

NON-ARCHIMEDEAN AMOEBAS AND TROPICAL VARIETIES

MANFRED EINSIEDLER, MIKHAIL KAPRANOV, AND DOUGLAS LIND

ABSTRACT. We study the non-archimedean counterpart to the complex amoeba of an algebraic variety, and show that it coincides with a polyhedral set defined by Bieri and Groves using valuations. For hypersurfaces this set is also the tropical variety of the defining polynomial. Using non-archimedean analysis and a recent result of Conrad we prove that the amoeba of an irreducible variety is connected. We introduce the notion of an adelic amoeba for varieties over global fields, and establish a form of the local-global principle for them. This principle is used to explain the calculation of the nonexpansive set for a related dynamical system.

1. AMOEBAS

1.1. Generalities. Let \mathbb{k} be a field. Recall [6, VI.6.1] that a norm (or absolute value) on \mathbb{k} is a function $a \mapsto |a|$ from \mathbb{k} to $\mathbb{R}_{\geq 0}$ such that

$$(1.1.1) \quad |a| = 0 \text{ if and only if } a = 0,$$

$$(1.1.2) \quad |ab| = |a| |b|,$$

$$(1.1.3) \quad |a + b| \leq |a| + |b|.$$

In this paper, unless otherwise specified, we will assume that \mathbb{k} is equipped with a nontrivial norm $|\cdot|$ and is complete with respect to it. If \mathbb{L}/\mathbb{k} is an extension of degree n , then the norm on \mathbb{k} extends to \mathbb{L} by the formula $|a| = |N_{\mathbb{L}/\mathbb{k}}(a)|^{1/n}$. By $\overline{\mathbb{k}}$ we denote a fixed algebraic closure of \mathbb{k} . Thus a norm on \mathbb{k} extends to $\overline{\mathbb{k}}$ and we have the map

$$(1.1.4) \quad \text{Log}: (\overline{\mathbb{k}}^\times)^d \rightarrow \mathbb{R}^d, \quad (a_1, \dots, a_d) \mapsto (\log |a_1|, \dots, \log |a_d|).$$

Note that the image of Log in (1.1.4) is dense in \mathbb{R}^d unless the norm is trivial (i.e., unless $|a| = 1$ for any $a \neq 0$).

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We denote by $\mathbb{G}_m = \mathbb{A}_{\mathbb{k}}^1 \setminus \{0\}$ the multiplicative group (punctured affine line) over \mathbb{k} , i.e., $\mathbb{G}_m = \text{Spec } \mathbb{k}[x, x^{-1}]$. The algebraic group \mathbb{G}_m^d will be referred to as the d -dimensional algebraic torus over \mathbb{k} .

Let $X \subset \mathbb{G}_m^d$ be a subscheme, i.e., $X = \text{Spec}(\mathbb{k}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]/I)$ where I is an ideal. Then $X(\overline{\mathbb{k}})$, the set of $\overline{\mathbb{k}}$ -points of X , is a subset in $(\overline{\mathbb{k}}^\times)^d$.

Definition 1.1.1. The *amoeba* of X , denoted $\mathcal{A}(X)$, is the closure of $\text{Log}(X(\overline{\mathbb{k}}))$ in \mathbb{R}^d .

Example 1.1.2. Let $\mathbb{k} = \mathbb{C}$ with its usual absolute value. Then $\overline{\mathbb{k}} = \mathbb{k}$ and $\mathcal{A}(X)$ coincides with $\text{Log}(X(\mathbb{C}))$ since $X(\mathbb{C}) \subset (\mathbb{C}^\times)^d$ is closed and Log is a continuous proper map. The case when X is a hypersurface in $(\mathbb{C}^\times)^d$ was first considered in [12]. More general complex amoebas were studied in [17, 18].

1.2. Non-archimedean norms. A norm $|\cdot|$ is called *non-archimedean* if it satisfies

$$(1.1.3') \quad |a + b| \leq \max\{|a|, |b|\}.$$

As is well known, non-archimedean norms on \mathbb{k} are in bijection with valuations, i.e., maps $v: \mathbb{k} \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying

$$(1.2.1) \quad v(a) = \infty \text{ if and only if } a = 0,$$

$$(1.2.2) \quad v(ab) = v(a) + v(b),$$

$$(1.2.3) \quad v(a + b) \geq \min\{v(a), v(b)\}.$$

Explicitly, $v(a) = -\log |a|$. We will use the same letter to denote the extension of v to $\overline{\mathbb{k}}$.

The connection to valuations is the reason why it will be more natural for us to work with the map $\text{val}: (\overline{\mathbb{k}}^\times)^d \rightarrow \mathbb{R}^d$ defined by

$$\text{val}(a_1, \dots, a_d) = (v(a_1), \dots, v(a_d)) = -\text{Log}(a_1, \dots, a_d).$$

Definition 1.2.1. The *tropical variety* $\mathcal{T}(X)$ of X is the closure of $\text{val}(X(\overline{\mathbb{k}}))$ in \mathbb{R}^d .

Clearly the amoeba and the tropical variety satisfy $\mathcal{T}(X) = -\mathcal{A}(X)$.

Examples 1.2.2. (a) The field $\mathbb{k} = \mathbb{Q}_p$ of p -adic numbers has the p -adic valuation v_p and the p -adic norm $|a|_p = p^{-v_p(a)}$, with respect to which it is complete. We sometimes write \mathbb{Q}_∞ for \mathbb{R} . Every variety defined over the rationals therefore has a p -adic amoeba for each $p \leq \infty$, where $p = \infty$ corresponds to the complex amoeba.

(b) Let \mathbb{K} be any field. The field $\mathbb{k} = \mathbb{K}((t))$ of formal Laurent series $g(t) = \sum_{j=m}^{\infty} g_j t^j$, where $g_j \in \mathbb{K}$ and $m \in \mathbb{Z}$, has a discrete valuation ord , given by $\text{ord}(g(t)) = \min\{j: g_j \neq 0\} \in \mathbb{Z} \subset \mathbb{R}$. If \mathbb{K} is algebraically closed and $\text{char}(\mathbb{K}) = 0$, then the field of Puiseux series

$$\bigcup_{n \geq 1} \mathbb{K}((t^{1/n}))$$

is algebraically closed [8] and thus coincides with $\bar{\mathbb{k}}$. It has a \mathbb{Q} -valued valuation ord defined similarly to the above.

Note that the assumption $\text{char}(\mathbb{K}) = 0$ is necessary. If $\text{char}(\mathbb{K}) = p$, then the equation $x^p - x = t^{-1}$ has no roots in any $\mathbb{K}((t^{1/n}))$.

(c) Let \mathbb{K} be an algebraically closed field of any characteristic. A *transfinite Puiseux series over \mathbb{K}* is a formal sum $g(t) = \sum_{q \in \mathbb{Q}} g_q t^q$, where $g_q \in \mathbb{K}$ are such that $\text{Supp}(g) = \{q : g_q \neq 0\}$ is well-ordered (i.e. every subset of it has a minimal element). Such series form a field $\mathbb{K}((t^{\mathbb{Q}}))$ that is always algebraically closed [19]. For example, the equation $x^p - x = t^{-1}$ in (b) has the root

$$x(t) = t^{-1} + t^{-1/p} + t^{-1/p^2} + \dots$$

One defines a valuation on $\mathbb{K}((t^{\mathbb{Q}}))$ by $v(g) = \min\{\text{Supp}(g)\}$.

1.3. Conventions. We let $\Gamma = v(\bar{\mathbb{k}}^{\times}) \subset \mathbb{R}$ denote the valuation group of v . It is a dense divisible subgroup of \mathbb{R} by our assumption that v is nontrivial. By a *convex polyhedron* in \mathbb{R}^d we mean a subset Δ given by a finite system of affine-linear inequalities

$$\sum_{j=1}^d b_{ij} u_j \geq c_i, \quad i = 1, \dots, r.$$

We say that Δ is Γ -rational if the inequalities above can be chosen such that $b_{ij} \in \mathbb{Z}$ and $c_i \in \Gamma$. In this case $\Delta \cap \Gamma^d$ is dense in Δ . Note that we do not require Δ to have full dimension. By a (Γ -rational) *polyhedral set* P we mean a finite union of (Γ -rational) convex polyhedra. We say that P is of *pure dimension* r if all the maximal polyhedra in P have dimension r .

