Newton numbers and residual measures of plurisubharmonic functions

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Abstract. We study the masses charged by $(dd^c u)^n$ at isolated singularity points of plurisubharmonic functions u. It is done by means of the local indicators of plurisubharmonic functions introduced in [15]. As a consequence, bounds for the masses are obtained in terms of the directional Lelong numbers of u, and the notion of the Newton number for a holomorphic mapping is extended to arbitrary plurisubharmonic functions. We also describe the local indicator of u as the logarithmic tangent to u.

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1 Introduction

The principal information on local behaviour of a subharmonic function u in the complex plane can be obtained by studying its Riesz measure μ_u. If u has a logarithmic singularity at a point x, the main term of its asymptotics near x is $\mu_u(\{x\}) \log|z-x|$. For plurisubharmonic functions u in $\mathbf{C}^n$, $n > 1$, the situation is not so simple. The local propoperties of u are controlled by the current $dd^c u$ (we use the notation $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/2\pi i$) which cannot charge isolated points. The trace measure $\sigma_u = dd^c u \wedge \beta_{n-1}$ of this current is precisely the Riesz measure of u; here $\beta_p = (p!)^{-1} 2^p (dd^c|z|^2)^p$ is the volume element of $\mathbf{C}^p$. A significant role is played by the Lelong numbers $\nu(u,x)$ of the function u at points x:

$$\nu(u,x) = \lim_{r\to 0} (\tau_{2n-2} r^{2n-2})^{-1} \sigma_u[B^{2n}(x,r)],$$

where τ_{2p} is the volume of the unit ball $B^{2p}(0,1)$ of $\mathbf{C}^p$. If $\nu(u,x) > 0$ then $\nu(u,x)\log|z-x|$ gives an upper bound for $u(z)$ near x, however the difference between these two functions can be comparable to $\log|z-x|$.

Another important object generated by the current $dd^c u$ is the Monge-Ampére measure $(dd^c u)^n$. For the definition and basic facts on the complex

Monge-Ampére operator $(dd^c)^n$ and Lelong numbers, we refer the reader to the books [12], [14] and [8], and for more advanced results, to [2]. Here we mention that $(dd^c u)^n$ cannot be defined for all plurisubharmonic functions u, however if $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega \setminus K)$ with $K \subset\subset \Omega$, then $(dd^c u)^n$ is well defined as a positive closed current of the bidimension $(0,0)$ (or, which is the same, as a positive measure) on Ω. This measure cannot charge pluripolar subsets of $\Omega \setminus K$, and it can have positive masses at points of K, e.g. $(dd^c \log|z|)^n = \delta(0)$, the Dirac measure at 0, $|z| = (\sum |z_j|^2)^{1/2}$. More generally, if $f : \Omega \to \mathbf{C}^N$, $N \geq n$, is a holomorphic mapping with isolated zeros at $x^{(k)} \in \Omega$ of multiplicities m_k, then $(dd^c \log|f|)^n|_{x^{(k)}} = m_k\, \delta(x^{(k)})$. So, the masses of $(dd^c u)^n$ at isolated points of singularity of u (the residual measures of u) are of especial importance.

Let a plurisubharmonic function u belong to $L^\infty_{loc}(\Omega \setminus \{x\})$; its residual mass at the point x will be denoted by $\tau(u,x)$:

$$\tau(u,x) = (dd^c u)^n|_{\{x\}}.$$

The problem under consideration is evaluation of this value.

The following well-known relation compares $\tau(u,x)$ with the Lelong number $\nu(u,x)$:

$$\tau(u,x) \geq [\nu(u,x)]^n. \tag{1}$$

The equality in (1) means that, roughly speaking, the function $u(z)$ behaves near x as $\nu(u,x)\log|z-x|$. In many cases however relation (1) is not optimal; e.g. for

$$u(z) = \sup\{\log|z_1|^{k_1}, \log|z_2|^{k_2}\}, \quad k_1 > k_2, \tag{2}$$

$\tau(u,0) = k_1 k_2 > k_2^2 = [\nu(u,0)]^2$.

As follows from the Comparison Theorem due to Demailly (see Theorem A below), the residual mass is determined by asymptotic behaviour of the function near its singularity, so one needs to find appropriate characteristics for the behaviour. To this end, a notion of local indicator was proposed in [15]. Note that $\nu(u,x)$ can be calculated as

$$\nu(u,x) = \lim_{r\to-\infty} r^{-1} \sup\{v(z) : \ |z-x| \leq e^r\} = \lim_{r\to-\infty} r^{-1} \mathcal{M}(u,x,r),$$

where $\mathcal{M}(u,x,r)$ is the mean value of u over the sphere $|z-x| = e^r$, see [4]. In [5], the *refined*, or *directional, Lelong numbers* were introduced as

$$\begin{aligned} \nu(u,x,a) &= \lim_{r\to-\infty} r^{-1} \sup\{v(z) : \ |z_k - x_k| \leq e^{ra_k},\ 1 \leq k \leq n\} \\ &= \lim_{r\to-\infty} r^{-1} g(u,x,ra), \end{aligned} \tag{3}$$

where $a = (a_1, \ldots, a_n) \in \mathbf{R}^n_+$ and $g(u,x,b)$ is the mean value of u over the set $\{z : |z_k - x_k| = \exp b_k,\ 1 \le k \le n\}$. For x fixed, the collection $\{\nu(u,x,a)\}_{a\in\mathbf{R}^n_+}$ gives a more detailed information about the function u near x than $\nu(u,x)$ does, so one can expect for a more precise bound for $\tau(u,x)$ in terms of the directional Lelong numbers. It was noticed already in [5] that $a \mapsto \nu(u,x,a)$ is a concave function on $\mathbf{R}^n_+$. In [15], it was observed that this function produces the following plurisubharmonic function $\Psi_{u,x}$ in the unit polydisk $D = \{y \in \mathbf{C}^n : |y_k| < 1,\ 1 \le k \le n\}$:

