arXiv:math/0511059v3 [math.AC] 22 Sep 2008

TROPICAL ALGEBRAIC SETS, IDEALS AND AN ALGEBRAIC NULLSTELLENSATZ

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Abstract. This paper introduces the foundations of the polynomial algebra and basic structures for algebraic geometry over the extended tropical semiring. Our development, which includes the tropical version for the fundamental theorem of algebra, leads to the reduced polynomial semiring – a structure that provides a basis for developing a tropical analogue to the classical theory of commutative algebra. The use of the new notion of tropical algebraic com-sets, built upon the complements of tropical algebraic sets, eventually yields the tropical algebraic Nullstellensatz.

Contents

Introduction

The notion of tropical mathematics was introduced only in the past decade [5, 18]. Since then this theory has developed rapidly and led to many applications [4, 6, 10, 12, 15, 17]. A survey can be found in [11]. Tropical mathematics is the mathematics over idempotent semirings, the ***tropical semiring*** is usually taken to be $(\mathbb{R} \cup \{-\infty\}, \max, +)$; the real numbers, together with the formal element $-\infty$, equipped by the operations of maximum and summation – addition and multiplication respectively [9]. The basic formalism of tropical geometry and been presented by Mikhalkin [13].

The main goal of this paper is the development of another approach to the basics of tropical polynomial algebra with a view to tropical algebraic geometry, which is built on the ***extended tropical semiring***,

Date: December 2007.
Key words and phrases. Idempotent semiring, max-plus algebra, tropical algebraic geometry, algebraic sets and com-sets, polynomial ideals, Nullstellensatz.
The author has been supported by the grant of the High Council for Scientific and Technological Cooperation between Israel and France.

$(\mathbb{T}, \oplus, \odot)$, as has been presented in [8]. This extension is obtained by taking two copies of the reals,

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad \bar{\mathbb{U}} = \mathbb{R}^\nu \cup \{-\infty\},$$

each is enlarged by $\{-\infty\}$, and gluing them along $-\infty$ to define the set $\mathbb{T} = \bar{\mathbb{R}} \cup \bar{\mathbb{U}}$. The correspondence $\nu : \mathbb{R} \to \mathbb{U}$ is the identity map, so we denote the image of $a \in \mathbb{R}$ by a^ν. Accordingly, elements of $\mathbb{U}$, which is called the ***ghost*** part of $\mathbb{T}$, are denoted as a^ν; $\mathbb{R}$ is called the ***tangible*** (or the ***real***) part of $\mathbb{T}$. The map ν is sometimes extended to whole $\mathbb{T}$,

$$\nu : \mathbb{T} \longrightarrow \bar{\mathbb{U}}, \tag{1}$$

by declaring $\nu : a^\nu \mapsto a^\nu$ and $\nu : -\infty \mapsto -\infty$; this map is called the ***ghost map***.

The set $\mathbb{T}$ is then provided with the following total order extending the usual order on $\mathbb{R}$:

(*i*) $-\infty \prec \alpha$, $\forall \alpha \in \mathbb{T}$;

(*ii*) for any real numbers $a < b$, we have $a \prec b$, $a \prec b^\nu$ and $a^\nu \prec b$, $a^\nu \prec b^\nu$;

(*iii*) $a \prec a^\nu$ for all $a \in \mathbb{R}$.

(We use the generic notation $a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{T}$.) Then $\mathbb{T}$ is endowed with the two operations $\oplus$ and $\odot$, defined as follows:

$$\begin{array}{rcl} \alpha \oplus \beta & = & \begin{cases} max_{(\prec)}\{\alpha, \beta\}, & \alpha \neq \beta, \\ \alpha^\nu, & \alpha = \beta \neq -\infty, \end{cases} \\ -\infty \oplus -\infty & = & -\infty, \\ a \odot b & = & a + b, \\ a^\nu \odot b & = & a \odot b^\nu = a^\nu \odot b^\nu = (a+b)^\nu, \\ (-\infty) \odot \alpha & = & \alpha \odot (-\infty) = -\infty. \end{array}$$

The semiring $(\mathbb{T}, \oplus, \odot)$ modifies the classical ***max-plus algebra*** and as has been proven, its arithmetic is commutative, associative, and distributive. Note that while the standard tropical semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$ is an idempotent semiring, since $a \oplus a = a^\nu$, the semiring $(\mathbb{T}, \oplus, \odot)$ is not an idempotent semiring. (The topology of $(\mathbb{T}, \oplus, \odot)$ is more complected than the Euclidean topology which is used on the standard tropical semiring, the details are brought below in Section 1.)

The connection with the standard tropical semiring is established by the natural epimorphism of semirings,

$$\pi : (\mathbb{T}, \oplus, \odot) \longrightarrow (\mathbb{R} \cup \{-\infty\}, \max, +), \tag{2}$$

given by $\pi : a^\nu \mapsto a$, $\pi : a \mapsto a$ for all $a \in \mathbb{R}$, and $\pi : -\infty \mapsto -\infty$. (We write $\pi(\alpha)$ for the image of $\alpha \in \mathbb{T}$ in $\bar{\mathbb{R}}$.) This epimorphism induces epimorphisms π_* of polynomial semirings, Laurent polynomial semirings, and tropical matrices.

The fact that $(\mathbb{R}, \odot)$ is a group and $(\bar{\mathbb{U}}, \oplus, \odot)$ is an ideal provides $\mathbb{T}$ with a structure to which much of the theory of commutative algebra (including polynomials and determinants) can be transferred, leading to applications in combinatorics, polynomials, Newton polytopes, algebraic geometry, and convex geometry.

We start our discussion by observing the difference between tropical polynomials and tropical polynomial functions, and study the relation, which is not one-to-one correspondence, between polynomials and functions. To overcome this miss-correspondence, we determine the reduced polynomial semiring $\tilde{\mathbb{T}}[x_1, \dots, x_n]$ which is well behaved and allows an analogous development of polynomial theory to that of the classical case. This study includes polynomial factorizations and, by introducing the tropical algebraic set

$$\mathcal{Z}_{\text{tr}}(f) = \{\mathbf{a} \in \mathbb{T}^{(n)} \mid f(\mathbf{a}) \in \bar{\mathbb{U}}\}, \qquad f \in \mathbb{T}[x_1, \dots, x_n],$$

one of our main results is the ***fundamental theorem of the tropical algebra*** – a tropical version that is similar to the classical theorem.

Theorem 2.5: *The tropical semiring $\mathbb{T}$ is algebraically closed (in tropical sense), that is, $\mathcal{Z}_{\text{tr}}(f) \neq \emptyset$ for any nonconstant $f \in \mathbb{T}[x_1, \dots, x_n]$.*

The new notion of tropical com-set, defined as

$$\mathcal{C}_{\text{tr}}(\mathfrak{a}) = \{D_f \mid D_f \text{ is a connected component of } \mathcal{Z}_{\text{tr}}(f)^c \text{ of } f \in \mathfrak{a}\},$$

which are built upon the complements, $\mathcal{Z}_{\rm tr}(f)^c$, of tropical algebraic set $\mathcal{Z}_{\rm tr}(f)^c$, is central in our development. The relation between com-sets and tropical ideals, is the focal point for the tropical Nullstellensatz:

Theorem 5.3: ***(Weak Nullstellensatz)*** *Let* $\mathfrak{a} \subset \tilde{\mathbb{T}}[x_1, \dots, x_n]$ *be a finitely generated proper ideal, then* $\mathcal{Z}_{\rm tr}(\mathfrak{a}) \neq \emptyset$. *Conversely, if* $\mathcal{Z}_{\rm tr}(\mathfrak{a}) = \emptyset$, *then* $\mathfrak{a} = \tilde{\mathbb{T}}[x_1, \dots, x_n]$.

Theorem 5.7: ***(Algebraic Nullstellensatz)*** *Let* $\mathfrak{a} \subset \tilde{\mathbb{T}}[x_1, \dots, x_n]$, *where* $\tilde{\mathbb{U}}[x_1, \dots, x_n] \subseteq \mathfrak{a}$, *be a finitely generated tropical ideal, then* $\sqrt{\mathfrak{a}} = I_{\rm tr}(\mathcal{C}_{\rm tr}(\mathfrak{a}))$.

A similar context of the issues appear in this paper has been raised in [1, Qu. : A.16, C.2.A] and in [2, Qu. 14].

Notations: In this paper we sometimes refer to the standard arithmetic operations. To distinguish these operations, the standard addition and the multiplication are signed by $+$ and $\cdot$ respectively. For short, we write ab for $a \odot b$.

Acknowledgement: The author would like to thank *Professor Eugenii Shustin* for his invaluable help. I'm deeply grateful to him for his support and the fertile discussions we had.

1. The Topology of $\mathbb{T}^{(n)}$

Introducing a topology for $\mathbb{T}^{(n)}$, obtained as the product topology on $\mathbb{T}$, in which the semiring's operations satisfy continuity is essential for our future development. Our topological setting is motivated by the following argument: given a point $a \in \mathbb{R}$ with a small neighborhood $\widehat{W} \subset \mathbb{T}$, $a \in \widehat{W}$, pick $b \in \widehat{W} \cap \mathbb{R}$, and consider the sum $a \oplus b$ when $b \to a$. Then, in order to preserve the continuity of $\oplus$, $\widehat{W}$ must contain also the corresponding ghost element $a^\nu \in \mathbb{U}$. Later, we also want our tropical sets to be closed sets.

Our auxiliary topology on the enlarged ghost part, $\bar{\mathbb{U}} = \mathbb{R}^\nu \cup \{-\infty\}$, is the Euclidean topology of the half line $[0, \infty)$ in which $\oplus$ and $\odot$ are continuous, and closed sets are defined. The tangible part is concerned also as having the Euclidean topology, but here, the topology is partial, since $\oplus$ is continuous only for different elements.

Given a subset $U \subset \bar{\mathbb{R}}$, we write U^ν for the the corresponding ghost subset $\{u^\nu \mid u \in U\} \subset \bar{\mathbb{U}}$, recall that we identify $(-\infty)^\nu$ with $-\infty$.

Definition 1.1. *A subset* $\widehat{W} \subset \mathbb{T}$ *is defined to be* ***closed set*** *if* $\widehat{W} = U \cup V^\nu$, *where* $U \subseteq \bar{\mathbb{R}}$ *and* $V^\nu \subseteq \bar{\mathbb{U}}$ *satisfy:*

(i) $U^\nu, V^\nu \subseteq \bar{\mathbb{U}}$ *are both closed sets and,*

(ii) $U^\nu \subseteq V^\nu$.

A set $W \subset \mathbb{T}$ *is said to be* ***open*** *if its complement is closed.*

(In particular, a closed set may consist only of ghost points, but when it includes a tangible point it must also contain its ghost. Conversely, an open set can be pure tangible subset of $\mathbb{R}$.)

Using the decomposition $\widehat{W} = U \cup V^\nu$, it easy to verify that finite unions and arbitrary intersections of the closed sets are also of this form, accordingly, these sets form the closed sets of our topology. Like in the standard case: the ***closure*** of a set W is the smallest closed set $\widehat{W}$ containing W, ***connected set*** W is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

Example 1.2.

(i) $\{a, a^\nu\}$ *and* $\{-\infty\}$ *are closed sets;*

(ii) $\{1 \prec \alpha \prec 2 \mid \alpha \in \mathbb{R}\}$ *is open set;*

(iii) $\{\alpha \mid a \succ \alpha \prec a^\nu\}$, *for some* $a \in \mathbb{R}$, *is open set;*

(iv) $\{0 \preceq \alpha \preceq 1 \mid \alpha \in \mathbb{R}\} \cup \{0^\nu \preceq \alpha \preceq 1^\nu \mid \alpha \in \mathbb{U}\}$ *is closed set;*

As mentioned earlier, having a topology on $\mathbb{T}$, we define the topology on $\mathbb{T}^{(n)}$ to be the product topology of $\mathbb{T}$.

2. Polynomials and Functions

2.1. **The tropical polynomial semiring.** The tropical semiring, $(\mathbb{T}[x], \oplus, \odot)$, of polynomials in one variable is defined to be all formal sums $f = \bigoplus_{i\in\mathbb{N}} \alpha_i x^i$, with $\alpha_i \in \mathbb{T}$, for which almost all $\alpha_i = -\infty$, where we define polynomial addition and multiplication in the usual way:

$$\left(\bigoplus_i \alpha_i x^i\right)\left(\bigoplus_j \beta_j x^j\right) = \bigoplus_k \left(\bigoplus_{i+j=k} \alpha_i \beta_{k-j}\right) x^k.$$

Accordingly, we write a polynomial $\bigoplus \alpha_i x^i$ as $\bigoplus_{i=0}^t \alpha_i x^i$, when $\alpha_i = -\infty$ for all $i > t$, and define its ***degree*** to be t. A term $\alpha_i x^i$ is said to be a monomial of f when $\alpha_i \neq -\infty$.

We sometimes write x^ν for $0^\nu x$. Note that, since 0 is the multiplicative unit of $\mathbb{T}$, we write x^i for $0x^i$ and say a polynomial is ***monic*** if its coefficient of highest degree, which we call the leading coefficient, is 0. We identify αx^0 with α, for each $\alpha \in \mathbb{T}$; thus we may view $\mathbb{T} \subset \mathbb{T}[x]$. The elements of $\mathbb{U}[x]$ form the ghost part of $\mathbb{T}[x]$ where $\mathbb{R}[x]$ is its tangible part; that is, polynomials of each part have, respectively, only ghost coefficients or only tangible coefficients.

The polynomial semiring $\mathbb{T}[x_1, \dots, x_n]$ is defined inductively, as $\mathbb{T}[x_1, \dots, x_{n-1}][x_n]$; a typical polynomial, as usual, is

$$f = \bigoplus \alpha_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

We write $\mathbf{i}$ for multi index $(i_1, \dots, i_n)$ and let $\mathbf{x} = (x_1, \dots, x_n)$, and thus write $f = \bigoplus \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$, for short. The ***support*** of a polynomial f is define to be those $\mathbf{i}$ for which $\alpha_{\mathbf{i}} \neq -\infty$, that is

$$\mathrm{Supp}(f) = \{\ \mathbf{i} \mid \alpha_{\mathbf{i}} \neq -\infty\}.$$

Given $f \in \mathbb{R}[x_1, \dots, x_n]$, i.e. $\alpha_{\mathbf{i}} = a_{\mathbf{i}} \in \mathbb{R}$ for all $\mathbf{i}$, the corresponding polynomial $f = \bigoplus a_{\mathbf{i}}^\nu \mathbf{x}^{\mathbf{i}}$ is denoted as f^ν. Moreover, f can be decomposed uniquely according to its tangible part f^{t} and its ghost part f^{g}, and written uniquely as $f = f^{\mathrm{t}} \oplus f^{\mathrm{g}}$. We call this ***(t,g)-decomposition*** of f, clearly this decomposition is unique. If $f = f^{\mathrm{t}}$ then f is said to be ***tangible polynomial***, and is said to be ***ghost polynomial*** when $f = f^{\mathrm{g}}$.