2. MAIN RESULTS AND EXAMPLES

2.1. Amoebas and tropical varieties of polynomials. Let

$$f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}, \quad \mathbf{n} = (n_1, \dots, n_d), \quad \mathbf{x}^{\mathbf{n}} = x_1^{n_1} \dots x_d^{n_d}$$

be a Laurent polynomial with coefficients $a_{\mathbf{n}} \in \mathbb{k}$, and let $X = X_f \subset \mathbb{G}_m^d$ be the hypersurface $\{f = 0\}$. We denote $\mathcal{T}(X_f)$ by $\mathcal{T}(f)$. For $\mathbf{u} \in \mathbb{R}^d$ define

$$(2.1.1) \quad f^{\tau}(\mathbf{u}) = \min_{\mathbf{n} \in \mathbb{Z}^d} \{v(a_{\mathbf{n}}) + \mathbf{u} \cdot \mathbf{n}\}.$$

Then f^{τ} is a convex piecewise-linear function on \mathbb{R}^d known as the *tropicalization* of f (see [20] and [23] for background). Note that for almost all \mathbf{n} we have $a_{\mathbf{n}} = 0$, so $v(a_{\mathbf{n}}) = +\infty$. Therefore f^{τ} is the minimum of finitely many affine-linear functions.

Theorem 2.1.1. *If $f \neq 0$ then $\mathcal{T}(f)$ is equal to the non-differentiability locus of f^{τ} . In particular, $\mathcal{T}(f)$ is either empty (when f is a monomial), or is a rational polyhedral set of pure dimension $d - 1$, or is all of \mathbb{R}^d (when $f = 0$).*

This can be reformulated as follows. Denote the convex hull of a set $E \in \mathbb{R}^d$ by $\text{Conv}(E)$. Then

$$\mathcal{N}(f) = \text{Conv}\{\mathbf{n} : a_{\mathbf{n}} \neq 0\} \subset \mathbb{R}^d$$

is the Newton polytope of f . Let

$$\tilde{\mathcal{N}}(f) = \text{Conv}\{(\mathbf{n}, u) \in \mathbb{Z}^d \times \mathbb{R} : u \geq v(a_{\mathbf{n}})\} \subset \mathbb{R}^{d+1}$$

be the *extended Newton polyhedron* of f . Then $\tilde{\mathcal{N}}(f)$ projects to $\mathcal{N}(f)$ by forgetting the last coordinate. The following is then an equivalent formulation of Theorem 2.1.1.

Corollary 2.1.2. (a) *The connected components of $\mathbb{R}^d \setminus \mathcal{T}(f)$ are in bijection with vertices of $\tilde{\mathcal{N}}(f)$.*

(b) *If $(\mathbf{n}, v(a_{\mathbf{n}}))$ is a vertex of $\tilde{\mathcal{N}}(f)$, then the corresponding component $C_{\mathbf{n}} \subset \mathbb{R}^d \setminus \mathcal{T}(f)$ consists of $\mathbf{u} \in \mathbb{R}^d$ such that*

$$\min_{\mathbf{m} \in \mathbb{Z}^d} \{v(a_{\mathbf{m}}) + \mathbf{u} \cdot \mathbf{m}\} = v(a_{\mathbf{n}}) + \mathbf{u} \cdot \mathbf{n}$$

and the minimum on the left side is achieved for exactly one \mathbf{m} , namely $\mathbf{m} = \mathbf{n}$.

(c) *The unbounded connected components of $\mathbb{R}^d \setminus \mathcal{T}(f)$ correspond to those vertices $(\mathbf{n}, v(a_{\mathbf{n}}))$ of $\tilde{\mathcal{N}}(f)$ that project to a vertex \mathbf{n} of $\mathcal{N}(f)$.*

Note that $C_{\mathbf{n}}$ is a convex polyhedral domain.

Example 2.1.3. The case $d = 1$ of Theorem 2.1.1, or, equivalently, Corollary 2.1.2 is well known (see [13, Thm. 6.4.7] or [6, Exer. VI.4.11]). If $f(x) = \sum_{j=r}^s a_j x^j$, then the Corollary says that the values $v(z_0)$ for roots $z_0 \in \bar{\mathbb{k}}^\times$ of $f(x)$ are precisely the negatives of the slopes of the non-vertical edges of the Newton polygon $\tilde{\mathcal{N}}(f)$.

Proof of Theorem 2.1.1. Let $\mathcal{T} = \mathcal{T}(f)$ and let \mathcal{S} be the non-differentiability locus of f^τ . Clearly \mathcal{S} is a Γ -rational polyhedral set.

Lemma 2.1.4. $\mathcal{T} \subset \mathcal{S}$.

Proof. Since \mathcal{S} is closed, it is enough to show that $\text{val}(X(\bar{\mathbb{k}})) \subset \mathcal{S}$. Let $\mathbf{u} = (u_1, \dots, u_d) \in \text{val}(X(\bar{\mathbb{k}}))$, i.e. $u_i = v(z_i)$ where $f(z_1, \dots, z_d) = 0$. Note that $v(a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}) = v(a_{\mathbf{n}}) + \mathbf{u} \cdot \mathbf{n}$. Recall that for non-archimedean absolute values, if $a_1, \dots, a_r \in \bar{\mathbb{k}}^\times$ with $a_1 + \dots + a_r = 0$, then there are at least two a_j with maximal $|a_j|$. Since $f(\mathbf{z}) = \sum a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} = 0$, it follows that there are at least two terms in the sum whose valuations are both equal to $\min\{v(a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}})\}$. This exactly means that f^τ is non-differentiable at \mathbf{u} : two affine-linear functionals from the set to be minimized achieve the same minimal value at \mathbf{u} . \square

Since \mathcal{S} is Γ -rational polyhedral, $\mathcal{S} \cap \Gamma^d$ is dense in \mathcal{S} . To prove that $\mathcal{S} \subset \mathcal{T}$ it is therefore enough to prove that $\mathcal{S} \cap \Gamma^d \subset \mathcal{T}$. Let $\mathbf{u} \in \mathcal{S} \cap \Gamma^d$. By changing the variables

$$(2.1.2) \quad z_i \mapsto z_i \cdot a_i, \quad a_i \in \bar{\mathbb{k}}^\times, \quad v(a_i) = u_i,$$

we reduce to the following.

Lemma 2.1.5. *If $\mathbf{0} \in \mathcal{S}$, then $\mathbf{0} \in \mathcal{T}$.*

Proof. We will find a root \mathbf{z}^0 of f of the form

$$z_i^0 = (t_0)^{b_i}, \quad t_0 \in \overline{\mathbb{k}}^\times, \quad v(t_0) = 0$$

for an appropriate choice of $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{Z}^d$. Indeed, let

$$f_{\mathbf{b}}(t) = f(t^{b_1}, \dots, t^{b_d}) = \sum_{\mathbf{n}} a_{\mathbf{n}} t^{\mathbf{b} \cdot \mathbf{n}} \in \mathbb{k}[t, t^{-1}].$$

The fact that $\mathbf{0} \in \mathcal{S}$ means that $\tilde{\mathcal{N}}(f)$ has a face of positive dimension which is horizontal and whose height u is minimal. Let F be the maximal face with this property. Assume that $\mathbf{b} \in \mathbb{Z}^d$ is generic in the following sense: for each edge $[(\mathbf{m}, u), (\mathbf{n}, u)]$ of F we have $\mathbf{b} \cdot (\mathbf{m} - \mathbf{n}) \neq 0$. Then the extended Newton polygon $\tilde{\mathcal{N}}(f_{\mathbf{b}}) \subset \mathbb{R}^2$ has a horizontal edge of minimal height u . By the classical result in Example 2.1.3, $f_{\mathbf{b}}$ has a root t_0 with $v(t_0) = 0$. \square

This completes the proof of Theorem 2.1.1. \square

We now extend the correspondence between components of $\mathbb{R}^d \setminus \mathcal{T}(f)$ and Laurent series expansions of $1/f$ to the non-archimedean case. This is similar to the known case $\mathbb{k} = \mathbb{C}$ described in [12, Ch. 6, Cor. 1.6].

Let $(\mathbf{n}, v(a_{\mathbf{n}}))$ be a vertex of $\tilde{\mathcal{N}}(f)$. We then write

$$f(\mathbf{x}) = a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} (1 + g(\mathbf{x})), \quad g(\mathbf{x}) = \sum_{\mathbf{m} \neq \mathbf{n}} \frac{a_{\mathbf{m}}}{a_{\mathbf{n}}} \mathbf{x}^{\mathbf{m} - \mathbf{n}}$$

and form the Laurent expansion

$$(2.1.3) \quad R_{\mathbf{n}}(\mathbf{x}) = \frac{1}{f(\mathbf{x})} = a_{\mathbf{n}}^{-1} \mathbf{x}^{-\mathbf{n}} \sum_{n=0}^{\infty} (-1)^n g(\mathbf{x})^n$$

by using geometric series.