$$\Psi_{u,x}(y) = -\nu(u,x,(-\log|y_k|)),$$

the *local indicator* of the function u at x. It is the largest negative plurisubharmonic function in D whose directional Lelong numbers at 0 coincide with those of u at x, $(dd^c\Psi_{u,x})^n = \tau(\Psi_{u,x},0)\,\delta(0)$, and finally,

$$\tau(u,x) \ge \tau(\Psi_{u,x},0), \tag{4}$$

so the singularity of u at x is controlled by its indicator $\Psi_{u,x}$.

Since $\tau(\Psi_{u,x},0) \ge [\nu(\Psi_{u,x},0)]^n = [\nu(u,x)]^n$, (4) is a refinement of (1). For the function u defined by (2), $\tau(\Psi_{u,0},0) = k_1k_2 = \tau(u,0) > [\nu(u,0)]^2$.

Being a function of a quite simple nature, the indicator can produce effective bounds for residual measures of plurisubharmonic functions. In Theorems 1–3 of the present paper we study the values $N(u,x) := \tau(\Psi_{u,x},0)$, *the Newton numbers of u at x*; the reason for this name is explained below. We obtain, in particular, the following bound for $\tau(u,x)$ (Theorem 4):

$$\tau(u,x) \ge \frac{[\nu(u,x,a)]^n}{a_1\ldots a_n} \quad \forall a \in \mathbf{R}^n_+;$$

it reduces to (1) when $a_1 = \ldots = a_n = 1$. For n plurisubharmonic functions $u_1, \ldots, u_n$ in general position (see the definition below), we estimate the measure $dd^c\Psi_{u_1,x}\wedge\ldots\wedge dd^c\Psi_{u_n,x}$ and prove the similar relation (Theorem 6)

$$dd^cu_1 \wedge \ldots \wedge dd^cu_n|_{\{x\}} \ge \frac{\prod_j \nu(u_j,x,a)}{a_1\ldots a_n} \quad \forall a \in \mathbf{R}^n_+. \tag{5}$$

The main tool used to obtain these bounds is the Comparison Theorem due to Demailly. To formulate it we give the following

Definition 1. A q-tuple of plurisubharmonic functions $u_1, \ldots, u_q$ is said to be *in general position* if their unboundedness loci $A_1, \ldots, A_q$ satisfy the following condition: for all choices of indices $j_1 < \ldots < j_k$, $k \le q$, the $(2q-2k+1)$-dimensional Hausdorff measure of $A_{j_1} \cap \ldots \cap A_{j_k}$ equals zero.

Theorem A (Comparison Theorem, [2], Th. 5.9). *Let n-tuples of plurisubharmonic functions $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be in general position* *on a neighbourhood of a point $x \in \mathbf{C}^n$. Suppose that $u_j(x) = -\infty,\ 1 \le j \le n$, and*

$$\limsup_{z\to x} \frac{v_j(z)}{u_j(z)} = l_j < \infty.$$

Then

$$dd^c v_1 \wedge \ldots \wedge dd^c v_n|_{\{x\}} \le l_1 \ldots l_n\, dd^c u_1 \wedge \ldots \wedge dd^c u_n|_{\{x\}}.$$

We also obtain a geometric interpretation for the value $N(u,x)$ (Theorem 7). Let $\Theta_{u,x}$ be the set of points $b \in \overline{\mathbf{R}^n_+}$ such that $\nu(u,x,a) \ge \langle b,a\rangle$ for some $a \in \mathbf{R}^n_+$, then

$$\tau(u,x) \ge N(u,x) = n!\, Vol(\Theta_{u,x}). \tag{6}$$

In many cases the folume of $\Theta_{u,x}$ can be easily calculated, so (6) gives an effective formula for $N(u,x)$.

To illustrate these results, consider functions $u = \log|f|$, $f = (f_1,\ldots,f_n)$ being an equidimensional holomorphic mapping with an isolated zero at a point x. It is probably the only class of functions whose residual measures were studied in details before. In this case, $\tau(u,x)$ equals m, the multiplicity of f at x, and

$$\nu(\log|f|,x,a) = I(f,x,a) := \inf\{\langle a,p\rangle :\ p \in \omega_x\} \tag{7}$$

where

$$\omega_x = \{p \in \mathbf{Z}^n_+ :\ \sum_j \left|\frac{\partial^p f_j}{\partial z^p}(x)\right| \ne 0\}$$

(see [13]). For polynomials $F : \mathbf{C}^n \to \mathbf{C}$, the value $I(F,x,a)$ is a known object (*the index of F at x with respect to the weight a*) used in number theory (see e.g. [11]).

Relation (4) gives us $m = \tau(\log|f|,x) \ge N(\log|f|,x)$. In general, the value $N(\log|f|,x)$ is not comparable to $m_1 \ldots m_n$ with m_j the multiplicity of the function f_j: for $f(z) = (z_1^2 + z_2, z_2)$ and $x = 0$, $m_1m_2 = 1 < 2 = N(\log|f|,x) = m$ while for $f(z) = (z_1^2 + z_2, z_2^3)$, $N(\log|f|,x) = 2 < 3 = m_1m_2 < 6 = m$. A more sharp bound for m can be obtained by (5) with $u_j = \log|f_j|,\ 1 \le j \le n$. In this case, the left-hand side of (5) equals m, and its right-hand side with $a_1 = \ldots = a_n$ equals $m_1 \ldots m_n$. For the both above examples of the mapping f, the supremum of the right-hand side of (5) over $a \in \mathbf{R}^n_+$ equals m. For $a_1,\ldots,a_n$ rational, relation (5) is a known bound for m via the multiplicities of weighted homogeneous initial Taylor polynomials of f_j with respect to the weights $(a_1,\ldots,a_n)$ ([1], Th. 22.7).

