lemma is the key step for the fixed point theorem of Theorem 1.1.

Lemma 2.1. *For $\mu,\ \nu \in \mathcal{P}_2^m(\mathbb{R}^d)$, we have*

$$W_2(g * (U\#\mu) * (U\#\mu), g * (U\#\nu) * (U\#\nu)) \le \frac{1}{\sqrt{2}} W_2(\mu, \nu). \tag{2.11}$$

Proof. It is enough to apply successively (2.9), (2.6) and (2.10). We have

$$\begin{aligned} W_2(g * (U\#\mu) * (U\#\mu), g * (U\#\nu) * (U\#\nu)) &\le W_2((U\#\mu) * (U\#\mu), (U\#\nu) * (U\#\nu)) \\ &\le \sqrt{2} W_2(U\#\mu, U\#\nu) \\ &\le \frac{1}{\sqrt{2}} W_2(\mu, \nu) \end{aligned}$$

■

This result is already present in Theorem 4.1 in ref. [11]. The proof presented here is different. We recall the following elementary fact that we prove in Appendix A.

Lemma 2.2. *The space $\mathcal{P}_2^m(\mathbb{R}^d)$ is a complete metric space for W_2.*

The following Lemma shows that a mild solution can be seen as a fixed point of some contracting non-linear operator.

Lemma 2.3. *Let $m \in \mathbb{R}^d$, $f_0 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and $g \in \mathcal{P}_2^0(\mathbb{R}^d)$. Define $E_{f_0} = \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ with $f(0,.) = f_0$ equipped with the uniform norm. For $f \in E_{f_0}$, we define the map $\Phi : E_{f_0} \longrightarrow E_{f_0}$ by*

$$\Phi[f](t,\,\cdot\,) := e^{-t} f(0,\,\cdot\,) + \int_0^t e^{-(t-s)} g * (U\#f(s,\,\cdot\,)) * (U\#f(s,\,\cdot\,)) ds. \tag{2.12}$$

Then for all $f_0^1,\ f_0^2 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and for $f^1 \in E_{f_0^1},\ f^2 \in E_{f_0^2}$, it holds for every $t \ge 0$

$$W_2(\Phi[f^1](t,\,\cdot\,), \Phi[f^2](t,\,\cdot\,))^2 \le e^{-t} W_2(f_0^1, f_0^2)^2 + \frac{1}{2} \int_0^t e^{-(t-s)} W_2(f^1(s,\,\cdot\,), f^2(s,\,\cdot\,))^2 ds. \tag{2.13}$$

Proof. Let $f \in E_{f_0}$. It is clear that $\Phi[f](0,\cdot) = f_0$. Since for all $t \geq 0$, $f(t,\cdot) \in \mathcal{P}_2^m(\mathbb{R}^d)$, $\rho(t) = 1$ and $u(t) = m$. So, by (2.3), $\Phi[f](t,\cdot) \in \mathcal{P}_2^m(\mathbb{R}^d)$ for all $t \geq 0$. By writing

$$\Phi[f](t,\cdot) = e^{-t} f_0 + (1-e^{-t}) \int_0^t \frac{e^{-(t-s)}}{1-e^{-t}} g * (U\#f(s,\cdot)) * (U\#f(s,\cdot)) ds,$$

the convexity of W_2 (2.4) gives that for all $f^1 \in E_{f_0^1}$, $f^2 \in E_{f_0^2}$,

$$W_2(\Phi[f^1](t,\cdot), \Phi[f^2](t,\cdot))^2 \leq e^{-t} W_2(f_0^1, f_0^2)^2$$
$$+ (1-e^{-t}) W_2\left(\int_0^t \frac{e^{-(t-s)}}{1-e^{-t}} g * (U\#f_s^1) * (U\#f_s^1) ds, \int_0^t \frac{e^{-(t-s)}}{1-e^{-t}} g * (U\#f_s^2) * (U\#f_s^2) ds \right)^2.$$

And using (2.8), it holds that

$$W_2(\Phi[f^1](t,\cdot), \Phi[f^2](t,\cdot))^2 \leq \int_0^t e^{-(t-s)} W_2(g * (U\#f_s^1) * (U\#f_s^1), g * (U\#f_s^2) * (U\#f_s^2))^2 ds$$
$$+ e^{-t} W_2(f_0^1, f_0^2)^2.$$

And by (2.11), we obtain (2.13). ■

To prove Lemma 2.3, we used arguments that are used several times in ref. [12].

Proof of item (1) in Theorem 1.1. Let $f_0 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and $g \in \mathcal{P}_2^0(\mathbb{R}^d)$. Consider the map Φ defined by (2.12). Hence for $f_1, f_2 \in E_{f_0}$, (2.13) leads to

$$W_2(\Phi[f_1](t,\cdot), \Phi[f_2](t,\cdot))^2 \leq \frac{1}{2} \int_0^t e^{-(t-s)} W_2(f_1(s,\cdot), f_2(s,\cdot))^2 ds.$$

By considering the supremum in time,

$$W_2(\Phi[f_1](t,\cdot), \Phi[f_2](t,\cdot))^2 \leq \frac{1-e^{-t}}{2} \sup_{t\in\mathbb{R}^+} W_2(f_1(t,\cdot), f_2(t,\cdot))^2.$$

So

$$\sup_{t\in\mathbb{R}^+} W_2(\Phi[f_1](t,\cdot), \Phi[f_2](t,\cdot)) \leq \frac{1}{\sqrt{2}} \sup_{t\in\mathbb{R}^+} W_2(f_1(t,\cdot), f_2(t,\cdot)). \tag{2.14}$$

Hence Φ preserves the space E_{f_0} and is a contraction. As $\mathcal{P}_2^m(\mathbb{R}^d)$ is a complete metric space for W_2, E_{f_0} is complete. Hence there exists a unique mild solution of the equation (1.4) belonging to E_{f_0}. ■

Proposition 2.2 (Properties of mild solutions). *Let $m \in \mathbb{R}^d$, $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ be the mild solution of equation* (1.4) *with $f(0,\cdot) = f_0$. Then we have the following properties.*

(1) f is a solution of equation (1.4) *in the sense of distributions (see Definition 1.1).*

(2) If f_0 and g are probability densities, then $f(t,\cdot)$ is a probability density for all $t \geq 0$.

(3) For all $t \geq 0$, denoting $\Sigma_f(t) := \int_{\mathbb{R}^d} (v-m)\,{}^t(v-m) f(t,dv)$,

$$\Sigma_f(t) = e^{-t/2} \Sigma_f(0) + 2(1-e^{-t/2}) \Sigma_g. \tag{2.15}$$

(4) For every mild solution f_1, $f_2 \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$, we have for all $t \geq 0$

$$W_2(f_1(t,\cdot), f_2(t,\cdot)) \leq e^{-t/4} W_2(f_1(0,\cdot), f_2(0,\cdot)). \tag{2.16}$$