Remark 2.1. *The tangible part $\mathbb{R}[x_1, \dots, x_n]$ is not closed under the semiring operations. Moreover, there are $f \in \mathbb{R}[x_1, \dots, x_n]$ for which $f^k \notin \mathbb{R}[x_1, \dots, x_n]$ for some positive $k \in \mathbb{N}$; for example take $f = x \oplus 1$ then $f^2 = x^2 \oplus 1^\nu x \oplus 2$ which is a non-tangible polynomial (but is not ghost polynomial).*

A power of a non-ghost polynomial, i.e. it has a tangible monomial, can be a ghost polynomial; for example take $f = 0^\nu x^2 \oplus 1x \oplus 2^\nu$, then

$$f^2 = (0^\nu x^2 \oplus 1x \oplus 2^\nu)^2 = 0^\nu(x^4 \oplus 1x^3 \oplus 2x^2 \oplus 3x \oplus 4),$$

which is a ghost polynomial. On the other hand, $\mathbb{U}[x_1, \dots, x_n]$ is closed under addition and under the multiplication with any element of $\mathbb{T}[x_1, \dots, x_n]$; therefore, as will be seen later, is a semiring ideal.

We note that whenever $fg = -\infty$ then $f = -\infty$ or $g = -\infty$, and thus the only element $f \in \mathbb{T}[x_1, \dots, x_n]$ for which $f^k = -\infty$ is $-\infty$ itself.

Recall that $\mathbb{T}$ lacks subtraction, and therefore we don't have cancelation of monomials; this property is expressed in degree computations that always satisfy the rules:

$$\deg(fg) = \deg(f) + \deg(g) \quad \text{ and } \quad \deg(f \oplus g) = \max\{\deg(f), \deg(g)\}.$$

(This is different from the classical theory in which $Deg(f + g) \leq max\{\deg(f), \deg(g)\}$, in the tropical case we always have equality.) For the same reason, for a given $f \in \mathbb{T}[x_1, \dots, x_n]$ one can define the "lower degree" to be

$$\underline{\deg}(f) = min\{\deg(h) \mid h \textit{ is a monomial of } f\}.$$

which then satisfies

$$\underline{\deg}(fg) = \underline{\deg}(f) + \underline{\deg}(g) \quad \text{ and } \quad \underline{\deg}(f \oplus g) = \min\{\underline{\deg}(f), \underline{\deg}(g)\}.$$

Clearly, we always have $\underline{\deg}(f) \leq \deg(f)$ and both can only increase by preforming operations over polynomials.

A ***tropical homomorphism*** of tropical polynomial semirings

$$\varphi : (\mathbb{T}[x_1, \dots, x_n], \oplus, \odot) \longrightarrow (\mathbb{T}[x_1, \dots, x_m], \oplus, \odot)$$

is a semiring homomorphism $\varphi : \mathbb{T}[x_1, \dots, x_n] \setminus \{-\infty\} \to \mathbb{T}[x_1, \dots, x_m] \setminus \{-\infty\}$ such that $\varphi(f^\nu) = (\varphi(f))^\nu$ for any $f \in \mathbb{T}[x_1, \dots, x_n]$; accordingly, we have $\varphi(\mathbb{U}[x_1, \dots, x_n]) \subset \mathbb{U}[x_1, \dots, x_m]$. (Within this definition we include the case of $m = 0$, that is $\varphi : (\mathbb{T}[x_1, \dots, x_n], \oplus, \odot) \to \mathbb{T}$.) The ***tropical kernel***, $\ker \varphi$, is the preimage of $\bar{\mathbb{U}}[x_1, \dots, x_m]$. We call φ a ***ghost injection*** if $\ker \varphi = \bar{\mathbb{U}}[x_1, \dots, x_n]$ and say that φ is a ***tropical injection*** if φ is 1:1 and is a ghost injection.

Given a point $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{T}^{(n)}$, there is a tropical homomorphism $\varphi_{\mathbf{c}} : \mathbb{T}[x_1, \dots, x_n] \to \mathbb{T}$, given by sending

$$\varphi_{\mathbf{c}} : \bigoplus_i \alpha_{i_1,\dots,i_n} {x_1}^{i_1} \cdots {x_n}^{i_n} \longmapsto \bigoplus_i \alpha_{i_1,\dots,i_n} {c_1}^{i_1} \cdots {c_n}^{i_n},$$

which we call the ***substitution homomorphism*** (with respect to $\mathbf{c}$). (Note that in tropical algebra, as usual, ${c_j}^{i_j}$ means the tropical product of c_j taken i_j times, which is just $i_j \cdot c_j$ in the classical notation.) We write $f(\mathbf{c})$ for the image of the polynomial f under substitution to $\mathbf{c}$.

In our philosophy, elements of $\bar{\mathbb{U}}$ are those to be ignored, accordingly we define a root of polynomial, in the tropical sense:

Definition 2.2. *An element $\mathbf{a} \in \mathbb{T}^{(n)}$ is a **root** of f if $f(\mathbf{a}) \in \bar{\mathbb{U}}$, i.e if $f \in \ker \varphi_{\mathbf{a}}$, where $\varphi_{\mathbf{a}}$ is the tropical substitution homomorphism $\varphi_{\mathbf{a}} : (x_1, \dots, x_n) \mapsto \mathbf{a}$.*

(Note that we include $-\infty$ as a proper root since later we want to study connectedness of sets of roots and their complements which coincides with our topological setting.) By this definition, given a ghost polynomial $f = f^{\mathrm{g}}$, then any $\mathbf{a} \in \mathbb{T}^{(n)}$ is a root of f. Note the we can also have non-ghost polynomials for which any $\mathbf{a} \in \mathbb{T}^{(n)}$ is a root (take for example $f = 0^\nu x^2 \oplus -1x \oplus 0^\nu$). However, our main interest is in non-ghost polynomials, mainly in tangible ones.

Remark 2.3. *Suppose $f = f^{\mathrm{t}} \oplus f^{\mathrm{g}}$ is the (t,g)-decomposition of a polynomial f into tangible and ghost parts. Then any root $\mathbf{a}$ of f^{t} is a root of f. Indeed, $f(\mathbf{a}) = f^{\mathrm{t}}(\mathbf{a}) \oplus f^{\mathrm{g}}(\mathbf{a})$, and each part is in $\bar{\mathbb{U}}$.*

Lemma 2.4. *For any nonconstant polynomial $f \in \mathbb{T}[x]$ without a constant monomial and for any $\mathrm{a}^\nu \neq -\infty$ in $\mathbb{U}$, there exists $r \in \mathbb{T}$ with $f(r) = \mathrm{a}^\nu \in \mathbb{U}$,*

Proof. Write $f = \bigoplus \alpha_i x^i$. For each $i > 0$, there is some $r_i \in \mathbb{R}$ such that $\alpha_i ({r_i}^i)^\nu = \mathrm{a}^\nu$. Indeed, assume α_i is tangible then using the ghost map (1) (written in the standard arithmetic) we have

$$\nu(r_i) = \frac{1}{i}(\mathrm{a} - \alpha_i)^\nu.$$

Now, take r among these r_i such that $\nu(r)$ is minimal among r_i^ν, $1 \leq i \leq t$. Then $(f(r))^\nu = \mathrm{a}^\nu$. $\square$

Using Lemma 2.4, we can state the ***Fundamental theorem of the tropical algebra*** – a tropical version that is similar to the classical theorem.

Theorem 2.5. *The tropical semiring $\mathbb{T}$ is algebraically closed in tropical sense, that is any nonconstant tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ has a root.*

Proof. Assume $f \in \mathbb{T}[x]$, if f has a single nonconstant monomial $h_i = \alpha_i x^i$, or $f = x^i g$, for some $i > 0$, then $h(\mathrm{a}^\nu) \in \mathbb{U}$ and respectively $(\mathrm{a}^\nu)^i g(\mathrm{a}^\nu) \in \mathbb{U}$, for any a^ν, and we are done. Suppose $f = \bigoplus_{i=0}^m \alpha_i x^i$, we may assume that $\alpha_0 \neq -\infty$. Write $f = g \oplus \alpha_0$. If $\alpha_0 \in \mathbb{U}$, then for any ghost a^ν, $f(\mathrm{a}^\nu) \in \mathbb{U}$, so a^ν is a root. Thus, we may assume that $\alpha_0 \notin \bar{\mathbb{U}}$. By lemma 2.4, there is some r such that $\nu(g(r)) = \alpha_0^\nu$, which implies $f(r) = \alpha_0^\nu \oplus \alpha_0 \in \mathbb{U}$. The generalization to $f \in \mathbb{T}[x_1, \dots, x_n]$ is clear. $\square$

Remark 2.6. *In the familiar tropical semiring $(\mathbb{R}, \max, +)$ roots are not defined directly and are realized as the points on which the evaluation of a polynomial is attained by at least two of its monomials. In other word, the roots are simply the domain of non-differentiability of the corresponding function. Unfortunately, using this notion, $(\mathbb{R}, \max, +)$ is not algebraically closed in the tropical sense; take for example a polynomial having a single monomial.*

2.2. **Tropical polynomial functions.** As mentioned earlier, in the tropical world the correspondence between polynomials and polynomial functions is not one-to-one, mainly due to convexity matters, and a function can have many polynomial descriptions; for example consider the family of the polynomials

$$f_t = x^2 \oplus tx \oplus 0,$$

where $t \leq 0$ serves as a parameter, all the members of this family describe the same function. We denote by ψ_f the tropical function corresponding to a polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$, that is $\psi_f : \mathbf{a} \mapsto f(\mathbf{a})$ written as $\psi_f(\mathbf{a})$. We denote by $\mathcal{F}(\mathbb{T}^{(n)})$ the semiring of polynomial functions

$$\mathcal{F}(\mathbb{T}^{(n)}) = \{\psi_f : \mathbb{T}^{(n)} \longrightarrow \mathbb{T} \mid f \in \mathbb{T}[x_1, \dots, x_n]\}.$$

(The operations

$$(\psi_f \odot \psi_g)(\mathbf{a}) = \psi_f(\mathbf{a}) \odot \psi_g(\mathbf{a}), \qquad \textit{and} \qquad (\psi_f \oplus \psi_g)(\mathbf{a}) = \psi_f(\mathbf{a}) \oplus \psi_g(\mathbf{a}),$$

of $\mathcal{F}(\mathbb{T}^{(n)})$ are defined point-wise.) Given a tropical function, the central idea for the further development is finding the best representative among all of its polynomials descriptions.

Definition 2.7. *Two polynomials $f, g \in \mathbb{T}[x_1, \dots, x_n]$ are said to be **equivalent**, denoted as $f \sim g$, if they take on the same values, that is $f(\mathbf{a}) = g(\mathbf{a})$, for any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{T}^{(n)}$ (i.e. in function view $\psi_f = \psi_g$).*

Example 2.8. *For all $a, b \in \mathbb{R}$, $a \neq b$, the following relations hold true:*

(i) $x \oplus a \nsim x \oplus a^\nu$,

(ii) $x \oplus a \nsim x \oplus b$,

(iii) $(x \oplus \alpha)^2 \sim x^2 \oplus \alpha^2$.

To prove (iii), write $f = (x \oplus \alpha)^2 = x^2 \oplus \alpha^\nu x \oplus a^2$ and $g = x^2 \oplus \alpha^2$. Suppose $f \nsim g$, this means that there is some $a \in \mathbb{T}$ for which $f(a) \neq g(a)$ and thus $f(a) = \alpha^\nu a \succ \alpha^2 \oplus a^2$. But, if $a \succ \alpha$ we have $a^2 \succ \alpha^\nu a$, and when $a \prec \alpha$ we get $\alpha^2 \succ \alpha^\nu a$. which is a contradiction. (In the case of $a = \alpha$ we have $f(\alpha) = g(a) = \alpha^\nu$.) Namely, $f(a) = \alpha^2 \oplus a^2$ for any $a \in \mathbb{T}$.

Given $f \in \mathbb{T}[x_1, \dots . x_n]$, the graph $\Gamma(f)$ of f is defined to be

$$\Gamma(f) = \{(\mathrm{a}_1, \dots, \mathrm{a}_n, f(\mathbf{a})) \;:\; \mathbf{a} = (\mathrm{a}_1, \dots, \mathrm{a}_n) \in \mathbb{T}^{(n)}\} \subset \mathbb{T}^{(n+1)}.$$

We write $\Gamma(f|_\Pi)$ for the restriction of to the subdomain $\Pi \subset \mathbb{T}^{(n)}$, that is

$$\Gamma(f|_\Pi) = \{(\mathrm{a}_1, \dots, \mathrm{a}_n, f(\mathbf{a})) \;:\; \mathbf{a} \in \Pi\}.$$

Accordingly, we have the following relations

Lemma 2.9. *Let $f \in \mathbb{T}[x_1, \dots . x_n]$ then*

$$f \sim f' \quad \Longleftrightarrow \quad \Gamma(f) = \Gamma(f')$$

and

$$\deg(f) = \deg(f') \quad \textit{and} \quad \underline{\deg}(f) = \underline{\deg}(f').$$

Proof. The first relation is by definitions, $f \sim f'$ if and only if $f(\mathbf{a}) = f'(\mathbf{a})$ for all $\mathbf{a} = (\mathrm{a}_1, \dots, \mathrm{a}_n) \in \mathbb{T}^{(n)}$ if and only if $(\mathrm{a}_1, \dots, \mathrm{a}_n, f(\mathbf{a})) = (\mathrm{a}_1, \dots, \mathrm{a}_n, f'(\mathbf{a}))$ for all $\mathbf{a} = (\mathrm{a}_1, \dots, \mathrm{a}_n) \in \mathbb{T}^{(n)}$ if and only if $\Gamma(f) = \Gamma(f')$.

Suppose $f \sim f'$ and $\deg(f) > \deg(f')$. Write $\mathbf{i} = (i_1, \dots, i_n)$, $\mathbf{j} = (j_1, \dots, j_n)$ for the multi-indices and let $\alpha_\mathbf{i}\mathbf{x}^\mathbf{i}$ and $\alpha_\mathbf{j}\mathbf{x}^\mathbf{j}$ be respectively the monomials of highest degree of f and f'. Then $i_s > j_s$ for some $s = 1, \dots, n$, which implies that for a point $\mathbf{a} \in \mathbb{T}^{(n)}$ whose s'th coordinate is sufficiently large we have $f(\mathbf{a}) \succ f'(\mathbf{a})$ – a contradiction. To prove that $\underline{\deg}(f) = \underline{\deg}(f')$, we use the same argument only by considering the monomials of lowest degree with respect to a point having a sufficiently small coordinate. □

Remark 2.10. *Assume $f \in \mathbb{T}[x_1, \dots, x_n]$ is tangible (resp. ghost) and $f \sim f'$, then f' needs not be also tangible (resp. ghost); for example $x^2 \oplus \alpha^\nu x \oplus \alpha^2 \sim x^2 \oplus \alpha^2$ (cf. Example 2.8).*

Instead of polynomials, we are interested in their equivalence classes. There is a natural representative for each equivalence class. Given a polynomial $f = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ having a monomial $h = \alpha_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$, we denote by $f \setminus h$ the polynomial $\bigoplus_{\mathbf{i} \neq \mathbf{j}} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$.