Recall that the convex hull $\Gamma_+(f,x)$ of the set $\bigcup_p\{p + \mathbf{R}^n_+\}$, $p \in \omega_x$ is called *the Newton polyhedron* of $(f_1,\ldots,f_n)$ at x, the union $\Gamma(f,x)$ of the compact faces of the boundary of $\Gamma_+(f,x)$ is called *the Newton boundary* of $(f_1,\dots,f_n)$ at x, and the value $N_{f,x} = n!\,Vol(\Gamma_-(f,x))$ with $\Gamma_-(f,x) = \{\lambda t : t\in\Gamma(f,x),\ 0\le\lambda\le 1\}$ is called *the Newton number* of $(f_1,\dots,f_n)$ at x (see [10], [1]). The relation

$$(8) \qquad m \ge N_{f,x}$$

was established by A.G. Kouchnirenko [9] (see also [1], Th. 22.8). Since $\Theta_{\log|f|,x} = \Gamma_-(f,x)$, (8) is a particular case of the relation (6). It is the reason to call $N(u,x)$ the Newton number of u at x.

These observations show that the technique of plurisubharmonic functions (and local indicators in particular) is quite a powerful tool to produce, in a unified and simple way, sharp bounds for the multiplicities of holomorphic mappings.

Finally, we obtain a description for the indicator $\Psi_{u,x}(z)$ as the weak limit of the functions $m^{-1}u(x_1+z_1^m,\dots,x_n+z_n^m)$ as $m\to\infty$ (Theorem 8), so $\Psi_{u,x}$ can be viewed as the tangent (in the logarithmic coordinates) for the function u at x. Using this approach we obtain a sufficient condition, in terms of $\mathcal{C}_{n-1}$-capacity, for the residual mass $\tau(u,x)$ to coincide with the Newton number of u at x (Theorem 9).

2 Indicators and their masses

We will use the following notations. For a domain Ω of $\mathbf{C}^n$, $PSH(\Omega)$ will denote the class of all plurisubharmonic functions on Ω, $PSH_-(\Omega)$ the subclass of the nonpositive functions, and $PSH(\Omega,x) = PSH(\Omega)\cap L^\infty_{loc}(\Omega\setminus\{x\})$ with $x\in\Omega$.

Let $D = \{z\in\mathbf{C}^n :\ |z_k|<1,\ 1\le k\le n\}$ be the unit polydisk, $D^* = \{z\in D:\ z_1\cdot\ldots\cdot z_n\ne 0\}$, $\mathbf{R}^n_\pm = \{t\in\mathbf{R}^n :\ \pm t_k>0\}$. By $CNVI_-(\mathbf{R}^n_-)$ we denote the collection of all nonpositive convex functions on $\mathbf{R}^n_-$ increasing in each variable t_k. The mapping $Log : D^*\to\mathbf{R}^n_-$ is defined as $Log(z) = (\log|z_1|,\dots,\log|z_n|)$, and $Exp:\mathbf{R}^n_-\to D^*$ is given by $Exp(t) = (\exp t_1,\dots,\exp t_n)$.

A function u on D^* is called *n-circled* if

$$(9) \qquad u(z) = u(|z_1|,\dots,|z_n|),$$

i.e. if $Log^*Exp^*u = u$. Any n-circled function $u\in PSH_-(D^*)$ has a unique extension to the whole polydisk D keeping the property (9). The class of such functions will be denoted by $PSH^c_-(D)$. The cones $CNVI_-(\mathbf{R}^n_-)$ and $PSH^c_-(D)$ are isomorphic: $u\in PSH^c_-(D) \iff Exp^*u\in CNVI_-(\mathbf{R}^n_-)$, $h\in CNVI_-(\mathbf{R}^n_-) \iff Log^*h\in PSH^c_-(D)$.

Definition 2 [15]. A function $\Psi \in PSH_-^c(D)$ is called an *indicator* if its convex image $Exp^*\Psi$ satisfies

$$Exp^*\Psi(ct) = c\,Exp^*\Psi(t) \quad \forall c > 0,\ \forall t \in \mathbf{R}_-^n. \tag{10}$$

The collection of all indicators will be denoted by I. It is a convex subcone of $PSH_-^c(D)$, closed in $\mathcal{D}'$ (or equivalently, in $L_{loc}^1(D)$). Besides, if $\Psi_1, \Psi_2 \in I$ then $\sup\{\Psi_1, \Psi_2\} \in I$, too.

Every indicator is locally bounded in D^*. In what follows we will often consider indicators locally bounded in $D \setminus \{0\}$; the class of such indicators will be denoted by I_0: $I_0 = I \cap PSH(D, 0)$.

An example of indicators can be given by the functions

$$\varphi_a(z) = \sup_k a_k \log|z_k|,\ a_k \geq 0.$$

If all $a_k > 0$, then $\varphi_a \in I_0$.