Proposition 2.1.6. (a) $R_{\mathbf{n}}(\mathbf{x})$ is a well-defined Laurent series.

(b) The domain of convergence of $R_{\mathbf{n}}(\mathbf{x})$ is $\text{val}^{-1}(C_{\mathbf{n}})$.

Proof. Assume without loss of generality that $\mathbf{0} \in C_{\mathbf{n}}$, for otherwise we can make the same change of variables as (2.1.2). Assume that $\mathbf{u} \in C_{\mathbf{n}}$. Then Corollary 2.1.2(b) implies that $v(b_{\mathbf{m}}) + \mathbf{u} \cdot \mathbf{m} > 0$ for all coefficients $b_{\mathbf{m}}$ of g . For $\mathbf{u} = \mathbf{0}$ this shows together with completeness of \mathbb{k} that the sum (possibly infinite) defining the coefficients in $R_{\mathbf{n}}(\mathbf{x})$ (as a Laurent series) at each $\mathbf{x}^{\mathbf{m}}$, $\mathbf{m} \in \mathbb{Z}^d$, converges as claimed in (a).

Suppose \mathbf{z} satisfies $\text{val}(\mathbf{z}) = \mathbf{u} \in C_{\mathbf{n}}$. Then $v(b_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}) > 0$ for all coefficients $b_{\mathbf{m}}$ of g . Therefore, $R_{\mathbf{n}}(\mathbf{z})$, considered as a series of Laurent polynomials but also as a Laurent series, converges. In other words the domain of convergence of $R_{\mathbf{n}}(\mathbf{z})$ contains $\text{val}^{-1}(C_{\mathbf{n}})$. Note that the domain of convergence of the Laurent series $R_{\mathbf{n}}(\mathbf{x})$ is $\text{val}^{-1}(D)$ for some convex $D \subset \mathbb{R}^d$. Furthermore, D

cannot contain any element of the boundary of $C_{\mathbf{n}}$ since $R_{\mathbf{n}}(\mathbf{z}) = 1/f(\mathbf{z})$ for any \mathbf{z} where $R_{\mathbf{n}}(\mathbf{z})$ converges. It follows that $D = C_{\mathbf{n}}$. \square

2.2. Amoebas and Bieri-Groves sets. Let A be a commutative ring with unit 1. Recall from [6, VI.3.1, Def. 1] that a (*ring*) *valuation* w on A is a map $w: A \rightarrow \mathbb{R} \cup \{\infty\}$ such that for all $a, b \in A$ we have that

$$(2.2.1) \quad w(ab) = w(a) + w(b),$$

$$(2.2.2) \quad w(a + b) \geq \min\{w(a), w(b)\},$$

$$(2.2.3) \quad w(0) = \infty \text{ and } w(1) = 0.$$

There may be nonzero elements a in A for which $w(a) = \infty$. However, $w^{-1}(\infty)$ is easily seen to be a prime ideal of A . Thus if A is a field, then w is a valuation in the usual sense of Section 1.2.

Let $\mathbb{k}, |\cdot|, v$, and $X \subset \mathbb{G}_m^d$ be as before. Let $A = \mathbb{k}[X]$ be the coordinate ring of X , generated by the coordinate functions $x_1^{\pm 1}, \dots, x_d^{\pm 1}$. Define $\mathcal{W}(A)$ to be the set of all ring valuations on A extending v on \mathbb{k} . Let $G = \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$. Then there is an embedding $X(\overline{\mathbb{k}})/G \rightarrow \mathcal{W}(A)$ given by $\mathbf{z} \mapsto w_{\mathbf{z}}$, where $w_{\mathbf{z}}(f) = v(f(\mathbf{z}))$. However, $\mathcal{W}(A)$ is usually much bigger than $X(\overline{\mathbb{k}})/G$.

Example 2.2.1. Let $d = 1$ and $X = \mathbb{G}_m$, so $A = \mathbb{k}[x, x^{-1}]$. Assume that $\Gamma = v(\overline{\mathbb{k}}^\times) \neq \mathbb{R}$. Fix $u_0 \in \mathbb{R} \setminus \Gamma$. For $f(x) = \sum_{j=-r}^s a_j x^j \in A$ define

$$w(f) = \min_{j \in \mathbb{Z}} \{v(a_j) + j u_0\},$$

which is a ring valuation on A (see Lemma 1 of [6, VI.10.1]). But w does not have the form w_z for any $z \in \overline{\mathbb{k}}^\times$ since $u_0 \notin \Gamma$. Indeed, an easy additional argument shows that even with no assumption on Γ this w cannot have the form w_z .

Define the map $\beta: \mathcal{W}(A) \rightarrow \mathbb{R}^d$ by

$$\beta(w) = (w(x_1), \dots, w(x_d)) \in \mathbb{R}^d.$$

Definition 2.2.2. The *Bieri-Groves set* of X is defined as

$$\mathcal{BG}(X) = \beta(\mathcal{W}(A)) \subset \mathbb{R}^d.$$

Theorem 2.2.3 (Bieri-Groves [2]). *Let $X \subset \mathbb{G}_m^d$ be an irreducible variety of dimension r . Then $\mathcal{BG}(X)$ is a Γ -rational polyhedral set of pure dimension r .*

Remark 2.2.4. Every variety X is a finite union $X = X_1 \cup \dots \cup X_s$ of irreducible varieties X_i . It is elementary to show that $\mathcal{BG}(X) = \mathcal{BG}(X_1) \cup \dots \cup \mathcal{BG}(X_s)$ (see [2, §2.2]), so that $\mathcal{BG}(X)$ is a closed set in \mathbb{R}^d . Hence $\mathcal{T}(X) \subset \mathcal{BG}(X)$ by using the valuations $w_{\mathbf{z}} \in \mathcal{W}(\mathbb{k}[X])$ with $\mathbf{z} \in X(\overline{\mathbb{k}})$ as above.

Our results relating Bieri-Groves sets to amoebas and tropical varieties are as follows.

Theorem 2.2.5. *If $X \subset \mathbb{G}_m^d$ is an irreducible variety of dimension r , then $\mathcal{T}(X) = \mathcal{BG}(X)$. In particular, $\mathcal{T}(X)$ and $\mathcal{A}(X)$ are Γ -rational polyhedral sets of pure dimension r .*

If $X = X_1 \cup \dots \cup X_s$, then clearly $\mathcal{T}(X) = \mathcal{T}(X_1) \cup \dots \cup \mathcal{T}(X_s)$. By Remark 2.2.4 we obtain the following.

Corollary 2.2.6. *Let $X \subset \mathbb{G}_m^d$ be an arbitrary variety. Then $\mathcal{T}(X)$ and $\mathcal{A}(X)$ are Γ -rational polyhedral sets.*

Theorem 2.2.7. *If X is irreducible, then $\mathcal{T}(X)$, and hence $\mathcal{BG}(X)$ and $\mathcal{A}(X)$, are connected.*

Remark 2.2.8. Let I be the ideal in $\mathbb{k}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ defining X . Then trivially $\mathcal{T}(X) \subset \mathcal{T}(f)$ for every $f \in I$. Speyer and Sturmfels [23, Thm. 2.1] have shown that

$$\mathcal{T}(X) = \bigcap_{f \in I} \mathcal{T}(f).$$

Furthermore, they describe in [23, Cor. 2.3] that the intersection can be taken over just those f in a (finite) universal Gröbner basis for I . Hence a tropical variety is always the intersection of a finite number of tropical hypersurfaces, each of which has an explicit description as a Γ -rational polyhedral set from Theorem 2.1.1. Their approach can be developed into an alternative proof of Theorem 2.2.5.

2.3. Adelic amoebas. Let \mathbb{F} be a field of one of the following two types:

- (a) A number field, i.e., a finite extension of \mathbb{Q} ,
- (b) A function field, i.e., $\mathbb{F} = \mathbb{K}(C)$ is the field of rational functions on a smooth projective algebraic curve C of a field \mathbb{K} (a detailed account of such fields is contained in [4]).