Definition 2.11. *A polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ **dominates** g if $f(\mathbf{a}) \oplus g(\mathbf{a}) = f(\mathbf{a})$ for all $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{T}^{(n)}$, i.e. $\psi_f \oplus \psi_g = \psi_f$. A monomial h of f dominated by $f \setminus h$ (which not empty) is called **inessential**; otherwise h is said to be **essential**. The **essential part** f^{e} of a polynomial $f = \bigoplus \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ is the sum of those monomials $\alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ that are essential, while its inessential part f^{i} consists of the sum of all inessential $\alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$. When $f = f^{\mathrm{e}}$, f is said to be an **essential polynomial**.*

For example, $x^2 \oplus 2$ is the essential part of $x^2 \oplus 0x \oplus 2$, where $0x$ is inessential monomial. In other words, the essential part consisting of all the monomials which are need to obtain a same polynomial function. Namely, from the function point of view, to obtain f^{e} we cancel out all the unnecessary monomials of f.

Lemma 2.12. *For any $f \in \mathbb{T}[x_1, \dots, x_n]$, $f \sim f^{\mathrm{e}}$.*

Proof. Let $f = \bigoplus_i h_i$ and assume that $f \nsim f^{\mathrm{e}}$. Then, there is some $\mathbf{a} \in \mathbb{T}^{(n)}$ for which $f(\mathbf{a}) \neq f^{\mathrm{e}}(\mathbf{a})$. This means that $f \setminus h_i$ does not dominate some monomial h_i and this monomial is not part of f^{e}. Namely, f contains an essential monomial h_i which is not in f^{e}. This contradicts the construction of f^{e}. □

Proposition 2.13. *The essential part, f^{e}, of a polynomial f is unique.*

Proof. Assume that f have two different essential parts, say f^{e} and $f^{\mathrm{e}\prime}$, then $\Gamma(f^{\mathrm{e}}) \neq \Gamma(f^{\mathrm{e}\prime})$. But then, by Lemma 2.12, $f \sim f^{\mathrm{e}}$ and $f \sim f^{\mathrm{e}\prime}$, and by Lemma 2.9, $\Gamma(f^{\mathrm{e}}) = \Gamma(f^{\mathrm{e}\prime})$ – a contradiction. □

Integrating Lemma 2.9, Lemma 2.12, and Proposition 2.13 we conclude:

Corollary 2.14. *$f \sim g$ if and only if $f^{\mathrm{e}} = g^{\mathrm{e}}$.*

Clearly, $\sim$ is an equivalence relation, so f^{e} serves as a canonical representative for the equivalence class

$$\mathcal{C}_f = \{f' \in \mathbb{T}[x_1, \dots, x_n] : \ f' \sim f\}.$$

Thus, each equivalence class under $\sim$ has a canonical (essential) representative. One can use these representatives to establish the one-to-one correspondence:

$$\mathbb{T}[x_1, \dots, x_n]/_{\sim} \longrightarrow \mathcal{F}(\mathbb{T}^{(n)}).$$

Yet, we are looking for a better representative since these representatives are not suitable for the purpose of factorization.

Note 2.15. *Assume the essential part of f is tangible and is comprised of m tangible monomials (i.e. $f^{\mathrm{e}} \in \mathbb{R}[x_1, \dots, x_n]$), considering f^{e} over $(\bar{\mathbb{R}}, max, +)$, then $\Gamma_{(\mathbb{R}, max, +)}(f^{\mathrm{e}}) \subset \mathbb{R}^{(n+1)}$ is a convex polyhedron having m faces $\overline{\mathcal{D}}_i$ of codimension 1. On the other hand, the tangible part of $\Gamma(f^{\mathrm{e}})$ over $\mathbb{R}^{(n)}$ (i.e. $\Gamma(f^{\mathrm{e}}) \cap \mathbb{R}^{(n+1)}$) consists of the same faces $\mathcal{D}_i$ as those of $\Gamma_{(\mathbb{R}, max, +)}(f^{\mathrm{e}})$ but without their boundaries (in the view of the Euclidean topology). These boundaries "pass" to $\mathbb{R}^{(n)} \times \mathbb{U}$, so in $\mathbb{R}^{(n+1)}$ the faces $\mathcal{D}_i$ are open sets. In other words, using the Euclidean topology for $\mathbb{R}^{(n+1)}$, $\Gamma_{(\mathbb{R}, max, +)}(f^{\mathrm{e}})$ is the closure of $\Gamma(f^{\mathrm{e}}|_{\mathbb{R}^{(n)}})$. Note that this is true only for tangible f^{e}, yet we always have the onto projection $\Gamma(f^{\mathrm{e}}) \longrightarrow \Gamma_{(\mathbb{R}, max, +)}(f^{\mathrm{e}})$ and for any $\Pi_i \in \{\mathbb{R}, \mathbb{U}\} \times \dots \times \{\mathbb{R}, \mathbb{U}\}$, we have the isomorphism*

$$\Gamma(f^{\mathrm{e}}|_{\Pi_i}) \stackrel{\sim}{\longrightarrow} \Gamma_{(\mathbb{R}, max, +)}(f^{\mathrm{e}}). \tag{3}$$

In general, over $\mathbb{T}^{(n)}$, we have 2^n subgraphs, $\Gamma(f^{\mathrm{e}}|_{\Pi_i})$, each is isomorphic to $\Gamma_{(\bar{\mathbb{R}}, max, +)}(f^{\mathrm{e}})$ in $(\bar{\mathbb{R}}, max, +)$.

Suppose $f = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \in \mathbb{T}[x_1, \dots, x_n]$, we identify each monomial $\alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ (for $\mathbf{i} = (i_1, \dots, i_n)$) with the point

$$(i_1, \dots, i_n, \pi(\alpha_{\mathbf{i}})) \in \mathbb{N}^{(n)} \times \mathbb{R} \subset \mathbb{R}^{(n+1)}.$$

Let C_f be the polyhedron determined by the points

$$\{(i_1, \dots, i_n, \pi(\alpha_{\mathbf{i}})) : \mathbf{i} \in \mathrm{Supp}(f)\},$$

which we call the ***vertices*** of C_f, and take the convex hull $\mathcal{CH}_f$ of these vertices. We say that a vertex is tangible (resp. ghost) vertex if it corresponds to a tangible monomial (resp. ghost monomial).

Let $A_{\mathbf{j}} \subset \mathcal{CH}_f$ be the set of points whose first n coordinates are equal. The point $\mathbf{a_j} = (j_1, \dots, j_n, a) \in A_{\mathbf{i}}$ whose $(n+1)$'th coordinate is maximal among all the points of $A_{\mathbf{j}}$ is said to be an ***upper point*** of $\mathcal{CH}_f$. The upper part of $\mathcal{CH}_f$, consisting of all the upper points in $\mathcal{CH}_f$, is called the ***essential complex*** of f and is denoted $\overline{\mathcal{CH}}_f$. The points of $\overline{\mathcal{CH}}_f$ of the form $\{(i_1, \dots, i_n, \pi(\alpha_{\mathbf{i}})) : \mathbf{i} \in \mathbb{N}^n\}$ are called ***lattice points***. For example, when $f = x^2 + 2$, the lattice points are (2,0), (1,1), and (0,2).

Note that the essential complex can be consisted of both tangible and ghost vertices, in particular the essentiality of a vertex (and of monomial, as will be seen later) is independent on being tangible or ghost.

In fact the structure described above can be understood in the more winder context of the Newton polytope [7]. Recall that the Newton polytope Δ_f, of $f = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ is the convex hull of the $\mathbf{i}$'s in Supp(f). By taking the onto projection, which is obtained by deleting the last coordinate, of the non-smooth part of $\overline{\mathcal{CH}}_f$ (that is a polyhedral complex) on Δ_f the induced polyhedral subdivision S_f of Δ_f is obtained. Thereby, a dual geometric object having combinatorial properties is produced. This object plays a major role in classical tropical theory and it being used in many applications [6, 12, 14].

Lemma 2.16. *There is a one-to-one correspondence between the vertices of the essential complex $\overline{\mathcal{CH}}_f$ of f and the essential monomials of f.*

Vertices of the essential complex $\overline{\mathcal{CH}}_f$ are in one-to-one correspondence with the vertices of the induced subdivision S_f of Newton polytope Δ_f. (The latter are precisely the projections of the vertices of $\overline{\mathcal{CH}}_f$ on Δ_f.) The proof is then obtained by the one-to-one correspondence between vertices of S_f and essential monomials of f [14].

Note that $\overline{\mathcal{CH}}_f$ may contain lattice points not corresponding to monomials of the original polynomial f. For instance, take $f = x^2 + 2$, then the lattice point (1,1) does not correspond to a monomial of f. In general, the inessential part of f does not appear in $\overline{\mathcal{CH}}_f$ as vertices but it may appear as points that lie on its faces. A vertex of $\overline{\mathcal{CH}}$ is called ***interior*** if its projection to Δ_f is not a vertex (but is still a vertex of S_f). We say the monomial $h_{\mathbf{i}} = \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ is ***quasi-essential*** for f if $(i_1, \dots, i_n, \pi(\alpha_{\mathbf{i}}))$ lies on $\overline{\mathcal{CH}}_f$ and is not a vertex. This has the following interpretation:

Lemma 2.17. *An inessential monomial is quasi-essential if any (arbitrarily small) increase of its coefficient makes it essential.*

Proof. Let $\alpha_i \mathbf{x}^{\mathbf{i}}$ be a quasi-essential monomial. Any arbitrarily small increasing of its coefficient $\alpha_{\mathbf{i}}$ makes the corresponding lattice point $(i_1, \dots, i_n, \pi(\alpha_{\mathbf{i}}))$ of $\overline{\mathcal{CH}}$ a vertex. Then, by Lemma 2.16, $\alpha_i \mathbf{x}^{\mathbf{i}}$ becomes essential. □

Remark 2.18. *Summarizing the above discussion, we see that the polynomial corresponding to the upper part of $\mathcal{CH}_f$ is precisely the essential part of f, and in particular $\overline{\mathcal{CH}}_{f^{\mathrm{e}}} = \overline{\mathcal{CH}}_f$. Thus, two polynomials are equivalent iff they have the same essential part iff their essential complexes, including their indicated tangible/ghost vertices, are identical.*

Any $f \in \mathbb{T}[x_1, \dots, x_n]$ can be written uniquely as

$$f = f_{\mathrm{r}} \oplus f_{\mathrm{u}}$$

with $f_{\mathrm{r}}, f_{\mathrm{u}} \in \mathbb{R}[x_1, \dots, x_n]$, we call this form the ***(r,u)–decomposition*** of f. To obtain this decomposition, just take each ghost monomial $\alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ (i.e. $\alpha_{\mathbf{i}} \in \mathbb{U}$) and replace it by the two tangible copies $\pi(\alpha_{\mathbf{i}}) \mathbf{x}^{\mathbf{i}}$, i.e.

$$\alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \ \rightsquigarrow \ \pi(\alpha_{\mathbf{i}}) \mathbf{x}^{\mathbf{i}} \oplus \pi(\alpha_{\mathbf{i}}) \mathbf{x}^{\mathbf{i}}. \tag{4}$$

Then, take one copy from each pair of these monomials to create f_{u}, the remaining monomials are ascribed to f_{r}, in particular $f_{\mathrm{u}} = \pi_*(f^{\mathrm{g}})$ and $f = f_{\mathrm{r}}$ if f is tangible. In this view we have the following:

Proposition 2.19. *$f \sim g$ if and only if $\overline{\mathcal{CH}}_{f_{\mathrm{r}}} = \overline{\mathcal{CH}}_{g_{\mathrm{r}}}$ and $\overline{\mathcal{CH}}_{f_{\mathrm{u}}} = \overline{\mathcal{CH}}_{g_{\mathrm{u}}}$.*

Proof. By Corollary 2.14 $f \sim g$ iff $f^{\mathrm{e}} = g^{\mathrm{e}}$, so we may assume f and g are essential. Since (r,u)-decomposition is unique, we get $f_{\mathrm{r}} = g_{\mathrm{r}}$ and $f_{\mathrm{u}} = g_{\mathrm{u}}$, where all f_{r}, g_{r}, f_{u}, and g_{u} are essentials. Thus, $\overline{\mathcal{CH}}_{f_{\mathrm{r}}} = \overline{\mathcal{CH}}_{g_{\mathrm{r}}}$ and $\overline{\mathcal{CH}}_{f_{\mathrm{u}}} = \overline{\mathcal{CH}}_{g_{\mathrm{u}}}$. □

2.3. **The representatives of polynomial classes.** Next we want to identify the best canonical representative of a class of equivalent polynomials. Note that we already have a canonical representative, which is the common essential part of all the class members. Yet, we are looking for a better representative which, as will be seen later, is useful for easy factorization; for this purpose we need the following:

Definition 2.20. *A polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ is called **full** if every lattice point lying on $\overline{\mathcal{CH}}_f$ corresponds to a monomial which is either essential or quasi-essential, and furthermore, every nonessential monomial is ghost; a full polynomial f is **tangible-full** if f^{e} is tangible. The **full closure** $\tilde{f}$ of f is the sum of f^{e} with all the quasi-essential monomials of f taken ghost.*

By this definition, the full closure is unique, and therefore $\tilde{f}$ is also canonical representative of $\mathcal{C}_f$. We call $\tilde{f}$ the ***full representative*** of $\mathcal{C}_f$, this representative plays a major role in our future development.

Remark 2.21. *When f is a polynomial consisting of a single monomial, then f is (full) essential and we always have $f = \tilde{f}$.*

Example 2.22.

(i) $x^2 \oplus 1^\nu x \oplus 0$ is tangible-full essential;

(ii) $x^2 \oplus 1^\nu x \oplus 0^\nu$ is full essential;

(iii) $x^2 \oplus 0^\nu x \oplus 0$ is full but not essential;

(iv) $x^2 \oplus 0x \oplus 0$ is not full since $0x$ is tangible;

(v) $x^2 \oplus 0^\nu x \oplus 0$ is the full closure of $x^2 \oplus 0x \oplus 0$ and $x^2 \oplus 0$.

By the construction of $\overline{\mathcal{CH}}_f$ and the fact that the full polynomials contain all the monomials corresponding to lattice points of their essential complexes we have the following:

Lemma 2.23. *Any full polynomial $f \in \mathbb{T}[x]$ (which is not a monomial) corresponds to a descending sequence of tangible elements $m_1, \dots, m_t$, where $t = \deg f - \underline{\deg} f$, which is defined uniquely by the slopes of the series of edges $e_1, \dots, e_t$ of $\overline{\mathcal{CH}}_f \subset \mathbb{R}^{(2)}$, each e_i is determined by the pair $(i-1, \pi(\alpha_{i-1}))$ and $(i, \pi(\alpha_i))$.*

The descending sequence of tangible elements $m_1, \dots, m_t$ is denoted by M_f. Note that M_f is not necessarily strictly descending and it might have identical adjacent elements. The sequence of edges is denoted by E_f.

