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Laurent Determinants and Arrangements of Hyperplane Amoebas

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We study amoebas associated with Laurent polynomials and obtain new results regarding the number and structure of the connected components of the complement of the amoeba. We also investigate the associated Laurent determinant. In the case of a hyperplane arrangement we perform explicit computations leading to a closed formula for the Laurent determinant.

INTRODUCTION

The notion of amoebas was introduced by Gelfand *et al.* in [7]. Given a Laurent polynomial f its amoeba $\mathscr{A}_f$ is the image of the hypersurface $\mathscr{Z}_f = f^{-1}(0)$ under the map $(z_1, ..., z_n) \mapsto (\log|z_1|, ..., \log|z_n|)$. It will typically be a semianalytic closed subset of $\mathbb{R}^n$ with tentacle-like asymptotes going off to infinity and separating the connected components of the complement ${}^c\mathscr{A}_f$. These components are convex and they reflect the structure of the Newton polytope $\mathscr{N}_f$ of the Laurent polynomial f. Furthermore, each such component corresponds to a specific Laurent series development of the rational function $1/f$. The problem of finding and describing the connected components of ${}^c\mathscr{A}_f$ was posed in [7]. In this paper we introduce what we call the order of a complement component, and we show that it

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provides a bijection between the family of components and a subset of $\mathcal{N}_f \cap \mathbb{Z}^n$. This implies in particular that the number of connected components of ${}^c\mathcal{A}_f$ is at most equal to the number of integer points in the Newton polytope. We then go on to introduce a certain matrix of Laurent coefficients of $1/f$. Even though the individual Laurent coefficients may be unwieldy hypergeometric functions, the (square of the) determinant of this matrix, which we call the Laurent determinant of f, appears to have a tractable structure. We devote the last part of the paper to the special situation where f is a polynomial that factors into linear forms. Its zero set is then a union of hyperplanes, and consequently the amoeba is a union, or an arrangement, of hyperplane amoebas. It is proved that when the coefficients of the linear functions lie outside a certain secondary amoeba, the number of components of the complement ${}^c\mathcal{A}_f$ is maximal, that is, equal to the number of integer points in the Newton polytope $\mathcal{N}_f$. We are also able in this case to compute the Laurent determinant of f explicitly, and it turns out to be exactly equal to the reciprocal of the polynomial defining the aforesaid secondary amoeba. We would like to acknowledge our gratitude towards the referees, whose suggestions have improved the exposition.

1. NOTATION AND BASIC DEFINITIONS

We denote by $\mathbb{T}^n$ the standard complex torus $(\mathbb{C}\setminus\{0\})^n$, and we consider it as a subset of complex projective space $\mathbb{P}^n$ in the usual manner. By f we shall mean a Laurent polynomial in $\mathbb{T}^n$, that is, an expression of the form $f(z_1, ..., z_n) = \sum_{\alpha \in A} a_\alpha z^\alpha$, where A is a finite subset of the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and each coefficient a_α is a nonzero complex number. Given such a Laurent polynomial f, the following associated objects will be of particular interest to us.

The Newton polytope. This is the convex hull in $\mathbb{R}^n$ of the index set A, and we denote it by $\mathcal{N}_f$. The set of integer points in $\mathcal{N}_f$ admits a natural partition $\mathcal{N}_f \cap \mathbb{Z}^n = \bigcup_\Gamma A_\Gamma$, where Γ is any face of $\mathcal{N}_f$ and A_Γ denotes the intersection of $\mathbb{Z}^n$ with the relative interior of Γ. The extreme cases occur when Γ is a vertex, which means that $A_\Gamma = \Gamma$, and when $\Gamma = \mathcal{N}_f$. For any integer point $v \in \mathcal{N}_f$ we shall also consider the dual cone C_v of $\mathcal{N}_f$ at v, which is defined as

$$C_v = \{s \in \mathbb{R}^n;\ \langle s, v\rangle = \max_{\alpha \in \mathcal{N}_f} \langle s, \alpha\rangle\}.$$

Notice that the dimension of the dual cone C_v is equal to $n - \dim \Gamma$, when $v \in A_\Gamma$. In particular, it has nonempty interior precisely if v is a vertex of $\mathcal{N}_f$, and it equals $\{0\}$ whenever v is an interior point of $\mathcal{N}_f$.

The amoeba. We denote by $\mathscr{A}_f$ the amoeba of f, which by definition is the image $\log(\mathscr{Z}_f) \subset \mathbb{R}^n$ of the zero set $\mathscr{Z}_f = \{z \in \mathbb{T}^n;\ f(z) = 0\}$ under the logarithmic modulus map $\log : \mathbb{T}^n \to \mathbb{R}^n$ given by

$$(z_1, ..., z_n) \mapsto (\log |z_n|, ..., \log |z_n|).$$

This terminology was motivated by the typical shape of the image $\mathscr{A}_f$, with thin tentacles going off to infinity. The complement ${}^c\mathscr{A}_f = \mathbb{R}^n \backslash \mathscr{A}_f$ consists of finitely many connected components, which are open and in fact convex.

The Laurent series. The rational function $1/f$ may be expanded in a Laurent series $\sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha$, with the coefficients given by the integral formula

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{\log^{-1}(u)} \frac{dz_1 \wedge \cdots \wedge dz_n}{f(z)\, z_1^{\alpha_1+1} \cdots z_n^{\alpha_n+1}}.$$

Here $u = (u_1, ..., u_n)$ should be a point in the amoeba complement ${}^c\mathscr{A}_f$ and $\log^{-1}(u)$ is the corresponding oriented n-cycle in $\mathbb{T}^n \backslash \mathscr{Z}_f$. We shall always choose the orientation so as to have $d \arg(z_1) \wedge \cdots \wedge d \arg(z_n) > 0$ on $\log^{-1}(u)$. Observe that the Laurent coefficient c_α really depends only on the choice of connected component of ${}^c\mathscr{A}_f$, for if u and v are in the same component then the cycles $\log^{-1}(u)$ and $\log^{-1}(v)$ are clearly homologous in $\mathbb{T}^n \backslash \mathscr{Z}_f$.

We quote the following simple result from [7]. See also [5] and [8].

THEOREM 1.1. *The connected components of the amoeba complement ${}^c\mathscr{A}_f$ are convex, and they are in bijective correspondence with the different Laurent expansions* (*centered at the origin*) *of the rational function* $1/f$.

The convexity here follows now from the general fact that the domains of convergence of Laurent series are exactly the logarithmically convex ones, that is, sets of the form $\log^{-1}(E)$, with $E \subset \mathbb{R}^n$ being a convex domain. Even though the Laurent expansions corresponding to different components E_ν are necessarily different, for they have different domains of convergence, it is not obvious that they are all linearly independent, see Section 7 below.

We shall be concerned with the problem of finding the components of the complement to an amoeba, a problem referred to in [7, Remark 1.10] as a difficult and interesting one. Our first observation is that such a component cannot suddenly disappear when the coefficients of f vary slightly.

PROPOSITION 1.2. *Let $f_a(z)=\sum_{\alpha\in A} a_\alpha z^\alpha$ be a Laurent polynomial. The mapping*

$$\mathbb{C}^A \ni a \mapsto \#\{\text{components of } {}^c\mathscr{A}_{f_a}\}$$

is then lower semicontinuous. Here we consider the coefficients $a=\{a_\alpha\}$ as variables.

Proof. Fix the coefficient vector a and choose one point u_ν in each connected component E_ν of the amoeba complement ${}^c\mathscr{A}_{f_a}$. It is obvious by continuity that every u_ν will still lie in the complement of $\mathscr{A}_{f_{a'}}$ if a' is close enough to a. To prove the proposition it therefore suffices to show that two different points u_ν cannot lie in the same component of ${}^c\mathscr{A}_{f_{a'}}$. But this follows from the fact that the Laurent coefficients c_α depend continuously on a, so two series that are different for the original value a must remain different also for nearby a'. ∎

2. THE ORDER OF A COMPLEMENT COMPONENT

We are going to use the argument principle to make a more detailed study of the structure of the connected components of the amoeba complement ${}^c\mathscr{A}_f$. To this end we introduce the following notion of order for points in ${}^c\mathscr{A}_f$.

DEFINITION 2.1. Let u be a point in the amoeba complement ${}^c\mathscr{A}_f$. The *order* of u is then defined as the vector $\nu\in\mathbb{Z}^n$ whose components are

$$\nu_j=\frac{1}{(2\pi i)^n}\int_{\log^{-1}(u)}\frac{z_j\,\partial_j f(z)}{f(z)}\,\frac{dz_1\wedge\cdots\wedge dz_n}{z_1\cdots z_n},\qquad j=1,\ldots,n.$$

Since the homology class of the cycle $\log^{-1}(u)$ is the same for all u in the same connected component E of ${}^c\mathscr{A}_f$, we may as well call ν the order of the component E. When we wish to emphasize the dependence on f and u we will write $\nu_j(f,u)$ rather than just ν_j.

That each ν_j is indeed an integer is easily seen as follows: Write the coordinates in polar form $z_k=\exp(u_k+i\theta_k)$, and consider, for fixed arguments θ_k, $k\neq j$, the ordinary contour integral

$$\frac{1}{2\pi i}\int_{|z_j|=e^{u_j}}\frac{\partial_j f(z)}{f(z)}\,dz_j.$$

By the classical argument principle this integral will be integer valued. Since it also depends continuously on the remaining arguments θ_k, it must in fact be independent of these, and it follows that its value is equal to v_j.

LEMMA 2.2. *For any vector* $s \in \mathbb{Z}^n \setminus \{0\}$ *the directional order* $\langle s, v(f, u) \rangle$ *is equal to the number of zeros (minus the order of the pole at the origin) of the one-variable Laurent polynomial*

$$w \mapsto f(c_1 w^{s_1}, ..., c_n w^{s_n})$$

inside the unit circle $|w| = 1$. *Here* $c \in \mathbb{T}^n$ *is any vector with* $\log(c) = u$.