Two norms on \mathbb{F} are said to be equivalent if they define the same topology. Let $S = S(\mathbb{F})$ be the set of equivalence classes of norms on \mathbb{F} (inducing the trivial norm on \mathbb{K} in the case (b)). It is well known that one can choose a representative $|\cdot|_p$ for each $p \in S$ such that:

$$(2.3.1) \quad \text{for every } a \in \mathbb{F}^\times \text{ we have } |a|_p = 1 \text{ for almost all } p \in S,$$

$$(2.3.2) \quad \prod_{p \in S} |a|_p = 1 \quad \text{for every } a \in \mathbb{F}^\times.$$

We assume that such a choice has been made.

Let $X \subset \mathbb{G}_m^d$ be an algebraic variety defined over \mathbb{F} . For every $p \in S$ we have the completion \mathbb{F}_p of \mathbb{F} with respect to $|\cdot|_p$. Thus we can form the amoeba

$$(2.3.3) \quad \mathcal{A}_p(X) = \text{closure}[\text{Log}(X(\overline{\mathbb{F}}_p))] \subset \mathbb{R}^d,$$

corresponding to X regarded as a variety over \mathbb{F}_p .

Definition 2.3.1. The *adelic amoeba* of X is the union

$$(2.3.4) \quad \mathcal{A}_{\mathbb{A}}(X) = \bigcup_{p \in S} \mathcal{A}_p(X).$$

Remark 2.3.2. Let I be the ideal in $\mathbb{F}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ defining X . By the result of Speyer and Sturmfels [23, Cor. 2.3] described in Remark 2.2.8, there is a finite set $\mathcal{F} \subset I$ such that for every p we have that $\mathcal{A}_p(X) = \bigcap_{f \in \mathcal{F}} \mathcal{A}_p(f)$. Furthermore, by Theorem 2.1.1, for each $f \in \mathcal{F}$ the sets $\mathcal{A}_p(f)$ are equal for almost every p . Hence the sets $\mathcal{A}_p(X)$ agree for almost every p , and so the union in (2.3.4) can therefore be expressed as a finite union. This shows that $\mathcal{A}_{\mathbb{A}}(X)$ is a closed polyhedral set.

Remark 2.3.3. Bieri and Groves explicitly introduced a “global” version of their sets by taking the union over all non-archimedean p (see [2, Thm. B]). However, for certain results (such as the following) the union over all p is needed.

Theorem 2.3.4. *Let $X \subset \mathbb{G}_m^d$ be a hypersurface defined over \mathbb{F} . Assume that $\mathbf{0} \notin \mathcal{A}_p(X)$ for at least one p . Then for any nonzero vector $\mathbf{v} \in \mathbb{R}^d$ the open half-line $(0, \infty) \cdot \mathbf{v}$ meets $\mathcal{A}_{\mathbb{A}}(X)$.*

A special case of the above theorem appears in [10, Prop. 5.5] motivated by algebraic dynamical systems (see Section 4), where the proof makes use of the notion of homoclinic points for such actions. An alternative proof for the case $d = 2$ is worked out in [22].

Example 2.3.5. Let $\mathbb{F} = \mathbb{Q}$. Then S consists of all prime numbers and ∞ , with $|x|_p = p^{-\text{ord}_p(x)}$ and $|x|_{\infty}$ the usual absolute value.

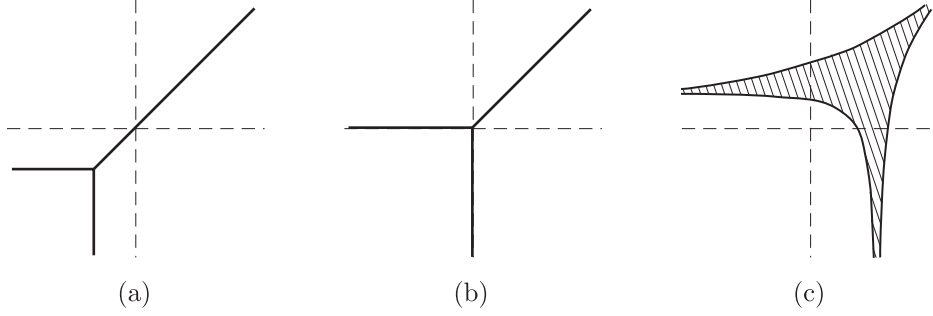
Let $d = 2$ and X be given by the equation $f(x, y) = 3 + x + y$. Theorem 2.1.1 implies that $\mathcal{A}_3(X)$ is the union of the three rays in Figure 1(a) meeting at the point $(-1, -1)$, and for $p \neq 3$ or ∞ the set $\mathcal{A}_p(X)$ is the union of the three rays starting at the origin shown in Figure 1(b). Finally, $\mathcal{A}_{\infty}(X)$ is shown in Figure 1(c). Observe that every open half-line starting at the origin hits at least one of these three amoebas, and in this example exactly one amoeba, except in the direction of $(1, 1)$.

We conjecture that a version of this result holds for lower-dimensional varieties as well.

Conjecture 2.3.6. *Let $X \subset \mathbb{G}_m^d$ be irreducible of dimension r such that $\mathcal{A}_{\mathbb{A}}(X)$ is not contained in any hyperplane. Then for any linear subspace $L \subset \mathbb{R}^d$ of co-dimension r and any relatively open half-space H in L whose boundary contains $\mathbf{0}$, we have that $H \cap \mathcal{A}_{\mathbb{A}}(X) \neq \emptyset$.*

Proof of Theorem 2.3.4. Let $f(\mathbf{x}) = \sum a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ be the equation of X , so $a_{\mathbf{n}} \in \mathbb{F}$. We write $\mathcal{A}_p(f)$ for $\mathcal{A}_p(X_f)$ and $\mathcal{T}_p(f)$ for $\mathcal{T}_p(X_f)$.

Let $S_{\text{gen}} \subset S$ be the set of all non-archimedean norms p for which $|a_{\mathbf{n}}|_p = 1$ for all nonzero $a_{\mathbf{n}}$. Since the number of archimedean norms is finite, by (2.3.1) we see that $S \setminus S_{\text{gen}}$ is finite, and so S_{gen} is nonempty.

FIGURE 1. The amoeba of $3 + x + y$ over different fields

Let $\mathcal{V}(f)$ denote the set of vertices of $\mathcal{N}(f)$. For $p \in S_{\text{gen}}$ the extended Newton polytope $\tilde{\mathcal{N}}(f)$ is the product $\mathcal{N}(f) \times [0, \infty)$. For each vertex $\mathbf{n} \in \mathcal{V}(f)$ put

$$D_{\mathbf{n}} = \{ \mathbf{u} \in \mathbb{R}^d : \mathbf{u} \cdot \mathbf{n} > \mathbf{u} \cdot \mathbf{w} \text{ for all } \mathbf{w} \in \mathcal{N}(f) \setminus \{ \mathbf{n} \} \},$$

which is the *normal cone* to $\mathcal{N}(f)$ at \mathbf{n} .

By Corollary 2.1.2(c), each vertex $\mathbf{n} \in \mathcal{V}(f)$ corresponds to an unbounded component $C_{\mathbf{n}}^{(p)}$ of $\mathbb{R}^d \setminus \mathcal{T}_p(f)$. Putting $D_{\mathbf{n}}^{(p)} = -C_{\mathbf{n}}^{(p)}$, using $\mathcal{A}_p(f)$ instead of $\mathcal{T}_p(f)$ and using Log instead of val , we see by Proposition 2.1.6 that the Laurent series $R_{\mathbf{n}}^{(p)}(\mathbf{x})$ in (2.1.3) for $1/f(\mathbf{x})$ has domain of convergence $\text{Log}^{-1}(D_{\mathbf{n}}^{(p)})$. Note also that the geometric series for $R_{\mathbf{n}}^{(p)}(\mathbf{x})$ produces Laurent coefficients that are only finite sums, so that $R_{\mathbf{n}}^{(p)}(\mathbf{x})$ has coefficients in \mathbb{F} . Hence $R_{\mathbf{n}}^{(p)}(\mathbf{x}) = R_{\mathbf{n}}(\mathbf{x})$ is independent of p . From Corollary 2.1.2(b) we see that $D_{\mathbf{n}}^{(p)} = D_{\mathbf{n}}$ for all $p \in S_{\text{gen}}$. In this case, all components of $\mathbb{R}^d \setminus \mathcal{A}_p(f)$ are unbounded, and $\mathcal{A}_p(f)$ is the complement of $\bigcup_{\mathbf{n} \in \mathcal{V}(f)} D_{\mathbf{n}}$ whenever $p \in S_{\text{gen}}$.

