AMOEBAS, MONGE-AMPÈRE MEASURES, AND TRIANGULATIONS OF THE NEWTON POLYTOPE

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Abstract

The amoeba of a holomorphic function f is, by definition, the image in $\mathbf{R}^n$ of the zero locus of f under the simple mapping that takes each coordinate to the logarithm of its modulus. The terminology was introduced in the 1990s by the famous (biologist and) mathematician Israel Gelfand and his coauthors Kapranov and Zelevinsky (GKZ). In this paper we study a natural convex potential function N_f with the property that its Monge-Ampère mass is concentrated to the amoeba of f. We obtain results of two kinds; by approximating N_f with a piecewise linear function, we get striking combinatorial information regarding the amoeba and the Newton polytope of f; by computing the Monge-Ampère measure, we find sharp bounds for the area of amoebas in $\mathbf{R}^2$. We also consider systems of functions $f_1, \ldots, f_n$ and prove a local version of the classical Bernstein theorem on the number of roots of systems of algebraic equations.

1. Introduction

The classical Jensen formula can be regarded as a relation between the zeros of a holomorphic function f and the properties of a certain convex function associated to f. In this paper we consider an analogue of this convex function, which we denote N_f, in the case where f is a function of several variables, and we search for relations with the zero locus of f.

The results we find are of two kinds. In Section 3, we show that approximation of N_f by a piecewise linear function leads to an approximation of the so-called amoeba by a polyhedral complex (Th. 1). When f is a polynomial, this polyhedral complex is dual (in a precise sense) to a certain subdivision of the Newton polytope of f. An explicit computation of the polyhedral complex amounts to evaluating certain parameters depending on the coefficients of f. It turns out that these satisfy a GKZ-type system of differential equations and can be expanded in a hypergeometric power series involving the coefficients (Th. 3).

DUKE MATHEMATICAL JOURNAL
Vol. 121, No. 3, © 2004
Received 18 September 2002. Revision received 3 April 2003.
2000 *Mathematics Subject Classification*. Primary 32A60; Secondary 52A41, 52B20.

In Section 5 we investigate the measures arising when Monge-Ampère or Laplace operators act on N_f. We find several relations between such measures and the hypersurface defined by f (Ths. 5 and 6). The former can be regarded as a local variant of the Bernstein formula for the number of solutions to a system of polynomial equations. We also obtain an estimate on the Monge-Ampère measure, which gives an upper bound on the area of the amoeba of a polynomial in two variables (Th. 7 and Cor. 1). This estimate was used in [8] to give a new characterization of so-called Harnack curves, which are fundamental for Hilbert's sixteenth problem in real algebraic geometry.

2. Background

Suppose that f is an entire function in the complex plane with zeros $a_1, a_2, a_3, \ldots$ ordered so that $|a_1| \le |a_2| \le \cdots$. Assume for simplicity that $f(0) \ne 0$. The Jensen formula states that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\,dt = \log|f(0)| + \sum_{k=1}^{m} \log\frac{r}{|a_k|},$$

where m is the largest index such that $|a_m| < r$. If this expression is considered as a function N_f of $\log r$, there is a strong connection between this function and the zeros of f. Thus it follows immediately that N_f is a piecewise linear convex function whose gradient is equal to the number of zeros of f inside the disc $\{|z| < r\}$. The second derivative of N_f, in the sense of distributions, is a sum of point masses at $\log|a_k|$, $k = 1, 2, 3, \ldots$. In this paper we consider a certain generalization of the function occurring in the Jensen formula to holomorphic functions of several variables.

Let Ω be a convex open set in $\mathbf{R}^n$, and let f be a holomorphic function defined in $\mathrm{Log}^{-1}(\Omega)$, where $\mathrm{Log} : (\mathbf{C} \setminus \{0\})^n \longrightarrow \mathbf{R}^n$ is the mapping $(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$. In [12] Ronkin considers the function N_f defined in Ω by the integral

$$N_f(x) = \frac{1}{(2\pi i)^n}\int_{\mathrm{Log}^{-1}(x)} \frac{\log|f(z_1, \ldots, z_n)|\,dz_1 \cdots dz_n}{z_1 \cdots z_n}. \tag{1}$$

As we will see, the function N_f retains some of its properties from the one-variable case, while others are lost or attain a new form. For example, N_f is a convex function, but it is no longer piecewise linear. In Section 3 we consider the consequences of approximating N_f by a piecewise linear function. In Section 5 we investigate the relation between local properties of N_f and the hypersurface $f^{-1}(0)$.

The properties of the function N_f are closely related to the amoeba of f. The amoeba of f, which we denote $\mathscr{A}_f$, is defined to be the image in Ω of the hypersurface $f^{-1}(0)$ under the map Log. The term amoeba was first used by Gelfand, Kapranov, and Zelevinsky [5] in the case where f is a polynomial.

Suppose that E is a connected component of the amoeba complement $\Omega \setminus \mathscr{A}_f$. It is not difficult to show that all such components are convex. For example, $\operatorname{Log}^{-1}(E)$ is the intersection of $\operatorname{Log}^{-1}(\Omega)$ with the domain of convergence of a certain Laurent series expansion of $1/f$, and domains of convergence of Laurent series are always logarithmically convex. In [4] Forsberg, Passare, and Tsikh defined the *order* of such a component to be the vector $\nu = (\nu_1, \ldots, \nu_n)$ given by the formula

$$\nu_j = \frac{1}{(2\pi i)^n} \int_{\operatorname{Log}^{-1}(x)} \frac{\partial f}{\partial z_j} \frac{z_j \, dz_1 \cdots dz_n}{f(z) z_1 \cdots z_n}, \quad x \in E. \tag{2}$$

Here x may be any point in E. They proved, for the case when f is a Laurent polynomial, that the order is an integer vector, that is, $\nu \in \mathbf{Z}^n$, that ν is in the Newton polytope of f, and that two distinct components always have different orders. These conclusions remain true, with essentially the same proofs, in the more general setting considered here.

Ronkin proved in [12] a theorem that in the language of amoebas amounts to the following statement.

THEOREM

Let f be a holomorphic function as above. Then N_f is a convex function. If $U \subset \Omega$ is a connected open set, then the restriction of N_f to U is affine linear if and only if U does not intersect the amoeba of f. If x is in the complement of the amoeba, then $\operatorname{grad} N_f(x)$ *is equal to the order of the complement component containing x.*

Sketch of proof

The convexity of N_f follows from a general theorem because $\log|f|$ is plurisubharmonic (see, e.g., [11, Cor. 1, p. 84]). Differentiation with respect to x_j under the integral sign in definition (1) of N_f yields precisely the real part of the integral (2) defining the order. However, the integral (2) is always real valued, and this shows immediately that N_f is affine linear in each connected component of $\Omega \setminus \mathscr{A}_f$. The fact that N_f is not linear on any open set intersecting the amoeba of f can be proved in several ways. It follows, for instance, from our results in Section 5.

3. Triangulations and polyhedral subdivisions

In his doctoral thesis [3], Mikael Forsberg noted that the amoebas of many polynomials in two variables have the appearance of slightly thickened graphs. Moreover, there seemed to be some kind of duality between these graphs and certain subdivisions (often triangulations) of the Newton polygon of the polynomial. We make this precise by associating to an amoeba $\mathscr{A}_f$ a polyhedral complex, which we call its *spine*. This is done as follows.

Assume that f is a holomorphic function defined in $(\mathbf{C} \setminus \{0\})^n$, and write

$$A = \left\{\alpha \in \mathbf{Z}^n;\, \mathbf{R}^n \setminus \mathscr{A}_f \text{ has a component of order } \alpha\right\}.$$

It may happen that A is empty, but we assume that this is not the case. The most interesting situation arises when A has plenty of points, for example, when the convex hull of A coincides with the Newton polyhedron of f. This always happens if f is a Laurent polynomial.

For $\alpha \in A$, let

$$c_\alpha = N_f(x) - \langle \alpha, x \rangle, \tag{3}$$

where x is any point in the complement component of order α, and let

$$S(x) = \max_{\alpha \in A} \big(c_\alpha + \langle \alpha, x \rangle\big). \tag{4}$$

This $S(x)$ is a piecewise linear function approximating the Ronkin function. The spine $\mathscr{S}_f$ is defined as the corner set of S, that is, the set of x where $S(x)$ is nonsmooth.

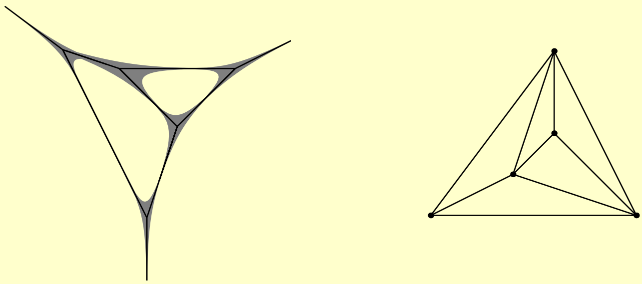

Figure 1. Amoeba of the polynomial
$1 + z_1^5 + 80z_1^2z_2 + 40z_1^3z_2^2 + z_1^3z_2^4$ (shaded) together with its spine
(solid) and the dual triangulation of the Newton polytope

We now define a precise meaning to the statement that the spine is dual to a certain subdivision of the convex hull of A. For an example, see Figure 1.

Definition 1

Let K be a convex set in $\mathbf{R}^n$. A collection T of nonempty closed convex subsets of K is called a *convex subdivision* if it satisfies the following conditions.

(i) The union of all sets in T is equal to K.

(ii) If $\sigma, \tau \in T$ and $\sigma \cap \tau$ is nonempty, then $\sigma \cap \tau \in T$.

(iii) If $\sigma \in T$ and τ is any subset of σ, then $\tau \in T$ if and only if τ is a face of σ.