Proof. Recall that since f is full, it has exactly $t+1$ monomials, and by the construction of $\overline{\mathcal{CH}}_f$ it also contains $t+1$ lattice points (not all of them need to be vertices). The sequence M_f is descending due to the convexity of $\overline{\mathcal{CH}}_f$. Since otherwise, assume $m_{i+1} > m_i$, for some $i = 1, \dots, t-1$, and observe the corresponding lattice points

$$(i-1, \pi(\alpha_{i-1})), \ (i, \pi(\alpha_i)), \ (i+1, \pi(\alpha_{i+1})),$$

which by assumption should satisfy

$$\pi(\alpha_{i+1}) - \pi(\alpha_i) > \pi(\alpha_i) - \pi(\alpha_{i-1}).$$

(Here use the standard notation to describe the slopes of the edges since we work only on $\mathbb{R}^{(2)}$.) But this means, due to the convexity of $\overline{\mathcal{CH}}_f$, that $(i, \pi(\alpha_i)) \notin \overline{\mathcal{CH}}_f$ and thus, is not a lattice point. □

Remark 2.24. *Clearly, the lemma holds true for the full closure of any $f \in \mathbb{T}[x]$. Moreover, one can state Lemma 2.23 for any essential polynomial $f \in \mathbb{T}[x]$, but in this case the number of monomials of f will be less or equal to t.*

Definition 2.25. *The **reduced domain** $\tilde{\mathbb{T}}[x_1, \dots, x_n]$ of $\mathbb{T}[x_1, \dots, x_n]$ is the set of full elements, where addition and multiplication are defined by taking the full representative of the respective sum or product in $\mathbb{T}[x_1, \dots, x_n]$. In other words, we define*

$$f \oplus g = \widetilde{f \oplus g}, \qquad f \odot g = \widetilde{f \odot g},$$

*for $f, g \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$, and sometimes call $\tilde{\mathbb{T}}[x_1, \dots, x_n]$ the **reduced polynomial semiring**. Accordingly, $\tilde{\mathbb{R}}[x_1, \dots, x_n]$ is the set of tangible-full elements, and $\tilde{\mathbb{U}}[x_1, \dots, x_n]$ is the set of full elements, all of whose coefficients are ghosts.*

Usually we omit the symbol $\odot$ and, for short, write $\tilde{f}\tilde{g}$ for $\tilde{f} \odot \tilde{g}$. Since $\tilde{f}$ is unique (and thus canonical) representative of a class $\mathcal{C}_f$, we have $\tilde{\mathbb{T}}[x_1, \dots, x_n] \cong \mathbb{T}[x_1, \dots, x_n]/_{\sim}$ and therefore get the one-to-one correspondence

$$\tilde{\mathbb{T}}[x_1, \dots, x_n] \longrightarrow \mathcal{F}(\mathbb{T}^{(n)})$$

between full polynomials and polynomial function. In the rest of our exposition we appeal to the reduced domain $\tilde{\mathbb{T}}[x_1, \dots, x_n]$.

Definition 2.26. *A polynomial $\tilde{f} \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$ is said to be **reducible** if $\tilde{f} = \tilde{g}\tilde{h}$ for some nonconstant $\tilde{g}, \tilde{h} \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$, otherwise $\tilde{f}$ called is **irreducible**. The product $\tilde{f} = \tilde{q}_1 \cdots \tilde{q}_s$ is called a maximal **factorization** of $\tilde{f}$ **into irreducibles** if each of the $\tilde{q}_i$'s is irreducible. We say that $\tilde{g} \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$ **divides** $\tilde{f}$ if $\tilde{f} = \tilde{q}\tilde{g}$ for some $\tilde{q} \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$.*

We instantly encounter new difficulties.

(i) Not every nonlinear polynomial $\tilde{f} \in \tilde{\mathbb{T}}[x]$ is reducible; for example one can easily check that $\tilde{f} = x^2 + 2^\nu x + 3$ is irreducible.

(ii) The factorization into irreducibles need not necessarily be unique; for example $x^2 \oplus 2^\nu = (x \oplus 1^\nu)^2$ and at the same time $x^2 \oplus 2^\nu = (x \oplus 1)(x \oplus 1^\nu)$, while $x \oplus 1^\nu \neq x \oplus 1$.

(iii) a can be a root of a polynomial f, but $(x \oplus a) \nmid f$, for example 1 is a root of $f = x^2 \oplus x \oplus 2$ but $(x \oplus 1) \nmid f$.

Proposition 2.27. *The polynomial $\tilde{g}$ divides $\tilde{f}$, i.e. $\tilde{g}|\tilde{f}$, iff the essential part of $\tilde{q}\tilde{g}$ is the essential part of $\tilde{f}$ for some $\tilde{q}$, which means $(\tilde{f} \oplus \tilde{q}\tilde{g})^e$ is ghost.*

Proof. $\tilde{g}|\tilde{f}$ iff $\tilde{f} = \tilde{g}\tilde{q}$, for some $\tilde{q}$, which means $f \sim gq$ for any $f \in C_{\tilde{f}}$, $gq \in C_{\tilde{g}\tilde{q}}$. Then, by Corollary 2.14 we get $f^e = (gq)^e$. ☐

2.4. **Tropical polynomials in one indeterminate.** The use of the reduced domain $\tilde{\mathbb{T}}[x]$ makes the development of the theory of polynomials in one intermediate quite close to the classical commutative theory. We start our exposition with tangible polynomials and then extend the results to whole $\tilde{\mathbb{T}}[x]$.

Remark 2.28. *Suppose α_i, α_j, $\alpha_k \in \mathbb{R}$ are three tangible coefficients of $f = \bigoplus_i \alpha_i x^i$ in $\mathbb{T}[x]$, where $i < j < k$, then $\alpha_j \in \overline{\mathcal{CH}}(f)$ only if*

$$\alpha_j \geq \frac{\alpha_i \cdot (k-j) + \alpha_k \cdot (j-i)}{k-i}.$$

(The arithmetic operations here are the classical ones.) This relation is simply derived form the convexity of $\mathcal{CH}$, and the fact that $\overline{\mathcal{CH}}$ is its upper part.

Theorem 2.29. *Any full-tangible polynomial $\tilde{f} \in \tilde{\mathbb{T}}[x]$ is factored uniquely into a product of tangible linear polynomials.*

Proof. Proof by induction on $n = \deg(\tilde{f})$. Dividing out by α_n, we may assume that $\tilde{f}$ monic. The assertion is obvious for $n = 1$. For $n = 2$, given $\tilde{f}(x) = x^2 \oplus \alpha_1 x \oplus \alpha_0$, cf. Remark 2.18, we have:

$$\tilde{f} = \begin{cases} (x \oplus \sqrt{\alpha_0})^2, & \alpha_1 \preceq \sqrt{\alpha_0}; \\ (x \oplus \frac{\alpha_1}{\alpha_0})(x \oplus \alpha_0), & \alpha_1 \succ \sqrt{\alpha_0}. \end{cases}$$

(Here, $\sqrt{\alpha}$ stands for the tropical square root, which, in the standard meaning, is just $\frac{\alpha}{2}$ up to ghost indication.)

Suppose $n > 2$, if $\tilde{f} = x^j \tilde{g}$, for some $j < n$ we are done by the induction assumption. Otherwise

$$\tilde{f} = x^n \oplus \alpha_{n-1} x^{n-1} \oplus \cdots \oplus \alpha_1 x \oplus \alpha_0,$$

with $\alpha_0 \neq -\infty$. Recall that since f is full, $\alpha_i \neq -\infty$ for all $i = 0, \dots, n$, and each (i, α_i) appears on $\overline{\mathcal{CH}}(\tilde{f})$, but (i, α_i) is not necessarily a vertex. We claim that

$$\tilde{f} = (x \oplus \alpha_{n-1}) \left(x^{n-1} \oplus \frac{\alpha_{n-2}}{\alpha_{n-1}} x^{n-2} \oplus \cdots \oplus \frac{\alpha_1}{\alpha_{n-1}} x \oplus \frac{\alpha_0}{\alpha_{n-1}} \right).$$

This completes the proof by induction.

To proof this we need to show that $\frac{\alpha_{i-1}}{\alpha_{n-1}} \prec \alpha_i$ for any $i = 1, \dots, n-1$. Recall that $\tilde{f}$ is monic, that is $\alpha_n = 0$. Assume $\frac{\alpha_{i-1}}{\alpha_{n-1}} \succeq \alpha_i = \frac{\alpha_i}{\alpha_n}$, and thus $\frac{\alpha_n}{\alpha_{n-1}} \succeq \frac{\alpha_i}{\alpha_{i-1}}$. If the inequality is equality, it contradicts the essentially of $\alpha_i x^i$ for $\tilde{f}$ (since then, it would be quasi-essential). Otherwise, it contradicts the proprieties in which the sequence M_f of the edges' slopes is descending (cf. Lemma 2.23).

Conversely, any different products of tangible linear polynomials clearly produces a different essential complex, and thus the factorization of a tangible-full polynomial into linear factors is unique. □

The above theorem can be implemented in the following algorithm:

Algorithm 2.30. (Decomposition algorithm) *Let $\tilde{f} = \bigoplus_i \alpha_i x^i$ be a full-tangible polynomial in $\tilde{\mathbb{T}}[x]$, the algorithm acts recursively:*

(i) if $\tilde{f}$ is not monic set $\tilde{f}^{(1)} = \bigoplus_i (\alpha_i/\alpha_n) x^i$ and apply the algorithm for $\tilde{f}^{(1)}$, otherwise

(ii) write $\tilde{f} = (x \oplus \alpha_{n-1})\tilde{f}^{(1)} = (x \oplus \alpha_{n-1})(x^{n-1} \oplus \frac{\alpha_{n-2}}{\alpha_{n-1}} x^{n-2} \oplus \cdots \oplus \frac{\alpha_1}{\alpha_{n-1}} x \oplus \frac{\alpha_0}{\alpha_{n-1}})$,

(iii) apply the algorithm again to $\tilde{f}^{(1)}$.

The algorithm is applied for full-tangible polynomial, therefore:

Corollary 2.31. *The factorization of full-tangible polynomials is unique, in particular, each is factored uniquely into linear terms.*

Remark 2.32. *Any linear factor of $\tilde{f}$ determines a root of $\tilde{f}$, indeed, assume $(x \oplus a)$ is a factor of $\tilde{f}$ then $\tilde{f} = (x \oplus a)\tilde{g}$ and thus $\tilde{f}(a) = (a \oplus a)\tilde{g}(a) \in \bar{\mathbb{U}}$. The factorization of $\tilde{f}$ may contain identical components, is such a case the multiplicity of a root is defined to be the number of the corresponding (identical) components in the factorization.*

Example 2.33. *The algorithm is simulated for $\tilde{f} = 2x^4 \oplus 5x^3 \oplus 5x^2 \oplus 3x \oplus 0$:*

(1) $\tilde{f} = 2(x^4 \oplus 3x^3 \oplus 3x^2 \oplus 1x \oplus (-2))$

(2) $\quad = 2(x \oplus 3)(x^3 \oplus x^2 \oplus (-2)x \oplus (-4))$

(3) $\quad = 2(x \oplus 3)(x^3 \oplus 0)(x^2 \oplus (-2)x \oplus (-4))$

(4) $\quad = 2(x \oplus 3)(x^3 \oplus 0)(x \oplus (-2))(x \oplus (-2))$.

Thus, 3, 0, and -2 are roots of $\tilde{f}$, where -2 has multiplicity 2. Since $\tilde{f}$ is full-tangible then the above factorization to product of linear terms is unique.

Next we look at nontangible full polynomials. Let us call a polynomial $\tilde{f} = \bigoplus_{i=0}^{t} \alpha_i x^i$ ***semitangible-full*** if $\tilde{f}$ is full with α_t and α_0 tangible, but α_i are ghost for all $0 < i < t$. Dividing out by α_t, we may assume that any semitangible-full polynomial is monic.

Observation 2.34. *Recall that the restriction of the epimorphism $\pi : \mathbb{T} \to \bar{\mathbb{R}}$ to $\bar{\mathbb{R}}$ is the identity map while for any $a^\nu \in \mathbb{U}$ is given by $\pi(a^\nu) = a$ (see (2)). Suppose $\tilde{f} = \bigoplus_{i=0}^{t} \alpha_i x^i$ is monic semitangible-full. Then taking $\beta_i = \pi(\frac{\alpha_{t-1}}{\alpha_i})$, we have*

$$\tilde{f} = (x^2 \oplus \alpha_{t-1} x \oplus \beta_{t-1})\tilde{g}, \tag{5}$$

where $\tilde{g} = x^{t-2} \oplus \bigoplus_{i=1}^{t-3} \beta_i^\nu x^i \oplus \frac{\alpha_0}{\beta_{t-1}}$. Note that this factorization is not unique; we could factor out any two roots of $\pi_*(\tilde{f})$ to produce the first factor, just as long as they are not both maximal or both minimal.

Suppose $\alpha_t = 0^\nu$, namely $\tilde{f}$ is not semitangible-full, and let $\beta = \pi(\alpha_{t-1})$ then

$$\tilde{f} = (x^\nu \oplus \beta) \bigoplus_{i=0}^{t-1} \frac{\alpha_i}{\beta} x^i.$$

Therefore, whenever the leading terms are ghost we can use Observation 2.34 to factor out linear factors $(x^\nu \oplus a)$ until we reach a tangible leading term. But if we do this twice, we observe for $a \succ b$ that

$$(x^\nu \oplus a)(x^\nu \oplus b) = 0^\nu x^2 \oplus a^\nu x \oplus ab = (x \oplus a)(x^\nu \oplus b).$$

Thus, we can always make sure that our factorization has at most one linear factor $x^\nu \oplus a$ (for a tangible, and this is the maximal a of those which appear in the linear factors $x^\nu \oplus a$).

Likewise, when the constant term is ghost we can factor out some linear factor $x \oplus b^{\nu}$, and arrange for the constant term to be tangible. Since, in the above notation, $a^{\nu} \succ b^{\nu}$, we also have

$$(x^{\nu} \oplus a)(x \oplus b^{\nu}) = 0^{\nu} x^2 \oplus ax \oplus (ab)^{\nu}.$$

Iterating, we have the following result:

Proposition 2.35. *Every full polynomial is the product of at most one linear factor of the form $(x^{\nu} \oplus a)$, at most one linear factor of the form $(x \oplus b^{\nu})$, and a semitangible-full polynomial (which can be factored as in (5)).*

Putting together Theorem 2.29 and Observation 2.34, we see that any irreducible full polynomial must have no tangible interior vertices, and at most one interior lattice point (which must be a nontangible vertex), and thus must be quadratic, of the form $\alpha_2 x^2 + \alpha_1^{\nu} x + \alpha_0$, where $\alpha_1^{\nu} x$ is essential. In conjunction with Corollary 2.31 and Proposition 2.35, we have proved the following result:

Theorem 2.36. *Any full polynomial is the unique product of a full tangible polynomial (which can be factored uniquely into tangible linear factors), a linear factor $(x^{\nu} \oplus a)$, a linear factor $(x \oplus a^{\nu})$, and a semitangible-full polynomial, and this factorization is unique.*

Proof. Just factor at each tangible monomial, and multiply together the full tangible factors. □

Note 2.37. *Recall that using the (r,u)-decomposition any polynomial $\tilde{f} \in \tilde{\mathbb{T}}[x]$ can be written uniquely as $\tilde{f} = f_{\mathrm{r}} \oplus f_{\mathrm{u}}$, where f_{r} and f_{u} are tangible polynomials. Using Theorem 2.29, each of these components can be factored uniquely to a product of linear factors, therefore $\tilde{f}$ can be written as*

$$\tilde{f}(x) = \bigodot_j (x \oplus a_j)^{i_j} \oplus \bigodot_k (x \oplus b_k)^{h_k}$$

and this decomposition is unique.