Fix $\mathbf{v} \neq \mathbf{0}$, and suppose that $(0, \infty) \cdot \mathbf{v}$ does not meet $\mathcal{A}_{\mathbb{A}}(f)$. If $\mathbf{v} \notin D_{\mathbf{n}}$ for all $\mathbf{n} \in \mathcal{V}(f)$, then $\mathbf{v} \in \mathcal{A}_p(f)$ for all $p \in S_{\text{gen}}$, and we are done. Hence we may assume that $\mathbf{v} \in D_{\mathbf{n}}$ for some $\mathbf{n} \in \mathcal{V}(f)$.

By hypothesis, there is a $p_0 \in S$ so that $\mathbf{0} \notin \mathcal{A}_{p_0}(f)$. Since

$$[(0, \infty) \cdot \mathbf{v}] \cap \mathcal{A}_{p_0}(f) \subset [(0, \infty) \cdot \mathbf{v}] \cap \mathcal{A}_{\mathbb{A}}(f) = \emptyset,$$

it follows that $(0, \infty) \cdot \mathbf{v} \subset D_{\mathbf{n}}^{(p_0)}$. Now $D_{\mathbf{n}}^{(p_0)}$ is convex and open, so there is an $\epsilon > 0$ so that $R_{\mathbf{n}}(\mathbf{x}) = \sum b_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ converges for all $\mathbf{x} \in \text{Log}^{-1}[(-\epsilon, \epsilon) \cdot \mathbf{v}]$. It follows there are $\theta > 1$ and $c > 0$ so that

$$|b_{\mathbf{m}}|_{p_0} < c \theta^{-\epsilon |\mathbf{m} \cdot \mathbf{v}|}.$$

Also, $|b_{\mathbf{m}}|_p \leq 1$ for all $p \in S_{\text{gen}}$. Let $r = |S \setminus S_{\text{gen}}|$. By the product formula (2.3.2), there is a $q \in S$ for which

$$|b_{\mathbf{m}}|_q \geq c^{1/r} \theta^{(\epsilon/r) |\mathbf{m} \cdot \mathbf{v}|}$$

for infinitely many \mathbf{m} . This implies that the series $\sum b_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ does not converge with respect to $|\cdot|_q$ for any point in $\text{Log}^{-1}[(-\epsilon/r, \epsilon/r) \cdot \mathbf{v}]$. Hence the unbounded component $D_{\mathbf{n}}^{(q)}$ does not meet $(-\epsilon/r, \epsilon/r) \cdot \mathbf{v}$, and so $(0, \infty) \cdot \mathbf{v}$ must meet $\mathcal{A}_q(f)$. \square

Remark 2.3.7. The proof of Theorem 2.3.4 depends crucially on the observation that the Laurent expansions for unbounded components have coefficients in the ground field. This can fail for bounded components. For example, let $\mathbb{F} = \mathbb{Q}$ and $f(x, y) = 4 - x - y - x^{-1}y^{-1}$. It is shown in [15, Exam. 5.8] that the constant term in the Laurent expansion of $1/f$ for the bounded component containing the origin is

$$\frac{1}{4} \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} 4^{-3n},$$

and is the value of a hypergeometric function known to be transcendental.

3. NON-ARCHIMEDEAN ANALYSIS AND PROOFS OF MAIN RESULTS

3.1. Affinoid algebras and affinoid varieties. Let

$$T_d = \mathbb{k}\langle\langle x_1, \dots, x_d \rangle\rangle \subset \mathbb{k}[[x_1, \dots, x_d]]$$

be the set of formal series $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ such that $|a_{\mathbf{n}}| \rightarrow 0$ as $\|\mathbf{n}\| \rightarrow \infty$, where $\|\mathbf{n}\| = |n_1| + \dots + |n_d|$. This set is a \mathbb{k} -algebra called the *Tate algebra*. Throughout this section G denotes the Galois group $\text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$.

Proposition 3.1.1. (a) *The ring T_d is Noetherian.*

(b) *The maximal ideals of T_d are in bijection with the points of the unit polydisk*

$$\{\mathbf{z} \in \overline{\mathbb{k}}^d : |z_j| \leq 1 \text{ for } 1 \leq j \leq d\}$$

modulo the action of the Galois group G . Explicitly, if $\mathbf{z} = (z_1, \dots, z_d)$ is such a point whose coordinates z_j generate the finite extension \mathbb{L} of \mathbb{k} , then for any $f \in T_d$ the series $f(\mathbf{z})$ converges in \mathbb{L} , yielding a surjective homomorphism $T_d \rightarrow \mathbb{L}$.

Proof. For (a) see Theorem 1 of Section 5.2.6 of [5], and for (b) see Proposition 1 of Section 7.1.1. of the same reference. \square

For $f(\mathbf{x}) = \sum a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in T_d$ we set $|f| = \max\{|a_{\mathbf{n}}|\}$. This makes T_d into a \mathbb{k} -Banach algebra, i.e., a complete non-archimedean normed algebra with norm extending that on \mathbb{k} .

Definition 3.1.2. An *affinoid algebra* is a \mathbb{k} -Banach algebra A admitting a continuous epimorphism $T_d \rightarrow A$ for some d .

Examples 3.1.3. (a) All ideals in T_d are closed (see Corollary 2 of [5, §5.2.7]). If $I \subset T_d$ is an ideal, then $A = T_d/I$ has the residue norm $|f+I| = \inf\{|f+g| : g \in I\}$, which makes it into an affinoid algebra via the projection

$T_d \rightarrow A$. Up to replacing a norm with an equivalent one (giving the same topology) this is a general form of an affinoid algebra.

(b) If B is an affinoid algebra, A a \mathbb{k} -Banach algebra, and $\varphi: B \rightarrow A$ a continuous homomorphism such that A is finitely generated as a B -module, then A is affinoid.

Example 3.1.4. Let $\Delta \subset \mathbb{R}^d$ be a bounded Γ -rational polyhedron. We define $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ to consist of formal Laurent series $f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ satisfying the condition

$$\lim_{\|\mathbf{n}\| \rightarrow \infty} \{v(a_{\mathbf{n}}) + \mathbf{n} \cdot \mathbf{u}\} = \infty \text{ for all } \mathbf{u} \in \Delta.$$

One sees directly that $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ is a ring and the norm corresponding to the valuation

$$v(f) = \inf \{v(a_{\mathbf{n}}) + \mathbf{n} \cdot \mathbf{u} : \mathbf{n} \in \mathbb{Z}^d, \mathbf{u} \in \Delta\}$$

makes it into a \mathbb{k} -Banach algebra.

Note that for any $\mathbf{z} \in (\overline{\mathbb{k}^\times})^d$ such that $\text{val}(\mathbf{z}) \in \Delta$ the series $f(\mathbf{z})$ converges for any $f \in \mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$.

Proposition 3.1.5. $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ is an affinoid algebra.

We will call such algebras *polyhedral affinoid algebras*.

Proof. Write the inequalities defining Δ in the form

$$\sum_{j=1}^d b_{ij} u_j \geq c_i, \quad i = 1, \dots, r$$

with $b_{ij} \in \mathbb{Z}$ and $c_i \in \Gamma = v(\overline{\mathbb{k}^\times})$. Without loss of generality we can assume $c_i \in v(\mathbb{k}^\times)$ by taking, if necessary, integer multiples of the inequalities. Let $\mathbf{b}_i = (b_{i1}, \dots, b_{id}) \in \mathbb{Z}^d$, and choose $w_i \in \mathbb{k}^\times$ with $c_i = v(w_i)$. Then the condition $\text{val}(\mathbf{x}) \in \Delta$ can be rewritten as $|\mathbf{x}^{\mathbf{b}_i} / w_i| \leq 1$ for $i = 1, \dots, r$. Thus there is a continuous homomorphism

$$T_r = \mathbb{k}\langle\langle y_1, \dots, y_r \rangle\rangle \xrightarrow{\varphi} \mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle, \quad y_i \mapsto \mathbf{x}^{\mathbf{b}_i} / w_i.$$

By Example 3.1.3(b) it is enough to prove that φ is finite. Let $S \subset \mathbb{Z}^d$ be the semigroup with $\mathbf{0}$ generated by \mathbf{b}_i , $i = 1, \dots, r$, and $C = \text{Conv}(S)$ its convex hull. Denote by $\mathbb{k}\llbracket S \rrbracket$ the set of all formal Laurent series $\sum_{\mathbf{n} \in S} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$, and by $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle_S$ the intersection $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle \cap \mathbb{k}\llbracket S \rrbracket$. Our statement reduces to the following statements.