Proof. Item (1). By direct computation

$$\begin{aligned}
-\left\langle f, \frac{\partial \varphi}{\partial t}\right\rangle &= \int_{\mathbb{R}^d} \varphi(0,v) f_0(dv) - \int_0^{+\infty}\int_{\mathbb{R}^d} e^{-t}\varphi(t,v) f_0(dv)dt \\
&- \int_0^{+\infty}\int_{\mathbb{R}^d} e^s \left(\int_s^{+\infty} e^{-t}\frac{\partial \varphi}{\partial t}(t,v)dt\right) g*(U\#f(s,\cdot))*(U\#f(s,\cdot))(dv)ds \\
&= \int_{\mathbb{R}^d} \varphi(0,v) f_0(dv) - \int_0^{+\infty}\int_{\mathbb{R}^d} e^{-t}\varphi(t,v) f_0(dv)dt \\
&+ \int_0^{+\infty}\int_{\mathbb{R}^d} \varphi(s,v) g*(U\#f(s,\cdot))*(U\#f(s,\cdot))(dv)ds \\
&- \int_0^{+\infty}\int_{\mathbb{R}^d}\int_s^{+\infty} \varphi(t,v)e^{-(t-s)} g*(U\#f(s,\cdot))*(U\#f(s,\cdot))(dv)dtds \\
&= \int_{\mathbb{R}^d} \varphi(0,v) f_0(dv) + \langle Q_+(f,f),\varphi\rangle - \int_0^{+\infty}\int_{\mathbb{R}^d} \varphi(t,v) f(t,dv)dt.
\end{aligned}$$

Since for any $t \geq 0$, $f(t,\cdot)$ is a probability measure, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^d} \varphi(t,v) f(t,dv)dt = \int_0^{+\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(t,v) f(t,dv) f(t,dv_\star)dt = \langle Q_-(f,f),\varphi\rangle.$$

Item (2). This is obvious because if g is a density, then $g*(U\#f(t,\cdot))*(U\#f(t,\cdot))$ is a density for all $t \geq 0$. By (1.6), we obtain that $f(t,\cdot)$ is a density for all t since f_0 is a density by hypothesis.

Item (3). According to (1.6) and (2.3), it holds that

$$\begin{aligned}
\int_{\mathbb{R}^d} v\,{}^t v f(t,dv) &= e^{-t}\int_{\mathbb{R}^d} v\,{}^t v f_0(dv) + \int_0^t e^{-(t-s)}\left(\int_{\mathbb{R}^d} v\,{}^t v g*(U\#f(s,\cdot))*(U\#f(s,\cdot))(dv)\right)ds \\
&= e^{-t}\int_{\mathbb{R}^d} v\,{}^t v f_0(dv) + \int_0^t e^{-(t-s)}\left(\Sigma_g + m\,{}^t m + \frac{\Sigma_f(s)}{2}\right)ds.
\end{aligned}$$

So we have

$$e^t\Sigma_f(t) = \Sigma_f(0) + (1-e^{-t})\Sigma_g + \int_0^t \frac{e^s}{2}\Sigma_f(s)ds.$$

The case of equality in Gronwall's lemma leads to

$$\begin{aligned}
e^t\Sigma_f(t) &= \Sigma_f(0) + (1-e^{-t})\Sigma_g + \int_0^t \frac{\Sigma_f(0)+(1-e^{-s})\Sigma_g}{2} e^{(t-s)/2}ds \\
&= \Sigma_f(0) + (1-e^{-t})\Sigma_g + (e^{t/2}-1)\Sigma_f(0) + (e^t+1+2e^{t/2})\Sigma_g.
\end{aligned}$$

Which implies formula (2.15).

Item (4). Let $f_1 \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ be the mild solution with initial condition $f_0^1 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and let $f_2 \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ be the mild solution with initial condition $f_0^2 \in \mathcal{P}_2^m(\mathbb{R}^d)$. By (2.13), it comes that

$$W_2(f_1(t,\cdot), f_2(t,\cdot))^2 \leq e^{-t}W_2(f_0^1, f_0^2)^2 + \frac{1}{2}\int_0^t e^{-(t-s)}W_2(f_1(s,\cdot), f_2(s,\cdot))^2 ds.$$

Gronwall's Lemma leads to

$$e^t W_2(f_1(t,\cdot), f_2(t,\cdot))^2 \leq e^{t/2}W_2(f_0^1, f_0^2)^2$$

and (2.16) follows. ■

3 Equilibrium state

This section is devoted to the determination of the equilibrium state of the collision operator of equation (1.4). For $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}^d))$, we set

$$\hat{f}(t,\xi) := \int_{\mathbb{R}^d} e^{-i\langle v,\xi\rangle} f(t,dv).$$

So if $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ is the mild solution of equation (1.4) (see Definition 1.2) with initial condition $f(0,\cdot) = f_0$, then $\hat{f}$ is solution of the fixed point equation

$$\hat{f}(t,\xi) = e^{-t}\hat{f}(0,\xi) + \int_0^t e^{-(t-s)}\hat{g}(\xi)\hat{f}\left(s,\frac{\xi}{2}\right)^2 ds. \tag{3.1}$$

Note that the changeover in Fourier variable can be performed because β is constant. By differentiation of (3.1), it comes that

$$\begin{cases} \dfrac{\partial \hat{f}}{\partial t} = \hat{g}(\xi)\hat{f}\left(t,\dfrac{\xi}{2}\right)^2 - \hat{f}(t,\xi) \\ \hat{f}(0,\xi) = \hat{f}_0(\xi). \end{cases} \tag{3.2}$$

We notice that equation (1.7) can be written equivalently as

$$f = g * (U\#f) * (U\#f)$$

where U is defined in (2.2). Thus, equation (1.7) is equivalent to

$$\hat{f}(\xi) = \hat{g}(\xi)\hat{f}\left(\frac{\xi}{2}\right)^2. \tag{3.3}$$

Item (2) of Theorem 1.1 is a consequence of the following proposition. Theorem 1 of ref. [4] is improved by choosing a probability measure $g(dv)$ instead of a density $g(v)dv$ and by proving the uniqueness of the equilibrium state in this general setting. Moreover we do not use Levi's Theorem to recognize a Fourier transform of a probability measure.

Proposition 3.1 (Theorem 1 in ref. [4])**.** *Let $g \in \mathcal{P}_2^0(\mathbb{R}^d)$. For all $m \in \mathbb{R}^d$, there exists a unique $f \in \mathcal{P}_2^m(\mathbb{R}^d)$ solution of* (1.7)*. In addition we have*

$$\hat{f}(\xi) = e^{-i\langle m,\xi\rangle} \prod_{n=0}^{+\infty} \hat{g}\left(\frac{\xi}{2^n}\right)^{2^n}. \tag{3.4}$$

Proof. For $f \in \mathcal{P}_2^m(\mathbb{R}^d)$, we define the map $\Phi : \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathcal{P}_2(\mathbb{R}^d)$ by

$$\Phi[f] := g * (U\#f) * (U\#f).$$

By (2.3), we check that Φ maps $\mathcal{P}_2^m(\mathbb{R}^d)$ into itself and is a contraction from (2.11). So there exists a unique $f \in \mathcal{P}_2^m(\mathbb{R}^d)$ such that $\Phi[f] = f$. Let us now show that $\hat{f}$ satisfies (3.4). By iterating the equation (3.3), a Taylor expansion leads to

$$\begin{aligned} \hat{f}(\xi) &= \left[\prod_{k=0}^{n-1} \hat{g}\left(\frac{\xi}{2^k}\right)^{2^k}\right] \hat{f}\left(\frac{\xi}{2^n}\right)^{2^n} \\ &= \left[\prod_{k=0}^{n-1} \hat{g}\left(\frac{\xi}{2^k}\right)^{2^k}\right] \left(1 + \frac{1}{2^n}\left(-i\langle m,\xi\rangle - \frac{{}^t\xi(\Sigma_f + m\,{}^tm)\xi}{2^{n+1}} + o\left(\frac{|\xi|^2}{2^n}\right)\right)\right)^{2^n}. \end{aligned}$$

The second factor on the right-hand side converges to $e^{-i\langle m,\xi\rangle}$ when n tends to infinity. Since if $(x_n)_n$ converges to x, then $(1+x_n/n)^n$ converges to e^x. We thus obtain (3.4) by letting n tend to infinity. ∎

We denotes by f_m^∞ (which is a density if g is a density) the unique solution of (1.7) in $\mathcal{P}_2^m(\mathbb{R}^d)$. In particular, $\hat{f}_m^\infty$ satisfies (3.4).