We say that T is locally finite if every compact set in K intersects only a finite number of $\sigma \in T$, and we say that T is polytopal if every $\sigma \in T$ is a polytope.

Here a face of a convex set σ means a set of the form $\{x \in \sigma; \langle \xi, x\rangle = \sup_{y\in\sigma} \langle \xi, y\rangle\}$ for some $\xi \in \mathbf{R}^n$.

If $\sigma, \tau \in T$, and τ is a proper subset of σ, then it follows from condition (iii) that $\dim \tau < \dim \sigma$. Hence any descending chain $\sigma_1 \supset \sigma_2 \supset \cdots$ in T stabilizes after a finite number of steps. Therefore the intersection of any collection of sets in T is also in T.

If $\tau \subset \sigma$ are convex sets, we define

$$\operatorname{cone}(\tau, \sigma) = \big\{t(x-y); x \in \sigma, y \in \tau, t \geq 0\big\}.$$

Clearly, this set is a convex cone. If C is a convex cone, its dual is defined to be the cone $C^\vee = \{\xi \in \mathbf{R}^n; \langle \xi, x\rangle \leq 0, \forall x \in C\}$. If C is closed, it is well known that $C^{\vee\vee} = C$.

Definition 2

Let K, K' be convex sets in $\mathbf{R}^n$, and let T, T' be convex subdivisions of K, K'. We say that T and T' are dual (to each other) if there exists a bijective map $T \to T'$, denoted $\sigma \mapsto \sigma^*$, satisfying the following conditions.

(i) For $\sigma, \tau \in T$, $\tau \subset \sigma$ if and only if $\sigma^* \subset \tau^*$.

(ii) If $\tau \subset \sigma$, then $\operatorname{cone}(\tau, \sigma)$ is dual to $\operatorname{cone}(\sigma^*, \tau^*)$.

Notice that $\operatorname{cone}(\sigma, \sigma)$ is the affine subspace spanned by σ. Hence the second condition implies that σ is orthogonal to σ^* and, in particular, that $\dim \sigma + \dim \sigma^* = n$.

Next, we show that a convex, piecewise linear function determines a convex subdivision and that its Legendre transform determines a dual subdivision. Let $S(x)$ be a function of the form (4), where A is now any discrete subset of $\mathbf{R}^n$ and where c_α are arbitrary numbers such that $S(x)$ is finite for all x. The Legendre transform of S is defined by

$$\tilde{S}(\xi) = \sup_{x\in\mathbf{R}^n} \big(\langle \xi, x\rangle - S(x)\big),$$

and it is again a piecewise linear convex function with finite values defined in the convex hull K of A.

Consider the function

$$P(\xi, x) = S(x) + \tilde{S}(\xi) - \langle \xi, x\rangle \tag{5}$$

defined on $K \times \mathbf{R}^n$.

LEMMA 1

The function $P(\xi, x)$ defined by (5) has the following properties.

(i) $P(\xi, x) \geq 0$.

(ii) *$P(\xi, x)$ is convex in each argument when the other is held fixed.*

(iii) *For $x, y \in \mathbf{R}^n$ and $\xi, \eta \in K$, $\langle \xi - \eta, x - y \rangle = -P(\xi, x) + P(\xi, y) + P(\eta, x) - P(\eta, y)$.*

(iv) *For every $x \in \mathbf{R}^n$ there is a $\xi \in K$ such that $P(\xi, x) = 0$, and for every $\xi \in K$ there is an $x \in \mathbf{R}^n$ such that $P(\xi, x) = 0$.*

(v) *If $x, y \in \mathbf{R}^n$, there is a $t_0 > 0$ such that $P(\xi, x + ty)$ is a linear function of t for $t \in [0, t_0]$. If $\xi \in K$ and $\eta \in \mathbf{R}^n$ is such that $\xi + t\eta \in K$ for small positive t, then there is a $t_0 > 0$ such that $P(\xi + t\eta, x)$ is a linear function of t for $t \in [0, t_0]$.*

Proof

Properties (i), (ii), and (iii) are trivial.

(iv) If $x \in \mathbf{R}^n$, there is an $\alpha \in A$ such that $S(x) = c_\alpha + \langle \alpha, x \rangle$. This implies that $\tilde{S}(\alpha) = -c_\alpha$, so $P(\alpha, x) = 0$.

Let $\xi \in K$ be given. Then we must show that the function $S(x) - \langle \xi, x \rangle$ attains its smallest value at some point x_0. Indeed, if this is the case, then $\tilde{S}(\xi) = \langle \xi, x_0 \rangle - S(x_0)$ which means that $P(\xi, x_0) = 0$.

To show that $S(x) - \langle \xi, x \rangle$ attains its smallest value, assume first that ξ is in the interior of K. Then $S(x) - \langle \xi, x \rangle \to +\infty$ when $x \to \infty$, and it follows that the function attains its smallest value. The same argument works if $\dim K < n$ and ξ is in the relative interior of K. So, suppose that ξ is on the boundary of K, take $y \in \mathbf{R}^n$ such that ξ is in the relative interior of the face $L = \{\eta \in K; \langle \eta, y \rangle = \sup_{\alpha \in A} \langle \alpha, y \rangle\}$, and let $B = A \cap L$. Define $S_B(x) = \max_{\alpha \in B}(c_\alpha + \langle \alpha, x \rangle)$. By the preceding argument, $S_B(x) - \langle \xi, x \rangle$ attains its infimum at some point x_0 and hence at $x_0 + ty$ for every $t \in \mathbf{R}$. Since $S(x)$ is finite for all x, there can be only a finite number of $\alpha \in A \setminus B$ such that $c_\alpha + \langle \alpha, x_0 \rangle > S_B(x_0)$. Moreover, for each $\alpha \in A \setminus B$, $S_B(x_0 + ty) - \langle \alpha, x_0 + ty \rangle$ is an increasing function of t which tends to $+\infty$ when $t \to +\infty$. This implies that $S_B(x_0 + ty) = S(x_0 + ty)$ for sufficiently large t. Since $S(x) \geq S_B(x)$, it follows that $S(x) - \langle \xi, x \rangle$ attains its smallest value at $x = x_0 + ty$ for sufficiently large t.

(v) Let $x \in \mathbf{R}^n$, let $A_1 = \{\alpha \in A; c_\alpha + \langle \alpha, x \rangle = S(x)\}$ and $A_2 = A \setminus A_1$, and let $S_j(y) = \max_{\alpha \in A_j}(c_\alpha + \langle \alpha, y \rangle)$ for $j = 1, 2$. Then $S(x) = S_1(x) > S_2(x)$, so it follows that $S(y) = S_1(y)$ for y in a neighborhood of x. Moreover, $S_1(x + ty) = S(x) + t \max_{\alpha \in A_1} \langle \alpha, y \rangle$ is linear in t for all $t > 0$. This proves the first part.

To prove the second part, let $\xi \in K$, let $\eta \in \mathbf{R}^n$, let $\sigma = \{x \in \mathbf{R}^n; P(\xi, x) = 0\}$, and take $x \in \sigma$ with $\langle \eta, x \rangle$ as large as possible. We claim that $\tilde{S}(\xi + t\eta) = \tilde{S}(\xi) + t\langle \eta, x \rangle$ for small positive t.

To prove this claim, note that $\tilde{S}(\xi + t\eta) \geq \langle \xi + t\eta, x_0 \rangle - S(x_0) = \tilde{S}(\xi) + t\langle \eta, x_0 \rangle$. On the other hand, if we write $S(x) = \max(S_1(x), S_2(x))$ as in the first part, then $S \geq S_1$, so $\tilde{S}(\xi + t\eta) \leq \tilde{S}_1(\xi + t\eta) = \tilde{S}(\xi) + t\langle \eta, x_0 \rangle$ for small positive t. □

Define T to be the collection of all sets $\sigma_\xi = \{x;\, P(\xi, x) = 0\}$ for $\xi \in K$. Define T' analogously as the collection of all sets $\sigma'_x = \{\xi \in K;\, P(\xi, x) = 0\}$. Then T and T' are dual convex subdivisions.

PROPOSITION 1
With notation as in the preceding paragraph, T and T' are dual convex subdivisions of $\mathbf{R}^n$ and K, where the correspondence between T and T' is given by $\sigma^* = \bigcap_{x\in\sigma} \sigma'_x = \{\xi \in K;\, P(\xi, x) = 0, \forall x \in \sigma\}$ and $\sigma'^* = \bigcap_{\xi\in\sigma'} \sigma_\xi = \{x \in \mathbf{R}^n;\, P(\xi, x) = 0, \forall \xi \in \sigma'\}$ for $\sigma \in T$, $\sigma' \in T'$. Moreover, T is locally finite and T' is polytopal.

Proof
First, we must check that T and T' are convex subdivisions. The cells σ_ξ and σ'_x are convex by properties (i) and (ii) of Lemma 1, and they are nonempty by property (iv). It also follows from property (iv) that $\bigcup_{\xi\in K} \sigma_\xi = \mathbf{R}^n$ and that $\bigcup_{x\in\mathbf{R}^n} \sigma'_x = K$. Hence condition (i) of Definition 1 is satisfied.