Proposition 1 *Let $\Psi \in I_0$, $\Psi \not\equiv 0$. Then*

(a) there exist $\nu_1, \ldots, \nu_n > 0$ such that

$$\Psi(z) \geq \varphi_\nu(z) \quad \forall z \in D; \tag{11}$$

(b) $\Psi \in C(\overline{D} \setminus \{0\})$, $\Psi|_{\partial D} = 0$;

(c) the directional Lelong numbers $\nu(\Psi, 0, a)$ of Ψ at the origin with respect to $a \in \mathbf{R}_+^n$ (3) are

$$\nu(\Psi, 0, a) = -\Psi(Exp(-a)), \tag{12}$$

and its Lelong number $\nu(\Psi, 0) = -\Psi(e^{-1}, \ldots, e^{-1})$;

(d) $(dd^c\Psi)^n = 0$ on $D \setminus \{0\}$.

Proof. Let $\Psi_k(z_k)$ denote the restriction of the indicator $\Psi(z)$ to the disk $D^{(k)} = \{z \in D : z_j = 0\ \forall j \neq k\}$. By monotonicity of $Exp^*\Psi$, $\Psi(z) \geq \Psi_k(z_k)$. Since Ψ_k is a nonzero indicator in the disk $D^{(k)} \subset \mathbf{C}$, $\Psi_k(z_k) = \nu_k \log|z_k|$ with some $\nu_k > 0$, and (a) follows.

As $Exp^*\Psi \in C(\mathbf{R}_-^n)$, $\Psi \in C(D^*)$. Its continuity in $D \setminus \{0\}$ can be shown by induction in n. For $n = 1$ it is obvious, so assuming it for $n \leq l$, consider any point $z^0 \neq 0$ with $z_j^0 = 0$. Let $z^s \to z^0$, then the points $\tilde{z}^s$ with $\tilde{z}_j^s = 0$ and $\tilde{z}_m^s = z_m^s$, $m \neq j$, also tend to z^0, and by the induction hypothesis, $\Psi(\tilde{z}^s) \to \Psi(\tilde{z}^0) = \Psi(z^0)$. So, $\liminf_{s\to\infty} \Psi(z^s) \geq \lim_{s\to\infty} \Psi(\tilde{z}^s) = \Psi(z^0)$, i.e.

Ψ is lower semicontinuous and hence continuous at z^0. Continuity of Ψ up to ∂D and the boundary condition follow from (11).

Equality (12) is an immediate consequence of the definition of the directional Lelong numbers (3) and the homogeneity condition (10). The relation $\nu(u,x) = \nu(u,x,(1,\ldots,1))$ [5] gives us the desired expression for $\nu(\Psi,0)$.

Finally, statement (d) follows from the homogeneity condition (10), see [15], Proposition 4.

For functions $\Psi \in I_0$, the complex Monge-Ampére operator $(dd^c\Psi)^n$ is well defined and gives a nonnegative measure on D. By Proposition 1,

$$(dd^c\Psi)^n = \tau(\Psi)\delta(0)$$

with some constant $\tau(\Psi) \geq 0$ which is strictly positive unless $\Psi \equiv 0$. In this section, we will study the value $\tau(\Psi)$.

An upper bound for $\tau(\Psi)$ is given by

Proposition 2 *For $\Psi \in I_0$,*

$$\tau(\Psi) \leq \nu_1 \ldots \nu_n \tag{13}$$

with $\nu_1, \ldots, \nu_n$ the same as in Proposition 1, (a).

Proof. The function $\varphi_\nu(z) \in I_0$, and (11) implies

$$\limsup_{z\to 0} \frac{\Psi(z)}{\varphi_\nu(z)} \leq 1,$$

so (13) follows by Theorem A.

To obtain a lower bound for $\tau(\Psi)$, we need a relation between $\Psi(z)$ and $\Psi(z^0)$ for $z, z^0 \in D$. Denote

$$\Phi(z,z^0) = \sup_k \frac{\log|z_k|}{|\log|z_k^0||}, \quad z \in D,\ z^0 \in D^*.$$

Being considered as a function of z with z^0 fixed, $\Phi(z,z^0) \in I_0$.

Proposition 3 *For any $\Psi \in I$, $\Psi(z) \leq |\Psi(z^0)|\Phi(z,z^0) \quad \forall z \in D,\ z^0 \in D^*$.*

Proof. For a fixed $z^0 \in D^*$ and $t^0 = Log(z^0)$, define $u = |\Psi(z^0)|^{-1} Exp^*\Psi$ and $v = Exp^*\Phi = \sup_k t_k/|t_k^0|$. It suffices to establish the inequality $u(t) \leq v(t)$ for all $t \in \mathbf{R}^n_-$ with $t_k^0 < t_k < 0$, $1 \leq k \leq n$. Given such a t, denote $\lambda_0 = [1 + v(t)]^{-1}$. Since $\{t^0 + \lambda(t - t^0) : 0 \leq \lambda \leq \lambda_0\} \subset \overline{\mathbf{R}^n_-}$, the functions

$u_t(\lambda) := u(t^0 + \lambda(t - t^0))$ and $v_t(\lambda) := v(t^0 + \lambda(t - t^0))$ are well defined on $[0, \lambda_0]$. Furthermore, u_t is convex and v_t is linear there, $u_t(0) = v_t(0) = -1$, $u_t(\lambda_0) \le v_t(\lambda_0) = 0$. It implies $u_t(\lambda) \le v_t(\lambda)$ for all $\lambda \in [0, \lambda_0]$. In particular, as $\lambda_0 > 1$, $u(t) = u_t(1) \le v_t(1) = v(t)$, that completes the proof.