2.5. **Tropical polynomials in several indeterminates.** Polynomials in $\tilde{\mathbb{T}}[x_1, \dots, x_n]$ have some special properties, mainly due to their combinatorial nature. (Recall that $\mathbf{i} = (i_1, \dots, i_n)$ stands for a multi-index and $\mathbf{x} = (x_1, \dots, x_n)$.)

Proposition 2.38. *Let $\tilde{f} = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ and let $\tilde{g} = \bigoplus_{\mathbf{i}} (\alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}})^k$ for some positive $k \in \mathbb{N}$. Given $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{T}^{(n)}$, assume $\tilde{f}(\mathbf{a}) = h_{\mathbf{i}}(\mathbf{a})$ for some monomial $h_{\mathbf{i}}$ of $\tilde{f}$, then $\tilde{g}(\mathbf{a}) = (h_{\mathbf{i}}(\mathbf{a}))^k$.*

Proof. Assume $(\alpha_{\mathbf{i}} \mathbf{a}^{\mathbf{i}})^k \prec (\alpha_{\mathbf{j}} \mathbf{a}^{\mathbf{j}})^k$, but this means $\alpha_{\mathbf{i}} \mathbf{a}^{\mathbf{i}} \prec \alpha_{\mathbf{j}} \mathbf{a}^{\mathbf{j}}$ – a contradiction. □

Proposition 2.39. *For any $\tilde{f}, \tilde{g} \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$ and any positive $k \in \mathbb{N}$, $(\tilde{f} \oplus \tilde{g})^k = \tilde{f}^k \oplus \tilde{g}^k$.*

Proof. Expand the product $(\tilde{f} \oplus \tilde{g})^k$ and observe a mixed component $\tilde{f}^i \tilde{g}^j$, with $i + j = k$ and $i, j \neq 0$. Pick $\mathbf{a} \in \mathbb{T}^{(n)}$ and assume $\tilde{f}(\mathbf{a}) \succeq \tilde{g}(\mathbf{a})$, then $\tilde{f}(\mathbf{a})^i \tilde{g}(\mathbf{a})^j \preceq \tilde{f}(\mathbf{a})^i \tilde{f}(\mathbf{a})^j = \tilde{f}(\mathbf{a})^k$. On the other hand, if $\tilde{f}(\mathbf{a}) \preceq \tilde{g}(\mathbf{a})$, then $\tilde{f}(\mathbf{a})^i \tilde{g}(\mathbf{a})^j \preceq \tilde{g}(\mathbf{a})^k$. This means $\tilde{f}^i \tilde{g}^j$ is inessential. □

Theorem 2.40. *Let $\tilde{f} = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ and let $\tilde{g} = \bigoplus_{\mathbf{i}} (\alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}})^k$ for some positive $k \in \mathbb{N}$, then $\tilde{f}^k = \tilde{g}$.*

Proof. By the law of polynomial multiplication, it is clear that as a polynomial $\tilde{f}^k$ has more monomials than $\tilde{g}$ (i.e. all the monomials of $\tilde{g}$ appear also in $\tilde{f}^k$). If f have a single monomial we are done. Otherwise, pick a monomial h_i of $\tilde{f}$ and write $\tilde{f} = h_{\mathbf{i}} \oplus \tilde{f}_1$. Using Proposition 2.39, $\tilde{f}^k = h_{\mathbf{i}}^k \oplus \tilde{f}_1^k$. Now proceed inductively on $\tilde{f}_1$ to complete the proof. □

Example 2.41. *Let $\tilde{f}(x, y) = x \oplus y$ then, by taking the full closures we have*

$$\tilde{f}^2(x, y) = (x \oplus y)^2 = x^2 \oplus 0^{\nu} xy \oplus y^2 = \widetilde{x^2 \oplus y^2}.$$

3. Tropical Algebraic Sets and Com-sets

As in the classical theory, using the notion of algebraic sets we establish the connection between polynomials and tropical geometry. It turns out that by introducing a new notion of tropical algebraic com-set the development becomes much easier and allows the formulation of tropical analogues to classical results, the tropical Nullstellensatz will be our main example.

Despite our main interest, from the point of view of commutative algebra, is mainly in the tropical reduced domain (cf. Definition 2.25), the development in this and in the next section is being made in the framework of the extended tropical polynomial semiring $\mathbb{T}[x_1, \dots, x_n]$ that is much wider.

3.1. Tropical algebraic sets.

Definition 3.1. *The* ***tropical algebraic set*** *of a non empty subset $F \subseteq \mathbb{T}[x_1, \dots, x_n]$ is defined to be*

$$\mathcal{Z}_{\rm tr}(F) = \{\mathbf{a} \in \mathbb{T}^{(n)} \mid f(\mathbf{a}) \in \bar{\mathbb{U}}, \ \ \forall f \in F\}. \tag{6}$$

$\mathcal{Z}_{\rm tr}(F)$ is sometimes called tropical set, for short, and we call its elements ***roots****, or* ***zeros****, of F. We say that a subset $Z \subset \mathbb{T}^{(n)}$ is* ***algebraic****, in the topical sense, if $Z = \mathcal{Z}_{\rm tr}(F)$ for a suitable $F \subseteq \mathbb{T}[x_1, \dots, x_n]$.*

Note that, if $\mathbf{a} \in \mathcal{Z}_{\rm tr}(F)$ we necessarily have $\mathbf{a}^\nu \in \mathcal{Z}_{\rm tr}(F)$, but the converse claim is not true.

Remark 3.2. *In our topology, over closed set, the operations $\oplus$ and $\odot$ are continuous, and the sets $\{-\infty\}$ and $\bar{\mathbb{U}}$ are closed. Accordingly, tropical polynomials are continuous as well, cf. Definition 1.1.*

Clearly, for any $f \in \mathbb{T}[x]$, $\mathcal{Z}_{\rm tr}(f)$ is just the set of roots of f. Analogously, we consider a tropical algebraic set $\mathcal{Z}_{\rm tr}(F)$ as the set of common solutions of all members of F. Therefore, when $f(\mathbf{a}) \in \bar{\mathbb{U}}$, we keep the familiar terminology and say that f ***vanishes*** at $\mathbf{a}$, or equivalently, that f gives value in $\bar{\mathbb{U}}$ at $\mathbf{a}$. When F has a single member, then $\mathcal{Z}_{\rm tr}(f) \subset \mathbb{T}^{(n)}$ is called a ***tropical hypersurface***; as an example see Fig. 1.

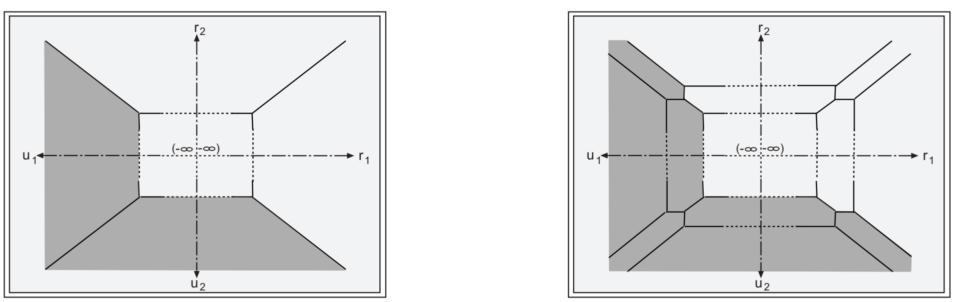

Figure 1. Tropical line and tropical conic in $\mathbb{T}^{(2)}$.

Example 3.3. *Let $f_1 = x_1 \oplus 1$ and $f_2 = x_2 \oplus 1$ be polynomials in $\mathbb{T}[x_1, x_2]$, then*

$$\mathcal{Z}_{\rm tr}(f_1) = \{(1, y) \mid y \in \mathbb{T}\} \ \cup \ \{(x, y) \mid 1 \preceq x \in \bar{\mathbb{U}}, y \in \mathbb{T}\},$$

while the tropical set of f_1 and f_2 is the union:

$$\begin{aligned}\mathcal{Z}_{\rm tr}(f_1, f_2) = \ \ & \{(1,1)\} \ \cup \ \{(1, y) \mid 1 \preceq y \in \bar{\mathbb{U}}\} \cup \\ & \{(x, 1) \mid 1 \preceq x \in \bar{\mathbb{U}}\} \ \cup \ \{(x, y) \mid 1 \preceq x, y \in \bar{\mathbb{U}}\}.\end{aligned}$$

Here, $(1, 1)$ is the only common tangible zero.

Lemma 3.4. *Assume Z is tropical algebraic set then Z is closed set in the topology of $\mathbb{T}^{(n)}$.*

Proof. We may assume $Z = \mathcal{Z}_{\rm tr}(f)$, for $f = \bigoplus_i f_i$ a sum of monomials f_i's, is a tropical hypersurface, otherwise $Z = \mathcal{Z}_{\rm tr}(F)$ will be an intersection of closed sets. Pick a point $\mathbf{a} \notin Z$ in the complement of Z, then we have $f(\mathbf{a}) = f_i(\mathbf{a}) \in \mathbb{R}$, for some monomial f_i. Assume first, that all the coordinates of $\mathbf{a}$ are tangible. In the classical sense f_i is smooth and linear, so there is an open neighborhood $U \subset \mathbb{R}^{(n)}$ of $\mathbf{a}$ such that $f(\mathbf{b}) = f_i(\mathbf{b})$ for each $\mathbf{b} \in U$. This implies the complement is open.

If $\mathbf{a}$ has a ghost coordinate then f_i is a tangible constant, since otherwise $\mathbf{a}$ would be in Z, them use the same argument of the previous paragraph. $\square$

The next lemma determines the operations on tropical algebraic sets.

Lemma 3.5. *Assume $Z', Z'' \subseteq \mathbb{T}^{(n)}$ are tropical sets, then so are $Z' \cap Z''$ and $Z' \cup Z''$.*

Proof. Suppose $Z' = \mathcal{Z}_{\mathrm{tr}}(F)$ and $Z'' = \mathcal{Z}_{\mathrm{tr}}(G)$, where $F, G \subset \mathbb{T}[x_1, \dots, x_n]$ are nonempty. We claim that

$$Z' \cap Z'' = \mathcal{Z}_{\mathrm{tr}}(F \cup G) \qquad \text{and} \qquad Z' \cup Z'' = \mathcal{Z}_{\mathrm{tr}}(fg \ : \ f \in F, g \in G).$$

The left part is by definition; assume $\mathbf{a} \in Z' \cap Z''$ then $f(\mathbf{a}) \in \bar{\mathbb{U}}$ and $g(\mathbf{a}) \in \bar{\mathbb{U}}$ for each $f \in F$ and $g \in G$, which is the same as all the members of $F \cup G$ give values in $\bar{\mathbb{U}}$.

For the right part, if $\mathbf{a} \in Z'$, then all the f's of F give values in $\bar{\mathbb{U}}$ at $\mathbf{a}$, which implies that at $\mathbf{a}$ all the products fg also give values in $\bar{\mathbb{U}}$. Thus $Z' \subset \mathcal{Z}_{\mathrm{tr}}(fg)$, and $Z'' \subset \mathcal{Z}_{\mathrm{tr}}(fg)$ follows similarly. This proves the containment $Z' \cup Z'' \subset \mathcal{Z}_{\mathrm{tr}}(fg)$. Conversely, assume $\mathbf{a} \in Z(fg)$. If $\mathbf{a} \in Z'$ we are done; otherwise $f'(\mathbf{a}) \notin \bar{\mathbb{U}}$ for some $f' \in F$, i.e. $f'(\mathbf{a}) \in \mathbb{R}$. But, since at $\mathbf{a}$, $f'g$ gives value in $\bar{\mathbb{U}}$ for all $g \in G$, then g must give value in $\bar{\mathbb{U}}$ at $\mathbf{a}$. This proves that $\mathbf{a} \in Z''$, and hence $\mathcal{Z}_{\mathrm{tr}}(fg) \subset Z' \cup Z''$. □

Remark 3.6. *From the Lemma and Proposition 2.38 we can conclude that $\mathcal{Z}_{\mathrm{tr}}(f) = \mathcal{Z}_{\mathrm{tr}}(f^k)$ for each $f \in \mathbb{T}[x_1, \dots, x_n]$ and any positive $k \in \mathbb{N}$.*

Remark 3.7. ***Tropicalization and tropical sets:*** *Based on Kapranov's Theorem* [3, 14]*, the classical tropical hypersurface over $(\bar{\mathbb{R}}, max, +)$ is the corner locus (i.e domain of non-smoothness) of a convex piecewise affine linear function of the form*

$$\mathcal{N}_f = \max_{\mathbf{i}}(Val(c_{\mathbf{i}}) + \mathbf{i}.\mathbf{x}) \tag{7}$$

where $\mathbf{i}.\mathbf{x}$ *stands for the standard inner product and the $c_{\mathbf{i}}$'s are coefficients of a "superior" polynomial $f \in \mathbb{K}[z_1, \dots, z_n]$ over a non Archimedean field $\mathbb{K}$ with a real valuation* Val. *Namely, a point* $\mathbf{a}$ *belongs to the corner locus exactly when two components of $\mathcal{N}_f$ simultaneously attain the maximum. This is precisely our interpretation of the tropical addition in view of Definition 3.1.*

In other words, one can consider $\mathcal{N}_f$ as a tangible polynomial in $\mathbb{T}[x_1, \dots, x_n]$, then its corner locus with respect to $(\bar{\mathbb{R}}, max, +)$, is exactly the restriction of $\mathcal{Z}_{\mathrm{tr}}(\mathcal{N}_f)$ to $\mathbb{R}^{(n)}$, cf. Note 2.15.

3.2. **Tropical algebraic com-sets.** The next object we introduce is central for our future development. Given a tropical algebraic set $Z \subset \mathbb{T}^{(n)}$ we denote the complement of Z by Z^c, that is

$$Z^c = \mathbb{T}^{(n)} \setminus Z \ .$$

Recall that Z is a closed set in the topology of $\mathbb{T}$, so for our purpose, connectedness of subsets of $\mathbb{T}^{(n)}$ is well defined.