If $\sigma_{\xi_1} \cap \sigma_{\xi_2}$ is nonempty, we claim that $\sigma_{\xi_1} \cap \sigma_{\xi_2} = \sigma_{(\xi_1+\xi_2)/2}$. Indeed, if $x \in \sigma_{\xi_1} \cap \sigma_{\xi_2}$, then $P(\xi_1, x) = P(\xi_2, x) = 0$. Since $P(\xi, x)$ is nonnegative and convex in ξ, it follows that $P((\xi_1 + \xi_2)/2, x) = 0$, so $x \in \sigma_{(\xi_1+\xi_2)/2}$. Conversely, let $x \in \sigma_{(\xi_1+\xi_2)/2}$, and let $x_0 \in \sigma_{\xi_1} \cap \sigma_{\xi_2}$. Then $P(\xi_1, x_0) = P(\xi_2, x_0) = P((\xi_1 + \xi_2)/2, x) = 0$, and by the preceding argument, $P((\xi_1 + \xi_2)/2, x_0) = 0$. By property (iii) of Lemma 1, with $y = x_0$, $\xi = (\xi_1 + \xi_2)/2$, and $\eta = \xi_1$ or ξ_2, it follows that $P(\xi_1, x) = \langle \xi_1 - \xi_2, x - x_0\rangle/2 = -P(\xi_2, x)$. Since P is nonnegative, it follows that $P(\xi_1, x) = P(\xi_2, x) = 0$, so $x \in \sigma_{\xi_1} \cap \sigma_{\xi_2}$. This verifies condition (ii) of Definition 1 for T.

Suppose that $\sigma_{\xi_1} \subset \sigma_{\xi_2}$. We must show that σ_{ξ_1} is a face of σ_{ξ_2}. Let $\eta = \xi_1 - \xi_2$, and let $\tau = \{x \in \sigma_{\xi_2};\, \langle \eta, x\rangle = \sup_{y\in\sigma_{\xi_2}} \langle \eta, y\rangle\}$ be the corresponding face of σ_{ξ_2}. We claim that $\sigma_{\xi_1} = \tau$. Indeed, if $x \in \sigma_{\xi_1}$ and $y \in \sigma_{\xi_2}$, it follows from property (iii) of Lemma 1 that $\langle \eta, x - y\rangle = P(\xi_1, y) \geq 0$, which shows that $\sigma_{\xi_1} \subset \tau$. On the other hand, if $x \in \tau$ and $y \in \sigma_{\xi_1} \subset \tau$, then $0 = \langle x - y, \eta\rangle = -P(\xi_1, x)$, so $x \in \sigma_{\xi_1}$, completing the proof that $\sigma_{\xi_1} = \tau$.

Conversely, let $\xi_2 \in K$, and let $\tau = \{x \in \sigma_{\xi_2};\, \langle \eta, x\rangle = \sup_{y\in\sigma_{\xi_2}} \langle \eta, y\rangle\}$ be a face of σ_{ξ_2}. We must find $\xi_1 \in K$ such that $\tau = \sigma_{\xi_1}$. Take $t_0 > 0$ so that $P(\xi_2 + t\eta, x)$ is linear for $t \in [0, t_0]$, and let $\xi_1 = \xi_2 + (t_0/2)\eta$. For all $x \notin \sigma_{\xi_2}$, $P(\xi_2, x) > 0$ and $P(\xi_2 + t_0\eta, x) \geq 0$. For all $x \in \sigma_{\xi_2} \setminus \tau$ and $y \in \tau$, $P(\xi_2 + t_0\eta, x) = t_0\langle \eta, y - x\rangle > 0$ and $P(\xi_2, x) = 0$. From the linearity of P, it follows that $P(\xi_1, x) > 0$ for all $x \notin \tau$. If $x, y \in \tau$, then $P(\xi_1, x) - P(\xi_1, y) = P(\xi_2, x) - P(\xi_2, y) - (t_0/2)\langle \eta, x - y\rangle = 0$. Since $P(\xi_1, x) = 0$ for some x, it follows that $\sigma_{\xi_1} = \tau$.

This completes the proof that T is a convex subdivision. The proof for T' is analogous.

Next, we check the duality property. It is clear that σ^* is either empty or a cell of T', that $\tau \subset \sigma \Rightarrow \sigma^* \subset \tau^*$, that $\xi \in \sigma_\xi^*$, and that $\sigma \subset \sigma^{**}$. Together with the corresponding statements for cells in T', this implies that $\sigma \mapsto \sigma^*$ is an inclusion reversing bijection from the cells of T to the cells of T'. Thus it remains to check condition (ii) of Definition 2.

Let $\tau \subset \sigma$ be cells in T. If $x - y \in \operatorname{cone}(\tau, \sigma)$ and $\eta - \xi \in \operatorname{cone}(\sigma^*, \tau^*)$, where $x \in \sigma$, $y \in \tau$, $\xi \in \sigma^*$, $\eta \in \tau^*$, it follows from property (iii) of Lemma 1 that $\langle \eta - \xi, x - y\rangle = -P(\eta, x) \leq 0$, which implies that $\operatorname{cone}(\sigma^*, \tau^*) \subset \operatorname{cone}(\tau, \sigma)^\vee$.

To prove the opposite inclusion, assume $\sigma = \sigma_\xi$, and let $\eta \in \operatorname{cone}(\tau, \sigma)^\vee$. Take $t_0 > 0$ such that $P(\xi + t\eta, x)$ is linear for $t \in [0, t_0]$. We claim that $\xi + (t_0/2)\eta \in \tau^*$. Since $\xi \in \sigma^*$, this implies that $\eta \in \operatorname{cone}(\sigma^*, \tau^*)$, as desired. To prove the claim, note that for all $x \notin \sigma$, $P(\xi, x) > 0$ and $P(\xi + t_0\eta, x) \geq 0$, so from the linearity of P, it follows that $P(\xi + (t_0/2)\eta, x) > 0$. From property (iv) of Lemma 1, it follows that $P(\xi + (t_0/2)\eta, x)$ must vanish for some $x \in \sigma$. But if $x \in \sigma$ and $y \in \tau$, it follows from the assumption $\eta \in \operatorname{cone}(\tau, \sigma)^\vee$ that

$$0 \geq \Big(\frac{t_0}{2}\Big)\langle \eta, x - y\rangle = P\Big(\xi + \Big(\frac{t_0}{2}\Big)\eta, y\Big) - P\Big(\xi + \Big(\frac{t_0}{2}\Big)\eta, x\Big),$$

so $P(\xi + (t_0/2)\eta, y) \leq P(\xi + (t_0/2)\eta, x)$. Hence $P(\xi + (t_0/2)\eta, y) = 0$ for all $y \in \tau$, which means that $\xi + (t_0/2)\eta \in \tau^*$, as was claimed. □

Now we specialize to the situation considered at the beginning of this section. Thus f is an entire holomorphic function, A is the set of orders of complement components of $\mathscr{A}_f$, which we assume is nonempty, and $S(x)$ is the piecewise linear approximation of the Ronkin function defined by (3) and (4).

THEOREM 1

Let f and A be as above. Then there exist a convex subdivision T of $\mathbf{R}^n$ and a dual convex subdivision T' of the convex hull of A with the following properties.

(i) *The spine $\mathscr{S}_f$ is the union of all cells in T of dimension less than n, and it is a subset of $\mathscr{A}_f$.*

(ii) *For $\alpha \in A$, the cell of T dual to the point $\{\alpha\} \in T'$ contains the complement component of order α.*

(iii) *If the convex hull of A coincides with the Newton polyhedron of f (e.g., if f is a Laurent polynomial), then the spine is a deformation retract of the amoeba.*

Proof

The dual subdivisions T and T' are constructed from S and its Legendre transform according to Proposition 1. Let E_α denote the complement component of order α, and let $F_\alpha = \{x; S(x) = c_\alpha + \langle \alpha, x\rangle\}$. Then $E_\alpha \subset F_\alpha$, and by definition, the spine is the union of the boundaries of the sets F_α. From this it follows that $\mathscr{S}_f \subset \mathscr{A}_f$. It is readily verified that the F_α are the maximal cells of T and that each F_α is dual to $\{\alpha\} \in T'$. This proves (ii), and it also follows that the cells of T of dimension less than n are precisely the proper faces of the F_α. The union of all such cells is the union of the boundaries of the F_α, that is, the spine. This proves (i).

To construct the deformation retraction in (iii), take a point p_α in each boundary component E_α, and consider the collection of all line segments from p_α to the boundary of F_α. We claim that the union of all these segments contains the amoeba. Therefore we can deform the amoeba along the segments up to the spine.

To prove the claim, consider one of these points p_α, and consider a ray $\gamma = \{p_\alpha + tv; t \geq 0\}$ which does not intersect the boundary of F_α. We must show that this ray does not intersect the amoeba. The assumption that γ does not intersect the boundary of F_α implies that $\langle \alpha, v\rangle \geq \langle \beta, v\rangle$ for every $\beta \in A$ and hence for every β in the Newton polyhedron of f.

Let $r > 0$ be any real number, and let B_r be the ball of radius r centered at p_α. We show that $\gamma \cap B_r$ does not intersect the amoeba for any ray γ not intersecting the boundary of F_α.

Let $\epsilon > 0$ be a small number, to be specified in a moment, and write $f = f_1 + f_2$, where f_1 is a Laurent polynomial whose Newton polytope is contained in the Newton polytope of f, such that $|f_2(z)| < \epsilon |f_1(w)|$ for all $z \in \mathrm{Log}^{-1}(B_r)$ and $w \in \mathrm{Log}^{-1}(p_\alpha)$. Let C be the set of exponent vectors occurring in f_1, and let s be an integer vector such that $\langle \alpha, s\rangle \geq \langle \beta, s\rangle$ for all $\beta \in C$.

Consider the function $\phi(\zeta) = f(a_1\zeta^{-s_1}, \ldots, a_n\zeta^{-s_n})$ of one complex variable ζ. Then $\zeta^{\langle \alpha, s\rangle}\phi(\zeta)$ is holomorphic in the unit disc. Since $p_\alpha \in E_\alpha$, the number of poles minus the number of zeros in the unit disc of ϕ is $\langle \alpha, s\rangle$; hence $\zeta^{\langle \alpha, s\rangle}\phi(\zeta)$ does not vanish in the unit disc. By applying the maximum principle to its reciprocal and letting a vary over $\mathrm{Log}^{-1}(p_\alpha)$, this implies that

$$\min_{\mathrm{Log}^{-1}(p_\alpha + ts)} |f(z)| \geq e^{\langle \alpha, ts\rangle} \min_{\mathrm{Log}^{-1}(p_\alpha)} |f(z)|.$$

By continuity, this inequality remains true if we drop the requirement that s be an integer vector, and, in particular, it holds for $s = v$. Now it follows that $p_\alpha + tv \notin \mathscr{A}_f$ for $|tv| < r$, provided that $\epsilon < e^{-r|\alpha|}$. □

Remark. Suppose that f is a polynomial. In this case, the convex hull of A coincides with the Newton polytope of f. The convex subdivision of the Newton polytope constructed in Theorem 1 is an example of the coherent triangulations that play an important role in the theory of discriminants (see [5]).