Proof. The usual argument principle in one variable tells us that the number of zeros minus the number of poles of $f(cw^s)$ is given by the integral

$$\frac{1}{2\pi i} \int_{|w|=1} \partial \log f(c_1 w^{s_1}, ..., c_n w^{s_n}).$$

But under the mapping $w \mapsto (c_1 w^{s_1}, ..., c_n w^{s_n})$ the image of the unit circle $|w| = 1$ is a loop contained in the n-torus $\log^{-1}(u)$. Moreover, this loop is homologous (in $\log^{-1}(u)$ and hence in $\mathbb{T}^n \setminus \mathscr{Z}_f$) to the sum $s_1 \gamma_1 + \cdots + s_n \gamma_n$, where γ_j denotes the plane circular 1-cycle $[0, 1] \ni t \mapsto (c_1, ..., c_j e^{2\pi i t}, ..., c_n)$, see for instance [9, Sect. 4.6].

We therefore have

$$\begin{aligned} \int_{|w|=1} \partial \log f(cw^s) &= \sum_{j=1}^{n} s_j \int_{\gamma_j} \partial \log f(z) \\ &= \sum_{j=1}^{n} s_j \int_{|z_j| = e^{u_j}} \frac{\partial_j f(z)}{f(z)} \, dz_j \\ &= 2\pi i \sum_{j=1}^{n} s_j v_j, \end{aligned}$$

and the lemma is proved. ∎

Remark 2.3. It is possible to rotate the amoeba by making monomial coordinate changes. Indeed, if the vector $s \in \mathbb{Z}^n \setminus \{0\}$ has relatively prime components, then s may be completed to a basis for the lattice $\mathbb{Z}^n$, that is, one can find a unimodular $(n \times n)$-matrix S with integer entries, having s as its first row. (That this is always possible follows for instance from the invariant factor theorem, see [3].) Denoting by S^*f the new Laurent polynomial $z \mapsto f(z^S)$ one then has the identity $\langle s, v(f, u) \rangle = v_1(S^*f, S^{-1}u)$.

PROPOSITION 2.4. *The order v of any component of ${}^c\mathscr{A}_f$ is contained in the Newton polytope $\mathscr{N}_f$.*

Proof. It is enough to show that $\langle s, v\rangle \leqslant \max_{\alpha \in \mathscr{N}_f} \langle s, \alpha\rangle$ for any vector $s \in \mathbb{Z}^n \setminus \{0\}$. From Lemma 2.2 we know that $\langle s, v\rangle$ is equal to the number of zeros of $w \mapsto f(cw^s)$ inside the unit circle. Since the top degree of this one-variable Laurent polynomial is equal to $\max_{\alpha \in \mathscr{N}_f} \langle s, \alpha\rangle$, the proposition follows. ▮

PROPOSITION 2.5. *Two different components E and E' of the complement ${}^c\mathscr{A}_f$ cannot have equal orders v and v'.*

Proof. Take two points u and u' in $\mathbb{Q}^n \setminus \mathscr{A}_f$ and let $s \in \mathbb{Z}^n \setminus \{0\}$ be the direction from u to u', so that $u' = u + rs$ for some $r > 0$. We shall show that $\langle s, v'\rangle > \langle s, v\rangle$. Indeed, by Lemma 2.2 these two numbers coincide with the number of zeros inside $|w| = 1$ of the one-variable polynomials $w \mapsto f(c'w^s)$ and $w \mapsto f(cw^s)$, respectively, where $\log(c') = u'$ and $\log(c) = u$. Now, since $c'_j/c_j = e^{rs_j}$, and hence $c'w^s = c(e^r w)^s$, we may also interpret $\langle s, v'\rangle$ as the number of zeros of $f(cw^s)$ inside the larger circle $|w| = e^r$. But if there would be no zero of this polynomial in the ring $1 < |w| < e^r$, then the line segment $[u, u']$ would not intersect the amoeba $\mathscr{A}_f$. ▮

PROPOSITION 2.6. *Let E be a component of the complement ${}^c\mathscr{A}_f$ and suppose that its order is $v \in \mathscr{N}_f$. The dual cone C_v of $\mathscr{N}_f$ at v is then equal to the recession cone of E. That is, for any $u \in E$ one has $u + C_v \subset E$ and no strictly larger cone is contained in E. (Notice that if v is in the k-skeleton of $\mathscr{N}_f$ then C_v has dimension $n-k$.)*

Proof. Take $u \in E$ and fix a direction vector $s \in \mathbb{Z}^n \setminus \{0\}$. What we must prove is that the ray $u + \mathbb{R}_+ s$ is disjoint from the amoeba $\mathscr{A}_f$ if and only if $\langle s, v\rangle = \max_{\alpha \in \mathscr{N}_f} \langle s, \alpha\rangle$. By invoking Lemma 2.2 we see that the ray avoids the amoeba precisely if the one-variable polynomial $w \mapsto f(cw^s)$ has all its zeros inside the unit circle. Since its degree is equal to $\max_{\alpha \in \mathscr{N}_f} \langle s, \alpha\rangle$ and since $\langle s, v\rangle$ counts its zeros inside the unit circle we arrive at the desired conclusion. ▮

See [11] for a background on recession cones of convex sets.

PROPOSITION 2.7. *Take v in $\mathscr{N}_f \cap \mathbb{Z}^n$ and suppose we can find a point $z \in \log^{-1}({}^c\mathscr{A}_f)$ such that $|a_v z^v| > |\sum_{\alpha \in \mathscr{N}_f \setminus \{v\}} a_\alpha z^\alpha|$. That is, the geometric series*

$$\frac{1}{f(z)} = \sum_{k=0}^{\infty} \frac{(a_v z^v - \sum_{\alpha \in \mathscr{N}_f} a_\alpha z^\alpha)^k}{(a_v z^v)^{k+1}}$$

converges in a polyring containing z. *Then the order of the component of* ${}^c\mathscr{A}_f$ *that contains* $\log(z)$ *is equal to* ν.

Proof. Denote by μ the order of the component containing $\log(z)$. We must show that $\mu = \nu$. Let us actually prove that $\langle s, \mu \rangle = \langle s, \nu \rangle$ for any vector $s \in \mathbb{Z}^n \setminus \{0\}$. Indeed, by Lemma 2.2 the quantity $\langle s, \mu \rangle$ is equal to the number of zeros of $w \mapsto f(cw^s)$ inside the unit circle. Now, if we write $g(w) = f(cw^s) - a_\nu c^\nu w^{\langle s, \nu \rangle}$, then we have $|g(w)| < |f(cw^s)|$ on the domain of integration. By the ordinary Rouché theorem we therefore get that the number of zeros of $f(cw^s)$ inside the unit circle is equal to the number of zeros of $w \mapsto f(cw^s) - g(w) = a_\nu c^\nu w^{\langle s, \nu \rangle}$ there. But this latter number is equal to $\langle s, \nu \rangle$ and the proof is complete. ∎

THEOREM 2.8. *The number of connected components of the amoeba complement* ${}^c\mathscr{A}_f$ *is at least equal to the number of vertices of the Newton polytope* $\mathscr{N}_f$ *and at most equal to the total number of integer points in* $\mathscr{N}_f \cap \mathbb{Z}^n$.

Proof. The lower bound has already been obtained in [7] and [8]. It is a consequence of the fact that when ν is a vertex of $\mathscr{N}_f$ one can always find z so that the monomial $a_\nu z^\nu$ is dominating in f. By Proposition 2.7 the point $\log(z)$ will then lie in a component of ${}^c\mathscr{A}_f$ with order ν. Hence there must at least be one such component for each vertex. The upper bound follows immediately from Propositions 2.4 and 2.5. ∎

DEFINITION 2.9. A Laurent polynomial f is said to be *optimal* if its amoeba complement ${}^c\mathscr{A}_f$ has the same number of components as the number of integer points in the Newton polytope $\mathscr{N}_f$, that is, if the upper bound is attained in Theorem 2.8. We then also say that the hypersurface $\mathscr{Z}_f$ is optimal. Letting Γ be any face of $\mathscr{N}_f$ we shall say that f is Γ-optimal if for every integer point ν in the relative interior of Γ there is a component in ${}^c\mathscr{A}_f$ having the order ν.

Remark 2.10. Using the patchworking technique of Viro one can find an optimal Laurent polynomial for any given Newton polytope. See [7, Sect. 11.5] and the forthcoming paper "Real algebraic curves, moment map and amoebas" by Grisha Mikhalkin.

3. LAURENT DETERMINANTS

We are now going to study the various Laurent expansions of $1/f$ more carefully. In analogy with the study of period integrals on complex manifolds we are led to consider a matrix of Laurent coefficients and its determinant.

DEFINITION 3.1. Let Q be the maximal subset of $\mathcal{N}_f \cap \mathbb{Z}^n$ such that for each $\nu \in Q$ there is a component E_ν of ${}^c\mathcal{A}_f$ having order ν. To each pair $(\alpha, \nu) \in Q \times Q$ we associate the Laurent coefficient

$$c_{-\alpha,\nu} = \frac{1}{(2\pi i)^n} \int_{\log^{-1}(u_\nu)} \frac{z^\alpha}{f(z)} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}, \qquad u_\nu \in E_\nu .$$

The determinant $\det(c_{-\alpha,\nu})$ shall be called the *Laurent determinant* of f, and we denote it by Δ_f. Letting Q_Γ be the intersection of Q with the relative interior of a general face Γ of $\mathcal{N}_f$, we denote by $\Delta_{f,\Gamma}$ the minor of Δ_f obtained by letting (α, ν) run through $Q_\Gamma \times Q_\Gamma$.