Consider now the function

$$P(z) = -\prod_{1\le k\le n} |\log|z_k||^{1/n} \in I.$$

Theorem 1 *For any* $\Psi \in I_0$,

$$\tau(\Psi) \ge \left|\frac{\Psi(z^0)}{P(z^0)}\right|^n \quad \forall z^0 \in D^*. \tag{14}$$

Proof. By Proposition 3,

$$\frac{\Psi(z)}{\Phi(z,z^0)} \le |\Psi(z^0)| \quad \forall z \in D,\ z^0 \in D^*.$$

By Theorem A,

$$(dd^c\Psi)^n \le |\Psi(z^0)|^n (dd^c\Phi(z,z^0))^n,$$

and the statement follows from the fact that

$$(dd^c\Phi(z,z^0))^n = \prod_{1\le k\le n} |\log|z_k^0||^{-1} = |P(z^0)|^{-n}.$$

Remarks. 1. One can consider the value

$$A_\Psi = \sup_{z\in D}\left|\frac{\Psi(z)}{P(z)}\right|^n; \tag{15}$$

by Theorem 1,

$$\tau(\Psi) \ge A_\Psi. \tag{16}$$

2. Let $I_{0,M} = \{\Psi \in I_0 :\ \tau(\Psi) \le M\},\ M > 0$. Then (14) gives the lower bound for the class $I_{0,M}$:

$$\Psi(z) \ge M^{1/n} P(z) \quad \forall z \in D,\ \forall \Psi \in I_{0,M}.$$

Let now $\Psi_1, \dots, \Psi_n \in I$ be in general position in the sense of Definition 1. Then the current $\bigwedge_k dd^c\Psi_k$ is well defined, as well as $(dd^c\Psi)^n$ with $\Psi = \sup_k \Psi_k$. Moreover, we have

Proposition 4 *If $\Psi_1,\dots,\Psi_n\in I$ are in general position, then*

$$\bigwedge_k dd^c\Psi_k = 0 \text{ on } D\setminus\{0\}. \tag{17}$$

Proof. For $\Psi_1,\dots,\Psi_n\in I_0$, the statement follows from Proposition 1, (d), and the polarization formula

$$\bigwedge_k dd^c\Psi_k = \frac{(-1)^n}{n!}\sum_{j=1}^n(-1)^j\sum_{1\le i_1<\ldots<i_j\le n}\left(dd^c\sum_{k=1}^j\Psi_{j_k}\right)^n. \tag{18}$$

When the only condition on $\{\Psi_k\}$ is to be in general position, we can replace $\Psi_k(z)$ with $\Psi_{k,N}(z)=\sup\{\Psi_k(z),N\sup_j\log|z_j|\}\in I_0$ for which $\bigwedge_k dd^c\Psi_{k,N}=0$ on $D\setminus\{0\}$. Since $\Psi_{k,N}\searrow\Psi_k$ as $N\to\infty$, it gives us (17).

The mass of $\bigwedge_k dd^c\Psi_k$ will be denoted by $\tau(\Psi_1,\dots,\Psi_n)$.

Theorem 2 *Let $\Psi_1,\dots,\Psi_n\in I$ be in general position, $\Psi=\sup_k\Psi_k$. Then*

(a) $\tau(\Psi)\le\tau(\Psi_1,\dots,\Psi_n)$;

(b) $\tau(\Psi_1,\dots,\Psi_n)\ge|P(z^0)|^{-n}\prod_k|\Psi_k(z^0)|\quad\forall z^0\in D^*$.

Proof. Since

$$\frac{\Psi(z)}{\Psi_k(z)}\le 1\ \forall z\ne 0,$$

statement (a) follows from Theorem A.

Statement (b) results from Proposition 3 exactly like the statement of Theorem 1.

3 Geometric interpretation

In this section we study the masses $\tau(\Psi)$ of indicators $\Psi\in I_0$ by means of their convex images $Exp^*\Psi\in CNVI_-(\mathbf{R}^n_-)$.

Let $V\in PSH^c_-(rD)\cap C^2(rD)$, $r<1$, and $v=Exp^*V\in CNVI_-((\mathbf{R}_-+\log r)^n)$. Since

$$\frac{\partial^2V(z)}{\partial z_j\partial\bar z_k}=\frac{1}{4z_j\bar z_k}\frac{\partial^2v(t)}{\partial t_j\partial t_k}\bigg|_{t=Log(z)},\quad z\in rD^*,$$

$$\det\left(\frac{\partial^2V(z)}{\partial z_j\partial\bar z_k}\right)=4^{-n}|z_1\ldots z_n|^{-2}\det\left(\frac{\partial^2v(t)}{\partial t_j\partial t_k}\right)\bigg|_{t=Log(z)}.$$

By setting $z_j = \exp\{t_j + i\theta_j\}$, $0 \le \theta \le 2\pi$, we get $\beta_n(z) = |z_1 \dots z_n|^2 dt\, d\theta$, so

$$(19)\quad (dd^cV)^n = n!\left(\frac{2}{\pi}\right)^n \det\left(\frac{\partial^2 V}{\partial z_j \partial \bar{z}_k}\right)\beta_n = n!\,(2\pi)^{-n}\det\left(\frac{\partial^2 v}{\partial t_j \partial t_k}\right) dt\, d\theta.$$

Every function $U \in PSH^c_-(D) \cap L^\infty(D)$ is the limit of a decreasing sequence of functions $U_l \in PSH^c_-(E) \cap C^2(E)$ on an n-circled domain $E \subset\subset D$, and by the convergence theorem for the complex Monge-Ampére operators,

$$(20)\qquad (dd^cU_l)^n|_E \longrightarrow (dd^cU)^n|_E.$$

On the other hand, for $u_l = Exp^*U_l$ and $u = Exp^*U$,

$$(21)\qquad \det\left(\frac{\partial^2 u_l}{\partial t_j \partial t_k}\right) dt\Bigg|_{Log(D^*\cap E)} \longrightarrow \mathcal{MA}[u]|_{Log(D^*\cap E)}\,,$$

the *real* Monge-Ampére operator of u [16].