Remark 3.1. *Let $m = 0$. By calculating the Hessian matrix $H_{\hat{f}_0^\infty}(0)$, we notice that if $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ is the mild solution of equation (1.4), then by (2.15), $\Sigma_f(t)$ converges to $2\Sigma_g$ when t goes to infinity. Which corresponds well to the covariance matrix of f_0^∞.*

Indeed, by differentiating the function $\xi \longmapsto \log(\hat{f}_0^\infty(\xi))$, we have since $\hat{f}_0^\infty$ satisfies (3.4) than

$$\nabla \hat{f}_0^\infty(\xi) = \hat{f}_0^\infty(\xi) \sum_{n=0}^{+\infty} \frac{\nabla \hat{g}(\xi/2^n)}{\hat{g}(\xi/2^n)}.$$

By differentiating the previous formula, we have

$$H_{\hat{f}_0^\infty}(\xi) = \hat{f}_0^\infty(\xi) \sum_{n=0}^{+\infty} \frac{\hat{g}(\xi/2^n) H_{\hat{g}}(\xi/2^n) - \nabla \hat{g}(\xi/2^n)\, {}^t\nabla \hat{g}(\xi/2^n)}{2^n (\hat{g}(\xi/2^n))^2} + \nabla \hat{f}_m^\infty(\xi)\, {}^t\!\left(\sum_{n=0}^{+\infty} \frac{\nabla \hat{g}(\xi/2^n)}{\hat{g}(\xi/2^n)}\right).$$

We obtain the covariance matrix of f_0^∞ by calculating this expression above at $\xi = 0$ since for a centered probability measure μ, $\hat{\mu}(0) = 1$, $i\nabla\hat{\mu}(0) = 0$, and $-H_{\hat{\mu}}(0) = \int v\, {}^t v d\mu(v)$.

Remark 3.2. *In the particular case where $g \in \mathcal{P}_2^0(\mathbb{R}^d)$ is a Gaussian (centered of covariance matrix Σ_g), then $\hat{g}(\xi) = \exp(-({}^t\xi\Sigma_g\xi)/2)$ and by (3.4),*

$$\hat{f}_m^\infty(\xi) = e^{-i\langle m,\xi\rangle} \prod_{n=0}^{+\infty} \exp\left(-\frac{{}^t\xi\Sigma_g v}{2^{n+1}}\right) = \exp\left(-i\langle m,\xi\rangle - \frac{{}^t\xi\Sigma_g\xi}{2} \sum_{n=0}^{+\infty} \frac{1}{2^n}\right) = \exp\left(-i\langle m,\xi\rangle - {}^t\xi\Sigma_g\xi\right).$$

So by the Fourier inverse transform, f_m^∞ is also a Gaussian with mean vector m and covariance matrix $2\Sigma_g$.

Proof of items (2) and (3) in Theorem 1.1. Item (2) Readily follows form Proposition 3.1.

Item (3). Let $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ be the mild solution of equation (1.4) with initial condition $f_0 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and let $f_m^\infty \in \mathcal{P}_2^m(\mathbb{R}^d)$ be the equilibrium state of equation (1.4). Note that

$$\forall t \geq 0, \quad f_m^\infty = e^{-t} f_m^\infty + \int_0^t e^{-(t-s)} g * (U\# f_m^\infty) * (U\# f_m^\infty) ds.$$

By taking the map Φ defined in (2.12), we have by (2.13)

$$\begin{aligned} W_2(f(t,\cdot), f_m^\infty)^2 &= W_2(\Phi[f](t,\cdot), \Phi[f_m^\infty])^2 \\ &\leq e^{-t} W_2(f_0, f_m^\infty)^2 + \frac{1}{2}\int_0^t e^{-(t-s)} W_2(f(s,\cdot), f_m^\infty)^2 ds. \end{aligned}$$

By Gronwall's Lemma, we have

$$e^t W_2(f(t,\cdot), f_m^\infty)^2 \leq e^{t/2} W_2(f_0, f_m^\infty)^2$$

and (1.8) follows. ■

We now make the link with the result obtained in ref. [8] for the discrete midpoint model corresponding to the equation (1.4) with $g = \delta_0$.

Proposition 3.2 (Proposition 2.3 in ref. [8])**.** *We consider equation (1.4) with $g = \delta_0$. For $f_0 \in \mathcal{P}_2^m(\mathbb{R}^d)$, there exists a unique mild solution of equation (1.4) in $\mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ with initial condition f_0. Moreover, we have the estimate*

$$W_2(f(t,\cdot), \delta_m) \leq e^{-t/4} W_2(f_0, \delta_m). \tag{3.5}$$

Proof. Existence and uniqueness of the solution follow from the item (1) in Theorem 1.1. By items (2) and (3) in Theorem 1.1, there exists a unique equilibrium state $f_m^\infty \in \mathcal{P}_2^m(\mathbb{R}^d)$ such that

$$W_2(f(t,\cdot), f_m^\infty) \leq e^{-t/4} W_2(f_0, f_m^\infty).$$

Since $g = \delta_0$, we have $\hat{g}(\xi) = 1$ for all ξ. And so by (3.4), we have

$$\hat{f}_m^\infty(\xi) = e^{-i\langle m,\xi\rangle}.$$

We recognize the Fourier transform of δ_m, so $f_m^\infty = \delta_m$ and (3.5) follows. ■

As mentioned in ref. [8], the conservation of the center of mass m has played a fundamental role in the functional space $\mathcal{P}_2^m(\mathbb{R}^d)$. It would be much more difficult to prove these estimates on the sphere $\mathbb{S}^{d-1}$ because the center of mass is no longer conserved.

4 Convergence for d_2

We introduce in this section the Fourier-Toscani-based distance between μ and $\nu \in \mathcal{P}_2^m(\mathbb{R}^d)$ having the same mean value by

$$d_2(\mu,\nu) := \sup_{\xi\in\mathbb{R}^d} \frac{|\hat{\mu}(\xi) - \hat{\nu}(\xi)|}{|\xi|^2}.$$

A Taylor expansion shows that this metric is well-defined for μ, $\nu \in \mathcal{P}_2^m(\mathbb{R}^d)$ and metrizes the weak topology on $\mathcal{P}_2^m(\mathbb{R}^d)$ (see ref. [13]). We proved in the previous section that $f_m^\infty \in \mathcal{P}_2^m(\mathbb{R}^d)$. So $d_2(f(t,\cdot), f_m^\infty)$ is well defined for all $t \geq 0$. The following result gives the exponential convergence of $f(t,\cdot)$ to f_m^∞ for the d_2 metric.