Notice that the sign of Δ_f is well defined as long as we agree to use the same ordering of the points in Q for both indices α and ν. Let us next show that the Laurent determinant naturally factors as a product of the smaller determinants $\Delta_{f,\Gamma}$.

PROPOSITION 3.2. *Let Γ be a face of $\mathcal{N}_f$ of any dimension between zero and* $\dim \mathcal{N}_f$, *and take $\nu \in Q_\Gamma$. Then the Laurent coefficient $c_{-\alpha,\nu}$ vanishes for all $\alpha \in Q \setminus (\Gamma \cap \mathbb{Z}^n)$.*

Proof. Choose a primitive vector $s \in \mathbb{Z}^n$ in the relative interior of the dual cone C_ν. This means that there is a strict inequality $\langle s, \nu \rangle > \langle s, \alpha \rangle$ for each $\alpha \in Q \setminus (\Gamma \cap \mathbb{Z}^n)$. Performing a rotation of the amoeba as explained in Remark 2.3 we may rewrite our Laurent coefficient as

$$c_{-\alpha,\nu} = \frac{1}{(2\pi i)^n} \int_{\log^{-1}(v_\nu)} \frac{z^{S\alpha}}{S^* f(z)} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n},$$

where the unimodular matrix S has s as its first row, and $v_\nu = S^{-1} u_\nu$. To compute the integral we apply the Fubini theorem and consider, for fixed $z_2, ..., z_n$, the one-variable integral

$$\int_{|z_1| = e^{v_{\nu 1}}} \frac{z_1^{\langle s,\alpha\rangle - 1}\, dz_1}{f(c_1' z_1^{s_1}, ..., c_n' z_1^{s_n})} = e^{-v_{\nu 1}\langle s,\alpha\rangle} \int_{|w|=1} \frac{w^{\langle s,\alpha\rangle - 1}\, dw}{f(c_1 w^{s_1}, ..., c_n w^{s_n})},$$

where the c_j' are monomials in $z_2, ..., z_n$, and in the last integral we made the substitutions $w = e^{v_{\nu 1}} z_1$ and $c_j = e^{-s_1 v_{\nu 1}} c_j'$. The fact that $\langle s, \nu \rangle = \max_{\alpha \in \mathcal{N}_f} \langle s, \alpha \rangle$ implies, just as in the proof of Proposition 2.6, that the one-variable Laurent polynomial $w \mapsto f(cw^s)$ has all its zeros contained inside the unit circle. Hence the value of the above integral is equal to the residue at infinity of the rational function $w \mapsto w^{\langle s,\alpha\rangle - 1}/f(cw^s)$. But here the denominator is of degree $\langle s, \nu \rangle$, for almost all values of c, and this is at least two more then the degree of the numerator whenever α is in

$Q\backslash(\Gamma\cap\mathbb{Z}^n)$. The residue at infinity is thus necessarily equal to zero, and hence so is the Laurent coefficient $c_{-\alpha,\nu}$. ∎

It follows from Proposition 3.2 that the matrix $(c_{-\alpha,\nu})$ is block-triangular with respect to the partial ordering given by the inclusion of faces. This implies the following factorization result.

COROLLARY 3.3. *The Laurent determinant admits the factorization*

$$\Lambda_f=\prod_{\Gamma}\Lambda_{f,\Gamma},$$

where the product is taken over all faces Γ of $\mathcal{N}_f$, including the vertices and the polytope $\mathcal{N}_f$ itself.

EXAMPLE 3.4. When Γ is a vertex of $\mathcal{N}_f$, say $\Gamma=\alpha\in\mathbb{Z}^n$, it is easy to compute the (1×1)-minor $\Lambda_{f,\Gamma}=c_{-\alpha,\alpha}$ explicitly. Indeed, we may then choose the cycle of integration so that the monomial $a_\alpha z^\alpha$ dominates in f, and by the Rouché theorem we thus get

$$\begin{aligned}c_{-\alpha,\alpha}&=\frac{1}{(2\pi i)^n}\int_{\log^{-1}(u_\alpha)}\frac{z^\alpha}{f(z)}\frac{dz}{z}\\&=\frac{1}{(2\pi i)^n}\int_{\log^{-1}(u_\alpha)}\frac{z^\alpha}{a_\alpha z^\alpha}\frac{dz}{z}=\frac{1}{a_\alpha}.\end{aligned}$$

EXAMPLE 3.5. When $n=1$ and $\Gamma=\mathcal{N}_f$ one generically has the explicit formula $\Lambda_{f,\Gamma}^2=1/\mathcal{D}_f$, where $\mathcal{D}_f$ is the classical discriminant of f. To see this, write

$$f(z)=a_0+a_1z+\cdots+a_Nz^N=a_N(z-b^1)\cdots(z-b^N),$$

with $b^1,\ldots,b^N$ denoting the roots of f. Assume that the b^k all have different absolute values, so that f is Γ-optimal. In Section 7 below we will prove a general formula (Theorem 7.1) which will imply that $\Lambda_{f,\Gamma}=\pm 1/(a_N^{N-1}\prod_{j<k}(b^j-b^k))$. Since $a_N^{2N-2}\prod_{j<k}(b^j-b^k)^2=\mathcal{D}_f$, the claimed identity $\Lambda_{f,\Gamma}^2=1/\mathcal{D}_f$ follows. Notice also that the full Laurent determinant in this case equals $\pm 1/(a_0a_N^N\prod_{j<k}(b^j-b^k))$.

The following example illustrates that the square of the Laurent determinant Λ_f is not always equal to the reciprocal of a polynomial in the coefficients of f. However, it does remain rational and the denominator is a product of generalized discriminants. This resembles the structure of the squared determinant of the generalized period matrix studied by Varchenko, see [1, Theorem 12.2].

EXAMPLE 3.6. Take $n=2$ and consider a third degree polynomial f, which factors as a product of affine functions:

$$\begin{aligned}f(z) &= \sum_{j+k\leqslant 3} a_{jk} z_1^j z_2^k \\ &= (b_0^1+b_1^1z_1+b_2^1z_2)(b_0^2+b_1^2z_1+b_2^2z_2)(b_0^3+b_1^3z_1+b_2^3z_2)\end{aligned}$$

Under the assumption that f is optimal, Theorem 7.1 will allow us to evaluate the Laurent determinant Δ_f explicitely, and we find that its square equals

$$\begin{aligned}&(b_0^1b_1^1b_2^1b_0^2b_1^2b_2^2b_0^3b_1^3b_2^3)^{-2}\,(b_0^1b_1^2-b_1^1b_0^2)^{-2}\,(b_0^1b_1^3-b_1^1b_0^3)^{-2}\,(b_0^2b_1^3-b_1^2b_0^3)^{-2}\\ &\quad\times(b_0^1b_2^2-b_2^1b_0^2)^{-2}\,(b_0^1b_2^3-b_2^1b_0^3)^{-2}\,(b_0^2b_2^3-b_2^2b_0^3)^{-2}\,(b_1^1b_2^2-b_2^1b_1^2)^{-2}\\ &\quad\times(b_1^1b_2^3-b_2^1b_1^3)^{-2}\,(b_1^2b_2^3-b_2^2b_1^3)^{-2}\\ &\quad\times(b_0^1b_1^2b_2^3+b_1^1b_2^2b_0^3+b_2^1b_0^2b_1^3-b_2^1b_1^2b_0^3-b_0^1b_2^2b_1^3-b_1^1b_0^2b_2^3)^{-2},\end{aligned}$$

which expressed in the coefficients of f becomes $a_{11}/(a_{00}a_{30}a_{03})^2\ A_1(a)\ A_2(a)\ A_3(a)\ B(a)$, where

$$\begin{aligned}A_1(a) &= a_{10}^2a_{20}^2+18a_{00}a_{30}a_{10}a_{20}-4(a_{00}a_{20}^3+a_{30}a_{10}^3)-27a_{00}^2a_{30}^2,\\ A_2(a) &= a_{01}^2a_{02}^2+18a_{00}a_{03}a_{01}a_{02}-4(a_{00}a_{02}^3+a_{03}a_{01}^3)-27a_{00}^2a_{03}^2,\\ A_3(a) &= a_{21}^2a_{12}^2+18a_{30}a_{03}a_{21}a_{12}-4(a_{30}a_{12}^3+a_{03}a_{21}^3)-27a_{30}^2a_{03}^2\end{aligned}$$

and

$$\begin{aligned}B(a) = a_{11}^3 &- 4a_{11}(a_{10}a_{12}+a_{20}a_{02}+a_{01}a_{21})\\ &-12(a_{00}a_{21}a_{12}+a_{30}a_{01}a_{02}+a_{03}a_{10}a_{20}-a_{10}a_{02}a_{21}-a_{20}a_{01}a_{12})\\ &+108a_{00}a_{30}a_{03}.\end{aligned}$$

Here the Newton polytope $\mathcal{N}_f$ has one interior point, namely $(1,1)$, and $a_{11}/B(a)$ is the square of the scalar determinant $\Delta_{f,\Gamma}$, where $\Gamma=\mathcal{N}_f$. Notice that the identities in this example only hold for a_{jk} that satisfy algebraic equations expressing the fact that f is factorizable.