Since $(dd^cU_l)^n$ and $(dd^cU)^n$ cannot charge pluripolar sets, (19) with $V = U_l$ and (20), (21) imply

$$(dd^cU)^n(E) = n!\,(2\pi)^{-n}\mathcal{MA}[u]\,d\theta\,(Log(E)\times[0,2\pi]^n)$$

for any n-circled Borel set $E \in D$, i.e.

$$(22)\qquad (dd^cU)^n(E) = n!\,\mathcal{MA}[u](Log(E)).$$

This relation allows us to calculate $\tau(\Psi)$ by using the technique of real Monge-Ampére operators in $\mathbf{R}^n$ (see [16]).

Let $\Psi \in I$. Consider the set

$$B_\Psi = \{a \in \mathbf{R}^n_+ :\ \langle a, t\rangle \le Exp^*\Psi(t) \quad \forall t \in \mathbf{R}^n_-\}$$

and define

$$\Theta_\Psi = \overline{\mathbf{R}^n_+ \setminus B_\Psi}.$$

Clearly, the set B_Ψ is convex, so $Exp^*\Psi$ is the restriction of its support function to $\mathbf{R}^n_-$. If $\Psi \in I_0$, the set Θ_Ψ is bounded. Indeed, $a \in \Theta_\Psi$ if and only if $\langle a, t^0\rangle \ge Exp^*\Psi(t^0)$ for some $t^0 \in \mathbf{R}^n_-$, that implies $|a_j| \le |Exp^*\Psi(t^0)/t^0_j|\ \forall j$. By Proposition 1, (a), $|Exp^*\Psi(t^0)| \le \nu_j|t_j|$ and therefore $|a_j| \le \nu_j\ \forall j$.

Given a set $F \in \mathbf{R}^n$, we denote its Eucledean volume by $Vol(F)$.

Theorem 3 *$\forall\Psi \in I_0$,*

$$(23)\qquad \tau(\Psi) = n!\,Vol(\Theta_\Psi).$$

Proof. Denote $U(z) = \sup\{\Psi(z), -1\} \in PSH^c_-(D) \cap C(D)$, $u = Exp^* U \in CNVI_-(\mathbf{R}^n_-)$. Since $U(z) = \Psi(z)$ near ∂D,

$$\tau(\Psi) = \int_D (dd^c U)^n.$$

Furthermore, as $(dd^c U)^n = 0$ outside the set $E = \{z \in D:\ \Psi(z) = -1\}$,

$$\tau(\Psi) = \int_E (dd^c U)^n. \tag{24}$$

In view of (22),

$$\int_E (dd^c U)^n = n! \int_{Log(E)} \mathcal{MA}[u]. \tag{25}$$

As was shown in [16], for any convex function v in a domain $\Omega \subset \mathbf{R}^n$,

$$\int_F \mathcal{MA}[v] = Vol(\omega(F, v)) \quad \forall F \subset \Omega, \tag{26}$$

where

$$\omega(F, v) = \bigcup_{t^0 \in F} \{a \in \mathbf{R}^n :\ v(t) \geq v(t^0) + \langle a, t - t^0\rangle \quad \forall t \in \Omega\}$$

is the gradient image of the set F for the surface $\{y = v(x),\ x \in \Omega\}$.

We claim that

$$\Theta_\Psi = \omega(Log(E), u). \tag{27}$$

Observe that

$$\Theta_\Psi = \{a \in \overline{\mathbf{R}^n_+} :\ \sup_{\psi(t) = -1} \langle a, t\rangle \geq -1\}$$

where $\psi = Exp^* \Psi$.

If $a \in \omega(Log(E), u)$, then for some $t^0 \in \mathbf{R}^n_-$ with $\psi(t^0) = 1$ we have $\langle a, t^0\rangle \geq \langle a, t\rangle$ for all $t \in \mathbf{R}^n_-$ such that $\psi(t) < -1$. Taking here $t_j \to -\infty$ we get $a_j \geq 0$, i.e. $a \in \overline{\mathbf{R}^n_+}$. Besides, $\langle a, t^0\rangle \geq \langle a, t\rangle - 1 - \psi(t)$ for all $t \in \mathbf{R}^n_-$ with $\psi(t) > -1$, and applying this for $t \to 0$ we derive $\langle a, t^0\rangle \geq -1$. Therefore, $a \in \Theta_\Psi$ and $\Theta_\Psi \supset \omega(Log(E), u)$.

Now we prove the converse inclusion. If $a \in \Theta_\Psi \cap \mathbf{R}^n_+$, then

$$\sup\{\langle a, t^0\rangle :\ t^0 \in Log(E)\} \geq -1.$$

Let t be such that $\psi(t) = -\delta > -1$, then $t/\delta \in Log(E)$ and thus

$$\begin{aligned} \langle a, t\rangle - 1 - \psi(t) &= \delta\langle a, t/\delta\rangle - 1 + \delta \leq \delta \sup_{t^0 \in Log(E)} \langle a, t^0\rangle - 1 + \delta \\ &\leq \sup_{t^0 \in Log(E)} \langle a, t^0\rangle = \sup_{z^0 \in E} \langle a, Log(z^0)\rangle. \end{aligned}$$

Since E is compact, the latter supremum is attained at some point $\hat{z}^0$. Furthermore, $\hat{z}^0 \in E \cap D^*$ because $a_k \neq 0$, $1 \leq k \leq n$. Hence $\sup_{t^0 \in Log(E)} \langle a, t^0 \rangle = \langle a, \hat{t}^0 \rangle$ with $\hat{t}^0 = Log(z^0) \in \mathbf{R}^n_-$, so that $a \in \omega(Log(E), u)$ and $\Theta_\Psi \cap \mathbf{R}^n_+ \subset \omega(Log(E), u)$. Since $\omega(Log(E), u)$ is closed, this implies $\Theta_\Psi = \omega(Log(E), u)$, and (27) follows.