Proposition 4.1 (Exponential convergence for d_2)**.** *Let $f_0 \in \mathcal{P}_2^m(\mathbb{R}^d)$ and $g \in \mathcal{P}_2^0(\mathbb{R}^d)$. If $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$ is the mild solution of equation* (1.4) *with $f(0,\cdot) = f_0$ and if $f_m^\infty \in \mathcal{P}_2^m(\mathbb{R}^d)$ is the equilibrium state of* (1.4) *with mean velocity m, then it holds that for all $t \geq 0$*

$$d_2(f(t,\cdot), f_m^\infty) \leq \frac{M_2(f_0) + 2M_2(g) + |m|^2}{2} e^{-t/2}. \tag{4.1}$$

Proof. For $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2^m(\mathbb{R}^d))$, mild solution of (1.4), we set for any $t \geq 0$ and $\xi \in \mathbb{R}$,

$$H(t,\xi) := \frac{\hat{f}(t,\xi) - \hat{f}_m^\infty(\xi)}{|\xi|^2}.$$

A Taylor expansion gives that

$$e^{-i\langle v,\xi\rangle} = 1 - i\langle v,\xi\rangle - \langle v,\xi\rangle^2 \int_0^1 (1-s) e^{-is\langle v,\xi\rangle} ds.$$

So

$$\begin{aligned}
|H(t,\xi)| &= \left| \int_{\mathbb{R}^d}\int_0^1 \frac{\langle v,\xi\rangle^2}{|\xi|^2}(1-s)e^{-is\langle v,\xi\rangle} f(t,v)\,ds\,dv - \int_{\mathbb{R}^d}\int_0^1 \frac{\langle v,\xi\rangle^2}{|\xi|^2}(1-s)e^{-is\langle v,\xi\rangle} f_m^\infty(v)\,ds\,dv \right| \\
&\leq \int_{\mathbb{R}^d}\int_0^1 \left| \frac{\langle v,\xi\rangle^2}{|\xi|^2}(1-s)e^{-is\langle v,\xi\rangle}(f(t,v) - f_m^\infty(v)) \right| ds\,dv.
\end{aligned}$$

Hence the previous inequality taken at $t = 0$ and Cauchy-Schwarz inequality leads to

$$|H(0,\xi)| \leq \frac{M_2(f_0) + 2M_2(g) + |m|^2}{2}. \tag{4.2}$$

We define now G by

$$G(t,\xi) := \frac{\hat{g}(\xi)}{4}\left(\hat{f}\left(t,\frac{\xi}{2}\right) + \hat{f}_m^\infty\left(\frac{\xi}{2}\right)\right).$$

Using (3.2) and (3.3), it holds that

$$\begin{aligned} G(t,\xi)H\left(t,\frac{\xi}{2}\right) - H(t,\xi) &= \frac{\hat{g}(\xi)}{|\xi|^2}\hat{f}\left(t,\frac{\xi}{2}\right)^2 - \frac{\hat{f}(t,\xi)}{|\xi|^2} - \left(\frac{\hat{g}(\xi)}{|\xi|^2}\hat{f}_m^\infty\left(\frac{\xi}{2}\right) - \frac{f_m^\infty(\xi)}{|\xi|^2}\right) \\ &= \frac{1}{|\xi|^2}\frac{\partial \hat{f}}{\partial t}. \end{aligned}$$

So H satisfies

$$\frac{\partial H}{\partial t} = G(t,\xi)H\left(t,\frac{\xi}{2}\right) - H(t,\xi)$$

and by Duhamel's formula, we get

$$H(t,\xi) = e^{-t}H(0,\xi) + \int_0^t e^{-(t-s)}G(s,\xi)H\left(s,\frac{\xi}{2}\right)ds.$$

For $R > 0$, we set

$$y(t) := e^t \sup_{|\xi|\le R} |H(t,\xi)|$$

since the map $\xi \longmapsto \xi/2$ maps $\{\xi,\ |\xi| \le R\}$ into $\{\xi,\ |\xi| \le R/2\}$. Since $G(t,\xi) \le 1/2$, Gronwall's Lemma applied to inequality

$$y(t) \le y(0) + \frac{1}{2}\int_0^t y(s)ds$$

gives that for all $R \ge 0$

$$\sup_{|\xi|\le R} |H(t,\xi)| \le |H(0,\xi)|e^{-t/2}.$$

So, by using the estimate (4.2) we get formula (4.1). ■

5 Convergence L^1

This section is devoted to the proof of item (4) in Theorem 1.1. The initial condition $f(0,\cdot)$ is assumed to be a regular function and we prove the exponential convergence in L^1 of $f(t,\cdot)$ to the equilibrium state determined in section 3. The regularity of the initial condition is measured in term of the Sobolev norm. For $s \ge 0$ and for $d \in \mathbb{N}^*$, the Sobolev norm in $\mathbb{R}^d$ of regularity s is given by

$$\|f\|_{H^s(\mathbb{R}^d)} := \sqrt{\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi}.$$

Lemma 5.1. *Let $s \ge 0$ and $f_0,\ g \in H^s(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$. Let $f \in \mathcal{C}^0(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}^d))$ be the mild solution* (1.4) *with initial condition f_0. Then for every $t \ge 0$, $f(t,\cdot) \in H^s(\mathbb{R}^d)$ and*

$$\|f(t,\cdot)\|_{H^s(\mathbb{R}^d)} \le e^{-t}\|f_0\|_{H^s(\mathbb{R}^d)} + (1-e^{-t})\|g\|_{H^s(\mathbb{R}^d)}. \tag{5.1}$$

Proof. Let $R > 0$. Define

$$Z_R(t) := \int_{|\xi|<R} (1+|\xi|^2)^s |\hat{f}(t,\xi)|^2 d\xi.$$

Using (3.2) and the inequality $|\hat{\mu}(\xi)| \leq 1$ for μ a probability measure, it comes that $\partial_t \hat{f}(t,\xi)$ is uniformly bounded by 2. By differentiation, it holds that

$$\begin{aligned}\frac{d}{dt} Z_R(t) &= 2\int_{|\xi|<R} (1+|\xi|^2)^s \Re\left(\overline{\hat{f}(t,\xi)}\frac{\partial \hat{f}}{\partial t}(t,\xi)\right) d\xi \\ &\leq 2\int_{|\xi|<R} (1+|\xi|^2)^s |\hat{g}(\xi)| \cdot |\hat{f}(t,\xi)| d\xi - 2Z_R(t).\end{aligned}$$

Cauchy-Schwarz inequality applied to the right-hand side gives that

$$\frac{d}{dt} Z_R(t) \leq 2\|g\|_{H^s(\mathbb{R}^d)} \sqrt{Z_R(t)} - 2Z_R(t).$$

Since $Z_R(t)$ never vanishes,

$$\frac{d}{dt}\sqrt{Z_R(t)} \leq \|g\|_{H^s(\mathbb{R}^d)} - \sqrt{Z_R(t)}.$$

And by Gronwall's Lemma,

$$\sqrt{Z_R(t)} \leq e^{-t}\sqrt{Z_R(0)} + (1-e^{-t})\|g\|_{H^s(\mathbb{R}^d)}.$$

We conclude the proof by letting R to infinity. ∎

Since g is a probability density, $\|\hat{g}\|_\infty \leq 1$, using the explicit definition $\hat{f}_m^\infty$ in (3.4), we obtain $|f_m^\infty| \leq |\hat{g}|$ and the following result.

Lemma 5.2. *For all $s \geq 0$, we have*

$$\|f_m^\infty\|_{H^s(\mathbb{R}^d)} \leq \|g\|_{H^s(\mathbb{R}^d)}. \tag{5.2}$$

We also need the following two interpolation inequalities (Theorem 4.1 and 4.2 in ref. [6]) that we prove for the reader convenience. We recall the notion of a moment of order $\alpha > 0$ of a density $f \in \mathcal{P}_\alpha(\mathbb{R}^d)$

$$M_\alpha(f) := \int_{\mathbb{R}^d} |v|^\alpha f(v) dv.$$

Lemma 5.3. *Let $\alpha > 0$ and $d \geq 1$ be an integer. Then there exists a constant $C(\alpha, d) > 0$ such that for every function $f \in L^2(\mathbb{R}^d) \cap \mathcal{P}_\alpha(\mathbb{R}^d)$*

$$\|f\|_{L^1(\mathbb{R}^d)} \leq C(\alpha,d) \|f\|_{L^2(\mathbb{R}^d)}^{\alpha'} M_\alpha(f)^{1-\alpha'}, \tag{5.3}$$

with $\alpha' := 2\alpha/(2\alpha+d)$.