EXAMPLE 3.7. Again let $n=2$ and suppose f is a polynomial of the special form

$$f(z)=a_{00}+a_{30}z_1^3+a_{03}z_2^3+a_{11}z_1z_2.$$

Observe that the Newton polytope here is the same as in Example 3.6. If we assume that the mixed term $a_{11}z_1z_2$ is dominating as in Proposition 2.7, then a simple geometric series computation, see [2, Prop. 14.6], shows that the coefficient $c_{-(1,1),(1,1)} = \Lambda_{f,\Gamma}$, with $\Gamma = \mathcal{N}_f$, is equal to

$$\frac{1}{a_{11}} \sum_{k=0}^{\infty} \frac{(3k)!}{(k!)^3} \left(\frac{a_{00} a_{30} a_{03}}{a_{11}} \right)^k.$$

More generally, we can for any n consider the polynomial

$$f(z) = a_{00\cdots 0} + a_{(n+1)e_1} z_1^{n+1} + \cdots + a_{(n+1)e_n} z_n^{n+1} + a_{11\cdots 1} z_1 z_2 \cdots z_n,$$

where e_j denotes the jth standard basis vector in $\mathbb{R}^n$. The Laurent determinant corresponding to the face $\Gamma = \mathcal{N}_f$, with the single interior integer point $(1, 1, ..., 1)$, will then be

$$a_{11\cdots 1}^{-1} \sum_{k \geqslant 0} ((n+1)\,k)!\; z^k/(k!)^{n+1},$$

with $z = a_{00\cdots 0} a_{(n+1)e_1} \cdots a_{(n+1)e_n} / a_{11\cdots 1}$. In particular, for $n = 4$ we encounter the special hypergeometric function that plays such a crucial role in the mirror symmetry of quintic Calabi–Yau threefolds.

4. THE AMOEBA OF A HYPERPLANE

In this section we study in some detail the simplest kind of amoebas, namely the images of hyperplanes in $\mathbb{P}^n$. That is, we let f be a first-order polynomial $b_0 + b_1 z_1 + \cdots + b_n z_n$, with at least one coefficient b_j different from zero. It will be advantageous to consider a compactified amoeba $\bar{\mathcal{A}}_f$, obtained as the image of the entire projective hyperplane. With respect to homogeneous coordinates $(Z_0, Z_1, ..., Z_n) \in \mathbb{C}^{n+1} \backslash \{0\}$ the hyperplane is defined by the homogeneous linear equation $\langle b, Z \rangle = 0$. Let Σ_n denote the closed standard simplex in $\mathbb{R}^{n+1}$, that is, the intersection of the closed first octant with the hyperplane $\{t_0 + t_1 + \cdots + t_n = 1\}$. The vertices of Σ_n coincide with the standard basis vectors $e_0, e_1, ..., e_n$ in $\mathbb{R}^{n+1}$. The natural homogeneous mapping $\mathbb{C}^{n+1} \backslash \{0\} \to \Sigma_n$ defined by

$$(Z_0, ..., Z_n) \mapsto \frac{(|Z_0|, ..., |Z_n|)}{|Z_0| + \cdots + |Z_n|}$$

then descends to a mapping $\tau \colon \mathbb{P}^n \to \Sigma_n$, sometimes referred to as a moment map. The image under τ of the torus $\mathbb{T}^n \subset \mathbb{P}^n$ is the interior of Σ_n, which is homeomorphic to $\mathbb{R}^n$. An explicit such homeomorphism given by

$$(t_0, ..., t_n) \mapsto \left(\log \frac{t_1}{t_0}, ..., \log \frac{t_n}{t_0} \right),$$

and composing τ with this homeomorphism we recover our mapping log: $\mathbb{T}^n \to \mathbb{R}^n$.

DEFINITION 4.1. Let f be a first-order polynomial $b_0 + b_1 z_1 + \cdots + b_n z_n$. The image of the corresponding projective hyperplane $\langle b, Z \rangle = 0$ under the mapping $\tau \colon \mathbb{P}^n \to \Sigma_n$ is called the *compactified amoeba* of f, and we denote it by $\bar{\mathscr{A}}_f$.

A nice feature of the compactified hyperplane amoeba is that it is a compact convex polytope in the standard simplex Σ_n.

PROPOSITION 4.2. *The compactified amoeba $\bar{\mathscr{A}}_f$ of a first order polynomial $f = b_0 + b_1 z_1 + \cdots + b_n z_n$ is equal to the convex polytope in Σ_n defined by the linear inequalities*

$$\beta_j t_j \leqslant \sum_{k \neq j} \beta_k t_k, \qquad j = 0, ..., n, \tag{1}$$

where β_j denotes the absolute value $|b_j|$. If $\beta_j + \beta_k \neq 0$, then the point $v_{jk} = (t_0, ..., t_n)$ with

$$t_j = \frac{\beta_k}{\beta_j + \beta_k}, \qquad t_k = \frac{\beta_j}{\beta_j + \beta_k}, \qquad t_l = 0, \qquad l \neq j, k,$$

belongs to $\bar{\mathscr{A}}_f$, and in fact $\bar{\mathscr{A}}_f$ coincides with the convex hull of these extreme points v_{jk}.

Proof. Suppose that $t \in \bar{\mathscr{A}}$. This means that we can find arguments θ_j so that the sum $\sum_{j=0}^{n} \beta_j e^{i\theta_j} t_j$ vanishes, and hence

$$\beta_j t_j = \left| \sum_{k \neq j} \beta_k e^{i\theta_k} t_k \right| \leqslant \sum_{k \neq j} \beta_k t_k.$$

Conversely, to prove that (1) is also sufficient for t to lie in $\bar{\mathscr{A}}$, we must show that arguments θ_j can be chosen so as to make the sum $\beta_0 t_0 e^{i\theta_0} + \beta_1 t_1 e^{i\theta_1} + \cdots + \beta_n t_2 e^{i\theta_n}$ equal to zero. Letting v denote the number of terms $\beta_j t_j$ which are nonzero, we see that our task is trivial for $v = 0$, whereas the case $v = 1$ is ruled out by (1). The next case $v = 2$ is also obvious, for then the system (1) reduces to a single equation $\beta_j t_j = \beta_k t_k$ and we can take

$\theta_j = 0$, $\theta_k = \pi$. When $v > 2$ we reorder the terms so that $\beta_n t_n$ is the largest term and then let m be the largest integer that satisfies

$$\sum_{j=0}^{m-1} \beta_j t_j < \sum_{j=m}^{n} \beta_j t_j.$$

We necessarily have $1 \leqslant m \leqslant n-1$, so we may consider the three positive numbers

$$a = \beta_0 t_0 + \cdots + \beta_{m-1} t_{m-1}, \qquad b = \beta_m t_m, \quad c = \beta_{m+1} t_{m+1} + \cdots + \beta_n t_n.$$

Recalling (1) one verifies the inequalities $a \leqslant b + c$, $b \leqslant a + c$ and $c \leqslant a + b$, so a, b, c are in fact the side lengths of a triangle, and thus

$$\beta_0 t_0 e^{i\theta_0} + \cdots + \beta_{m-1} t_{m-1} e^{i\theta_0} + \beta_m t_m e^{i\theta_1} + \beta_{m+1} t_{m+1} e^{i\theta_2} + \cdots + \beta_n t_n e^{i\theta_2} = 0$$

for a suitable choice of θ_0, θ_1, θ_2.

A straightforward inspection shows that the coordinates of v_{jk} satisfy the inequalities (1), and $\bar{\mathscr{A}}_f$ being convex, it follows that the convex hull of the points v_{jk} is contained in $\bar{\mathscr{A}}_f$.

It remains to be shown that all extreme points of $\bar{\mathscr{A}}_f$ are among the v_{jk}. Consider the 1-skeleton of Σ_n. It is the union of all edges e_{jk} connecting any two vertices e_j and e_k. If $\beta_j + \beta_k \neq 0$ then $\bar{\mathscr{A}}_f$ intersects e_{jk} precisely in the single point v_{jk}. If $\beta_j + \beta_k = 0$ then (1) contains no condition on t_j and t_k, so the entire edge e_{jk} is contained in $\bar{\mathscr{A}}_f$, but then $e_j = v_{jl}$ and $e_k = v_{kl}$ for any l such that $\beta_l \neq 0$. So in either case the extreme points of $e_{jk} \cap \bar{\mathscr{A}}_f$ are among the v_{jk}, and hence it will now suffice to show that all extreme points of $\bar{\mathscr{A}}_f$ lie in the 1-skeleton of Σ_n. Any point $t \in \bar{\mathscr{A}}_f$ not in the 1-skeleton must have three nonzero coordinates, say t_j, t_k, $t_l > 0$. If one of the corresponding coordinates of β vanishes, say $\beta_j = 0$, then (1) involves no condition on t_j. It follows that t lies on a line segment in $\bar{\mathscr{A}}_f$ from the vertex e_j to a point t' with $t'_j = 0$, and hence t is not an extreme point. On the other hand, if $\beta_j t_j$, $\beta_k t_k$ and $\beta_l t_l$ are all positive, then equality can hold in at most one of the inequalities (1), for this particular point t. This shows that t lies in the relative interior of the intersection of $\bar{\mathscr{A}}_f$ with a hyperplane, so again it cannot be an extreme point. We conclude that the v_{jk} comprise all extreme points of $\bar{\mathscr{A}}_f$, which is therefore equal to the convex hull of the v_{jk} as claimed. ∎

Applying the homeomorphism $(t_0, ..., t_n) \mapsto (\log(t_1/t_0), ..., \log(t_n/t_0))$ we now easily deduce the inequalities that define the amoeba $\mathscr{A}_f$ in $\mathbb{R}^n$. Notice that if the hyperplane is of the very special form $Z_j = 0$ for some j with $0 \leqslant j \leqslant n$, then it does not intersect the torus $\mathbb{T}^n$, so its amoeba A_f is actually empty. This happens precisely when the compactified amoeba is equal to the corresponding face $t_j = 0$ of Σ_n.