Definition 3.8. *Given a tropical algebraic set $\mathcal{Z}_{\mathrm{tr}}(f) \subset \mathbb{T}^{(n)}$, $f \in \mathbb{T}[x_1, \dots, x_n]$, the set*

$$\mathcal{C}_{\mathrm{tr}}(f) = \{D_f \mid D_f \text{ is a connected component of } \mathcal{Z}_{\mathrm{tr}}(f)^c\}, \tag{8}$$

is defined to be the ***tropical algebraic com-set*** *(or tropical com-set, for short) of f. A set $C = \{D_t \mid D_t \subset \mathbb{T}^{(n)}\}$ is said to be algebraic com-set if $C = \mathcal{C}_{\mathrm{tr}}(f)$ for some $f \in \mathbb{T}[x_1, \dots, x_n]$.*

Accordingly, any member D_f of $\mathcal{C}_{\mathrm{tr}}(f)$ is an open set, cf. Remark 3.2. Since the two enlarged copies of $\mathbb{R}$ (i.e. $\bar{\mathbb{R}}$ and $\bar{\mathbb{U}}$) are glued along $-\infty$, the connectivity of components may comprise paths through $-\infty$; for instance, the set

$$D_f = \{x \in \mathbb{T} \mid a^{\nu} \succ x \prec b\}$$

is a proper connected component with $-\infty \in D_f$.

Example 3.9. *The tropical algebraic com-set of $f = x \oplus a$ is*

$$\mathcal{C}_{\mathrm{tr}}(f) = \{\{x \in \mathbb{T} \mid a^{\nu} \succ x \prec a\}, \ \{x \in \mathbb{R} \mid a \prec x\}\} \, .$$

Remark 3.10. *In view of Definition 4.10, over each $D_f \cap \mathbb{R}^{(n)}$, $D_f \in \mathcal{C}_{\mathrm{tr}}(f)$, f is a continuous smooth linear function (in the standard meaning), where for all $\mathbf{a} \in D_f$, either $f(\mathbf{a}) \in \mathbb{R}$ or $f(\mathbf{a}) \in \mathbb{U}$.*

To emphasize, a tropical com-set is the set of connected components (each is a set by itself) of the complement of a tropical algebraic set. For the forthcoming development, we define the union

$$\widetilde{\mathcal{C}_{\rm tr}}(f) = \bigcup_{D_f \in \mathcal{C}_{\rm tr}(f)} D_f \tag{9}$$

of all the members of $\mathcal{C}_{\rm tr}(f)$. Therefore, $\widetilde{\mathcal{C}_{\rm tr}}(f) \subseteq \mathbb{T}^{(n)}$ and $\mathcal{Z}_{\rm tr}(f)^c = \widetilde{\mathcal{C}_{\rm tr}}(f)$.

Example 3.11. *Here are some typical cases, assume $f \in \mathbb{T}[x]$ then:*

(i) if f is a tangible constant, i.e. $f \in \mathbb{R}$, then $\mathcal{Z}_{\rm tr}(f) = \emptyset$ and $\mathcal{C}_{\rm tr}(f) = \{\mathbb{T}\}$;

(ii) if f is a ghost polynomial, $\mathcal{Z}_{\rm tr}(f) = \mathbb{T}$ and $\mathcal{C}_{\rm tr}(f) = \{\emptyset\}$;

(iii) if $f = x$ then $\mathcal{Z}_{\rm tr}(f) = \bar{\mathbb{U}}$ and $\mathcal{C}_{\rm tr}(f) = \{\mathbb{R}\}$;

(iv) when $f = -\infty$ then $\mathcal{Z}_{\rm tr}(f) = \mathbb{T}$ and $\mathcal{C}_{\rm tr}(f) = \{\emptyset\}$.

We also have the analogous properties to that of tropical algebraic sets:

Lemma 3.12. *For any $f, g \in \mathbb{T}[x_1, \dots, x_n]$:*

(i) $\mathcal{C}_{\rm tr}(f) = \mathcal{C}_{\rm tr}(f^k)$;

(ii) $\mathcal{C}_{\rm tr}(fg) = \{D_f \cap D_g \neq \emptyset \mid D_f \in \mathcal{C}_{\rm tr}(f), D_g \in \mathcal{C}_{\rm tr}(g)\}$;

(iii) for any $D_{fg} \in \mathcal{C}_{\rm tr}(fg)$ there exists $D_f \in \mathcal{C}_{\rm tr}(f)$ such that $D_{fg} \subseteq D_f$.

Proof. (i) is obtained directly from the equality $\mathcal{Z}_{\rm tr}(f) = \mathcal{Z}_{\rm tr}(f^k)$ (cf. Remark 3.6). (ii) By definition:

$$\mathcal{C}_{\rm tr}(fg) = \{D \mid D \text{ is a connected component of } \mathcal{Z}^c_{\rm tr}(fg)\},$$

and thus, $\mathcal{Z}_{\rm tr}(fg)^c = \mathcal{Z}_{\rm tr}(f)^c \cap \mathcal{Z}_{\rm tr}(g)^c$. Since $\mathcal{Z}_{\rm tr}(fg) = \mathcal{Z}_{\rm tr}(f) \cup \mathcal{Z}_{\rm tr}(g)$, cf. Lemma 3.5, then $\mathcal{Z}_{\rm tr}(fg)^c$ consists of all nonempty intersections of connected components from $\mathcal{Z}_{\rm tr}(f)^c$ and from $\mathcal{Z}_{\rm tr}(g)^c$; (iii) is then obtained directly from (ii). □

We generalize Definition 4.10 as follows:

Definition 3.13. *The tropical algebraic com-set of a nonempty $F \subseteq \mathbb{T}[x_1, \dots, x_n]$ is defined as*

$$\mathcal{C}_{\rm tr}(F) = \bigcup_{f \in F} \mathcal{C}_{\rm tr}(f).$$

(This union is not a disjoint union and identical components have a single instance in $\mathcal{C}_{\rm tr}(F)$.) We say that $C \subseteq \mathbb{T}^{(n)}$ is tropical algebraic com-set if $C = \mathcal{C}_{\rm tr}(F)$ for a suitable $F \subseteq \mathbb{T}[x_1, \dots, x_n]$.

Definition 3.14. *Given tropical algebraic com-sets $C', C'' \subset \mathbb{T}^{(n)}$ we define the intersection $\sqcap$ to be*

$$C' \sqcap C'' = \{D' \cap D'' \neq \emptyset \mid D' \in C', D'' \in C''\}. \tag{10}$$

The inclusion $\sqsubseteq$ is defined by the rule:

$$C' \sqsubseteq C'' \iff \text{for each } D' \in C' \text{ there exists } D'' \in C'' \text{ such that } D' \subseteq D''. \tag{11}$$

Lemma 3.15. *Assume $C', C'' \subset \mathbb{T}^{(n)}$ are tropical algebraic com-sets, then so are $C' \cup C''$ and $C' \sqcap C''$.*

Proof. Suppose $C' = \mathcal{C}_{\rm tr}(F)$ and $C'' = \mathcal{C}_{\rm tr}(G)$, $F, G \subset \mathbb{T}[x_1, \dots, x_n]$, are not empties, we claim that

$$C' \cup C'' = C(F \cup G), \qquad \textit{and} \qquad C' \sqcap C'' = \mathcal{C}_{\rm tr}(fg \ : \ f \in F, g \in G).$$

Indeed, the left equality is by definition while the right is the generalization of Lemma 3.12 in terms of Equation (10). □

4. Tropical Ideals

Ideals are main structure in the classical theory; we develop this notion in the tropical sense. As will be seen, the tropical ideal is an analogous of the classical one. Later, we will study the main properties of tropical ideals and realize how they relate to tropical sets and com-sets.

4.1. **Definition and properties.**

Definition 4.1. *A subset $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ is a* ***tropical ideal*** *of polynomials if it satisfies:*

(i) $-\infty \in \mathfrak{a}$*;*

(ii) if $f, g \in \mathfrak{a}$*, then* $f \oplus g \in \mathfrak{a}$*;*

(iii) if $f \in \mathfrak{a}$*, and* $h \in \mathbb{T}[x_1, \dots, x_n]$*, then* $hf \in \mathfrak{a}$*.*

A ideal is called ***tangible ideal*** *if all of its elements are tangible and is called* ***ghost ideal*** *when all of its elements are ghost.*

As an example, one can easily verify that $\bar{\mathbb{U}}[x_1, \dots, x_n]$ is a proper tropical ideal of $\mathbb{T}[x_1, \dots, x_n]$. (Note that we may have ideal which are neither, tangible ideal nor ghost ideal.)

An immediate conclusion is:

Corollary 4.2. *There exists only a single proper maximal ideal* $\mathfrak{m} \subset \mathbb{T}[x_1, \dots, x_n]$*.*

Proof. We identify the maximal ideal as $\mathfrak{m} = \mathbb{T}[x_1, \dots, x_n] \setminus \mathbb{R}$, that is the set of all polynomials in n indeterminate $x_1, \dots, x_n$ except constant tangible polynomials. Assume that $\mathfrak{m}$ can be enlarged further, say by $a \in \mathbb{R}$. Now, if $a \in \mathfrak{m}$ then $(-a) \in \mathfrak{m}$, and hence $0 \in \mathfrak{m}$, which is the multiplicative unit of $\mathbb{T}[x_1, \dots, x_n]$. But then, for any $f \in \mathbb{T}[x_1, \dots, x_n]$ we have $0f = f$ which means that $f \in \mathfrak{m}$, thus $\mathfrak{m}$ is no more a proper ideal. Clearly, for any other proper ideal $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ we have $\mathfrak{a} \subseteq \mathfrak{m}$ since otherwise $\mathfrak{a}$ must contain a constant tangible polynomial and by the previous argument it would not be proper. □

The operations between ideals and a polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ are defined in terms of elements:

$$f \oplus \mathfrak{a} = \{f \oplus g \mid g \in \mathfrak{a}\} \qquad \textit{and} \qquad f \odot \mathfrak{a} = \{fg \mid g \in \mathfrak{a}\}.$$

Clearly, from the latter operation we have $f \odot \mathfrak{a} \subset \mathfrak{a}$ for any $f \in \mathbb{T}[x_1, \dots, x_n]$. The first natural construction of an ideal is the ideal generated by a finite number of polynomials.

Definition 4.3. *Let $f_1, \dots, f_s$ be a collection of polynomials in $\mathbb{T}[x_1, \dots, x_n]$, then we set*

$$\langle f_1, \dots, f_s \rangle = \left\{ \bigoplus_i h_i f_i \mid h_1, \dots, h_s \in \mathbb{T}[x_1, \dots, x_n] \right\}$$

to be the ***ideal generated by*** *$f_1, \dots, f_s$. When $s = 1$ the ideal is called* ***principal ideal****. Given an ideal $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ we say that $\mathfrak{a}$ is* ***finitely generated*** *if there exist $f_1, \dots, f_s \in \mathbb{T}[x_1, \dots, x_n]$ such that $\mathfrak{a} = \langle f_1, \dots, f_s \rangle$, or equivalently, we say that $f_1, \dots, f_s$ are the* ***tropical generating set*** *of $\mathfrak{a}$.*

As in the classical case, a tropical ideal may have many different generating sets.

Claim 4.4. *The set $\langle f_1, \dots, f_s \rangle$ is indeed a tropical ideal.*

Proof. $-\infty \in \langle f_1, \dots, f_s \rangle$ since $\bigoplus_i -\infty f_i = -\infty$. Suppose $f = \bigoplus_i p_i f_i$, $g = \bigoplus_i q_i f_i$ and let $h \in \mathbb{T}[x_1, \dots, x_n]$. Then, using the polynomial rules, the equations

$$f \oplus g = \bigoplus_i (p_i \oplus q_i) f_i, \qquad hf = \bigoplus_i (hp_i) f_i$$

complete the proof. □

Given an ideal $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$, as has been done previously for subsets of $\mathbb{T}[x_1, \dots, x_n]$, we define the tropical algebraic set of $\mathfrak{a}$ to be

$$\mathcal{Z}_{\text{tr}}(\mathfrak{a}) = \{\mathbf{a} \in \mathbb{T}^{(n)} \mid f(\mathbf{a}) \in \bar{\mathbb{U}}, \forall f \in \mathfrak{a}\}. \tag{12}$$

Proposition 4.5. *For any tropical ideals $\mathfrak{a} \subseteq \mathfrak{b}$ we have the inverse inclusion $\mathcal{Z}_{\text{tr}}(\mathfrak{b}) \subseteq \mathcal{Z}_{\text{tr}}(\mathfrak{a})$.*

The proof is technically straightforward, so we omit the proofs' details.

Earlier, we have shown how tropical sets are obtained from ideals, but we also have the converse direction in which tropical algebraic sets give rise to ideals.

Definition 4.6. *The ideal of a tropical algebraic set $Z \subseteq \mathbb{T}^{(n)}$ is defined to be*

$$I_{\text{tr}}(Z) = \{f \in \mathbb{T}[x_1, \dots, x_n] \mid f(\mathbf{a}) \in \bar{\mathbb{U}}, \ \forall \mathbf{a} \in Z\}.$$

The crucial observation is that $I_{\mathrm{tr}}(Z)$ is indeed a tropical ideal.

Lemma 4.7. *Let $Z \subset \mathbb{T}^{(n)}$ be a tropical algebraic set, then $I_{\mathrm{tr}}(Z) \subset \mathbb{T}[x_1, \dots, x_n]$ is a tropical ideal.*

Proof. $-\infty \in I_{\mathrm{tr}}(Z)$ by definition. Assume $f, g \in I_{\mathrm{tr}}(Z)$, $h \in \mathbb{T}[x_1, \dots, x_n]$, and $-\infty \neq \mathbf{a} \in Z$; then

$$(f \oplus g)(\mathbf{a}) = f(\mathbf{a}) \oplus g(\mathbf{a}) = x^\nu \oplus y^\nu \in \bar{\mathbb{U}}, \qquad (hf)(\mathbf{a}) = h(\mathbf{a})f(\mathbf{a}) = (h(\mathbf{a}))x^\nu \in \bar{\mathbb{U}},$$

where $f(\mathbf{a}) = x^\nu$ and $g(\mathbf{a}) = y^\nu$, and it follows that $I_{\mathrm{tr}}(Z)$ is an ideal. □

Lemma 4.8. *Let $f_1, \dots, f_s \in \mathbb{T}[x_1, \dots, x_n]$, then $\langle f_1, \dots, f_s \rangle \subset I_{\mathrm{tr}}(\mathcal{Z}_{\mathrm{tr}}(f_1, \dots, f_s))$.*

Proof. For $f \in \langle f_1, \dots, f_s \rangle$ we have $f = \bigoplus_i, h_i f_i$ where the h_i's are polynomials in $\mathbb{T}[x_1, \dots, x_n]$. Since all $f_1, \dots, f_s$ give values in $\bar{\mathbb{U}}$ on $\mathcal{Z}_{\mathrm{tr}}(f_1, \dots, f_s)$, so does $f = \bigoplus_i, h_i f_i$, which proves that $f \in I_{\mathrm{tr}}(\mathcal{Z}_{\mathrm{tr}}(f_1, \dots, f_s))$. □

Proposition 4.9. *Let Z' and Z'' be tropical algebraic sets then,*

(i) $Z' \subset Z''$ if and only if $I_{\mathrm{tr}}(Z') \supset I_{\mathrm{tr}}(Z'')$;

(ii) $Z' = Z''$ if and only if $I_{\mathrm{tr}}(Z') = I_{\mathrm{tr}}(Z'')$.