Lemma 3.1.6. (a) $\text{Im}(\varphi) = \mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle_S$.

(b) $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ is finite over $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle_S$.

Proof. (a) Let $f(\mathbf{x}) = \sum_{\mathbf{n} \in S} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ lie in $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle_S$. For any $\mathbf{n} \in S$ choose $\mathbf{m}(\mathbf{n}) = (m_1(\mathbf{n}), \dots, m_r(\mathbf{n})) \in \mathbb{Z}_+^r$ such that $m_1(\mathbf{n})\mathbf{b}_1 + \dots + m_r(\mathbf{n})\mathbf{b}_r = \mathbf{n}$. Then the series $g(\mathbf{y}) = \sum_{\mathbf{n} \in S} a_{\mathbf{n}} \mathbf{y}^{\mathbf{m}(\mathbf{n})}$ lies in T_r and $\varphi(g) = f$.

(b) Let $\mathbb{k}[S]$ and $\mathbb{k}[C \cap \mathbb{Z}^d]$ be the semigroup algebras of S and of $C \cap \mathbb{Z}^d$. Then it is well known that $\mathbb{k}[C \cap \mathbb{Z}^d]$ is finite over $\mathbb{k}[S]$. A system of module generators of $\mathbb{k}[C \cap \mathbb{Z}^d]$ over $\mathbb{k}[S]$ will be a system of module generators of $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ over $\mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle_S$. \square

This completes the proof of Proposition 3.1.5 \square

Example 3.1.7. A particular case of a polyhedral affinoid algebra is $\Delta = \{\mathbf{0}\}$. The algebra $\mathbb{k}\langle\langle \text{val}^{-1}(\mathbf{0}) \rangle\rangle$ consists of Laurent series $\sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ with $|a_{\mathbf{n}}| \rightarrow 0$ as $\|\mathbf{n}\| \rightarrow \infty$. It is the quotient of $T_{2d} = \mathbb{k}\langle\langle z_1, w_1, \dots, z_d, w_d \rangle\rangle$ by the ideal generated by the elements $z_i w_i - 1$, $i = 1, \dots, d$.

For an affinoid algebra A we denote by $\text{Max}(A)$ the set of its maximal ideals. Recall that $G = \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$.

Proposition 3.1.8. (a) $\text{Max}(A) \neq \emptyset$ unless $A = 0$.

(b) If $A = \mathbb{k}\langle\langle z_1, \dots, z_d \rangle\rangle / \langle\langle f_1, \dots, f_r \rangle\rangle$, then $\text{Max}(A)$ is identified with the set of G -orbits on

$$\{\mathbf{z} \in \overline{\mathbb{k}}^d : |z_j| \leq 1 \text{ for } 1 \leq j \leq d \text{ and } f_i(\mathbf{z}) = 0 \text{ for } 1 \leq i \leq r\}.$$

(c) If $A = \mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$, then $\text{Max}(A)$ is identified with the set of G -orbits on

$$\text{val}^{-1}(\Delta) \stackrel{\text{def}}{=} \{\mathbf{z} \in (\overline{\mathbb{k}}^\times)^d : \text{val}(\mathbf{z}) \in \Delta\}.$$

(d) If $A = \mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle / \langle\langle f_1, \dots, f_r \rangle\rangle$, then $\text{Max}(A)$ is identified with the set of G -orbits on

$$\{\mathbf{z} \in \text{val}^{-1}(\Delta) : f_i(\mathbf{z}) = 0 \text{ for } 1 \leq i \leq r\}.$$

Proof. (a) is standard, (b) follows from Proposition 3.1.1, (c) follows from (b) and the proof of Proposition 3.1.5, and (d) follows from (c). \square

3.2. Proof of Theorem 2.2.5. Let $X \subset \mathbb{G}_m^d$ be an irreducible variety. We have already observed that $\mathcal{T}(X) \subset \mathcal{BG}(X)$. Since $\mathcal{BG}(X)$ is a Γ -rational polyhedral set, we may reduce, as in the proof of Theorem 2.1.1, to the following case.

Lemma 3.2.1. If $\mathbf{0} \in \mathcal{BG}(X)$, then $\mathbf{0} \in \mathcal{T}(X)$.

Proof. Suppose that $\mathbf{0} \notin \mathcal{T}(X)$. Then $X(\overline{\mathbb{k}}) \cap \text{val}^{-1}(\mathbf{0}) = \emptyset$. Let X be defined by the polynomials f_1, \dots, f_r . Put $\widehat{A} = \mathbb{k}\langle\langle \text{val}^{-1}(\mathbf{0}) \rangle\rangle / \langle\langle f_1, \dots, f_r \rangle\rangle$. By Proposition 3.1.8(d),

$$\text{Max}(\widehat{A}) = (X(\overline{\mathbb{k}}) \cap \text{val}^{-1}(\mathbf{0})) / G = \emptyset,$$

so that $\widehat{A} = 0$. Hence there are $g_1, \dots, g_r \in \mathbb{k}\langle\langle \text{val}^{-1}(\mathbf{0}) \rangle\rangle$ such that $1 = f_1 g_1 + \dots + f_r g_r$.

On the other hand, suppose that $\mathbf{0} \in \mathcal{BG}(X)$. Then there is a valuation w on $A = \mathbb{k}[x_1^{\pm 1}, \dots, x_d^{\pm 1}] / \langle\langle f_1, \dots, f_r \rangle\rangle = \mathbb{k}[X]$ such that $w(x_i) = 0$ for $1 \leq i \leq d$. We can consider w as a ring valuation on $\mathbb{k}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ that

equals ∞ on $\langle f_1, \dots, f_r \rangle$. Since $w(x_i) = 0$, we can extend w by continuity to a valuation \widehat{w} on $\mathbb{k}\langle\langle \text{val}^{-1}(\mathbf{0}) \rangle\rangle$ via

$$\widehat{w}\left(\sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}\right) = \lim_{N \rightarrow \infty} w\left(\sum_{\mathbf{n} \in [-N, N]^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}\right).$$

But since $\widehat{w}(f_j) = w(f_j) = \infty$, we obtain that

$$\begin{aligned} 0 &= \widehat{w}(1) = \widehat{w}(f_1 g_1 + \dots + f_r g_r) \geq \min_j \{\widehat{w}(f_j g_j)\} \\ &= \min_j \{w(f_j) + \widehat{w}(g_j)\} = \infty. \end{aligned}$$

This contradiction proves the lemma, and hence the theorem. \square

Remark 3.2.2. The non-archimedean analytic point of view allows one to simplify several proofs in the original Bieri-Groves treatment of their sets. Thus, the property of total concavity of $\mathcal{BG}(X)$ (see [2]) follows at once, if formulated for $\mathcal{T}(X)$ instead, from the maximum modulus principle for affinoid sets (see Proposition 4 of [5, Sec. 6.2.1]).

3.3. Reminder on rigid analytic spaces. The basic reference for this section is [5], which contains a complete and accessible treatment of the ideas we use there.

Let A be an affinoid algebra. The set $\text{Max}(A)$ has the following structures (see [5, Chap. 9]):

- (1) A Grothendieck topology \mathcal{T} (the strong G -topology of [5, Sec. 9.1.4]), i.e.,
 - (1a) A family of subsets $U \subset \text{Max}(A)$ called *admissible open*, and
 - (1b) For any admissible open U , a family $\text{Cov}(U)$ of coverings of U by admissible open sets contained in U , satisfying the axioms of [5, Sec. 9.1.1]. Such coverings are called *admissible*.
- (2) A sheaf of local rings \mathcal{O} on the above topology such that $\mathcal{O}(\text{Max}(A)) = A$.

The construction of these objects is given a detailed treatment in [5]. We give here some instructive examples.

Examples 3.3.1. (a) If $f_1, \dots, f_r \in A$, then

$$U_{f_1, \dots, f_r} = \{x \in \text{Max}(A) : |f_i(x)| \leq 1, i = 1, \dots, r\}$$

is an admissible open subset called a *Weierstrass domain* [5, Sec. 7.2.3].

(b) If $f_1, \dots, f_r \in A^*$ then the covering of $\text{Max}(A)$ by domains of the form

$$U_{\epsilon_1, \dots, \epsilon_r} = \{x : |f_i(x)|^{\epsilon_i} \leq 1, i = 1, \dots, r\}, \quad \epsilon_i = \pm 1$$

is admissible. It is called a *Laurent covering* [5, Sec. 8.2.2].