Proof. For every $R > 0$, it holds that

$$\begin{aligned}\int_{\mathbb{R}} |f(v)| dv &\leq \int_{|v|\leq R} |f(v)| dv + \frac{1}{R^\alpha}\int_{\mathbb{R}^d} |v|^\alpha |f(v)| dv \\ &\leq R^{d/2}\sqrt{\mathrm{Vol}(B_d)}\|f\|_{L^2(\mathbb{R}^d)} + \frac{1}{R^\alpha} M_\alpha(f)\end{aligned} \tag{5.4}$$

where $\mathrm{Vol}(B_d)$ is the euclidian volume of the unit ball of $\mathbb{R}^d$. R is chosen such as

$$R^{d/2}\sqrt{\mathrm{Vol}(B_d)}\|f\|_{L^2(\mathbb{R}^d)} = \frac{1}{R^\alpha} M_\alpha(f).$$

This defines R as

$$R = \left(\frac{M_\alpha(f)}{\sqrt{\mathrm{Vol}(B_d)}\|f\|_{L^2(\mathbb{R}^d)}}\right)^{2/(2\alpha+d)}. \tag{5.5}$$

By taking R as in (5.5) in (5.4), we obtain (5.3) for $C(\alpha, d) = 2\mathrm{Vol}(B_d)^{\alpha/(2\alpha+d)}$. ∎

Lemma 5.4. *Let $s \geq 0$ and $d \geq 1$ be an integer. For every $s' > d/2 + 2s + 2$ there exists a constant $C(s, s', d) > 0$ such that for every $f_1,\ f_2 \in H^{s'}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} v f_1(v)dv = \int_{\mathbb{R}^d} v f_2(v)dv$*

$$\|f_1 - f_2\|_{H^s(\mathbb{R}^d)} \leq C(s, s', d)\sqrt{d_2(f_1, f_2)}\sqrt{\|f_1 - f_2\|_{H^{s'}(\mathbb{R}^d)}}. \tag{5.6}$$

Proof. For $s' > d/2 + 2s + 2$, we have

$$\int_{\mathbb{R}^d}(1 + |\xi|^2)^s|\hat{f}_1(\xi) - \hat{f}_2(\xi)|^2 d\xi \leq \int_{\mathbb{R}^d} \frac{|\hat{f}_1(\xi) - \hat{f}_2(\xi)|}{|\xi|^2} \frac{(1 + |\xi|^2)^{s+1}}{(1 + |\xi|^2)^{s'/2}}(1 + |\xi|^2)^{s'/2}|\hat{f}_1(\xi) - \hat{f}_2(\xi)|d\xi.$$

By Cauchy-Schwarz inequality, it holds that

$$\int_{\mathbb{R}^d}(1 + |\xi|^2)^s|\hat{f}_1(\xi) - \hat{f}_2(\xi)|^2 d\xi \leq d_2(f_1, f_2)\left(\int_{\mathbb{R}^d} \frac{d\xi}{(1 + |\xi|^2)^{s'-2s-2}}\right)^{1/2} \|f_1 - f_2\|_{H^{s'}(\mathbb{R}^d)}.$$

The assumption on s' implies that the integral in the right-hand side is finite. ■

Proof of item (4) in Theorem 1.1. Applying (5.3) with $v \longmapsto |f(t, v) - f_m^\infty(v)|$ and with $\alpha = 2$, leads to

$$\begin{aligned}\|f(t,\cdot) - f_m^\infty\|_{L^1(\mathbb{R}^d)} &\leq C(2, d)\|f(t,\cdot) - f_m^\infty\|_{L^2(\mathbb{R}^d)}^{4/(d+4)} M_2(|f(t,\cdot) - f_m^\infty|)^{d/(d+4)} \\ &\leq C(2, d)\|f(t,\cdot) - f_m^\infty\|_{L^2(\mathbb{R}^d)}^{4/(d+4)}(M_2(f(t,.)) + 2M_2(g) + |m|^2)^{d/(d+4)}.\end{aligned}$$

By (2.15), $M_2(f(t,.)) \leq M_2(f_0) + 2M_2(g)$ for all $t \geq 0$. So

$$\|f(t,\cdot) - f_m^\infty\|_{L^1(\mathbb{R}^d)} \leq C_1\|f(t,\cdot) - f_m^\infty\|_{L^2(\mathbb{R}^d)}^{4/(d+4)}(M_2(f_0) + M_2(g) + |m|^2)^{1/5} \tag{5.7}$$

with $C_1 = 4^{d/(d+4)}C(2, d)$. Then by (5.6) with $s = 0$, and $s' > 2 + d/2$, it comes that

$$\begin{aligned}\|f(t,\cdot) - f_m^\infty\|_{L^2(\mathbb{R}^d)}^2 &\leq C(0, s', d)^2 d_2(f(t,\cdot), f_m^\infty)\|f(t,\cdot) - f_m^\infty\|_{H^{s'}(\mathbb{R}^d)} \\ &\leq C(0, s', d)^2 d_2(f(t,\cdot), f_m^\infty)(\|f(t,\cdot)\|_{H^{s'}(\mathbb{R}^d)} + \|f_m^\infty\|_{H^{s'}(\mathbb{R}^d)}).\end{aligned}$$

By (5.1) and (5.2), we have $\|f(t,\cdot)\|_{H^{s'}(\mathbb{R}^d)} + \|f_m^\infty\|_{H^{s'}(\mathbb{R}^d)} \leq \|f_0\|_{H^{s'}(\mathbb{R}^d)} + 2\|g\|_{H^{s'}(\mathbb{R}^d)}$. So

$$\|f(t,\cdot) - f_m^\infty\|_{L^2(\mathbb{R}^d)}^{4/(d+4)} \leq C_2 d_2(f(t,\cdot), f_m^\infty)^{2/(d+4)}(\|f_0\|_{H^{s'}(\mathbb{R}^d)} + \|g\|_{H^{s'}(\mathbb{R}^d)})^{2/(d+4)} \tag{5.8}$$

with $C_2 = 2^{2/(d+4)}C(0, s', d)^{4/(d+4)}$. Substituting (5.8) in (5.7), leads to

$$\begin{aligned}\|f(t,\cdot) - f_m^\infty\|_{L^1(\mathbb{R}^d)} \leq C_1C_2(\|f_0\|_{H^{s'}(\mathbb{R}^d)} + &\|g\|_{H^{s'}(\mathbb{R}^d)})^{2/(d+4)} \\ &(M_2(f_0) + M_2(g) + |m|^2)^{d/(d+4)} d_2(f(t,\cdot), f_m^\infty)^{2/(d+4)}.\end{aligned}$$

And then by (4.1),

$$\begin{aligned}d_2(f(t,\cdot), f_m^\infty)^{2/(d+4)} &\leq \left(\frac{M_2(f_0) + 2M_2(g) + |m|^2}{2}\right)^{2/(d+4)} e^{-t/(d+4)} \\ &\leq (M_2(f_0) + M_2(g) + |m|^2)^{2/(d+4)} e^{-\frac{t}{(d+4)}}.\end{aligned}$$

Thus, we obtain (1.9) with a constant

$$C = C_1C_2(\|f_0\|_{H^{s'}(\mathbb{R}^d)} + \|g\|_{H^{s'}(\mathbb{R}^d)})^{2/(d+4)}(\sigma_f^2(0) + \sigma_g^2 + m^2)^{(d+2)/(d+4)}.$$