4. Dependence of the spine on the polynomial

In this section we restrict attention to Laurent polynomials f. The spine $\mathscr{S}_f$ of its amoeba is easy to compute once we know the coefficients c_α from (3). We study here how the c_α depend on the coefficients of the polynomial f. The main result is an expansion of the coefficients c_α as a hypergeometric series in the coefficients of the polynomial. We also show that the coefficient c_α depends only on the truncation of f to the smallest face of its Newton polytope containing α, and we work out a few examples.

Consider a Laurent polynomial

$$f(z) = \sum_{\alpha \in C} w_\alpha z^\alpha$$

with coefficients w_α, where C is a finite subset of the lattice $\mathbf{Z}^n$. To study the dependence of c_α on the coefficients, it is useful to introduce the functions

$$\Phi_\alpha(w) = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{\log(f(z)/z^\alpha)\, dz_1 \cdots dz_n}{z_1 \cdots z_n}, \tag{6}$$

where x is in the component of order α. This means that

$$c_\alpha = \operatorname{Re} \Phi_\alpha.$$

Notice that Φ_α is a holomorphic function in the coefficients w with values in $\mathbf{C}/2\pi i\mathbf{Z}$, defined whenever the complement of the amoeba of f has a component of order α.

Remark. The functions Φ_α, or rather $\partial\Phi_\alpha/\partial w_\alpha$, were used in [2] in the study of constant terms in powers of Laurent polynomials. The main result was obtained by showing, essentially, that the second term in representation (9) is nonconstant along every complex line parallel to the w_α-axis, provided that α is not a vertex of the Newton polytope of f. These functions were also used in [13] to prove the existence of amoebas with prescribed complement components.

Example

Consider a one-variable polynomial $f(z) = (z + a_1) \cdots (z + a_N) = w_0 + \cdots + w_{N-1} z^{N-1} + z^N$. Let $0 \le k \le N$, and assume that $|a_1| \le \cdots \le |a_k| < r < |a_{k+1}| \le$

$\cdots \leq |a_N|$. Then one finds that

$$\begin{aligned}\Phi_k(w) &= \frac{1}{2\pi i}\int_{|z|=r}\frac{\log(f(z)/z^k)\,dz}{z}\\ &= \frac{1}{2\pi i}\sum_{j=1}^{k}\int_{|z|=r}\frac{\log((z+a_j)/z)\,dz}{z} + \frac{1}{2\pi i}\sum_{j=k+1}^{N}\int_{|z|=r}\frac{\log(z+a_j)\,dz}{z}\\ &= \sum_{j=k+1}^{N}\log a_j = \log(a_{k+1}\cdots a_N).\end{aligned}$$

In this case, we observe that $\Phi_k(w)$ may be continued as a finitely branched holomorphic function whose branches correspond to various permutations of the roots of f. Incidentally, the sum of all branches of $\exp\Phi_k(w)$ is equal to w_k. For polynomials of several variables, the analytic continuation of $\exp\Phi_\alpha(w)$ is in general infinitely branched.

Let Γ be a face of the Newton polytope of f, and let $f|_\Gamma$ denote the truncation of f to Γ; that is, let

$$f|_\Gamma(z) = \sum_{\alpha\in\Gamma} w_\alpha z^\alpha.$$

It is known that when $\alpha \in \Gamma \cap \mathbf{Z}^n$, the complement of the amoeba of f has a component of order α precisely if the complement of the amoeba of $f|_\Gamma$ does (see [4, Prop. 2.6] and also [5]).

THEOREM 2

Let f be a Laurent polynomial, and let Γ be a face of the Newton polytope of f. If $\alpha \in \Gamma$ and the complement of $\mathscr{A}_f$ has a component of order α, then $\Phi_\alpha(f) = \Phi_\alpha(f|_\Gamma)$. In particular, if α is a vertex of the Newton polytope of f, then $\Phi_\alpha(w) = \log w_\alpha$.

Proof

Take an outward normal v to the Newton polytope at Γ. If $\alpha \in \Gamma$ and x is in the component of $\mathbf{R}^n \setminus \mathscr{A}_f$ of order α, it is known that $x + tv$ is also in that component for all $t > 0$. Therefore

$$\Phi_\alpha(f) - \Phi_\alpha(f|_\Gamma) = \frac{1}{(2\pi i)^n}\int_{\mathrm{Log}^{-1}(x+tv)}\log\frac{f(z)}{f|_\Gamma(z)}\frac{dz_1\cdots dz_n}{z_1\cdots z_n},$$

and here the integrand clearly tends to zero when $t \to \infty$. □

The main result of this section is that the functions Φ_α very nearly satisfy a system of GKZ hypergeometric equations. Let A be a finite set in $\mathbf{Z}^{n+1}$ which is assumed to lie in the affine sublattice defined by setting the first coordinate equal to 1. Denote by $\mathbf{C}^A$ the set of all tuples $(w_\alpha;\, \alpha \in A,\, w_\alpha \in \mathbf{C})$. A GKZ system associated to A is a system of differential equations of the form

$$(\partial^u - \partial^v)\Phi = 0 \quad \text{if } u, v \in \mathbf{N}^A,\ \sum_{\alpha \in A}(u_\alpha - v_\alpha)\alpha = 0 \tag{7}$$

and

$$\sum_{\alpha \in A} \alpha w_\alpha \partial_\alpha \Phi = \lambda \Phi. \tag{8}$$

Here $\partial_\alpha = \partial/\partial w_\alpha$, and ∂^u is the obvious multi-index notation for a higher partial derivative. The constant λ denotes an arbitrary vector in $\mathbf{C}^{n+1}$.

The system of differential equations in the following theorem is very similar to a GKZ system where $A = \{(1, \gamma);\, \gamma \in C\}$ and $\lambda = 0$. The only difference is that we introduce a nonhomogeneous term in equation (8). It is easy to see that all partial derivatives of Φ_α, which are actually coefficients in a Laurent series expansion of $1/f$, satisfy a true GKZ hypergeometric system.

THEOREM 3
Let $f(z) = \sum_{\gamma \in C} w_\gamma z^\gamma$ be a Laurent polynomial with C being a fixed finite subset of $\mathbf{Z}^n$. Then the holomorphic functions Φ_α have the power series expansion

$$\Phi_\alpha(w) = \log w_\alpha + \sum_{k \in K_\alpha} \frac{(-k_\alpha - 1)!}{\prod_{\beta \neq \alpha} k_\beta!} (-1)^{k_\alpha - 1} w^k, \tag{9}$$

where

$$K_\alpha = \Big\{ k \in \mathbf{Z}^C;\, k_\alpha < 0,\, k_\beta \geq 0 \text{ if } \beta \neq \alpha,\, \sum_\gamma k_\gamma = 0,\, \sum_\gamma \gamma k_\gamma = 0 \Big\}. \tag{10}$$

The series converges, for example, when $|w_\alpha| > \sum_{\beta \neq \alpha} |w_\beta|$. Moreover, Φ_α satisfies the differential equations

$$(\partial^u - \partial^v)\Phi_\alpha = 0 \quad \text{if } \sum_\gamma (u_\gamma - v_\gamma) = 0 \text{ and } \sum_\gamma \gamma (u_\gamma - v_\gamma) = 0 \tag{11}$$

and

$$\sum_\gamma w_\gamma \partial_\gamma \Phi_\alpha = 1, \tag{12}$$

$$\sum_\gamma \gamma w_\gamma \partial_\gamma \Phi_\alpha = \alpha. \tag{13}$$

Proof
Use the power series expansion of the logarithm function to write

$$\log\big(f(z)/z^\alpha\big) = \log w_\alpha + \sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\Big(\sum_{\beta\neq\alpha} \frac{w_\beta z^\beta}{w_\alpha z^\alpha}\Big)^m.$$

Now

$$\sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\Big(\sum_{\beta\neq\alpha} \frac{w_\beta z^\beta}{w_\alpha z^\alpha}\Big)^m = \sum_{m\geq 1}\sum_{\Sigma k_\beta = m} \frac{(-1)^{m-1}}{m}\frac{m!}{\prod k_\beta!}\frac{\prod w_\beta^{k_\beta} z^{k_\beta \beta}}{w_\alpha^m z^{m\alpha}}$$
$$= \sum_{L_\alpha} (-1)^{k_\alpha - 1}\frac{(-k_\alpha - 1)!}{\prod k_\beta!} w^k z^{\Sigma k_\gamma \gamma}.$$

Here all sums and products indexed by β are taken over $\beta \in C \setminus \{\alpha\}$ while γ ranges over all of C and $L_\alpha = \{k \in \mathbf{Z}^C; k_\alpha < 0, k_\beta \geq 0, \sum k_\gamma = 0\}$. The constant terms in this expression, considered as monomials in the z-variables, are precisely those corresponding to the set K_α, and we have proved (9).