Proof. (i) Suppose $Z' \subset Z''$, then any polynomial that gives value in $\bar{\mathbb{U}}$ on Z'' must also give value in $\bar{\mathbb{U}}$ on Z'. This proves $I_{\mathrm{tr}}(Z') \supset I_{\mathrm{tr}}(Z'')$. Assume that $I_{\mathrm{tr}}(Z') \supset I_{\mathrm{tr}}(Z'')$, we know that Z'' is the tropical algebraic set defined by a set $G \subset \mathbb{T}[x_1, \dots, x_n]$, and it follows that $g \in I_{\mathrm{tr}}(Z'') \subset I_{\mathrm{tr}}(Z')$ for any $g \in G$. Hence, the g's give values in $\bar{\mathbb{U}}$ on Z'. Since Z'' consists of all common solutions of the g's, it follows that $Z' \subset Z''$. (ii) is an immediate consequence of (i). □

Earlier we showed that a tropical set determines a tropical ideal, next we will show that the same is also valid for com-sets. But first, let's specify the tropical com-set of an ideal. Given an ideal $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$, its tropical algebraic com-set is defined to be the set of connected components

$$\mathcal{C}_{\mathrm{tr}}(\mathfrak{a}) = \{D_f \mid f \in \mathfrak{a}\} = \bigcup_{f \in \mathfrak{a}} \mathcal{C}_{\mathrm{tr}}(f).$$

Defining $\widetilde{\mathcal{C}_{\mathrm{tr}}}(\mathfrak{a}) = \bigcup_{D_\mathfrak{a} \in \mathcal{C}_{\mathrm{tr}}(\mathfrak{a})} D_\mathfrak{a}$, we have $\mathcal{Z}_{\mathrm{tr}}(\mathfrak{a}) = \mathbb{T}^{(n)} \setminus \widetilde{\mathcal{C}_{\mathrm{tr}}}(\mathfrak{a})$; namely, $\mathcal{Z}_{\mathrm{tr}}(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a}} (\mathbb{T}^{(n)} \setminus \widetilde{\mathcal{C}_{\mathrm{tr}}}(f))$.

We also have the converse direction in which tropical algebraic com-sets give rise to ideals.

Definition 4.10. *Let C be a tropical algebraic com-set, the tropical ideal $I_{\mathrm{tr}}(C)$ is defined as*

$$I_{\mathrm{tr}}(C) = \{f \in \mathbb{T}[x_1, \dots, x_n] \mid \forall D_f \in \mathcal{C}_{\mathrm{tr}}(f), \exists D_o \in C \text{ s.t. } D_f \subseteq D_o\}.$$

Proposition 4.11. *$I_{\mathrm{tr}}(C)$ is indeed a tropical ideal.*

Proof. Whether C contains a nonempty set or not, i.e. $C = \{\emptyset\}$, $\emptyset \subseteq D$ for any $D \in C$. Since $\mathcal{C}_{\mathrm{tr}}(-\infty) = \emptyset$, by Example 3.11, we have $-\infty \in I_{\mathrm{tr}}(C)$.

Given $f, g \in I_{\mathrm{tr}}(C)$, we need to show that $f \oplus g \in I_{\mathrm{tr}}(C)$. By the way contradiction, assume $f \oplus g \notin I_{\mathrm{tr}}(C)$; this means there exists $D_o \in \mathcal{C}_{\mathrm{tr}}(f \oplus g)$ that is not contained in any member of C. Clearly, $D_o \cap D_f \neq \emptyset$ or $D_o \cap D_g \neq \emptyset$ for some $D_f \in \mathcal{C}_{\mathrm{tr}}(f)$ or $D_g \in \mathcal{C}_{\mathrm{tr}}(g)$, otherwise $(f \oplus g)(\mathbf{a}) \in \bar{\mathbb{U}}$ for all $\mathbf{a} \in D_o$.

Denote the closure of D_f by $\overline{D}_f$ and let $\partial\overline{D}_f$ be the boundary of $\overline{D}_f$. Suppose $D_o \cap D_f \neq \emptyset$, then there is some $\mathbf{a} \in D_o \cap \partial\overline{D}_f$, and in particular $\mathbf{a} \in \mathcal{Z}_{\mathrm{tr}}(f)$. But then, there is $D_g \in \mathcal{C}_{\mathrm{tr}}(g)$, $D_o \cap D_g \neq \emptyset$, such that $\mathbf{a} \in D_o \cap D_g$. Now, since $D_o \nsubseteq D_g$, there exits $\mathbf{b} \in D_o \cap \partial\overline{D}_g$, and thus $\mathbf{b} \in \mathcal{Z}_{\mathrm{tr}}(g)$. Moreover, the intersection $\partial\overline{D}_f \cap \partial\overline{D}_g \neq \emptyset$ is contained in $\mathcal{Z}_{\mathrm{tr}}(f) \cap \mathcal{Z}_{\mathrm{tr}}(g)$, so we necessarily have $\partial\overline{D}_f \cap \partial\overline{D}_g \cap D_o \neq \emptyset$. Therefore, there is $\mathbf{c} \in D_o$ on which both f and g give values in $\bar{\mathbb{U}}$ – a contradiction. The last condition in which if $f \in I_{\mathrm{tr}}(\mathcal{C}_{\mathrm{tr}})$ and $h \in \mathbb{T}[x_1, \dots, x_n]$, then $fh \in I_{\mathrm{tr}}(C)$ is derived immediately as a result of Lemma 3.12. □

Lemma 4.12. *Let $f_1, \dots, f_s \in \mathbb{T}[x_1, \dots, x_n]$, then $\langle f_1, \dots, f_s \rangle \subset I_{\mathrm{tr}}(\mathcal{C}_{\mathrm{tr}}(f_1, \dots, f_s))$.*

Proof. For $f \in \langle f_1, \dots, f_s \rangle$, we have $f = \bigoplus_i, h_i f_i$ with $h_i \in \mathbb{T}[x_1, \dots, x_n]$. f is smooth and linear (in the usual sense) on any $D_f \in \mathcal{C}_{\mathrm{tr}}(f)$ and is equal to $h_j f_j$ for some $1 \leq j \leq s$. Hence, $D_f \in \mathcal{C}_{\mathrm{tr}}(h_j f_j)$ and there is $D_{f_j} \in \mathcal{C}_{\mathrm{tr}}(f_j)$ such that $D_f \subseteq D_{f_j}$, cf. Lemma 3.12. □

Proposition 4.13. *Let C' and C'' be tropical algebraic com-sets, then*

(i) $C' \sqsubseteq C''$ if and only if $I_{\text{tr}}(C') \subseteq I_{\text{tr}}(C'')$;

(ii) $C' = C''$ if and only if $I_{\text{tr}}(C') = I_{\text{tr}}(C'')$.

Proof. Suppose $C' \sqsubseteq C''$ and $f \in I_{\text{tr}}(C')$, then for each $D_f \in \mathcal{C}_{\text{tr}}(f)$ there exists $D' \in C'$ such that $D_f \subseteq D'$. Since $C' \sqsubseteq C''$ then there is $D'' \in C''$ such that $D' \subseteq D''$, and in particular $D_f \subseteq D''$, hence $f \in I_{\text{tr}}(C'')$. Conversely, assume (11) is not satisfied, then there exists $D'_o \in C'$ such that $D'_o \nsubseteq D''$ for any $D'' \in C''$. We know that $C' = \mathcal{C}_{\text{tr}}(F)$ for some $F \subset \mathbb{T}[x_1, \dots, x_n]$, thus $D'_o \in \mathcal{C}_{\text{tr}}(f)$ for some $f \in F$. In particularly, $f \in I_{\text{tr}}(C')$. But, since D'_o is not contained in any member of C'', then $f \notin I_{\text{tr}}(C'')$. □

4.2. **Radical ideals.** We turn to deal with special types of ideals.

Definition 4.14. *The **radical** $\sqrt{\mathfrak{a}}$ of an ideal $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ is defined to be the set of all $f \in \mathbb{T}[x_1, \dots, x_n]$ for which $f^k \in \mathfrak{a}$ for some positive $k \in \mathbb{N}$. An ideal $\mathfrak{a}$ is called a **radical ideal** if $\sqrt{\mathfrak{a}} = \mathfrak{a}$. An ideal $\mathfrak{p} \subset \mathbb{T}[x_1, \dots, x_n]$ is said to be **prime** ideal if when $fg \in \mathfrak{p}$, then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.*

Any ideal $\mathfrak{a}$ is contained in some prime ideal $\mathfrak{p}$. We can simply complete it to prime ideal: whenever an element $h = (fg) \in \mathfrak{a}$ and both f and g are not in $\mathfrak{a}$, add one of them (including its multiples) to $\mathfrak{a}$. By this construction, $\mathfrak{a}$ is completed to be a prime ideal $\mathfrak{p}$. We can conclude that:

Corollary 4.15. *Every topical prime ideal is a tropical radical ideal.*

The next two propositions are immediate.

Proposition 4.16. *The radical of a tropical ideal $\mathfrak{a}$ is again a tropical ideal.*

Proof. Suppose $f, g \in \sqrt{\mathfrak{a}}$, thus $f^k \in \mathfrak{a}$ and $g^m \in \mathfrak{a}$ for some positive integers k, m. Then

$$(f \oplus g)^{k+m} = \bigoplus_{i=0}^{k+m} h_i f^i g^{k+m-i},$$

where $h_i \in \mathbb{T}[x_1, \dots, x_n]$. In each term either $i \geq k$ or $k + m - i \geq n$. In the first case, $f^i \in \mathfrak{a}$, and in the second case, $g^{k+m-i} \in \mathfrak{a}$. Since $\mathbb{T}[x_1, \dots, x_n]$ is commutative and $\mathfrak{a}$ is an ideal, the sum of these terms is again in $\mathfrak{a}$, and hence $f \oplus g \in \mathfrak{a}$. To see that $\mathfrak{a}$ is closed under multiplication by elements $\mathbb{T}[x_1, \dots, x_n]$; let $h \in \mathbb{T}[x_1, \dots, x_n]$, then $(hf)^m = h^m f^m \in \mathfrak{a}$, so $hf \in \sqrt{\mathfrak{a}}$. □

Proposition 4.17. *The radical of $\sqrt{\mathfrak{a}}$ is equal to $\sqrt{\mathfrak{a}}$.*

Proof. Clearly, $\sqrt{\mathfrak{a}}$ is contained in the radical of $\sqrt{\mathfrak{a}}$. To see the reverse inclusion, assume $f \in \sqrt{\mathfrak{a}}$, then $f^k \in \sqrt{\mathfrak{a}}$ for a positive $k \in \mathbb{N}$, which means that $(f^k)^m \in \sqrt{\mathfrak{a}}$ for some positive $m \in \mathbb{N}$. Since $f^{km} \in \sqrt{\mathfrak{a}}$, we see that $f \in \sqrt{\mathfrak{a}}$. □

Definition 4.18. *A polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ is called **ghost-potent** if $f^k \in \mathbb{U}[x_1, \dots, x_n]$ for some positive $k \in \mathbb{N}$. A **ghost-radical** of a ghost ideal $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ is defined to be the set of all $f \in \mathbb{T}[x_1, \dots, x_n]$ for which $f^k \in \mathfrak{a}$ for some positive $k \in \mathbb{N}$. The ghost-radical of $\mathbb{U}[x_1, \dots, x_n]$, denoted rad$(\mathbb{U})$, is the set of all ghost-potent elements in $\mathbb{T}[x_1, \dots, x_n]$.*

(We may extend this definition by joining $-\infty$ to the ghost ideal.)

Clearly, any ghost element is ghost-potent. Restricting ourselves to the reduced domain $\tilde{\mathbb{T}}[x_1, \dots, x_n]$ we have the following:

Proposition 4.19. *The ghost-radical rad$(\mathbb{U})$ is unique and is equal to $\tilde{\mathbb{U}}[x_1, \dots, x_n]$.*

Proof. Assume, $f \notin \tilde{\mathbb{U}}[x_1, \dots, x_n]$ and $f^k \in \tilde{\mathbb{U}}[x_1, \dots, x_n]$ for some positive $k \in \mathbb{N}$. Since $f \notin \tilde{\mathbb{U}}[x_1, \dots, x_n]$, it has at least one essential tangible monomial $\alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$. But then f^k has also at least one essential tangible monomial, specifically $(\alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{i}})^k$ (cf. Theorem 2.40) – a contradiction ($f^k \notin \tilde{\mathbb{U}}[x_1, \dots, x_n]$). □

This statement is true only for the reduced domain, because we "ignore" inessential monomials, and is not true for the non-reduced polynomial semiring $\mathbb{T}[x_1, \dots, x_n]$, take for instance $0^\nu x^2 \oplus x \oplus 0^\nu$ is not ghost while $(0^\nu x^2 \oplus x \oplus 0^\nu)^k$, for each $k \in \mathbb{N}$, is ghost.

Theorem 4.20. *Let $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ be an ideal, and let P be the set of all prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$, then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in P} \mathfrak{p}.$$

In particular, rad$(\mathbb{U})$ is the intersection of all prime ideals in $\mathbb{T}[x_1, \dots, x_n]$ that contain $\mathbb{U}[x_1, \dots, x_n]$.

Proof. Denoting $\mathcal{P}_\cap = \bigcap_{\mathfrak{p} \in P} \mathfrak{p}$, we show $\sqrt{\mathfrak{a}} = \mathcal{P}_\cap$ by cross inclusion.

($\subseteq$) Let $f \in \sqrt{\mathfrak{a}}$, that is $f^k \in \mathfrak{a}$ for some positive integer $k \in \mathbb{N}$, and take k to be the least k for which this is true. Let $\mathfrak{p} \subset \mathbb{T}[x_1, \dots, x_n]$ be a prime ideal containing $\sqrt{\mathfrak{a}}$, then $f^k \in \mathfrak{p}$. Write $f^k = f f^{k-1}$, since $\mathfrak{p}$ is prime then either f in $\mathfrak{p}$ or f^{k-1} in $\mathfrak{p}$. If $f \in \mathfrak{p}$ we are done, otherwise $f \notin \mathfrak{p}$ and thus $f^{k-1} \in \mathfrak{p}$. But this contradicts the assumption that k is minimal. To see this, just repeat the decomposition inductively to obtain $f^2 \in \mathfrak{p}$, namely $f \in \mathfrak{p}$ – a contradiction. Thus, f is contained in every prime ideal $\mathfrak{p} \supseteq \sqrt{\mathfrak{a}}$.