And consequently the exponential convergence of $f(t,\cdot)$ towards f_m^∞ is obtained since $f_0 \in H^{s'}(\mathbb{R}^d) \cap \mathcal{P}_2^m(\mathbb{R}^d)$ and $g \in H^{s'}(\mathbb{R}^d) \cap \mathcal{P}_2^0(\mathbb{R}^d)$ with $s' > 2 + d/2$. ■

6 Numerical results

This section is devoted to the numerical resolution of (1.4) in dimension $d = 1$. We will present six tests cases for two different initial conditions f_0 and with three different values of g, where g is a density. For each test case, the solution f is depicted for different values of t and compared with the equilibrium state f_m^∞ theoretically found in order to characterize the exponential rate of convergence for the strong-norm L^1. To represent the solution of (1.4) numerically, we use an Euler scheme in time for $\Delta t = 0.015$ followed by a Simpson rule on the interval $[-10, 10]$ with a uniform step $\Delta x = 0.1$. We will therefore represent numerically the solutions of the equation

$$\begin{cases} \dfrac{\partial f}{\partial t} = \displaystyle\iint_{\mathbb{R}\times\mathbb{R}} g\left(v - \frac{v' + v'_\star}{2}\right) f(t, dv')f(t, dv'_\star) - f(t, \cdot\,) \int_{\mathbb{R}} f(t, dv') \\ \\ f(0, \cdot\,) = f_0. \end{cases} \tag{6.1}$$

In the first case, g is a centered Gaussian of variance $\sigma_g^2 = 1$,

$$g(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right). \tag{6.2}$$

In the second case where g is an indicator function

$$g(v) = \frac{1}{2}\mathbb{1}_{[-1,1]}(v). \tag{6.3}$$

In the third case, g writes

$$g(v) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(v - m_i)^2}{2\sigma_i^2}\right), \tag{6.4}$$

with $\sum_{i=1}^{n} m_i = 0$ since g is zero mean. We take in (6.4), $n = 3$, $m_1 = 3$, $m_2 = m_3 = -3/2$, $\sigma_1^2 = 1$, $\sigma_2^2 = 2$ and $\sigma_3^2 = 4$. In other words, g is a normalized sum of three Gaussians.

For each g, we consider two different initial conditions f_0 defined as follows

- The first the initial condition f_0 is a normalized Gaussian of mean 2 given by

$$f_0(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(v - 2)^2}{2}\right). \tag{6.5}$$

- The second initial condition f_0 is equal to g defined by (6.4), with $n = 3$, $m_1 = 3$, $m_2 = m_3 = -3/2$, $\sigma_1^2 = 1$, $\sigma_2^2 = 2$ and $\sigma_3^2 = 4$.

Test case 1

For the first test case, g is a normalized centered Gaussian (6.2). Since g is a Gaussian, Proposition 3.1 gives an explicit formula for the equilibrium state. In that case, f_m^∞ is a Gaussian of variance $2\sigma_g^2 = 2$ with the same mean as the initial condition f_0. Hence for f_0 defined by (6.5), f_m^∞ writes

$$f_m^\infty(v) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{(v - 2)^2}{4}\right). \tag{6.6}$$

From f_0 defined by (6.4), f_m^∞ writes

$$f_m^\infty(v) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{v^2}{4}\right) \tag{6.7}$$

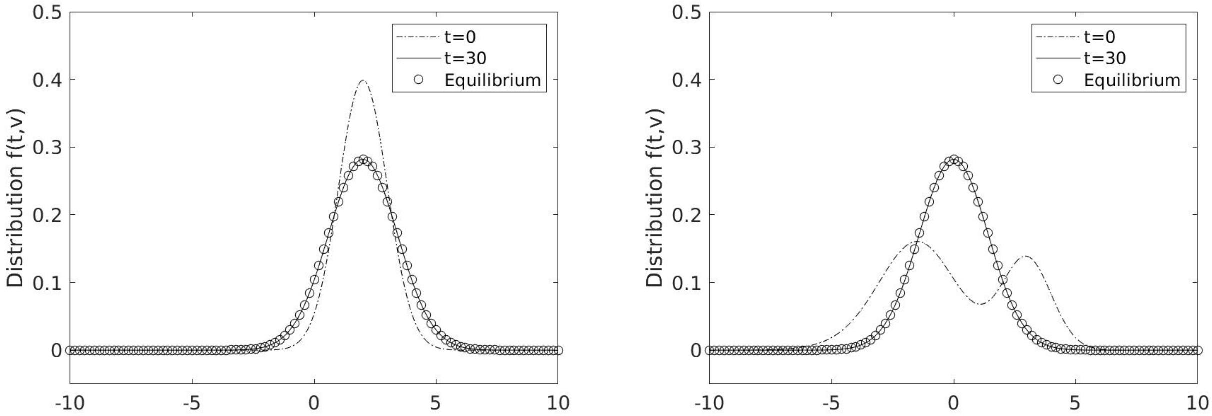

Figure 1: Distribution function of the solution of (6.1) with g defined by (6.2). On the left, solution for initial condition (6.5) at times $t = 0$, $t = 30$ and on the right, solution for initial condition (6.4) at times $t = 0$, $t = 30$. Equilibrium state given in (3.4) (in circle).

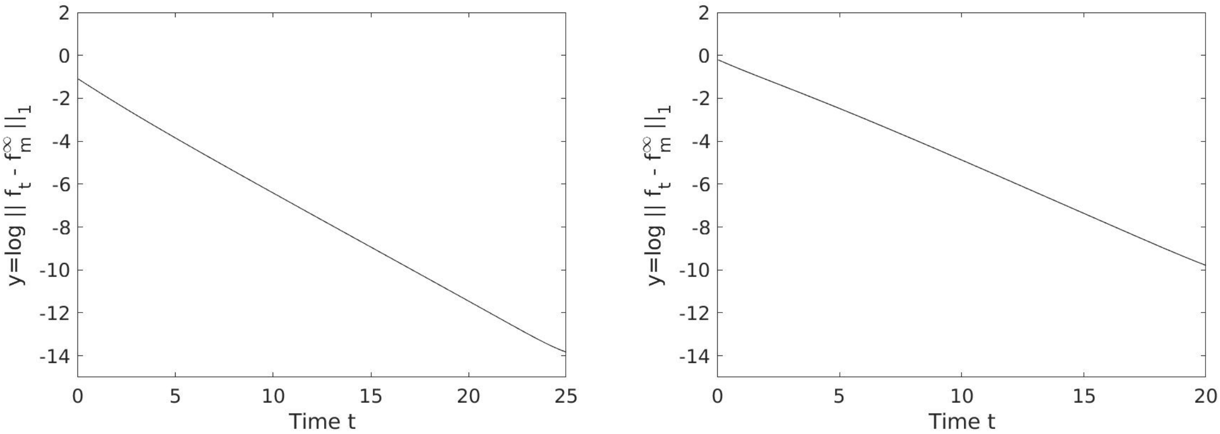

Figure 2: Function $t \longmapsto \log \|f(t,\cdot) - f_m^\infty\|_{L^1}$ where f is the solution of (6.1) with g defined by (6.2). On the left, $f(t,\cdot)$ for initial condition (6.5) and f_m^∞ given by (6.6). On the right, $f(t,\cdot)$ for condition (6.4) and f_m^∞ is given by (6.7).