To verify the differential equations, we differentiate under the sign of integration defining Φ_α. By a simple computation,

$$\partial^u \log\big(f(z)/z^\alpha\big) = -\Big(\sum u_\gamma - 1\Big)!\, z^{\Sigma u_\gamma \gamma}\big(-f(z)\big)^{-\Sigma u_\gamma},$$

which depends only on $\sum u_\gamma$ and $\sum u_\gamma \gamma$. Also,

$$\sum w_\gamma \partial_\gamma \log\big(f(z)/z^\alpha\big) = \sum \frac{w_\gamma z^\gamma}{f(z)} = 1.$$

This verifies (11) and (12). Finally,

$$\sum \gamma_j w_\gamma \partial_\gamma \log\big(f(z)/z^\alpha\big) = \sum \frac{\gamma_j w_\gamma z^\gamma}{f(z)} = \frac{z_j \partial f/\partial z_j}{f(z)}.$$

Comparing this to definition (2) of the order of a component proves relation (13). □

Example
Let us consider the polynomial $f(z) = 1 + z_1^{n+1} + \cdots + z_n^{n+1} + a z_1 \cdots z_n$ in n variables. The Newton polytope of this polynomial is a simplex, and the terms correspond to the vertices and the barycenter of the Newton polytope. The complement of the amoeba of f has at least $n+1$ components corresponding to the vertices of the Newton polytope, and in addition there may be a component corresponding to the single interior point $(1, \ldots, 1)$. For the case of $n = 2$, see Figure 2. Let K_n denote the set of $a \in \mathbf{C}$ such that $\mathbf{R}^n \setminus \mathscr{A}_f$ does not have a component of order $(1, \ldots, 1)$.

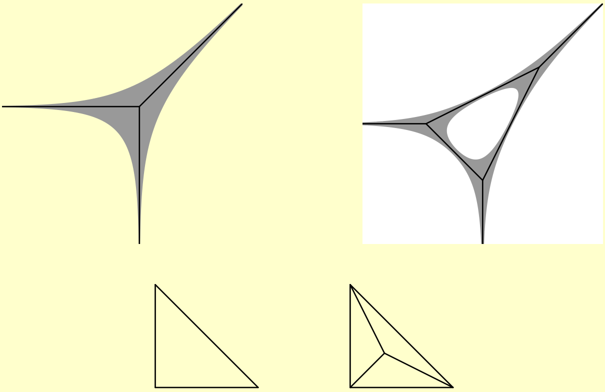

Figure 2. Amoebas, spines, and triangulated Newton polytopes of the polynomial $1 + z_1^3 + z_2^3 + az_1 z_2$ for $a = 0$ and $a = -6$

PROPOSITION 2

Let $f(z) = 1 + z_1^{n+1} + \cdots + z_n^{n+1} + az_1 \cdots z_n$. Then $\mathbf{R}^n \setminus \mathscr{A}_f$ has a component of order $(1, \ldots, 1)$ if and only if $\mathscr{A}_f$ does not contain the origin. Hence $\mathbf{R}^n \setminus \mathscr{A}_f$ has such a component precisely if

$$a \notin K_n = \big\{ -t_0 - \cdots - t_n ;\, |t_0| = \cdots = |t_n| = 1, t_0 \cdots t_n = 1 \big\}.$$

Proof

Let M be an invertible $(n \times n)$-matrix with integer entries, and let $f(z) = \sum_\alpha a_\alpha z^\alpha$ be any Laurent polynomial. Define a new Laurent polynomial $Mf = \sum_\alpha a_\alpha z^{M\alpha}$ where a multi-index α is regarded as a column vector. It is not difficult to show that the linear mapping $x \mapsto M^T x$ (where T denotes transpose) takes the amoeba of Mf onto the amoeba of f.

Now let f be the special polynomial in the proposition, and consider the M that map the Newton polytope of f onto a translate of itself. The set of such M can be identified with the symmetric group S_{n+1} via its action on the vertices. Then Mf coincides with f up to an invertible factor; hence they have the same amoeba. In particular, M^T maps the component of order $(1, \ldots, 1)$ in the complement of $\mathscr{A}_f$ (if it exists) onto itself. If x is any point in that component, then the convex hull of the points $M^T x$, where M ranges over S_{n+1}, contains the origin. Since every component in the complement of the amoeba is convex, it follows that the component of order $(1, \ldots, 1)$ contains the origin. Conversely, if some component E contains the origin, then $M^T E$ also contains the origin and hence coincides with E. This implies that E has order $(1, \ldots, 1)$.

Now the amoeba of f contains the origin precisely if

$$a = -(1 + z_1^{n+1} + \cdots + z_n^{n+1})/z_1 \cdots z_n$$

with $|z_1| = \cdots = |z_n| = 1$ or, equivalently,

$$a \in \{ -t_0 - \cdots - t_n;\, |t_0| = \cdots = |t_n| = 1, t_0 \cdots t_n = 1 \}. \qquad \square$$

We depict the sets K_n for $n = 2$ and $n = 3$ in Figure 3. Note that the cusps on the boundary correspond to polynomials defining singular hypersurfaces.

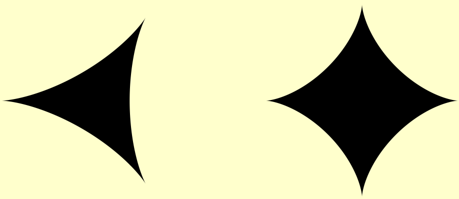

Figure 3. The sets K_n for $n = 2$ and 3

Let us, in particular, consider the case of $n = 2$. The coefficients c_α are equal to zero when α is a vertex of the Newton polytope, while

$$c_{(1,1)} = \log|a| + \operatorname{Re} \sum_{k>0} \frac{(3k-1)!}{(k!)^3} (-1)^{k-1} a^{-3k}$$

when $|a| > 3$ by Theorem 2. In fact, it can easily be checked that the series converges even when $|a| = 3$. What happens with the spine when a approaches the boundary of K_2 from the outside? For example, when $a \to -3$, a numerical computation of the power series shows that $c_{(1,1)}$ converges to a limit with the approximate value 0.9693. This means that the complement of the spine has a rather large component that suddenly disappears when a enters K_2.

5. Associated Monge-Ampère measures

Let f be a holomorphic function in $\operatorname{Log}^{-1}(\Omega)$, where Ω is a convex domain in $\mathbf{R}^n$. We have seen that N_f is a convex function that is linear in each component of $\Omega \setminus \mathscr{A}_f$. Now we associate another object with f, namely, the Monge-Ampère measure μ_f of the Ronkin function. This is a positive measure with support on the amoeba $\mathscr{A}_f$.

In Section 6 this measure is used to compute the maximal area of a plane amoeba. The goal of this section is to establish a few general results about μ_f. First, we compute the total mass of μ_f when f is a Laurent polynomial. Then we give an interpretation of μ_f in terms of solutions to certain systems of equations. Let us remark that it might also be interesting to study the Monge-Ampère measure of the Legendre transform of the Ronkin function. An investigation of such measures is not pursued in the present paper.

If u is a smooth convex function, its Hessian $\mathrm{Hess}(u)$ is a positive semidefinite matrix that in a certain sense measures how convex u is near a given point. In particular, the determinant of the Hessian is a nonnegative function. The product of this function with ordinary Lebesgue measure is known as the real Monge-Ampère measure of u, which we denote by Mu. In fact, the Monge-Ampère operator can be extended as a continuous operator from the space of all convex functions with the topology of locally uniform convergence to the space of measures with the weak topology. In general, if u is a nonsmooth convex function, Mu will be a positive Borel measure (see [10]). We denote the Monge-Ampère measure of N_f by μ_f. Hence μ_f is a positive measure supported on the amoeba of f.

It is interesting to consider also a generalization of the Monge-Ampère operator. Notice that on smooth functions u, the operator M is actually the restriction to the diagonal of a symmetric multilinear operator $\tilde{M}$ on n functions $u_1, \ldots, u_n$. Conversely, $\tilde{M}$ can be recovered from M by the polarization formula

$$\tilde{M}(u_1, \ldots, u_n) = \frac{1}{n!} \sum_{k=1}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} (-1)^{n-k} M(u_{j_1} + \cdots + u_{j_k}). \tag{14}$$

This expression still makes sense if $u_1, \ldots, u_n$ are arbitrary convex functions, and by approximating $u_1, \ldots, u_n$ with smooth convex functions it follows that $\tilde{M}(u_1, \ldots, u_n)$ is a positive measure that depends multilinearly on the arguments u_j. We call $\tilde{M}$ the *mixed real Monge-Ampère operator*.

The Monge-Ampère operator can be given the following geometric interpretation. Let u be a convex function defined in a neighborhood of a Borel set $E \subset \mathbf{R}^n$. Consider the set of $\xi \in \mathbf{R}^n$ such that the function $\langle \xi, x \rangle - u(x)$ attains its global maximum at some $x \in E$. Then the Lebesgue measure of this set equals $Mu(E)$.

In the case of $E = \mathbf{R}^n$, it is possible to give a similar characterization of the mixed Monge-Ampère operator. If u is a convex function defined in $\mathbf{R}^n$, we let K_u denote the set of all $\xi \in \mathbf{R}^n$ such that the function $\langle \xi, x \rangle - u(x)$ is bounded from above; in other words, the Legendre transform $\tilde{u}(\xi)$ is finite.