Now relation (23) is a consequence of (24)–(27). The theorem is proved.

Note that the value $\tau(\Psi_1, \ldots, \Psi_n)$ also can be expressed in geometric terms. Namely, if $\Psi_1, \ldots, \Psi_n \in I_0$, the polarization formula (18) gives us, by Theorem 3,

$$\tau(\Psi_1, \ldots, \Psi_n) = (-1)^n \sum_{j=1}^{n} (-1)^j \sum_{1 \leq i_1 < \ldots < i_j \leq n} Vol(\Theta_{\sum_k \Psi_{j_k}}).$$

We can also give an interpretation for the bound (16). Write A_Ψ from (15) as

$$A_\Psi = \sup_{a \in \mathbf{R}^n_+} \frac{|\psi(-a)|^n}{a_1 \ldots a_n} = \sup_{a \in \mathbf{R}^n_+} |\psi(-a/a_1) \ldots \psi(-a/a_n)|\,, \tag{28}$$

$\psi = Exp^* \Psi$. For any $a \in \mathbf{R}^n_+$, the point $a^{(j)}$ whose jth coordinate equals $|\psi(-a/a_j)|$ and the others are zero, has the property $\langle a^{(j)}, -a \rangle = \psi(-a)$. This remains true for every convex combination $\sum \rho_j a^{(j)}$ of the points $a^{(j)}$, and thus $r \sum \rho_j a^{(j)} \in \Theta_\Psi$ with any $r \in [0, 1]$. Since

$(n!)^{-1} |\psi(-a/a_1) \ldots \psi(-a/a_n)|$ is the volume of the simplex generated by the points $0, a^{(1)}, \ldots, a^{(n)}$, we see from (28) that $(n!)^{-1} A_\Psi$ is the supremum of the volumes of all simplices contained in Θ_Ψ.

Besides, $(n!)^{-1}[\nu(\Psi, 0)]^n$ is the volume of the simplex

$$\{a \in \overline{\mathbf{R}^n_+} : \langle a, (1, \ldots, 1) \rangle \leq \nu(\Psi, 0)\} \subset \Theta_\Psi.$$

It is a geometric description for the "standard" bound $\tau(\Psi) \geq [\nu(\Psi, 0)]^n$.

4 Singularities of plurisubharmonic functions

Let u be a plurisubharmonic function in a domain $\Omega \subset \mathbf{C}^n$, and $\nu(u, x, a)$ be its directional Lelong number (3) at $x \in \Omega$ with respect to $a \in \mathbf{R}^n_+$. Fix a point x. As is known [5], the function $a \mapsto \nu(u, x, a)$ is a concave function on $\mathbf{R}^n_+$. So, the function

$$\psi_{u,x}(t) := -\nu(u, x, -t), \quad t \in \mathbf{R}^n_-,$$

belongs to $CNVI_-(\mathbf{R}^n_-)$ and thus

$$\Psi_{u,x} := Log^*\psi_{u,x} \in PSH^c_-(D).$$

Moreover, due to the positive homogeneity of $\nu(u,x,a)$ in a, $\Psi_{u,x} \in I$. The function $\Psi_{u,x}$ was introduced in [15] as *(local) indicator* of u at x. According to (3),

$$\begin{aligned}\Psi_{u,x}(z) &= \lim_{R\to+\infty} R^{-1}\sup\{u(y):\ |y_k - x_k| \le |z_k|^R,\ 1\le k\le n\}\\ &= \lim_{R\to+\infty} R^{-1}\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} u(x_k + |z_k|^R e^{i\theta_k})\,d\theta_1\dots d\theta_n.\end{aligned}$$

Clearly, $\Psi_{u,x}\equiv 0$ if and only if $\nu(u,x) = 0$. It is easy to see that $\Psi(\Phi,0) = \Phi\ \forall\Phi\in I$. In particular,

$$\nu(u,x,a) = \nu(\Psi_{u,x},0,a) = -\Psi_{u,x}(Exp(-a)) \quad \forall a\in\mathbf{R}^n_+. \tag{29}$$

So, the results of the previous sections can be applied to study the directional Lelong numbers of arbitrary plurisubharmonic functions.

Proposition 5 (cf. [7], Pr. 5.3) *For any $u\in PSH(\Omega)$,*

$$\nu(u,x,a) \ge \nu(u,x,b)\sup_k \frac{a_k}{b_k} \quad \forall x\in\Omega,\ \forall a,b\in\mathbf{R}^n_+.$$

Proof. In view of (29), the relation follows from Proposition 3.

Given $r\in\mathbf{R}^n_+$ and $z\in\mathbf{C}^n$, we denote $r^{-1} = (r_1^{-1},\dots,r_n^{-1})$ and $r\cdot z = (r_1 z_1,\dots,r_n z_n)$.

Proposition 6 ([15]). *If $u\in PSH(\Omega)$ then*

$$u(z) \le \Psi_{u,x}(r^{-1}\cdot z) + \sup\{u(y):\ y\in D_r(x)\} \tag{30}$$

for all $z\in D_r(x) = \{y:\ |y_k - x_k|\le r_k,\ 1\le k\le n\} \subset\subset \Omega$.