COROLLARY 4.3. *The amoeba $\mathscr{A}_f$ of an affine function $f(z) = b_0 + b_1 z_1 + \cdots + b_n z_n$ is equal to the closed (possibly empty) subset in $\mathbb{R}^n$ defined by the inequalities*

$$\log \beta_0 \leqslant \log \left(\sum_{k=1}^{n} \beta_k e^{u_k} \right)$$

$$u_j + \log \beta_j \leqslant \log \left(\beta_0 + \sum_{k \neq j} \beta_k e^{u_k} \right), \qquad j = 1, ..., n,$$

where $\beta_k = |b_k|$.

The following observation is also a direct consequence of Proposition 4.2.

COROLLARY 4.4. *A compactified hyperplane amoeba $\bar{\mathscr{A}}_f$ always intersects every edge e_{jk} in the 1-skeleton of the simplex Σ_n. If e_j and e_k are two different vertices of Σ_n not contained in $\bar{\mathscr{A}}_f$, then they necessarily lie in different connected components of the complement $\Sigma_n \backslash \bar{\mathscr{A}}_f$.*

We thus see that an affine polynomial f is optimal if and only if its compactified amoeba does not contain any vertex e_j of Σ_n. This means exactly that all coefficients b_j are nonzero. This is a special instance of Proposition 6.3 below.

COROLLARY 4.5. *Let the first-order polynomial f be optimal. Then the complement $\Sigma_n \backslash \bar{\mathscr{A}}_f$ consists of $n+1$ disjoint simplices, each containing one of the vertices e_j.*

Even though an optimal hyperplane amoeba $\mathscr{A}_f \in \mathbb{R}^n$ is itself not convex when $n > 1$, it does contain $(n-1)$-dimensional convex cones. Actually, one can find $n(n+1)/2$ such cones contained in $\mathscr{A}_f$, so that their union divides $\mathbb{R}^n$ into $n+1$ connected components. We call this union the *spine* of the amoeba $\mathscr{A}_f$.

PROPOSITION 4.6. *Let $\mathscr{A}_f$ be the amoeba of an optimal affine polynomial $b_0 + b_1 z_1 + \cdots + b_n z_n = 0$, that is $\beta_j = |b_j| \neq 0$ for every j. Then $\mathscr{A}_f$ contains the union of the $n(n+1)/2$ convex $(n-1)$-dimensional cones*

$$\{u \in \mathbb{R}^n;\ L_j(u) = L_k(u) \geqslant L_l(u),\ l \neq j, k\}, \tag{2}$$

for $0 \leqslant j < k \leqslant n$. Here $L_0(u) = \log \beta_0$ and $L_j(u) = u_j + \log \beta_j$, $j = 1, ..., n$. The point $c = (\log(\beta_0/\beta_1), ..., \log(\beta_0/\beta_n))$ is the common vertex of all these cones.

Proof. It is clear that each system (2) defines an $(n-1)$-dimensional convex cone with vertex at c, namely the convex hull of the $n-1$ rays from c through $c+p_l$, $l \neq j, k$, where

$$p_0 = (1, 1, ..., 1), \; p_1 = (-1, 0, ..., 0), ..., \; p_n = (0, ..., 0, -1).$$

Checking with the inequalities in Corollary 4.3 one sees that all these cones are contained in the amoeba $\mathscr{A}_f$. ▮

5. ARRANGEMENTS OF FEW HYPERPLANE AMOEBAS

Having understood the structure of a single hyperplane amoeba we now turn to the case of a union, or arrangement, of such amoebas. In other words, we consider amoebas defined by products $\langle b^1, Z\rangle \cdots \langle b^N, Z\rangle$ of linear polynomials. Here each b^j denotes a vector $(b^j_0, ..., b^j_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. In this section we restrict our attention to the case where N, the number of hyperplanes, does not exceed $n+1$. First, we observe that fewer than $n+1$ hyperplane amoebas always have a common intersection, at least if we compactify them.

PROPOSITION 5.1. *The intersection of n compactified hyperplane amoebas $\bar{\mathscr{A}}^1, ..., \bar{\mathscr{A}}^n$ is never empty in Σ_n.*

Proof. Since any linear mapping $A: \mathbb{C}^{n+1} \to \mathbb{C}^n$ must necessarily have a nontrivial kernel, we se that the system

$$\begin{cases} b^1_0 Z_0 + \cdots + b^1_n Z_n = 0 \\ \cdot \\ \cdot \\ b^n_0 Z_0 + b^n_n Z_n = 0 \end{cases}$$

has a nonzero solution $Z \in \mathbb{C}^{n+1}$. But then $\tau(Z) \in \bar{\mathscr{A}}^1 \cap \cdots \cap \bar{\mathscr{A}}^n$. ▮

When we increase the number of amoebas to $n+1$ there need not be a common intersection any longer. But it is not enough to assume that the hyperplanes $\{\langle b^j, Z\rangle = 0\}$, $j = 0, ..., n$, have no point in common. Indeed, this amounts to the condition $\det B \neq 0$, where B is the square matrix with rows $b^0, ..., b^n$. The following result shows that a stronger determinant condition is required.

PROPOSITION 5.2. *The intersection of $n+1$ compactified hyperplane amoebas $\bar{\mathscr{A}}^0, \ldots, \bar{\mathscr{A}}^n$ is empty in Σ_n if and only if the determinant*

$$\det B(\theta) = \begin{vmatrix} b_0^0 e^{i\theta_0^0} & \cdots & b_n^0 e^{i\theta_n^0} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_0^n e^{i\theta_0^n} & \cdots & b_n^n e^{i\theta_n^n} \end{vmatrix}$$

is nonzero for any choice of arguments $\theta_k^j \in [0, 2\pi]$.

Proof. Assume first that the intersection is not empty, and take $t \in \bar{\mathscr{A}}^0 \cap \cdots \cap \bar{\mathscr{A}}^n$. This means that for each j we can find an argument vector $\theta^j \in [0, 2\pi]^{n+1}$ so that

$$t(\theta^j) = (e^{i\theta_0^j} t_0, \ldots, e^{i\theta_n^j} t_n) \in \mathbb{C}^{n+1} \backslash \{0\}$$

satisfies $\langle b^j, t(\theta^j) \rangle = 0$, $j = 0, \ldots, n$. But these equations may be written as the matrix identity $B(\theta)\, t = 0$, and since $t \neq 0$ we conclude that $\det B(\theta) = 0$.

Suppose now that $\det B(\theta) = 0$ for some choice of arguments θ_k^j, and take a corresponding $Z \in \mathbb{C}^{n+1} \backslash \{0\}$ with $B(\theta)\, Z = 0$. Then we have $\langle b^j, Z(\theta^j) \rangle = 0$, $j = 0, \ldots, n$, and it follows that the vector $t = \tau(Z(\theta^j)) = \tau(Z)$ belongs to each amoeba $\bar{\mathscr{A}}^j$. Hence the intersection $\bar{\mathscr{A}}^0 \cap \cdots \cap \bar{\mathscr{A}}^n$ is not empty. ▮

For us it will be important to know when the complement of an arrangement of hyperplane amoebas contains a connected component *in the interior* of Σ_n. This is because components that intersect the boundary of Σ_n may be studied by considering lower-dimensional amoebas $\bar{\mathscr{A}}^j \cap \{t_k = 0\}$ instead. Our next result shows that $n+1$ compactified amoebas without common intersection determine one unique such interior complement component.

PROPOSITION 5.3. *If the intersection of $n+1$ compactified amoebas $\bar{\mathscr{A}}^0, \ldots, \bar{\mathscr{A}}^n$ is empty in Σ_n, then their complement $\Sigma_n \backslash (\bar{\mathscr{A}}^0 \cup \cdots \cup \bar{\mathscr{A}}^n)$ contains a unique connected component not intersecting the boundary $\partial \Sigma_n$. This component is an open simplex, and there is in fact a unique permutation σ such that it is defined by the inequalities*

$$\begin{cases} \beta_0^{\sigma(0)} t_0 > \beta_1^{\sigma(0)} t_1 + \cdots + \beta_n^{\sigma(0)} t_n \\ \beta_1^{\sigma(1)} t_1 > \beta_0^{\sigma(1)} t_0 + \beta_2^{\sigma(1)} t_2 + \cdots + \beta_n^{\sigma(1)} t_n \\ \cdot \\ \cdot \\ \beta_n^{\sigma(n)} t_n > \beta_0^{\sigma(n)} t_0 + \cdots + \beta_{n-1}^{\sigma(n)} t_{n-1} \end{cases} \qquad . \tag{3}$$

Proof. For each j, $k = 0, \ldots, n$ we consider the convex compact subset $K^j_k \subset \Sigma_n$ defined by

$$K^j_k = \{t \in \Sigma_n;\, \beta^j_k t_k \leqslant \beta^j_0 t_0 + \cdots \hat{k} \cdots + \beta^j_n t_n\}.$$

All the K^j_k are contained in the n-dimensional affine space $\{t_0 + t_1 + \cdots + t_n = 1\}$, and by hypothesis they have no common point of intersection. By Helly's theorem (see [4]) there must then be some collection of $n+1$ sets K^j_k having an empty intersection. Now, if we pick $n+1$ of the sets K^j_k in such a way that some upper index j never occurs, then their intersection will contain the intersection of the n amoebas $\bar{\mathscr{A}}^0, \ldots \hat{j} \ldots, \bar{\mathscr{A}}^n$, which by Proposition 5.1 is nonempty. On the other hand, if the $n+1$ sets are chosen so that some lower index k is missing, then the vertex e_k of Σ_n is readily seen to satisfy all the corresponding inequalities, and again the intersection is not empty. We conclude that there must be some permutation σ for which the system

$$\begin{cases} \beta^{\sigma(0)}_0 t_0 \leqslant \beta^{\sigma(0)}_1 t_1 + \cdots + \beta^{\sigma(0)}_n t_n \\ \beta^{\sigma(1)}_1 t_1 \leqslant \beta^{\sigma(1)}_0 t_0 + \beta^{\sigma(1)}_2 t_2 + \cdots + \beta^{\sigma(1)}_n t_n \\ \cdot \\ \cdot \\ \beta^{\sigma(n)}_n t_n \leqslant \beta^{\sigma(n)}_0 t_0 + \cdots + \beta^{\sigma(n)}_{n-1} t_{n-1} \end{cases} \tag{4}$$

has no solution in Σ_n.