PROPOSITION 3

If $u, u_1, \ldots, u_n$ are convex functions, then the following statements hold:

(i) K_u *is a convex set;*

(ii) $K_{u_1+u_2} = K_{u_1} + K_{u_2}$*;*

(iii) *if K_u is bounded, then its volume equals the total mass of Mu;*

(iv) *if $K_{u_1}, \dots, K_{u_n}$ are all bounded, then their mixed volume equals the total mass of $\tilde{M}(u_1, \dots, u_n)$.*

Proof

Convexity of K_u and the inclusion $K_{u_1} + K_{u_2} \subset K_{u_1+u_2}$ are trivial. It is well known that the Legendre transform of $u_1 + u_2$ is the lower semicontinuous regularization of the function

$$\xi \mapsto \inf_{\xi_1+\xi_2=\xi} \big(\tilde{u}_1(\xi_1) + \tilde{u}_2(\xi_2)\big)$$

(see [6, Th. 2.2.5]). If $\xi \in K_{u_1+u_2}$, it follows that there is a constant C such that there exist ξ_1, ξ_2 with $\tilde{u}_1(\xi_1)+\tilde{u}_2(\xi_2) \le C$ and $\xi_1+\xi_2$ arbitrarily close to ξ. Since $\tilde{u}_1, \tilde{u}_2$ are bounded from below, we may assume (after possibly increasing C) that $\tilde{u}_j(\xi_j) \le C$. The set L_j of ξ_j satisfying this inequality is closed by lower semicontinuity of $\tilde{u}_j$ and bounded because u_j is bounded in a neighborhood of the origin. By a compactness argument, we therefore obtain the existence of points $\xi_j \in L_j \subset K_{u_j}$ with $\xi_1+\xi_2 = \xi$. This proves that $K_{u_1+u_2} = K_{u_1} + K_{u_2}$.

Let G denote the set of ξ such that $\langle \xi, x\rangle - u(x)$ attains a maximal value. We claim that $\operatorname{int} K_u \subset G \subset K_u$. The second inclusion is obvious; for if a real-valued function attains a minimal value, then it certainly is bounded from below. To prove the first inclusion, take a point ξ in the interior of K_u, and choose points $\xi_0, \dots, \xi_n$ in K_u whose convex hull contains ξ in its interior. Thus we have $u(x) - \langle \xi_j, x\rangle \ge C_j$ for $j = 0, \dots, n$. It follows that

$$u(x) - \langle \xi, x\rangle \ge \max_j \big(C_j - \langle \xi_j - \xi, x\rangle\big).$$

Since the right-hand side goes to ∞ when $x \to \infty$, it follows that the left-hand side attains a minimal value. Hence $\xi \in G$.

Since K_u is convex, we have $\mathrm{Vol}(\operatorname{int} K_u) = \mathrm{Vol}(K_u)$, and it follows that $\mathrm{Vol}(K_u) = \mathrm{Vol}(G) = Mu(\mathbf{R}^n)$. This proves (iii). It follows immediately that (iv) is true when $u_1 = \cdots = u_n$. Since both sides are multilinear and symmetric, it follows that equality holds in general. □

Now we can compute the total mass of the Monge-Ampère measure μ_f for a Laurent polynomial f.

THEOREM 4

If f is a Laurent polynomial, then the total mass of μ_f is equal to the volume of the Newton polytope of f. If $f_1, \dots, f_n$ are Laurent polynomials, then the total mass of $\tilde{M}(N_{f_1}, \dots, N_{f_n})$ is equal to the mixed volume of the Newton polytopes of $f_1, \dots, f_n$.

Proof
We need only show that K_{N_f} is the Newton polytope of f. Let $\mathscr{N}$ denote the Newton polytope. If ξ is in $\mathscr{N}$, then $\langle \xi, x\rangle - N_f(x) \le \langle \xi, x\rangle - S(x)$, where $S(x)$ is defined by (3) and (4), and this latter function is certainly bounded from above. Conversely, if ξ is outside $\mathscr{N}$, take $v \in \mathbf{R}^n$ so that $\langle \xi, v\rangle > \sup_{\eta\in\mathscr{N}}\langle \eta, v\rangle$ and a vertex α of $\mathscr{N}$ where this supremum is attained. If x belongs to the complement component of order α, then $x + tv$ is also in that component for all $t > 0$. Hence $\langle \xi, x + tv\rangle - N_f(x + tv) = \langle \xi - \alpha, x + tv\rangle - c_\alpha \to \infty$ as $t \to \infty$, so ξ is not in K_{N_f}. □

Our next result depends on the following relations between the real and complex Monge-Ampère operators, due to Rashkovskii [9]. Suppose that $u_1, \dots, u_n$ are smooth convex functions on the domain Ω. Let $U_1, \dots, U_n$ be plurisubharmonic functions on $\operatorname{Log}^{-1}(\Omega)$ defined by $U_j(z) = u_j(\operatorname{Log} z)$. Then

$$n! \int_E \tilde{M}(u_1, \dots, u_n) = \int_{\operatorname{Log}^{-1}(E)} dd^c U_1 \wedge \cdots \wedge dd^c U_n, \tag{15}$$

where $d^c = (\partial - \bar{\partial})/2\pi i$. This remains true if one of the functions U_j, say, U_1, is allowed to be an arbitrary smooth plurisubharmonic function in $\operatorname{Log}^{-1}(\Omega)$, u_1 being defined by

$$u_1(x) = \frac{1}{(2\pi i)^n} \int_{\operatorname{Log}^{-1}(x)} \frac{U_1(z)\, dz_1 \cdots dz_n}{z_1 \cdots z_n}.$$

More generally, if $U_1, \dots, U_n$ are arbitrary smooth plurisubharmonic functions on $\operatorname{Log}^{-1}(\Omega)$, and

$$u_j(x) = \frac{1}{(2\pi i)^n} \int_{\operatorname{Log}^{-1}(x)} \frac{U_j(z)\, dz_1 \cdots dz_n}{z_1 \cdots z_n}, \tag{16}$$

then

$$n! \int_E \tilde{M}(u_1, \dots, u_n) = \int_{T^{n^2}} \int_{\operatorname{Log}^{-1}(E)} dd^c U_1(t^{(1)}z) \wedge \cdots \wedge dd^c U_n(t^{(n)}z)\, d\lambda(t). \tag{17}$$

Here T^{n^2} denotes the real n^2-dimensional torus $\{t = (t_j^{(k)});\ |t_j^{(k)}| = 1,\ j, k = 1, \dots n\}$ equipped with the usual normalized Haar measure λ, and each $t^{(k)} = (t_1^{(k)}, \dots, t_n^{(k)})$ acts on $\mathbf{C}^n$ by componentwise multiplication. These formulas can be checked by direct computation. A proof of (15) when $u_1 = \cdots = u_n$ can be found in [9]. The general case follows by polarization since both sides are multilinear. Formula (17) follows by reversing the order of integration on the right. Since the inner integral is constant along certain n-dimensional submanifolds of T^{n^2}, it is actually possible to omit some of the variables in the outer integration. This also proves the generalized version of (15) with U_1 an arbitrary smooth plurisubharmonic function.

The following result follows from formula (17) and the Poincaré-Lelong formula.

THEOREM 5

Let $f_1, \ldots, f_n$ *be holomorphic functions in* $\mathrm{Log}^{-1}(\Omega)$, *and let* $E \subset \Omega$ *be a Borel set. Then* $n!\tilde{M}(N_{f_1}, \ldots, N_{f_n})(E)$ *is equal to the average number of solutions in* $\mathrm{Log}^{-1}(E)$ *to the system of equations*

$$f_j(t_1^{(j)} z_1, \ldots, t_n^{(j)} z_n) = 0, \quad j = 1, \ldots, n, \tag{18}$$

as $t = (t_k^{(j)})$ *ranges over the torus* $\{t; |t_k^{(j)}| = 1, j, k = 1, \ldots, n\}$.

Proof

If U_j are smooth plurisubharmonic functions that converge to $\log|f_j|$, then u_j defined by (16) converge to N_{f_j}. By the general properties of the real Monge-Ampère operator, this implies that $\tilde{M}(u_1, \ldots, u_n)$ converges weakly to $\tilde{M}(N_{f_1}, \ldots, N_{f_n})$. Also, $dd^c U_1(t^{(1)}z) \wedge \cdots \wedge dd^c U_n(t^{(n)}z)$ converges weakly to the sum of point masses at the solutions of $f_1(t^{(1)}z) = \cdots = f_n(t^{(n)}z) = 0$. Hence the theorem follows by passing to the limit in (17) if we show only that

$$\int_{\mathrm{Log}^{-1}(E)} dd^c U_1(t^{(1)}z) \wedge \cdots \wedge dd^c U_n(t^{(n)}z)$$

remains uniformly bounded as $U_j \to \log|f_j|$. Here we may assume that E is compact and that U_j is of the form $U_j = \psi(\log|f_j|)$, where ψ is a convex function, constant near $-\infty$, which will converge to the identity function. Let $f_t(z) = (f_1(t^{(1)}z), \ldots, f_n(t^{(n)}z))$. Then $f_t(z)$ is a holomorphic function in z and t defined for z in a neighborhood of $\mathrm{Log}^{-1}(E)$ and t in a complex neighborhood of T^{n^2}. Using compactness arguments, it is not difficult to show that there exists a constant C such that the number of solutions in $\mathrm{Log}^{-1}(E)$ to the equation $f_t(z) = w$ is bounded above by C for almost all $t \in T^{n^2}$ and $w \in \mathbf{C}^n$. Since $\eta = dd^c\psi(\log|w_1|) \wedge \cdots \wedge dd^c\psi(\log|w_n|)$ induces a positive measure on $\mathbf{C}^n$ with total mass 1, it follows that

$$0 \le \int_{\mathrm{Log}^{-1}(E)} f_t^* \eta \le C$$

for almost all t, and this completes the proof. □

Notice that an alternative proof of Theorem 4 is to use Theorem 5 and the Bernstein theorem (see [1]) on the number of solutions to a system of polynomial equations. On the other hand, the Bernstein theorem follows from Theorems 4 and 5 if it is assumed that the number of solutions is a constant depending only on the Newton polytopes. In fact, Theorem 5 can be regarded as a localized version of the Bernstein theorem which also gives some information about where the solutions are likely to be.

Finally, we note that the method used to prove Theorem 5 can also be used to derive an interpretation of the Laplacian of the Ronkin function.