Recall that $G = \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$.

Proposition 3.3.2. *Let $A = \mathbb{k}\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ for a bounded Γ -rational polyhedron Δ as in Example 3.1.4, so that $\text{Max}(A) = \text{val}^{-1}(\Delta)/G$. For any Γ -rational subpolyhedron $\Sigma \subset \Delta$ the subset $\text{val}^{-1}(\Sigma)/G$ is admissible open.*

Proof. Any additional Γ -rational inequalities defining Σ can be written in the form $|w \cdot z_1^{b_1} \dots z_d^{b_d}| \leq 1$, so the subset in question is a Weierstrass domain. \square

By definition a (rigid) analytic space over \mathbb{k} is a system $Z = (\underline{Z}, \mathcal{T}, \mathcal{O}_Z)$ consisting of a set \underline{Z} , a Grothendieck topology \mathcal{T} on \underline{Z} , and a sheaf \mathcal{O}_Z of rings on \mathcal{T} such that locally on \mathcal{T} it is isomorphic to $\text{Max}(A)$ where A is an affinoid algebra with its Grothendieck topology and sheaf \mathcal{O} .

Example 3.3.3. Every scheme X of finite type over \mathbb{k} gives rise to an analytic space X^{an} with $\underline{X}^{\text{an}} = X(\overline{\mathbb{k}})/G$ (see [5, Sec. 9.3.4]). We are particularly interested in the case when $X \subset \mathbb{G}_m^d$ is a closed subscheme. In this case for each bounded Γ -rational polyhedron $\Delta \subset \mathbb{R}^d$ the intersection

$$(X(\overline{\mathbb{k}}) \cap \text{val}^{-1}(\Delta))/G$$

is an admissible subset in $\underline{X}^{\text{an}}$.

3.4. Connectedness and irreducibility for analytic spaces. Proof of Theorem 2.2.7. A rigid analytic space Z is called *disconnected* if it admits an admissible open covering consisting of two disjoint subspaces. Otherwise Z is called *connected*. One says that Z is *irreducible* if the normalization of Z (see [9, Sec. 2.1]) is connected. In particular an irreducible analytic space is connected. The following result of Conrad will be crucial for us.

Theorem 3.4.1 (Thm 2.3.1 of [9]). *Let X be an irreducible algebraic variety over \mathbb{k} . Then the analytic space X^{an} is irreducible and, in particular, is connected.*

We now assume that X is a closed subvariety in \mathbb{G}_m^d , and let $\mathcal{T} = \mathcal{T}(X) \subset \mathbb{R}^d$ be its tropical variety. Suppose that \mathcal{T} is disconnected: $\mathcal{T} = \mathcal{B} \sqcup \mathcal{C}$ where \mathcal{B} and \mathcal{C} are open and closed in \mathcal{T} . Define subsets $B, C \subset \underline{X}^{\text{an}} = X(\overline{\mathbb{k}})/G$ by

$$B = (X(\overline{\mathbb{k}}) \cap \text{val}^{-1}(\mathcal{B}))/G$$

and similarly for C . Then, clearly, $\underline{X}^{\text{an}} = B \sqcup C$. To establish Theorem 2.2.7 it is therefore enough to prove the following.

Proposition 3.4.2. (a) *B and C are admissible open in $\underline{X}^{\text{an}}$.*

(b) *The covering of $\underline{X}^{\text{an}}$ by B and C is admissible.*

Proof. We recall two properties which hold for the Grothendieck topology of any rigid analytic space Z ([5, p. 339]).

(G1) Let U be admissible open in \underline{Z} and $V \subset U$ be a subset. Assume there exists an admissible covering $\{U_i\}$ of U such that $V \cap U_i$ is admissible open in \underline{Z} for all i . Then V is admissible open in \underline{Z} .

(G2) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of an admissible open $U \subset \underline{Z}$ such that U_i is admissible open in \underline{Z} for each i . Assume that \mathcal{U} has a refinement which is an admissible covering of U . Then \mathcal{U} itself is an admissible covering of U .

Let $\mathbb{R}^d = \bigcup_{i \in I} \Delta_i$ be a decomposition of \mathbb{R}^d into parallel cubes of sufficiently small size ϵ . Then $\{\text{val}^{-1}(\Delta_i)/G : i \in I\}$ is an admissible covering of $(\overline{\mathbb{k}}^\times)^d/G$ and therefore the sets

$$D_i = (X(\overline{\mathbb{k}}) \cap \text{val}^{-1}(\Delta_i))/G$$

form an admissible covering of $\underline{X}^{\text{an}}$. Let $J_B = \{i \in I : \Delta_i \cap \mathcal{B} \neq \emptyset\}$, $J_C = \{i \in I : \Delta_i \cap \mathcal{C} \neq \emptyset\}$, and $J = J_B \cup J_C$. Since \mathcal{B} and \mathcal{C} are polyhedral, by taking ϵ small enough, we can assume $J_B \cap J_C = \emptyset$. As $D_i = \emptyset$ for $i \notin J$, we have $D_i, i \in J$ form an admissible covering of $\underline{X}^{\text{an}}$. Now, applying (G1) to $Z = U = \underline{X}^{\text{an}}$, $V = B$, $U_i = D_i, i \in J$ we find that $V \cap U_i = U_i$ for $i \in J_B$, and $V \cap U_i = \emptyset$ for $i \notin J_B$, so $V = B$ is admissible open. Similarly for C . This proves part (a) of the proposition. Part (b) follows at once from (G2), as the covering $\underline{X}^{\text{an}} = B \cup C$ has a refinement $\underline{X} = \bigcup_{i \in J} D_i$ which is admissible. \square

4. ADELIC AMOEBAS IN ALGEBRA AND DYNAMICS

In this section we briefly describe two situations in which adelic amoebas have already implicitly appeared.

4.1. The Bieri-Strebel geometric invariant. Let R be a commutative ring with 1, and $R[\mathbf{x}^\pm]$ denote the ring $R[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ of Laurent polynomials over R . Let \mathbf{S}_{d-1} denote the unit sphere in \mathbb{R}^d . For each $\mathbf{u} \in \mathbf{S}_{d-1}$ let $H_{\mathbf{u}} = \{\mathbf{v} \in \mathbb{R}^d : \mathbf{u} \cdot \mathbf{v} \leq 0\}$ be the half-space with outward normal \mathbf{u} . There is a continuum of subrings $R_{\mathbf{u}} = R[\mathbf{x}^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d \cap H_{\mathbf{u}}]$ of R as \mathbf{u} varies over \mathbf{S}_{d-1} .

Suppose that M is a finitely generated $R[\mathbf{x}^\pm]$ -module. For which $\mathbf{u} \in \mathbf{S}_{d-1}$ does M remain finitely generated over $R_{\mathbf{u}}$? This question led Bieri and Strebel [3] to define their geometric invariant for $R[\mathbf{x}^\pm]$ -modules M as

$$\Sigma_M^c = \{\mathbf{u} \in \mathbf{S}_{d-1} : M \text{ is not finitely generated over } R_{-\mathbf{u}}\}$$

(we have used a negative sign in $R_{-\mathbf{u}}$ since Bieri-Strebel use inward normals). This invariant has proved crucial in answering a number of important algebraic questions. For example, they show that certain $R[\mathbf{x}^\pm]$ -modules M are finitely presented if and only if Σ_M^c does not contain any pair of antipodal points (a condition reminiscent of the dynamical notion of totally non-symplectic).

The geometric invariant Σ_M^c can be obtained from Bieri-Groves sets as follows. Let v be a (ring) valuation on R . For an ideal \mathfrak{a} in $R[\mathbf{x}^\pm]$ define $\mathcal{W}(R[\mathbf{x}^\pm]/\mathfrak{a})$ to be the set of all valuations w on $R[\mathbf{x}^\pm]/\mathfrak{a}$ extending v such that $w(x_i) < \infty$ for $1 \leq i \leq d$. Define the Bieri-Groves set to be

$$\mathcal{BG}_v(R[\mathbf{x}^\pm]/\mathfrak{a}) = \{(w(x_1), \dots, w(x_d)) : w \in \mathcal{W}(R[\mathbf{x}^\pm]/\mathfrak{a})\}.$$

Let $\rho_0: \mathbb{R}^d \setminus \{\mathbf{0}\} \rightarrow \mathbf{S}_{d-1}$ be radial projection, and extend ρ_0 to subsets of \mathbb{R}^d by $\rho_0(E) = \rho_0(E \setminus \{\mathbf{0}\})$. Then Bieri and Groves [2, Thm. 8.1] showed the following.