($\supseteq$) We show that if $f \notin \sqrt{\mathfrak{a}}$, then there exists $\mathfrak{p} \subset \mathbb{T}[x_1, \dots, x_n]$ such that $f \notin \mathfrak{p}$ and hence $f \notin \mathcal{P}_\cap$. This will be done by constructing a prime ideal that does not contain f. Let $f \in \mathbb{T}[x_1, \dots, x_n]$ such that $f \notin \sqrt{\mathfrak{a}}$, since $-\infty \in \sqrt{\mathfrak{a}}$, then $f \neq -\infty$. Let $\mathcal{S}$ be the family of ideals of $\mathbb{T}[x_1, \dots, x_n]$ that do not contain any power of f and do contain $\mathfrak{a}$. This family $\mathcal{S}$ is not empty because $\mathfrak{a} \in \mathcal{S}$. Also, we see that chains of ideals in $\mathcal{S}$ have upper bounds because if f^k is not in any ideal of a given chain, then it is also not in the union of the ideals in that chain. So, we can now apply Zorn's Lemma to see that there is some maximal element $\mathfrak{p}_{(max)}$ of $\mathcal{S}$. Since $\mathfrak{p}_{(max)}$ is in $\mathcal{S}$, $\mathfrak{p}_{(max)}$ does not contain f.

We will now show that $\mathfrak{p}_{(max)}$ is prime. By way of contradiction, assume $g, h \in \mathbb{T}[x_1, \dots, x_n]$ are not in $\mathfrak{p}_{(max)}$ but such that $gh \in \mathfrak{p}_{(max)}$. Since $\mathfrak{p}_{(max)}$ is a maximal element of $\mathcal{S}$, we see that for some positive integers k and m, $f^k \in (g) \oplus \mathfrak{p}_{(max)}$ and $f^m \in (h) \oplus \mathfrak{p}_{(max)}$. But then $f^{k+m} \in (gh) \oplus \mathfrak{p}_{(max)} = \mathfrak{p}_{(max)}$, contradicting the fact that $\mathfrak{p}_{(max)} \in \mathcal{S}$. Thus $\mathfrak{p}_{(max)}$ is indeed a prime ideal, and so $f \notin \mathcal{P}_\cap$. □

5. An Algebraic Tropical Nullstellensatz

All our previous development leads to the foundation of an algebraic tropical nullstellensatz – the weak version and the strong version; the weak version is stated in terms of both, tropical algebraic sets and tropical algebraic com-sets, while the strong version is phrased only in terms of tropical algebraic com-sets. The latter is an algebraic rephrasing, enabled due to our semiring structure, of the tropical nullstellensatz that appeared in [16] and part of our development is based on this theorem.

5.1. **Weak Nullstellensatz.**

Theorem 5.1. *Let $f_1, \dots, f_s \in \mathbb{T}[x_1, \dots, x_n]$ be nonconstant polynomials, then $\mathcal{Z}_{\text{tr}}(f_1, \dots, f_s) \neq \emptyset$.*

In fact we can also allow constant ghost polynomials, but the tropical algebraic set of these polynomial is $\mathbb{T}^{(n)}$.

Proof. Suppose $n = 1$. For each $i = 1, \dots, s$, assume $h_i = \alpha_i x^i$ is the least significant monomial of f_i; dividing out by $\pi(\alpha_i) x^i$, then f_i has the form

$$f_i(x) = \underbrace{\alpha'_n x^n \oplus \cdots \oplus \alpha'_1 x}_{= g_i(x)} \oplus \beta, \qquad \beta \in \{0, 0^\nu\}.$$

We may assume g_i is nonconstant, since otherwise f_i has a single nonconstant monomial, which means that $\mathcal{Z}_{\text{tr}}(f_i) = \bar{\mathbb{U}}$. According to Lemma 2.4, for each i, there is $r_i \in \mathbb{T}$ for which $g_i(r_i) = 0^\nu$. Take r to be the ghost of the maximal r_i's, then, for each i, $g_i(r) = a_i^\nu \succeq 0$ and $g_i(r) \oplus 0 \in \mathbb{U}$. The generalization to $n > 1$ is obvious, just pick $a \in \mathbb{R}$, fix $x_2 = \cdots = x_n = a$, and apply the above argument for x_1. □

Corollary 5.2. *Let $f_1, \dots, f_s \in \mathbb{T}[x_1, \dots, x_n]$, then $\mathcal{Z}_{\text{tr}}(f_1, \dots, f_s) = \emptyset$ if and only if one of the f_i's is a constant tangible, i.e. $f_i = c \in \mathbb{R}$.*

The corollary is derived directly from Theorem 5.1.

Theorem 5.3. ***(Weak Nullstellensatz)*** *Let $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ be a proper finitely generated ideal, then $\mathcal{Z}_{\text{tr}}(\mathfrak{a}) \neq \emptyset$. Equivalently, if $\mathcal{Z}_{\text{tr}}(\mathfrak{a}) = \emptyset$, then $\mathfrak{a} = \mathbb{T}[x_1, \dots, x_n]$.*

Proof. Assume $\mathcal{Z}_{\text{tr}}(\mathfrak{a}) = \emptyset$, by Corollary 5.2 there exists a constant tangible polynomial $f \in \mathfrak{a}$, i.e. $f = a \in \mathbb{R}$. Then, $a^{-1} = 0/a \in \mathbb{T}$ and thus $0 \in \mathfrak{a}$, which means $0g = g \in \mathfrak{a}$ for each $g \in \mathbb{T}[x_1, \dots, x_n]$. This shows that $\mathfrak{a} = \mathbb{T}[x_1, \dots, x_n]$. □

Corollary 5.4. *Let $\mathfrak{a} \subset \mathbb{T}[x_1, \dots, x_n]$ be a tropical ideal, then $\mathbb{T}^{(n)} \in \mathcal{C}_{\mathrm{tr}}(\mathfrak{a})$ if and only if $\mathfrak{a} = \mathbb{T}[x_1, \dots, x_n]$.*

Proof. Immediate, by Theorem 5.3 and the relation: $\mathcal{Z}_{\mathrm{tr}}(\mathfrak{a}) = \emptyset$ if and only if $\mathbb{T}^{(n)} \in \mathcal{C}_{\mathrm{tr}}(\mathfrak{a})$. □

5.2. **Strong Nullstellensatz.** The use of the reduced tropical domain $\tilde{\mathbb{T}}[x_1, \cdot, x_n]$ allows us an easy algebraic formulation of geometric ideas, which lead to the tropical Nullstellensatz.

Remark 5.5. *Let $f = \bigoplus_i f_i$ and assume $D_f \in \mathcal{C}_{\mathrm{tr}}(f)$. Then, $f|_D = f_i|_{D_f}$ for some monomial $f_i = \alpha_i x_1^{i_1} \cdots x_n^{i_n}$. Suppose $i_t = 0$, for some $t = 1, \dots, n$, then if the t'th coordinate of a point $\mathbf{a} \in D_f$ has a tangible value a_t then, by the connectedness of D_f the point $\mathbf{a}'$, obtained by replacing the coordinate a_t by a_t^{ν}, is also in D_f.*

Theorem 5.6. *Let $\tilde{f} \in \tilde{\mathbb{R}}[x_1, \dots, x_n]$, $g_1, \dots, g_k \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$, and let $\mathfrak{a}$ be the ideal generated by $\tilde{g}_1, \dots, \tilde{g}_k$. Then $\tilde{f} \in \sqrt{\mathfrak{a}}$ if and only $\mathcal{C}_{\mathrm{tr}}(\tilde{f}) \sqsubseteq \mathcal{C}_{\mathrm{tr}}(\mathfrak{a})$.*

Please note that here we work on the reduced tropical semiring $\tilde{\mathbb{T}}[x_1, \dots, x_n]$, in other word polynomials are identified with polynomials functions. When the notations are clear from the context, for short, we write $D \in \mathcal{C}_{\mathrm{tr}}(\tilde{f})$ for a connected component $D_{\tilde{f}} \in \mathcal{C}_{\mathrm{tr}}(\tilde{f})$. (The proof of this theorem follows after the arguments of [16, Theorem 3.5 and its Corollary].)

Proof. ($\Rightarrow$) Assume $\tilde{f} \in \sqrt{\mathfrak{a}}$, then $\tilde{f}^m = \sum_i \tilde{h}_i \tilde{g}_i$, where $\tilde{h}_i \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$, $m \in \mathbb{N}$. Suppose $D \in \mathcal{C}_{\mathrm{tr}}(\tilde{f}^m)$, then $\tilde{f}^m|_D$ must coincide with one of the terms $(\tilde{h}_i \tilde{g}_i)|_D$ in the expression $(\bigoplus_i \tilde{h}_i \tilde{g}_i)|_D$, since otherwise the the latter function would have a ghost value inside D. Then, by definition, both $\tilde{h}_i|_D$ and $\tilde{g}_i|_D$ don't have ghost evaluations over D, which means $D \subset D'$ for some $D' \in \mathcal{C}_{\mathrm{tr}}(\tilde{g}_i)$. (Recall that $\mathcal{C}_{\mathrm{tr}}(\tilde{f}) = \mathcal{C}_{\mathrm{tr}}(\tilde{f}^m)$.)

($\Leftarrow$) Distribute the connected components of $\mathcal{C}_{\mathrm{tr}}(\tilde{f})$ into disjoint subsets Π_j, $j \in J$, where $J \subset \{1, \dots, k\}$, such that, for any $j \in J$ and $D \in \Pi_j$, we have $D \subset D'$ for some $D' \in \mathcal{C}_{\mathrm{tr}}(g_j)$. Fix some $j \in J$, pick $D \in \Pi_j$, and assume $\tilde{f}|_D = \tilde{f}_i|_D$ for some monomial $\tilde{f}_i = \alpha_i x^{i_1} \cdots x^{i_n}$. Similarly, we may assume $\tilde{g}_j|_{D'} = \tilde{g}_{j,r}|_{D'}$ for some monomial $g_{j,r} = \beta_r x^{r_1} \cdots x^{r_n}$ of $\tilde{g}_j$ (in particular $\tilde{g}_j|_D = \tilde{g}_{j,r}|_D$ where $\tilde{g}_{j,r}$ is a tangible monomial).

We claim that for any $t = 1, \dots, n$

$$i_t > 0 \quad \text{whenever} \quad r_t > 0; \tag{13}$$

otherwise, i.e. $i_t = 0$ and $r_t > 0$, take a point $\mathbf{a} \in D \subset \mathbb{T}^{(n)}$ having only tangible coordinates except the t'th coordinate which has a ghost value (Remark 5.5). Then, $f(\mathbf{a}) \in \bar{\mathbb{U}}$ on D while $\tilde{g}_j(\mathbf{a}) \in \mathbb{R}$ and thus $D \not\subseteq D'$.

Condition (13) yields that there is m_1 such that, for any $m \geq m_1$ and $D \in \Pi_j$, one has

$$m \cdot i_t \geq r_t, \qquad t = 1, \dots, n. \tag{14}$$

Accordingly, we define the function

$$F_{D,m}|_D = \frac{\tilde{f}^m|_D}{\tilde{g}_j|_D} = \frac{\tilde{f}_i^m|_D}{\tilde{g}_{j,r}|_D} = \frac{\alpha_i}{\beta_{j,r}} x_1^{mi_1 - r_1} \cdots x_n^{mi_n - r_n}|_D, \quad D \in \Pi_j,\ m \geq m_1, \tag{15}$$

for which $m \cdot i_t - r_t$ have always nonnegative integral values for any $t = 1, \dots, n$. (Note that, due to (14), over D this function is described by a proper polynomial.)

We claim that there exists m_2, such that for any $D \in \Pi_j$, in the complement of the closure of D (denoted $\overline{D}^c$), we have

$$\tilde{f}^m > F_{D,m} \tilde{g}_j \quad \text{whenever} \quad m \geq m_2 \ . \tag{16}$$

Indeed, write $\tilde{f} = FG$, $\tilde{g}_j = F'G'$, where F, F' are monomials, and $G = \bigoplus_{\mathbf{k}} \gamma_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$, $G' = \bigoplus_{\mathbf{k}'} \gamma'_{\mathbf{k}'} \mathbf{x}^{\mathbf{k}'}$ are polynomials (referred to as functions) equal 0 along D. (In particular, as functions, G and G' are convex functions.) Then $F_{D,m} = \frac{F^m}{F'}$, which is clearly a monomial on D, and thus $\frac{\tilde{f}^m}{F_{D,m}\tilde{g}} = \frac{G^m}{G'}$. By the convexity of G, and the fact it equal 0 on D, we have $G|_{\overline{D}^c} > 0$. Since, $G > 0$ and respectively $k_t \geq 1$, $t = 1, ..., n$, outside $\overline{D}$, we obtain (16) when m_2 exceeds all the values of the k'_t's of $\mathbf{k}'$ with respect to G'.

Define $\tilde{h}_i = \bigoplus_{D \in \Pi_J} F_{D,m}$. This is a tropical polynomial as $m \geq m_1$ and, due to Equations (15) and (16), it satisfies

$$(\tilde{h}_i \tilde{g}_j)\big|_D = \tilde{f}^m\big|_D, \quad (\tilde{h}_i \tilde{g}_j)\big|_{\overline{D}^c} \ < \ \tilde{f}^m\big|_{\overline{D}^c}, \quad D \in \Pi_j, \quad m \geq m_2 \ ,$$

where $\overline{D}^c$ is the complement of the closure of $D \in \mathcal{C}_{\text{tr}}(\tilde{f})$. ☐

Theorem 5.7. ***(Algebraic Nullstellensatz)*** *Let $\mathfrak{a} \subset \tilde{\mathbb{R}}[x_1, \dots, x_n]$ be a finitely generated tropical ideal, where $\tilde{\mathbb{U}}[x_1, \dots, x_n] \subseteq \mathfrak{a}$, then*

$$\sqrt{\mathfrak{a}} = \mathcal{I}_{\text{tr}}(\mathcal{C}_{\text{tr}}(\mathfrak{a})).$$

Proof. ($\subseteq$) Assume $\tilde{f} \in \sqrt{\mathfrak{a}}$, then $\tilde{f}^m \in \mathfrak{a}$ for some positive $m \in \mathbb{N}$, and hence $\mathcal{C}_{\text{tr}}(\tilde{f}^m) \sqsubseteq \mathcal{C}_{\text{tr}}(\mathfrak{a})$. By Lemma 3.12, $\mathcal{C}_{\text{tr}}(\tilde{f}) = \mathcal{C}_{\text{tr}}(\tilde{f}^m)$ and, since $\mathcal{C}_{\text{tr}}(\tilde{f}) \sqsubseteq \mathcal{C}_{\text{tr}}(\mathfrak{a})$, then $\tilde{f} \in \mathcal{I}_{\text{tr}}(\mathcal{C}_{\text{tr}}(\mathfrak{a}))$.

($\supseteq$) When $\tilde{f} \in \mathcal{I}_{\text{tr}}(\mathcal{C}_{\text{tr}}(\mathfrak{a}))$ it means that $\mathcal{C}_{\text{tr}}(\tilde{f}) \sqsubseteq \mathcal{C}_{\text{tr}}(\mathfrak{a})$, namely, each $D_{\tilde{f}} \in \mathcal{C}_{\text{tr}}(\tilde{f})$ is contained in some component $D_{\mathfrak{a}} \in \mathcal{C}_{\text{tr}}(\mathfrak{a})$ and hence in some component $D_{\tilde{f}_i} \in \mathcal{C}_{\text{tr}}(\tilde{f}_i)$ of some $\tilde{f}_i \in \mathfrak{a}$. The proof is then completed by applying Theorem 5.6. ☐

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