THEOREM 6
Let E be any Borel set in Ω, and let Δ denote the Laplace operator. Then

$$(n-1)!\int_E \Delta N_f = \int_{\mathrm{Log}^{-1}(E)\cap f^{-1}(0)} \omega^{n-1},$$

where $\omega = (|z_1|^{-2}d\bar{z}_1 \wedge dz_1 + \cdots + |z_n|^{-2}d\bar{z}_n \wedge dz_n)/4\pi i$.

Proof
Since $\mathrm{Hess}(|x|^2/2)$ is the identity matrix, it follows that

$$\Delta u = n\tilde{M}(u, |x|^2/2, \ldots, |x|^2/2)$$

for any convex function u. Now $\omega = dd^c|\operatorname{Log} z|^2/2$, and since $dd^c \log|f|$ is equal to the current of integration along $f^{-1}(0)$, it follows from (15) that

$$\begin{aligned} n!\int_E \Delta N_f &= n\int_{\mathrm{Log}^{-1}(E)} dd^c \log|f| \wedge \omega^{n-1} \\ &= n\int_{\mathrm{Log}^{-1}(E)\cap f^{-1}(0)} \omega^{n-1}. \end{aligned}$$

□

6. Monge-Ampère measures for planar curves

We now give some results on the Monge-Ampère measure which are specific to the two-variable case. For $n = 1$, the situation is of course trivial since the amoeba is a discrete point set and μ_f is a sum of point masses at these points. For $n = 2$, we have the following estimate on the Monge-Ampère measure.

THEOREM 7
Let f be a holomorphic function in two variables defined on a circular domain $\mathrm{Log}^{-1}(\Omega)$. *Then μ_f is greater than or equal to π^{-2} times the Lebesgue measure on $\mathscr{A}_f$.*

Proof
Let F denote the set of critical values of the mapping $\mathrm{Log} : f^{-1}(0) \to \Omega$. We include in F the images of any singular points of $f^{-1}(0)$. Since F is a null set for the Lebesgue measure, it is sufficient to prove the inequality in $\mathscr{A}_f \setminus F$.

Take a small neighborhood U of some point in $\mathscr{A}_f \setminus F$ such that there exist smooth functions ϕ_k, ψ_k defined in U with the property that

$$\mathrm{Log}^{-1}(U) \cap f^{-1}(0) = \bigcup_k \big\{(e^{x_1+i\phi_k(x)}, e^{x_2+i\psi_k(x)});\, x = (x_1, x_2) \in U\big\}.$$

We claim that

$$\operatorname{Hess} N_f = \frac{1}{2\pi} \sum_k \pm \begin{pmatrix} \partial\psi_k/\partial x_1 & \partial\psi_k/\partial x_2 \\ -\partial\phi_k/\partial x_1 & -\partial\phi_k/\partial x_2 \end{pmatrix}. \tag{19}$$

Differentiating the integral (1) defining N_f with respect to x_1, we obtain

$$\begin{aligned} \frac{\partial N_f}{\partial x_1} &= \operatorname{Re} \frac{1}{(2\pi i)^2} \int_{\operatorname{Log}^{-1}(x)} \frac{\partial f/\partial z_1\, dz_1\, dz_2}{f(z) z_2} \\ &= \frac{1}{2\pi i} \int_{\log|z_2|=x_2} n\big(f(\cdot, z_2), x_1\big) \frac{dz_2}{z_2} \\ &= \frac{1}{2\pi} \int_0^{2\pi} n\big(f(\cdot, e^{x_2+iy_2}), x_1\big)\, dy_2. \end{aligned}$$

Here $n(f(\cdot, z_2), x_1)$ is the number of zeros minus the number of poles of the function $z_1 \mapsto f(z_1, z_2)$ inside the disc $\{\log|z_1| < x_1\}$ provided that it is meromorphic in that domain. In general, $n(f(\cdot, z_2), x_1) - n(f(\cdot, z_2), x_1')$ is equal to the number of zeros in the annulus $\{x_1' < \log|z_1| < x_1\}$ when $x_1' < x_1$. The integrand in the last integral is a piecewise constant function with jumps of magnitude 1 at $y_2 = \psi_k(x)$. It follows that the gradient of $\partial N_f/\partial x_1$ is given by a sum of terms $\pm(2\pi)^{-1} \operatorname{grad} \psi_k$. This proves the first row of identity (19), up to sign changes. The correct sign of each term can be found by observing that $n(f(\cdot, e^{x_2+iy_2}), x_1)$ is increasing as a function of x_1; hence all the terms contributing to $\partial^2 N_f/\partial x_1^2$ should be positive. A similar computation involving $\partial N_f/\partial x_2$ proves the second row. However, we have not yet shown that the choices of signs in the two rows are consistent.

We now prove that all the terms on the right-hand side of (19) are symmetric, positive definite matrices with determinant equal to 1. Take a point x and an index k. Differentiating the expression $f(e^{x_1+i\phi_k(x)}, e^{x_2+i\psi_k(x)}) = 0$ with respect to x_1 and x_2 yields the equations

$$\begin{aligned} z_1 \frac{\partial f}{\partial z_1}\Big(1 + i\frac{\partial\phi_k}{\partial x_1}\Big) + z_2 \frac{\partial f}{\partial z_2} \cdot i\frac{\partial\psi_k}{\partial x_1} &= 0, \\ z_1 \frac{\partial f}{\partial z_1} \cdot i\frac{\partial\phi_k}{\partial x_2} + z_2 \frac{\partial f}{\partial z_2}\Big(1 + i\frac{\partial\psi_k}{\partial x_2}\Big) &= 0. \end{aligned}$$

Writing $a = z_1\partial f/\partial z_1$, $b = z_2\partial f/\partial z_2$, these equations have the solution

$$\begin{pmatrix} \partial\psi_k/\partial x_1 & \partial\psi_k/\partial x_2 \\ -\partial\phi_k/\partial x_1 & -\partial\phi_k/\partial x_2 \end{pmatrix} = \frac{1}{\operatorname{Im}(\bar{a}b)} \begin{pmatrix} |a|^2 & \operatorname{Re}(\bar{a}b) \\ \operatorname{Re}(\bar{a}b) & |b|^2 \end{pmatrix}.$$

This matrix clearly has determinant 1. Changing the sign if $\operatorname{Im}(\bar{a}b) < 0$, we also have that the diagonal elements are positive, so the matrix is positive definite. Since we have already observed that the diagonal elements in the right-hand side of (19) are positive, it follows that they are positive definite with determinant equal to 1.

The inequality now follows from the following lemma since $\operatorname{Log}^{-1}(x)$ intersects $f^{-1}(0)$ in at least two points for all x in $\mathscr{A}_f \setminus F$. $\square$

LEMMA 2
If A_1, A_2 are 2×2 positive definite matrices, then $\det(A_1 + A_2) \geq \det A_1 + \det A_2 + 2\sqrt{\det A_1 \det A_2}$. *Equality holds if and only if A_1 and A_2 are real multiples of one another.*

Proof
To see this, write $A_j = \begin{pmatrix} a_j & b_j \\ b_j & c_j \end{pmatrix}$ and apply the Cauchy-Schwarz inequality to the vectors $(b_j, \sqrt{a_j c_j - b_j^2})$ to obtain $b_1 b_2 + \sqrt{\det A_1 \det A_2} \leq \sqrt{a_1 a_2 c_1 c_2}$. Then it follows that $\det(A_1 + A_2) - \det A_1 - \det A_2 = a_1 c_2 + c_1 a_2 - 2b_1 b_2 \geq 2\sqrt{a_1 a_2 c_1 c_2} - 2b_1 b_2 \geq 2\sqrt{\det A_1 \det A_2}$. The conditions for equality are that $(b_1^2, a_1 c_1 - b_1^2)$ is proportional to $(b_2^2, a_2 c_2 - b_2^2)$ and that (a_1, c_1) is proportional to (a_2, c_2), which clearly is equivalent to A_1 being proportional to A_2. $\square$

As an immediate consequence of Theorems 4 and 7, we have the following estimate.

COROLLARY 1
Let f be a Laurent polynomial in two variables. Then the area of the amoeba of f is not greater than π^2 times the area of the Newton polytope of f.

On the contrary, when $n \geq 3$, the volume of the amoeba of a polynomial is almost always infinite.

We now give a few simple examples of polynomials for which the inequality in Theorem 7 becomes an equality, at least at most points in the amoeba. For a detailed account of the cases where equality holds, and the amoeba hence has maximal area, we refer to [8]. There it is shown that the polynomials with amoebas of maximal area are (up to certain changes of variables) precisely the real polynomials defining a special kind of so-called Harnack curves in the real plane.

Example
Consider the polynomial $f(z_1, z_2) = 1 + z_1 + z_2$. It is not difficult to see that for this polynomial, equality holds in all the computations in the proof of Theorem 7. Hence the area of $\mathscr{A}_f$ is equal to $\pi^2/2$. This can also be established by a direct computation. The amoeba $\mathscr{A}_f$ is bounded by the curves $e^{x_1} + e^{x_2} = 1$ and $|e^{x_1} - e^{x_2}| = 1$. For reasons of symmetry (cf. the proof of Prop. 2), the spine $\mathscr{S}_f$ divides the amoeba into three parts with equal area. The spine consists of the negative halves of the coordinate axes and the positive diagonal $x_1 = x_2 \geq 0$. We compute the area of the part of $\mathscr{A}_f$

in the third quadrant $\{x_1 \le 0, x_2 \le 0\}$. There the amoeba is bounded by the curve $x_2 = \log(1 - e^{x_1})$; hence its area is

$$\int_{-\infty}^{0} -\log(1-e^t)\,dt = \sum_{k=1}^{\infty} \int_{-\infty}^{0} \frac{e^{kt}}{k}\,dt = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Closely related to this is the classical Euler dilogarithm function Li_2, given for $-1 \le x \le 1$ by the two equivalent expressions

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_{-\infty}^{\log|x|} \log\bigl(1 - \operatorname{sgn}(x)\, e^t\bigr)\,dt.$$

In particular, one has $\mathrm{Li}_2(0) = 0$ and $\mathrm{Li}_2(1) = \pi^2/6$. In our next example we have use for the odd part Li_2^- of the dilogarithm, which admits the representations

$$\mathrm{Li}_2^-(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)^2} = \frac{\operatorname{sgn}(x)}{2} \int_{-\infty}^{\log|x|} \log \frac{1+e^t}{1-e^t}\,dt$$

and satisfies $\mathrm{Li}_2^-(0) = 0$ and $\mathrm{Li}_2^-(1) = \pi^2/8$.