Since we have shown that $f(t,.)$ converges exponentially to f_m^∞ for the strong-norm L^1, then the function $t \longmapsto \log \|f(t,.) - f_m^\infty\|_{L^1}$ must be bounded by an affine function. This is the case in Figure 2 but with a ratio 1/2 and not 1/5 theoretically found.

Test case 2

For the second test case, g defined by (6.3). Since g is not a Gaussian, the expression of the equilibrium state f_m^∞ is not explicit. Then f_m^∞ is approached by $f(t,\cdot)$ at time $t = 35$ corresponding to a converged result.

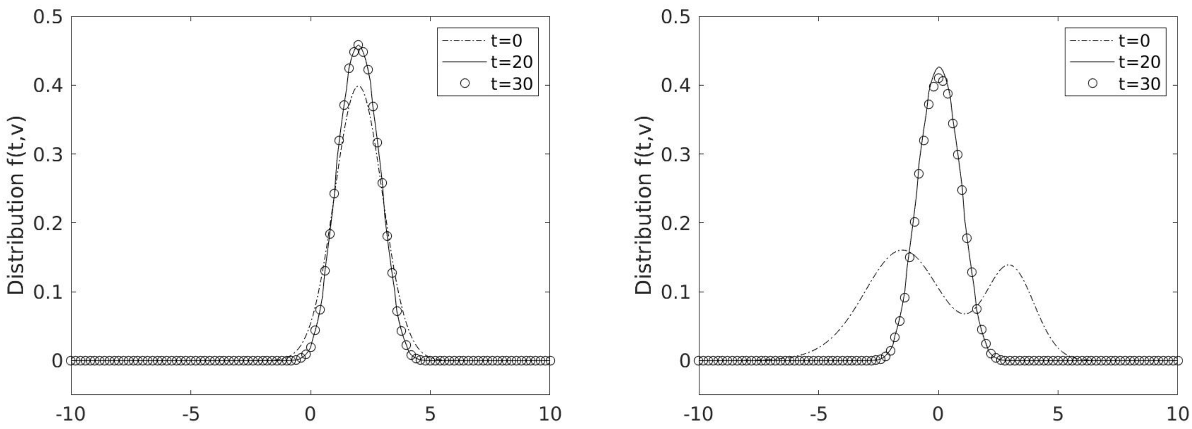

Figure 3: Distribution function of the solution of (6.1) with g defined by (6.3). On the left, solutions for initial condition (6.5) at times $t = 0$, $t = 20$, $t = 30$ and on the right, solutions for initial condition (6.4) at times $t = 0$, $t = 20$, $t = 30$.

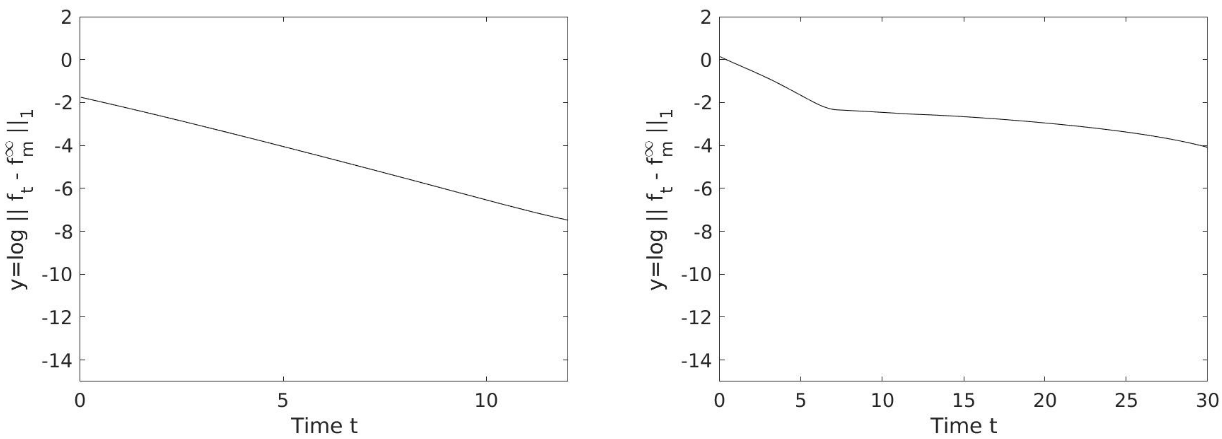

Figure 4: Function $t \longmapsto \log \|f(t,\cdot) - f_m^\infty\|_{L^1}$ for f the solution of (6.1) with g defined by (6.3). On the left, $f(t,\cdot)$ for initial condition (6.5) and f_m^∞ is replaced by a converged solution $f(t,\cdot)$ at time $t = 35$. On the right, $f(t,\cdot)$ for initial condition (6.4) and f_m^∞ replaced by a converged solution $f(t,\cdot)$ at time $t = 35$.

Fig 3 shows that $f(t,\cdot)$ goes towards the same asymptotic limit for the two different initial conditions. This numerical result is consistent with Proposition 3.1 claiming that the equilibrium state depends only on g. In Figure 4, the curve on the right is not rectilinear because there are two phenomenons. First of all, the distribution goes towards a Maxwellian distribution and next to the right one. However Fig. 4 shows that the convergence remains with an exponential rate.

Test case 3

The third test case is devoted to g defined by (6.4). Since g is not a Gaussian, no explicit formula are again available for the equilibrium state f_m^∞. Hence f_m^∞ is again approached by converged solution obtained at time $t = 35$ as for test case 2. The two different initial conditions lead to the same asymptotic state that theoretically only depends on the distribution g.

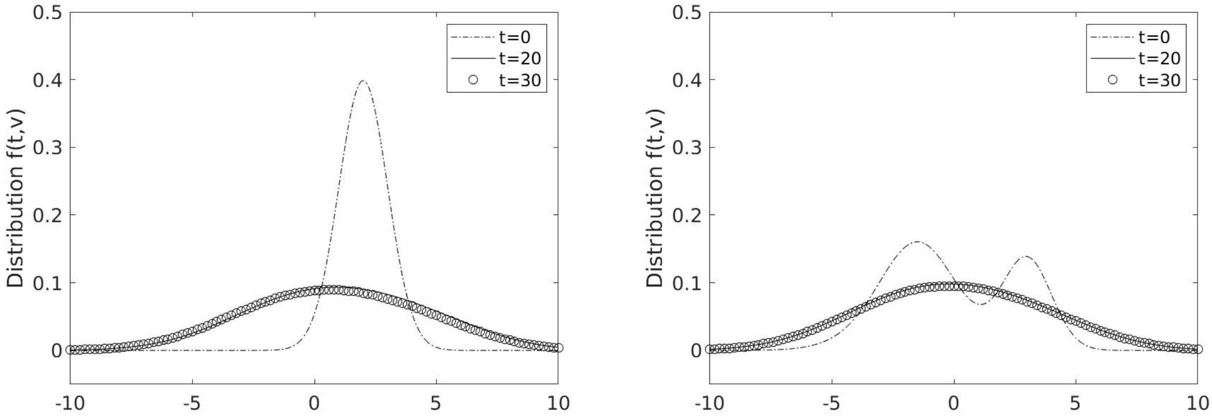

Figure 5: Distribution function of the solution of (6.1) with g defined by (6.4). On the left, solutions with initial condition (6.5) at times $t = 0$, $t = 20$, $t = 30$ and on the right, solutions with initial condition (6.4) at times $t = 0$, $t = 20$, $t = 30$.