Let us show that this permutation σ is unique. Indeed, assume that the system (4′) corresponding to some other permutation σ' also lacks a solution in Σ_n. Fix j and suppose that $\sigma(j) = m = \sigma'(i)$. Consider the nonempty convex intersection K of the n amoebas $\bar{\mathscr{A}}^l$, where l runs over all indices different from m. Since (4) has no solution K must lie in the complement component of $\bar{\mathscr{A}}^m$ given by

$$\beta^m_j t_j > \beta^m_0 t_0 + \cdots \hat{j} \cdots + \beta^m_n t_n.$$

Similarly, since (4′) has no solution each point of K must satisfy the inequality

$$\beta^m_i t_i > \beta^m_0 t_0 + \cdots \hat{\imath} \cdots + \beta^m_n t_n.$$

However, these two strict inequalities are incompatible unless $j = i$. Since j was arbitrary, we find that $\sigma = \sigma'$, and the uniqueness of σ is thereby proven.

Our next claim is that the fact that (4) has no solution in Σ_n implies that the reversed system (3) does have solutions in Σ_n. Letting F_j be the intersection between Σ_n and the closed half-space

$$\beta^{\sigma(j)}_j t_j \leqslant \beta^{\sigma(0)}_0 t_0 + \cdots \hat{j} \cdots + \beta^{\sigma(n)}_n t_n,$$

we see that F_j contains the entire face $\{t_j=0\}$. Our claim is therefore a consequence of Lemma 5.4 below. Since (3) clearly defines an open simplex contained in the interior of Σ_n, the proof is completed. ∎

LEMMA 5.4. *Let Σ be a nondegenerate closed simplex in $\mathbb{R}^n$, with vertices $v_0, ..., v_n$. For $j=0, ..., n$, let F_j be the intersection of Σ with a closed half-space containing the face opposite to v_j. Then, if the union $F_0 \cup \cdots \cup F_n$ covers Σ, the intersection $F_0 \cap \cdots \cap F_n$ is not empty.*

Proof. We argue by induction on the dimension n. The result is obvious for $n=1$, so we let n be arbitrary, and suppose the lemma has been proven for all dimensions $<n$. If F_0 contains the vertex v_0 then this vertex belongs to all the sets F_j and we are done. Assume therefore that v_0 is not contained in F_0, and let v_{0k} be the unique point of intersection between the edge from v_0 to v_k and the hyperplane that defines F_0. Let Σ' be the closed simplex with vertices v_0 and $v_{01}, ..., v_{0n}$ and let Σ'' be the closed $(n-1)$-dimensional simplex with vertices $v_{01}, ..., v_{0n}$. Then $\Sigma \backslash F_0 = \Sigma' \backslash \Sigma''$. Since the closed set $F_1 \cup \cdots \cup F_n$ covers $\Sigma \backslash F_0$ it must also cover its closure Σ'. Hence, the sets $F_j'' = F_j \cap \Sigma''$ together cover Σ''. Since F_j'' contains the face of Σ'' which is opposite to v_{0j}, we conclude from the induction hypothesis that $F_1 \cap \cdots \cap F_n \cap \Sigma''$ is nonempty. But Σ'' is a subset of F_0 and the lemma follows. ∎

We end this section by computing certain residue integrals associated to an arrangement of n hyperplane amoebas. This result will be used in Section 7.

PROPOSITION 5.5. *Let $\mathscr{A} = \mathscr{A}^1 \cup \cdots \cup \mathscr{A}^n$ be an arrangement of amoebas corresponding to the hyperplanes $L^k(z) = b_0^k + b_1^k z_1 + \cdots + b_n^k z_n = 0$, where $k=1, ..., n$. The integral*

$$\frac{1}{(2\pi i)^n} \int_{\log^{-1}(u)} \frac{dL^1(z) \wedge \cdots \wedge dL^n(z)}{L^1(z) \cdots L^n(z)}$$

is then equal to one if the point u belongs to the component of ${}^c\mathscr{A}$ having order $v=(1, 1, ..., 1)$. It is equal to zero if u lies in any other component of ${}^c\mathscr{A}$.

Proof. Since the order v of any component of ${}^c\mathscr{A}$ must have non-negative entries and satisfies $|v| \leqslant n$, we see that if $v \neq (1, 1, ..., 1)$ then $v_j = 0$ for some j. This means that the cycle $\log^{-1}(u)$ is homologous to zero in $\mathbb{C}^n \backslash \{L^1 \cdots L^n = 0\}$, and hence the integral is zero then. On the other hand, if $v=(1, 1, ..., 1)$, then for each j there is precisely one L^k in which the term $b_j^k z_j$ dominates (in the sense that (1) is violated). We may then explicitly calculate the integral $(1/2\pi i)^n \int_{\log^{-1}(u)} dz_1 \wedge \cdots \wedge dz_n / L^1(z) \cdots L^n(z)$ by applying Fubini's theorem, using the fact that at each step there will only be *one* pole inside the circle of integration. ∎

This result is a very special instance of a general method for reducing integrals of rational functions to (sums of) local residues. It is called the method of separating cycles, see [10, pp. 46–55].

6. GENERAL ARRANGEMENTS OF HYPERPLANE AMOEBAS

In this section we consider arrangements where the number N of hyperplane amoebas is allowed to be arbitrarily large. Our main objective is to understand the structure of the complement $\mathbb{R}^n \setminus (\mathscr{A}^1 \cup \cdots \cup \mathscr{A}^N)$, in particular to find all its connected components. We know by Proposition 1.2 that the number of components vary in an upper semicontinuous fashion, and we now want to find a condition which guarantees that the amoeba arrangement is optimal in the sense that the maximal number $\binom{N+n}{n}$ of components is attained.

Definition 6.1. An arrangement $\bar{\mathscr{A}}^1 \cup \cdots \cup \bar{\mathscr{A}}^N$ of compactified hyperplane amoebas is said to be *in optimal position* if no intersection $\bar{\mathscr{A}}^{k_1} \cap \cdots \cap \bar{\mathscr{A}}^{k_m}$ meets the $(m-1)$-skeleton of the simplex Σ_n. We then also say that the corresponding arrangement $\mathscr{A}^1 \cup \cdots \cup \mathscr{A}^N \subset \mathbb{R}^n$ is in optimal position.

Notice that for $N=1$ this definition agrees with the one we used in Section 4 above. For $m \geqslant n+1$ the condition means that the intersection $\bar{\mathscr{A}}^{k_1} \cap \cdots \cap \bar{\mathscr{A}}^{k_m}$ should be empty. The notion of optimal position may also be interpreted as a condition on the coefficients of the various linear forms $\langle b^k, Z \rangle$ that define the amoebas. In order to formulate this condition we first introduce a useful polynomial.

Definition 6.2. Given a $p \times q$-matrix B with entries b_j^k, $j=1, ..., p$, $k=1, ..., q$, we let $\Delta(B)$ denote the product of all subdeterminants of B. That is,

$$\Delta(B) = \prod_{l=1}^{\min\{p, q\}} \prod_{|J|=|K|=l} \Delta_{JK},$$

where $J=(j_1, ..., j_l)$, $K=(k_1, ..., k_l)$, with $j_1 < j_2 < \cdots < j_l$; $k_1 < k_2 < \cdots < k_l$, and Δ_{JK} is the determinant

$$\begin{vmatrix} b_{j_1}^{k_1} & \cdots & b_{j_l}^{k_1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{j_1}^{k_l} & \cdots & b_{j_l}^{k_l} \end{vmatrix}.$$

If the matrix B is regarded as a vector in $\mathbb{C}^{pq}\setminus\{0\}$ with variable coordinates b_j^k, then $\Delta(B)$ is a homogeneous polynomial of degree

$$\prod_{l=1}^{\min\{p,q\}} l\binom{p}{l}\binom{q}{l}.$$

In particular it defines an algebraic hypersurface in $\mathbb{P}^{pq-1}$.

It is clear that the condition $\Delta(B)\neq 0$ is equivalent to the hyperplanes $V^1, ..., V^N$ being in general position in $\mathbb{T}^n$, but in order to have the amoebas in optimal position we need to require more.

Proposition 6.3. *Consider, for $j=1, ..., N$, the hyperplanes $V^j=\{\langle b^j, z\rangle=0\}$ and the corresponding amoebas $\mathscr{A}^j$. Let B be the $(n+1)\times N$-matrix with rows $b^1, ..., b^N$, and regard B as a vector in $\mathbb{C}^{(n+1)N}\setminus\{0\}$, so that $\log(B)$ is a point in $\mathbb{R}^{(n+1)N-1}$. The amoeba arrangement $\mathscr{A}=\mathscr{A}^1\cup\cdots\cup\mathscr{A}^N$ is then in optimal position if and only if $\log(B)$ does not belong to the amoeba of the polynomial function $B\mapsto\Delta(B)$ defined above.*

Proof. Take $m\leqslant n$ and consider a typical intersection $\bar{\mathscr{A}}^{j_1}\cap\cdots\cap\bar{\mathscr{A}}^{j_m}\cap\{t_{k_0}=\cdots=t_{k_{n-m}}=0\}$. Use Proposition 5.2. ∎

Definition 6.4. Let $\bar{\mathscr{A}}=\bar{\mathscr{A}}^1\cup\cdots\cup\bar{\mathscr{A}}^N$ be an arrangement of compactified hyperplane amoebas, and let E be a connected component of the complement $\Sigma_n\setminus\mathscr{A}$. The *index* of E is then defined as

$$\iota=(\iota_1, ..., \iota_N)\in\{0, ..., n\}^N,$$

with $\iota_k=j$ if E is in the same component of $\Sigma_n\setminus\bar{\mathscr{A}}^k$ as the vertex e_j of Σ_n.