Example (see Fig. 4)
Here we let a be a real number and consider the polynomials $f(z_1, z_2) = f_a(z_1, z_2) = a + z_1 + z_2 + z_1 z_2$.

We want to compute the number of points in $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ for a given point $x \in \mathbf{R}^n$. For points z in this set, it must hold that $|a + z_1| = |1 + z_1| e^{x_2}$. Conversely, if this equation holds, then $(z_1, (a+z_1)/(1+z_1))$ is in this set. If $\theta = \arg z_1$, the equation can be rewritten $2e^{x_1}(e^{2x_2} - a)\cos\theta = a^2 + e^{2x_1} - e^{2x_2} - e^{2x_1+2x_2}$. From this it follows immediately that the amoeba of f is defined by the inequality $4e^{2x_1}(e^{2x_2} - a)^2 \ge (a^2 + e^{2x_1} - e^{2x_2} - e^{2x_1+2x_2})^2$. We now consider separately the cases of $a < 0$ and $a > 0$.

If $a < 0$, then the amoeba is bounded by the four curves

$$(1 \pm e^{x_1})(1 \pm e^{x_2}) = 1 - a. \tag{20}$$

If x is in the interior of the amoeba, then $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ has precisely two points, whereas for x in the boundary of the amoeba $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ consists of a single point. By Theorem 7, μ_f is greater than or equal to π^{-2} times the Lebesgue measure on the amoeba of f. For x in the interior of $\mathscr{A}_f$, the sum (19) representing $\mathrm{Hess}(N_f)$ consists of two terms. Since f has real coefficients, the curve $f^{-1}(0)$ is invariant under complex conjugation and it follows that $(\phi_1, \psi_1) = -(\phi_2, \psi_2)$; hence the two

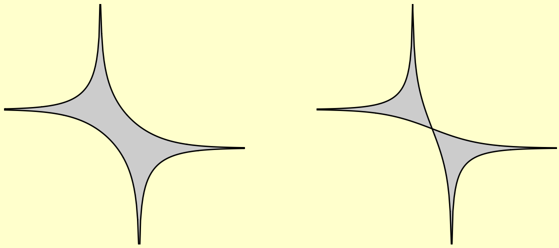

Figure 4. The amoebas of $a + z_1 + z_2 + z_1 z_2$ for $a = -5$ and $a = 5$

matrices are equal. This means that all inequalities in the proof of Theorem 7 actually become equalities. It follows also from Theorem 5 that μ_f has no mass on the boundary of the amoeba. Hence the area of the amoeba is equal to π^2.

If $a > 0$, the amoeba is still bounded by the curves defined by (20), but in this case a choice of the $+$ sign in both factors gives an empty set. The equation with $-$ signs in both factors defines a curve with two components bounding two complement components of the amoeba, while the curves defined by the two remaining equations intersect at a pinch in the middle of the amoeba. In this case too, $\mathrm{Log}^{-1}(0) \cap f^{-1}(0)$ has exactly two points for all x in the interior of the amoeba. However, if $x = (\log a, \log a)/2$, then $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ contains a real curve. For all other points x on the boundary of the amoeba, $\mathrm{Log}^{-1}(x)$ intersects $f^{-1}(0)$ in exactly one point. It follows just as in the case of $a < 0$ that μ_{f_a} is equal to π^{-2} times the Lebesgue measure in the interior of the amoeba. However, $\mathscr{A}_{f_a}$ is strictly smaller than $\mathscr{A}_{f_{-a}}$, so its area is smaller than π^2. The remaining mass of μ_{f_a}, which must have the same total mass as $\mu_{f_{-a}}$, resides as a point mass at the pinch $(\log a, \log a)/2$. We compute the size of this point mass in two different ways. For simplicity, we assume that $a > 1$.

First, we may compute the difference in area of the amoebas $\mathscr{A}_{f_a}$ and $\mathscr{A}_{f_{-a}}$. Note that $\mathscr{A}_{f_{-a}} \setminus \mathscr{A}_{f_a}$ consists of four regions. The lower right-hand part is bounded by the curves $x_2 = \log(e^{x_1} - a) - \log(e^{x_1} + 1)$ and $x_2 = \log(e^{x_1} - a) - \log(e^{x_1} - 1)$, where x_1 ranges from $\log a$ to ∞. Hence its area is

$$\int_{\log a}^{\infty} \log \frac{e^{x_1} + 1}{e^{x_1} - 1}\, dx_1.$$

The upper right-hand part is bounded above for $0 < x_1 < \infty$ by the curve $x_2 = \log(e^{x_1} + a) - \log(e^{x_1} - 1)$, and it is bounded below by the two curves $x_2 = \log(a - e^{x_1}) - \log(e^{x_1} - 1)$ for $0 < x_1 < (\log a)/2$ and $x_2 = \log(e^{x_1} + a) - \log(e^{x_1} + 1)$ for

$(\log a)/2 < x_1 < \infty$. Hence its area is

$$\int_0^{(\log a)/2} \log \frac{a+e^{x_1}}{a-e^{x_1}}\, dx_1 + \int_{(\log a)/2}^{\infty} \log \frac{e^{x_1}+1}{e^{x_1}-1}\, dx_1$$
$$= \int_{(\log a)/2}^{\log a} \log \frac{e^t+1}{e^t-1}\, dt + \int_{(\log a)/2}^{\infty} \log \frac{e^t+1}{e^t-1}\, dt.$$

The two remaining parts are congruent to the ones described above. Hence the size of the point mass at $(\log a, \log a)/2$ is equal to

$$\frac{4}{\pi^2} \int_{(\log a)/2}^{\infty} \log \frac{e^t+1}{e^t-1}\, dt = \frac{8}{\pi^2} \operatorname{Li}_2^-\Big(\frac{1}{\sqrt{a}}\Big).$$

Let us now present an alternative calculation of the same point mass by using instead our Theorem 5. Consider the system of equations

$$a + z_1 + z_2 + z_1 z_2 = a + t_1 z_1 + t_2 z_2 + t_1 t_2 z_1 z_2 = 0, \tag{21}$$

where t_1, t_2 are complex parameters with modulus 1. We want to determine, for all values of the parameters, the number of solutions to this system in $\operatorname{Log}^{-1}((\log a, \log a)/2)$. Eliminating z_2 from the system, we obtain the equation $t_1(t_2-1)z_1^2 + (at_1t_2 - a - t_1 + t_2)z_1 + a(t_2-1) = 0$. This equation has either two roots on the circle $|z_1| = \sqrt{a}$ or one root inside and one root outside this circle. In the former case, it is easy to see that the solutions for z_2 also have modulus $\sqrt{a}$. Now the two roots of the equation $\alpha\zeta^2 + \beta\zeta + \gamma = 0$ have the same modulus precisely if $\bar{\beta}^2(\beta^2 - 4\alpha\gamma)$ is real and negative. Applying this to our equation, we obtain the condition $(a\bar{t_1}\bar{t_2} - a - \bar{t_1} + \bar{t_2})^2((at_1t_2 - a - t_1 + t_2)^2 - 4at_1(t_2-1)^2) \le 0$. Since $(a\bar{t_1}\bar{t_2} - a - \bar{t_1} + \bar{t_2})^2/(\bar{t_1}\bar{t_2}) \le 0$ for all t_1, t_2, this is equivalent to $\bar{t_1}\bar{t_2}((at_1t_2 - a - t_1 + t_2)^2 - 4at_1(t_2-1)^2) \ge 0$. Writing $t_1 = -e^{i(\phi+\psi)}$, $t_2 = -e^{i(\phi-\psi)}$, this may be reformulated as

$$(a\cos\phi + \cos\psi)^2 \ge (a-1)^2. \tag{22}$$

When ϕ, ψ range over the square $|\phi| + |\psi| < \pi$, the parameters t_1, t_2 sweep out the torus minus a null set. Restricting ϕ, ψ to this square, it is clear that inequality (22) will be satisfied precisely if $a\cos\phi + \cos\psi \ge a - 1$ or, equivalently, $|\sin(\phi/2)| \le \cos(\psi/2)/\sqrt{a}$. To each choice of ϕ, ψ in the interior of this region, there correspond two solutions of the system (21) in $\operatorname{Log}^{-1}((\log a, \log a)/2)$. Hence the average number of solutions is equal to

$$\frac{16}{\pi^2} \int_0^{\pi/2} \arcsin \frac{\cos t}{\sqrt{a}}\, dt,$$

and by Theorem 5, the size of the point mass is half of this quantity.

The fact that these two calculations do indeed yield the same result now amounts precisely to the identity

$$\mathrm{Li}_2^-(x) = \int_0^{\pi/2} \arcsin(x\ \cos t)\, dt, \quad |x| \leq 1.$$

To verify this identity directly, one can first observe that both sides vanish for $x = 0$, and then one can perform explicit differentiations showing that their derivatives both equal $x^{-1} \log \sqrt{(1+x)/(1-x)}$.

In the particular case of $a = 1$, there is a factorization $f(z) = (z_1 + 1)(z_2 + 1)$, and the amoeba consists of the two lines $\{x_1 = 0\}$ and $\{x_2 = 0\}$. The Monge-Ampère measure μ_f then degenerates into a single point mass at the origin.

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