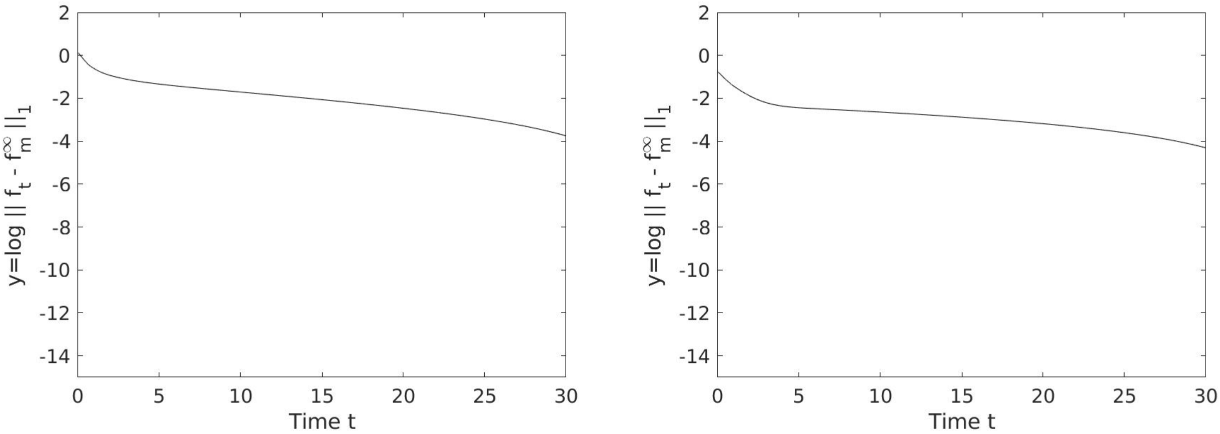

Figure 6: Function $t \longmapsto \log \|f(t,\cdot) - f_m^\infty\|_{L^1}$ where $f : t \longmapsto f(t,\cdot)$ is the solution of (6.1) with g defined by (6.4) and f_m^∞ the equilibrium state. On the left, $f(t,\cdot)$ with initial condition (6.5) and f_m^∞ is represent by the converged solution $f(t,\cdot)$ at time $t = 35$. On the right, $f(t,\cdot)$ for initial condition (6.4) and f_m^∞ is replaced by the converged solution $f(t,\cdot)$ at time $t = 35$.

Conclusion

We have shown in this paper the existence of a unique mild solution of the equation (1.4) and the exponential convergence toward the equilibrium state for the W_2 metric and for the strong-norm L^1, in the case $\beta = 1$. The result elaborated in ref. [8] has been extended to the cases $g \in \mathcal{P}_2^0(\mathbb{R})$. The extension of these results for a non-constant β and the inhomogeneous case is postponed to a following paper.

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The authors would like to thank J. A. Carrillo for suggesting the study of such a model during his visit in Bordeaux in 2019.

Appendix

A Complement on the Wasserstein metric

Proof of Proposition 2.1. (1) Let π_1 be an optimal coupling of (μ_1, ν_1) and π_2 be an optimal coupling of (μ_2, ν_2). For $\alpha \in [0,1]$, we set $\pi = \alpha\pi_1 + (1-\alpha)\pi_2$. It is easy to check that π is a coupling of $(\alpha\mu_1 + (1-\alpha)\mu_2, \alpha\nu_1 + (1-\alpha)\nu_2)$ and therefore

$$\begin{aligned}
W_2(\alpha\mu_1 + (1-\alpha)\mu_2, \alpha\nu_1 + (1-\alpha)\nu_2)^2 &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y) \\
&= \alpha \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi_1(x,y) \\
&+ (1-\alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi_2(x,y) \\
&= \alpha W_2(\mu_1,\nu_1)^2 + (1-\alpha) W_2(\mu_2,\nu_2)^2.
\end{aligned}$$

(2) For $v \in \mathbb{R}^d$, we consider an optimal coupling π_v of $(P_1(v,\cdot), P_2(v,\cdot))$ such that the map $v \longmapsto \pi_v$ is measurable. We set π as the coupling defined for any measurable function φ by

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x,y) d\pi(x,y) = \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \varphi(x,y) d\pi_v(x,y) d\mu(v).$$

It is easy to check that π is a coupling of $(\int_{\mathbb{R}^d} P_1(v,\cdot) d\mu(v), \int_{\mathbb{R}^d} P_2(v,\cdot) d\mu(v))$ and therefore

$$\begin{aligned}
W_2\left(\int_{\mathbb{R}^d} P_1(v,\cdot) d\mu(v), \int_{\mathbb{R}^d} P_2(v,\cdot) d\mu(v)\right)^2 &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y) \\
&= \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi_v(x,y) d\mu(v) \\
&= \int_{\mathbb{R}^d} W_2(P_1(v,\cdot), P_2(v,\cdot))^2 d\mu(v).
\end{aligned}$$

(3) Let π_1 be an optimal coupling of (μ_1, ν_1) and π_2 be an optimal coupling of (μ_2, ν_2). We set $\pi = \pi_1 * \pi_2$. It is easy to check that π is a coupling of $(\mu_1 * \mu_2, \nu_1 * \nu_2)$ and therefore

$$\begin{aligned}
W_2(\mu_1 * \mu_2, \nu_1 * \nu_2)^2 &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y) \\
&= \iiiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |x+u-y-v|^2 d\pi_1(x,y) d\pi_2(u,v).
\end{aligned}$$

Using the classical equality $|x+y|^2 = |x|^2 + |y|^2 + 2\langle x, y\rangle$, it holds that

$$\begin{aligned}
&W_2(\mu_1 * \mu_2, \nu_1 * \nu_2)^2 \\
&\leq W_2(\mu_1,\nu_1)^2 + W_2(\mu_2,\nu_2)^2 + 2 \iiiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \langle x-y, u-v\rangle d\pi_1(x,y) d\pi_2(u,v) \\
&= W_2(\mu_1,\nu_1)^2 + W_2(\mu_2,\nu_2)^2 + 2 \left\langle \iint_{\mathbb{R}^d \times \mathbb{R}^d} (x-y) d\pi_1(x,y), \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u-v) d\pi_2(u,v) \right\rangle \\
&= W_2(\mu_1,\nu_1)^2 + W_2(\mu_2,\nu_2)^2.
\end{aligned}$$

(4) Let π be a coupling of (μ, ν). By setting $\pi_0 = f\#\pi$, π_0 is a coupling of $(f\#\mu, f\#\nu)$. Thus

$$W_2(f\#\mu, f\#\nu)^2 \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi_0(x,y) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)-f(y)|^2 d\pi(x,y).$$

■

Proof of Lemma 2.2. Define the map T by

$$T : \mu \in \mathcal{P}_2(\mathbb{R}^d) \longmapsto \int_{\mathbb{R}^d} v d\mu(v).$$

Let π be a coupling of (μ, ν) with μ, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$. So

$$|T(\mu) - T(\nu)| = \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v - u) d\pi(u, v) \right| \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v - u| d\pi(u, v).$$

By taking the infimum over π, we obtain $|T(\mu) - T(\nu)| \leq W_1(\mu, \nu)$ and the Hölder inequality gives $W_1(\mu, \nu) \leq W_2(\mu, \nu)$. Therefore T is continuous. Let us now show that the space $\mathcal{P}_2^m(\mathbb{R}^d)$ is closed in $\mathcal{P}_2(\mathbb{R}^d)$. Let $(\mu_n)_n$ be a sequence in $\mathcal{P}_2^m(\mathbb{R}^d)$ converging to μ for W_2. $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ by completeness and the continuity of T gives that $T(\mu_n)$ converges to $T(\mu)$. So $T(\mu) = m$ and $\mu \in \mathcal{P}_2^m(\mathbb{R}^d)$. ■

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