Theorem 4.1.1. *Let M be a Noetherian $R[\mathbf{x}^\pm]$ -module and \mathfrak{a} be the annihilator of M . Then*

$$\Sigma_M^c = \bigcup_{v(R) \geq 0} \rho(\mathcal{BG}_v(R[\mathbf{x}^\pm]/\mathfrak{a})),$$

where the union is over all valuations on R which are nonnegative.

In the special case $R = \mathbb{Z}$ the geometric invariant Σ_M^c is related to the part of the adelic amoeba that corresponds to the finite primes p .

Corollary 4.1.2. *Let M be a Noetherian $\mathbb{Z}[\mathbf{x}^\pm]$ -module and \mathfrak{a} be the annihilator of M . Suppose M (or equivalently $R[\mathbf{x}^\pm]/\mathfrak{a}$) is torsion-free as a module over \mathbb{Z} , and define $X \subset \mathbb{G}_m^d$ to be the algebraic variety defined by \mathfrak{a} over the field \mathbb{Q} . Then*

$$\Sigma_M^c = \bigcup_{p < \infty} \rho(\mathcal{T}(X(\overline{\mathbb{Q}}_p))),$$

where the union is over all rational prime numbers, and $X(\overline{\mathbb{Q}}_p)$ is the variety defined by \mathfrak{a} over the algebraic closure $\overline{\mathbb{Q}}_p$ of the p -adic rationals.

In other words the radial projections of the negatives of the p -adic amoebas of \mathfrak{a} describe Σ_M^c completely. Since there are only finitely many distinct p -adic amoebas, this is actually a finite union. An explicit algorithm for computing this union, using universal Gröbner bases and Fitting ideals, is described in Proposition 6.6 of [10].

Proof. By Theorem 4.1.1 Σ_M^c can be calculated via the Bieri-Groves sets. By Theorem 2.2.5 we know that $\mathcal{BG}(X) = \mathcal{T}(X)$ over every fixed field \mathbb{k} with some fixed valuation v . However, Theorem 4.1.1 uses ring valuations w , that are allowed to have $w(n) = \infty$ for nonzero $n \in \mathbb{Z}$. For the corollary we need to show that the restriction to the p -adic valuations does not change the statement (under the torsion-free assumption).

So assume that w is a valuation on $\mathbb{Z}[\mathbf{x}^\pm]/\mathfrak{a}$ with $\rho(w(x_1), \dots, w(x_d)) = \mathbf{v} \neq \mathbf{0}$, and $w(p) = \infty$ for some prime number p . We claim that the direction \mathbf{v} is also captured by the p -adic valuation.

Suppose $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are the associated prime ideals to $\mathbb{Z}[\mathbf{x}^\pm]/\mathfrak{a}$. Then $w(\mathfrak{p}_i) = \{\infty\}$ for some i . Let $\mathfrak{p} = \mathfrak{p}_i$, and let \mathbb{F} be the field of fractions of $\mathbb{Z}[\mathbf{x}^\pm]/\mathfrak{p}$. Note that \mathbb{F} has characteristic zero by assumption. Then [2, Thm. C2] describes $\mathcal{BG}_p(\mathbb{Z}[\mathbf{x}^\pm]/\mathfrak{p})$ (using the p -adic valuation) near the origin, and implies that $\{r\mathbf{v} : r \in [0, \epsilon]\} \subset \mathcal{BG}_p(\mathbb{Z}[\mathbf{x}^\pm]/\mathfrak{p})$ for some $\epsilon > 0$. Since $\mathcal{BG}_p(\mathbb{Z}[\mathbf{x}^\pm]/\mathfrak{p}) \subset \mathcal{T}(X(\overline{\mathbb{Q}}_p))$, the claim follows. \square

4.2. Expansive subdynamics of algebraic actions. We now turn to dynamics. Again let M be a module over $\mathbb{Z}[\mathbf{x}^\pm]$. There is a corresponding algebraic \mathbb{Z}^d -action α_M on a compact group X_M defined as follows. Consider M as a discrete abelian group, and let X_M be its compact Pontryagin dual group. For $\mathbf{n} \in \mathbb{Z}^d$ let $\alpha_M^{\mathbf{n}}$ be the automorphism of X_M dual to the

automorphism of M given by multiplication by $\mathbf{x}^{\mathbf{n}}$. This process can be reversed, so that given an algebraic \mathbb{Z}^d -action by automorphisms of a compact abelian group, there is a corresponding $\mathbb{Z}[\mathbf{x}^{\pm}]$ -module via duality. The book [21] contains all necessary background and a wealth of examples.

A framework for studying general topological \mathbb{Z}^d -actions was developed in [7], focusing on the key idea of expansiveness along half-spaces. Fix a metric δ on X_M compatible with its topology. Then α_M is called *expansive along $H_{\mathbf{u}}$* if there is an $\epsilon > 0$ so that if ξ and η are two points in X_M with $\delta(\alpha_M^{\mathbf{n}}(\xi), \alpha_M^{\mathbf{n}}(\eta)) < \epsilon$ for all $\mathbf{n} \in H_{\mathbf{u}} \cap \mathbb{Z}^d$, then $\xi = \eta$. The *nonexpansive set* of α_M is

$$N(\alpha_M) = \{ \mathbf{u} \in S_{d-1} : \alpha_M \text{ is not expansive along } H_{\mathbf{u}} \}.$$

This set turns out to be closed in S_{d-1} . The expansive subdynamics philosophy advocated in [7] says that dynamical properties of α_M restricted to subspaces are either constant or vary nicely within a connected component of the complement of $N(\alpha_M)$, but typically change abruptly when passing from one connected component to another, analogous to a phase transition. The description of lower dimensional entropy in [7, Sec. 6] is an example of this philosophy in action.

It is therefore natural to ask for an explicit calculation for the nonexpansive set for an algebraic \mathbb{Z}^d -action. This was done in [10, Prop. 4.9] using the complex amoeba and Σ_M^c . Combining this with Corollary 4.1.2 shows that the nonexpansive set is the radial projection of an adelic amoeba.

Theorem 4.2.1. *Let M be a Noetherian $\mathbb{Z}[\mathbf{x}^{\pm}]$ -module and \mathfrak{a} be its annihilator. Suppose that X_M is connected (or, equivalently, that $\mathbb{Z}[\mathbf{x}^{\pm}]/\mathfrak{a}$ is torsion-free over \mathbb{Z}). Then*

$$N(\alpha_M) = \bigcup_{p \leq \infty} \rho(\mathcal{A}_{\overline{\mathbb{Q}_p}}(\mathfrak{a})) = \rho(\mathcal{A}_{\mathbb{A}}(\mathfrak{a})).$$

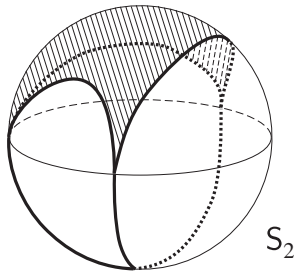


FIGURE 2. The nonexpansive set of $\langle 1 + x + y, z - 2 \rangle$

Example 4.2.2. Let $d = 3$, $\mathfrak{a} = \langle 1 + x + y, z - 2 \rangle$, and $M = \mathbb{Z}[\mathbf{x}^{\pm}]/\mathfrak{a}$. Then $N(\alpha_M)$ is depicted in Figure 2. The portion above the equator is the radial projection of the complex amoeba of \mathfrak{a} , the part below the equator

is the radial projection of the 2-adic amoeba, and the three points on the equator come from the p -adic amoeba for $p \neq 2$ (which also form Bergman's logarithmic limit set [1] of $V(\mathfrak{a})$). Here the expansive components are the three lobes of S_2 in the complement of $N(\alpha_M)$. It is perhaps interesting to note that entropy considerations show that none of these components can contain a pair of antipodal points.

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MANFRED EINSIEDLER, DEPARTMENT OF MATHEMATICS, BOX 354350, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195–4350

E-mail address: `einsiedl@math.washington.edu`

MIKHAIL KAPRANOV, DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06520

E-mail address: `mikhail.kapranov@yale.edu`

DOUGLAS LIND, DEPARTMENT OF MATHEMATICS, BOX 354350, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195–4350

E-mail address: `lind@math.washington.edu`