We let the order of a connected component of the complement of a compactified amoeba be the order of the corresponding component in the complement of the corresponding usual amoeba.

Proposition 6.5. *Let $\bar{\mathscr{A}}=\bar{\mathscr{A}}^1\cup\cdots\cup\bar{\mathscr{A}}^N$ be an arrangement of compactified hyperplane amoebas, and let E be a connected component of the complement $\Sigma_n\setminus\bar{\mathscr{A}}$. The order of E is then equal to*

$$\nu=(\nu_1, ..., \nu_n)\in\mathbb{N}^n,$$

with $\nu_k=\#\{j;\,\iota_j=k\}$.

Proof. The logarithmic derivative $\partial_j f/f$ is equal to the sum $\sum_j b_k^j/\langle b^j, z\rangle$, and a simple residue computation shows that

$$\int_{|z_k|=t_k} \frac{b_k^j}{\langle b^j, z\rangle}\, dz_k = \begin{cases} 2\pi i, & \text{if } \quad \iota_j = k, \\ 0, & \text{if } \quad \iota_j \neq k, \end{cases}.$$

This result is independent of the value of the other variables z_j, $j \neq k$, and hence the result follows by Fubini's theorem. ▮

THEOREM 6.6. *Let $\bar{\mathscr{A}} = \bar{\mathscr{A}}^1 \cup \cdots \cup \bar{\mathscr{A}}^N$ be an arrangement of compactified hyperplane amoebas in optimal position. Then the number of connected components of the complement $\Sigma_n \backslash \bar{\mathscr{A}}$ is equal to $\binom{N+n}{n}$.*

Proof. Let $L^k(z) = \langle b^k, z\rangle$, $k = 1, ..., N$, be the linear form corresponding to the hyperplane amoeba $\bar{\mathscr{A}}^k$, and add the n exceptional forms $L^{N+j}(z) = z_j$, $j = 1, ..., n$, with compactified amoebas $\bar{\mathscr{A}}^{N+j} = \{t_j = 0\}$. Now choose any subset $K = \{k_1, ..., k_n\} \subset \{1, ..., N+n\}$. This means we have picked n of the forms $L^1, ..., L^{N+n}$ and we now consider the corresponding lower dimensional amoebas $\bar{\mathscr{B}}^{k_j} = \bar{\mathscr{A}}^{k_j} \cap \{t_0 = 0\}$. By the optimal position assumption the intersection $\bar{\mathscr{B}}^{k_1} \cap \cdots \cap \bar{\mathscr{B}}^{k_n}$ is empty, so by Proposition 5.3 the complement of their union, which lies in the $(n-1)$-simplex $\Sigma_n \cap \{t_0 = 0\}$, contains a unique interior component. More precisely, there is a unique permutation σ of $\{1, 2, ..., n\}$ such that the system

$$\begin{cases} \beta_1^{k_{\sigma(1)}} t_1 > \beta_2^{k_{\sigma(1)}} t_2 + \cdots + \beta_n^{\sigma_{\sigma(1)}} t_n \\ \beta_2^{k_{\sigma(2)}} t_2 > \beta_1^{k_{\sigma(2)}} t_1 + \beta_3^{k_{\sigma(2)}} t_3 + \cdots + \beta_n^{k_{\sigma(2)}} t_n \\ \cdot \\ \cdot \\ \beta_n^{k_{\sigma(n)}} t_n > \beta_1^{k_{\sigma(n)}} t_1 + \cdots + \beta_{n-1}^{k_{\sigma(n)}} t_{n-1} \end{cases} \tag{6}$$

has solutions in $\Sigma_n \cap \{t_0 = 0\}$. The corresponding system of equations

$$\begin{cases} \beta_1^{k_{\sigma(1)}} t_1 = \beta_0^{k_{\sigma(1)}} t_0 + \beta_2^{k_{\sigma(1)}} t_2 + \cdots + \beta_n^{k_{\sigma(1)}} t_n \\ \beta_2^{k_{\sigma(2)}} t_2 = \beta_0^{k_{\sigma(2)}} t_0 + \beta_1^{k_{\sigma(2)}} t_1 + \cdots + \beta_n^{k_{\sigma(2)}} t_n \\ \cdot \\ \cdot \\ \beta_n^{k_{\sigma(n)}} t_n = \beta_0^{k_{\sigma(n)}} t_0 + \cdots + \beta_{n-1}^{k_{\sigma(n)}} t_{n-1} \end{cases}$$

then has a unique solution $(t_0, t_1, ..., t_n) \in \bar{\mathscr{A}}^{k_1} \cap \cdots \cap \bar{\mathscr{A}}^{k_n}$. Let us call this point t^K. Since $n+1$ amoebas never intersect we have $t^K \neq t^{K'}$ whenever $K \neq K'$, so the number of different points t^K is equal to $\binom{N+n}{n}$. Together with the simplex in $\{t_0 = 0\}$ defined by (6), the point t^K spans an open

n-simplex E^K, which is one of the connected components of $\Sigma_n \setminus (\bar{\mathscr{A}}^{k_1} \cap \cdots \cap \bar{\mathscr{A}}^{k_n})$. Now, all points in E^K sufficiently close to the vertex t^K will also lie in a certain connected component of the complement $\Sigma_n \setminus \bar{\mathscr{A}}$. What remains is to show that different points t^K give rise to different components of $\Sigma_n \setminus \bar{\mathscr{A}}$. It is enough to verify that if $K \neq K'$ then the component at t^K cannot have the same index $\imath$ as the component at $t^{K'}$, for the components are uniquely determined by their indices. The idea is that there must be some amoeba $\bar{\mathscr{A}}^j$ lying between t^K and $t^{K'}$. Indeed, if there are points $u^K \in E^K$ and $u^{K'} \in E^{K'}$, having identical index $\imath$, and lying arbitrarily close to t^K and $t^{K'}$ respectively, then t^K must belong to the closure $\bar{E}^{K'}$ and vice versa. On the other hand, the vertex t^K is the unique point in $\bar{E}^K$ which satisfies $t_0 \geqslant t_0^K$, and similarly $t^{K'}$ maximizes the value of t_0 in $\bar{E}^{K'}$. Without loss of generality we can assume $t_0^{K'} \geqslant t_0^K$, from which it now follows that $t^{K'} = t^K$, and hence $K' = K$. ∎

We end this section with a stronger version of Theorem 6.6, where one only assumes optimal position with respect to one face of the Newton polytope. Of course one can then only get the existence of complement components of the corresponding kind.

THEOREM 6.7. *Let $\bar{\mathscr{A}} = \bar{\mathscr{A}}^1 \cup \cdots \cup \bar{\mathscr{A}}^N$ be an arrangement of compactified hyperplane amoebas. Fix a subset $J \subset \{0, 1, ..., n\}$ and denote by m its cardinality. Let e_J be the corresponding face of the simplex Σ_n, that is, the convex hull of the vertices $e_{j_1}, ..., e_{j_m}$, and assume that no intersection $\bar{\mathscr{A}}^{k_1} \cap \cdots \cap \bar{\mathscr{A}}^{k_m}$ of m compactified hyperplane amoebas intersects e_J. Then, for each integer vector v in the relative interior of the corresponding face $\mathscr{N}_J$ of the Newton polytope, there is a (unique) connected component of the complement $\Sigma_n \setminus \bar{\mathscr{A}}$ of order v.*

Proof. Notice first that when $m = 1$ the conclusion clearly holds, for the vertex e_{j_1} will then lie in a component of $\Sigma_n \setminus \bar{\mathscr{A}}$ having an order that coincides with the corresponding vertex of the Newton polytope $\mathscr{N}_f$. So from now on we assume that $m \geqslant 2$.

By intersecting each amoeba $\bar{A}^k$ with e_J, that is, setting $t_i = 0$ for all $i \notin J$, we obtain a similar problem in dimension $m - 1$. We may therefore without loss of generality assume that $m = n + 1$ to start with. We can certainly also assume that $N > n$, for otherwise there will be no integer points in the interior of $\mathscr{N}_f$. What we must prove is now that there are $\binom{N-1}{n}$ components of $\Sigma_n \setminus \bar{\mathscr{A}}$ having orders v with no entry v_j equal to zero.

For every subset $K \subset \{1, 2, ..., N\}$ of cardinality n, we consider the intersection

$$\bar{\mathscr{A}}^K = \bar{\mathscr{A}}^{k_1} \cap \cdots \cap \bar{\mathscr{A}}^{k_n}.$$

We have $\binom{N}{n}$ such intersections. They are all nonempty (Proposition 5.1) and by our assumption they are also mutually disjoint. Let us associate to each $\bar{\mathscr{A}}^K$ an order $\mu_K \in \mathbb{N}^{n+1}$, where μ_K simply is the order, with respect to the remaining arrangement $\bigcup_{j \notin K} \bar{\mathscr{A}}^j$, of the complement component containing $\bar{\mathscr{A}}^K$. Clearly then, one has $|\mu_K| = N-n$, and we now want to show that each of the $\binom{N}{n}$ possible values of μ_K is actually achieved. This will follow if we can prove the implication

$$\mu_K = \mu_{K'} \Rightarrow K = K'.$$

So let K and K' be such that $\mu_K = \mu_{K'}$. Since $|\mu_K| = |\mu_{K'}| = N-n > 0$ there must be some nonzero entry, say the zeroth one, in the vector $\mu_K = \mu_{K'}$. Then we can find $l, l' \in \{1, ..., N\}$ so that $\bar{A}^K$ lies in the same component of $\Sigma_n \backslash \bar{A}^l$ as the vertex e_0 and similarly for $\bar{A}^K$ and $\Sigma_n \backslash \bar{A}^{l'}$. By Proposition 4.3 the amoebas $\bar{A}^{k_1}, ..., \bar{A}^{k_n}$ and $\bar{A}^l$ together bound a simplex contained in the interior of Σ_n. This means that there are points in $\Sigma_n \backslash \bar{\mathscr{A}}$ near $\bar{A}^K$ having order $v = \mu_K + (0, 1, ..., 1)$. Similarly, there are points near $\bar{A}^{K'}$ having this same order. By Proposition 2.5 these points must then lie in the same component of the complement $\Sigma_n \backslash \bar{\mathscr{A}}$. Using the notation from the proof of Theorem 6.6 we then get that $t^K = t^{K'} \in \bar{\mathscr{A}}^K \cap \bar{\mathscr{A}}^{K'}$, and since $n+1$ amoebas never intersect, we conclude that $K = K'$.

Consider now those K for which the zeroth entry of μ_K is nonzero. There are $\binom{N-1}{n}$ such subsets K. As in the proof of Theorem 6.6 we see that the corresponding points t_K are the t_0-maximal vertices of *different* connected components of $\Sigma_n \backslash \bar{\mathscr{A}}$. We have already seen that the order of such a component is $v = \mu_K + (0, 1, ..., 1)$ and since the zeroth entry in μ_K is $\geqslant 1$ we see that v is an interior point of the Newton polytope $\mathscr{N}_f$. But the number of integer vectors in the interior of $\mathscr{N}_f$ is precisely $\binom{N-1}{n}$. ∎

7. LINEAR INDEPENDENCE OF LAURENT SERIES

In one variable a Laurent series converges in an annulus, and a finite linear combination of such series cannot converge anywhere, unless all the corresponding annuli (which we assume to be nonempty) have a common intersection. This is easily seen by dividing each series into a sum of two power series and using the explicit formula for the radius of convergence. It follows in particular that series with mutually disjoint nonempty domains of convergence are necessarily linearly independent. In several variables this linear independence is not automatic anymore. Indeed, it suffices to consider the simple series

$$S_1=\sum_{j=1}^{\infty}(2/z)^j-\sum_{k=1}^{\infty}(2/w)^k, \qquad S_2=-\sum_{j=1}^{\infty}(2z)^j+\sum_{k=1}^{\infty}(2/w)^k,$$

$$S_3=\sum_{j=1}^{\infty}(2z)^j-\sum_{k=1}^{\infty}(2w)^k, \qquad S_4=-\sum_{j=1}^{\infty}(2/z)^j+\sum_{k=1}^{\infty}(2w)^k.$$

We see that the domains of convergence are pairwise disjoint, but the sum $S_1+\cdots+S_4$ has all its coefficients equal to zero. The purpose of this section is to show that the different Laurent series corresponding to an arrangement of hyperplane amoebas in optimal position are always linearly independent. We shall actually do more and prove an explicit formula for the Laurent determinant of $f(z)=\prod_{k=1}^{N}(b_0^k+b_1^k z_1+\cdots+b_n^k z_n)$. The fact that this expression turns out never to be zero implies in particular that the various Laurent series of $1/f$ are all linearly independent.

THEOREM 7.1. *Let $\bar{\mathscr{A}}=\bar{\mathscr{A}}^1\cup\cdots\cup\bar{\mathscr{A}}^N$ be an arrangement of compactified hyperplane amoebas in optimal position. Then its Laurent determinant is equal to $\pm 1/\Delta(B)$, where $\Delta(B)$ is as in Definition* 6.2, *and B is the $N\times(n+1)$-matrix whose kth row is $b^k=(b_0^k, b_1^k, ..., b_n^k)$, where $\langle b^k, Z\rangle=0$ defines $\bar{\mathscr{A}}^k$. (Notice that, by Position* 6.3, *the fact that $\bar{\mathscr{A}}$ is in optimal position means that $\Delta(B)\neq 0$.)*

Proof. Let us first introduce the notation

$$\begin{cases} L^k(z)=b_0^k+b_1^k z_1+\cdots+b_n^k z_n, & 1\leqslant k\leqslant N,\\ L^{n+k}(z)=z_k, & 1\leqslant k\leqslant n. \end{cases}$$

In order to compute the Laurent determinant, which in this case is of size $\binom{N+n}{n}$, it is convenient to replace the usual basis z^α, $|\alpha|\leqslant N$, for the monomials by the one given by all products of the form

$$L^{i_1}(z)\cdots L^{i_N}(z).$$

Since the monomials occur in the numerators in the integrals defining the $c_{-\alpha,\nu}$, we get from multilinearity that the Laurent determinant is equal to

$$\frac{1}{\det X}\det\left(\frac{1}{(2\pi i)^n}\int_{\log^{-1}(u_\nu)}\frac{L^{i_1}(z)\cdots L^{i_N}(z)\,dz_1\wedge\cdots\wedge dz_n}{f(z)\,z_1\cdots z_n}\right),$$

where X denotes the matrix which describes the new basis elements, that is, the polynomials $L^{i_1}(z)\cdots L^{i_N}(z)$ in terms of the original basis monomials z^α.

We claim that $\det X = \pm\prod_{0\in J} \Delta_{IJ}$. Both sides are homogeneous polynomials in the variables b_j^k, and the degrees of the left and right hand sides are equal to

$$\sum_{k=0}^{n} \binom{n}{k}\binom{N}{k}(N-k) \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k}\binom{N}{k+1}(k+1)$$

respectively. Since $\binom{N}{k}(N-k) = \binom{N}{k+1}(k+1)$ the degrees do indeed coincide. Now we shall use the fact that each determinant Δ_{IJ} is an *irreducible* homogeneous polynomial in the variables b_j^k. Indeed, suppose it factors as a product $P(b)\,Q(b)$. Consider the degree of P and Q with respect to the variables from the first row only. Since the degrees add up under multiplication, one factor, say P, must be independent of these variables, while Q depends on all of them. The same goes for any column, and it follows that P must in fact be a constant. To justify the claim it now suffices to observe that $\det X$ vanishes whenever one of the Δ_{IJ} does. Indeed, we have that $\Delta_{IJ}=0$, for some index J with $0\in J$, precisely if $n+1$ of the linear forms $L^1, ..., L^{N+n}$ are linearly dependent, say $\lambda_0 L^{k_0} + \cdots + \lambda_n L^{k_n} = 0$. In that case it follows that the $n+1$ rows of the matrix X corresponding to the products $L^{i_1}\cdots L^{i_N}$ which contain exactly one of the L^{k_l} are also linearly dependent, with the same coefficients λ_l. Hence $\det X = 0$ then.

We have thus shown that the Laurent determinant may be written

$$\begin{aligned}
&\frac{1}{\prod_{0\in J}\Delta_{IJ}} \det\left(\frac{1}{(2\pi i)^n}\int_{\log^{-1}(u_\nu)} \frac{dz_1\wedge\cdots\wedge dz_n}{L^{k_1}(z)\cdots L^{k_n}(z)}\right)\\
&\quad= \frac{1}{\prod_{0\in J}\Delta_{IJ}}\,\frac{1}{\prod_{0\notin J}\Delta_{IJ}}\\
&\qquad\times\det\left(\frac{1}{(2\pi i)^n}\int_{\log^{-1}(u_\nu)} \frac{dL^{k_1}(z)\wedge\cdots\wedge dL^{k_n}(z)}{L^{k_1}(z)\cdots L^{k_n}(z)}\right).
\end{aligned}$$

From the proof of Theorem 6.6 we have a natural bijection between the orders ν and the n-tuples $K=(k_1, ..., k_n)$, by associating to K the order ν of the component at the point t^K. We now claim that if the rows ν and the columns K of the matrix

$$\left(\frac{1}{(2\pi i)^n}\int_{\log^{-1}(u_\nu)} \frac{dL^{k_1}(z)\wedge\cdots\wedge dL^{k_n}(z)}{L^{k_1}(z)\cdots L^{k_n}(z)}\right) \tag{7}$$

are ordered according to this bijection and so that $|\nu|$ increases, then it is actually a lower triangular matrix with ones along the diagonal. But this is a simple consequence of Proposition 5.5, for that result immediately tells us that the diagonal entries are equal to 1, and also that the integral is zero unless u_ν is in the component of $\mathbb{R}^n \backslash (\mathscr{A}^{k_1} \cup \cdots \cup \mathscr{A}^{k_n})$ having order $(1, 1, ..., 1)$ with respect to the partial arrangement $\mathscr{A}^{k_1} \cup \cdots \cup \mathscr{A}^{k_n}$. But then the component of ${}^c\mathscr{A}$ that contains u_ν must have an order ν which is either the same as the order ν^K of the component at t^K (in which case we are considering a diagonal entry), or else strictly larger, in the sense that $|\nu| > |\nu^K|$. This means that (7) has all its entries above the diagonal equal to zero as claimed. Its determinant is therefore equal to one, and the proof is complete. ∎

It is a known fact that the dimension of the middle (compactly supported) homology group $H_n(\mathbb{T}^n \backslash \{L^1 \cdots L^N = 0\})$ is equal to $\binom{N+n}{n}$. This follows for example from the results of [6]. We thus have the following corollary.

COROLLARY 7.2. *The various Laurent series associated to a rational function* $1/f$, *where* $\mathscr{A}_f$ *is an arrangement of* N *hyperplane amoebas in optimal position in* $\mathbb{R}^n$, *are all linearly independent, and the corresponding cycles* $\log^{-1}(u_\nu)$ *constitute a basis for the homology group* $H_n(\mathbb{T}^n \backslash \mathscr{Z}_f)$.

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