Proof. Let us assume for simplicity $x = 0$, $D_r(0) = D_r$. Consider the function $v(z) = u(r\cdot z) - \sup\{u(y):\ y\in D_r\} \in PSH_-(D)$. The function $g_v(R,t) := \sup\{v(z):\ |z_k|\le \exp\{Rt_k\},\ 1\le k\le n\}$ is convex in $R>0$ and $t\in\mathbf{R}^n_-$, so for $R\to\infty$

$$\frac{g_v(R,t) - g_v(R_1,t)}{R - R_1} \nearrow \psi_{v,0}(t), \tag{31}$$

$\psi_{v,0} = Exp^* \Psi_{v,0}$.

For $R = 1,\ R_1 \to 0$, (31) gives us $g_v(1,t) \le \psi_{v,0}(t)$ and thus (30). The proposition is proved.

Let $\Omega_k(x)$ be the connected component of the set $\Omega \cap \{z \in \mathbf{C}^n :\ z_j = x_j \quad \forall j \ne k\}$ containing the point x. If for some $x \in \Omega$, $u|_{\Omega_k(x)} \not\equiv -\infty \quad \forall k$, then $\Psi_{u,x} \in I_0$. For example, it is the case for $u \in PSH(\Omega, x)$.

If $u \in PSH(\Omega, x)$, the measure $(dd^c u)^n$ is defined on Ω. Its residual mass at x will be denoted by $\tau(u,x)$:

$$\tau(u,x) = (dd^c u)^n|_{\{x\}}.$$

Besides, the idicator $\Psi_{u,x} \in I_0$. Denote $N(u,x) = \tau(\Psi_{u,x})$.

Proposition 7 ([15], Th. 1). *If $u \in PSH(\Omega, x)$, then $\tau(u,x) \ge N(u,x)$.*

Proof. Inequality (30) implies

$$\limsup_{z \to x} \frac{\Psi_{u,x}(r^{-1} \cdot (z - x))}{u(z)} \le 1,$$

and since

$$\lim_{y \to 0} \frac{\Psi_{u,x}(r^{-1} \cdot y))}{\Psi_{u,x}(y)} = 1 \quad \forall r \in \mathbf{R}_+^n,$$

the statement follows from Theorem A.

So, to estimate $\tau(u,x)$ we may apply the bounds for $\tau(\Psi_{u,x})$ from the previous section.

Theorem 4 *If $u \in PSH(\Omega, x)$, then*

$$\tau(u,x) \ge \frac{[\nu(u,x,a)]^n}{a_1 \dots a_n} \quad \forall a \in \mathbf{R}_+^n;$$

in other words, $\tau(u,x) \ge A_{u,x}$ where $A_{u,x} = A_{\Psi_{u,x}}$ is defined by (15).

Proof. The result follows from Theorem 1 and Proposition 7.

Let now $u_1, \dots, u_n \in PSH(\Omega)$ be in general position in the sense of Definition 1. Then the current $\bigwedge_k dd^c u_k$ is defined on Ω ([2], Th. 2.5); denote its residual mass at a point x by $\tau(u_1, \dots, u_n; x)$. Besides, the n-tuple of the indicators $\Psi_{u_k,x}$ is in general position, too, that implies $\bigwedge_k dd^c \Psi_{u_k,x} = \tau(\Psi_{u_1,x}, \dots, \Psi_{u_n,x})\, \delta(0)$ (Proposition 4).

In view of Theorem A and Proposition 6 we have

Theorem 5 $\tau(u_1,\ldots,u_n;x)\ge\tau(\Psi_{u_1,x},\ldots,\Psi_{u_n,x})$.

Now Theorems 2 and 5 give us

Theorem 6

$$\tau(u_1,\ldots,u_n;x)\ge\frac{\prod_j\nu(u_j,x,a)}{a_1\ldots a_n}\quad\forall a\in\mathbf{R}^n_+. \tag{32}$$

Remark. For $a_1=\ldots=a_n=1$, inequality (32) is proved in [2], Cor. 5.10.

By combination of Proposition 7 and Theorem 3 we get

Theorem 7 *For* $u\in PSH(\Omega,x)$,

$$\tau(u,x)\ge N(u,x)=n!\,V(\Theta_{u,x}) \tag{33}$$

with

$$\Theta_{u,x}=\{b\in\mathbf{R}^n_+:\ \sup_{\sum a_k=1}[\nu(u,x,a)-\langle b,a\rangle]\ge 0\}.$$

Remark on holomorphic mappings. Let $f=(f_1,\ldots,f_n)$ be a holomorphic mapping of a neighbourhood Ω of the origin into $\mathbf{C}^n$, $f(0)=0$ be its isolated zero. Then in a subdomain $\Omega'\subset\Omega$ the zero sets A_j of the functions f_j satisfy the conditions

$$A_1\cap\ldots\cap A_n\cap\Omega'=\{0\},\quad \operatorname{codim} A_{j_1}\cap\ldots\cap A_{j_k}\cap\Omega'\ge k$$

for all choices of indices $j_1<\ldots<j_k,\ k\le n$. Denote $u=\log|f|,\ u_j=\log|f_j|$. Then, as is known, $\tau(u,0)=\tau(u_1,\ldots,u_n;0)=m_f$, the multiplicity of f at 0. For $a=(1,\ldots,1)$, $\nu(u_j,0,a)$ equals m_j, the multiplicity of the function f_j at 0. Therefore, (32) with $a=(1,\ldots,1)$ gives us the standard bound $m_f\ge m_1\ldots m_n$.

For a_j rational, (32) is the known estimate of m_f via the multiplicities of weighted homogeneous initial Taylor polynomials for f_j (see e.g. [1], Th. 22.7). Indeed, due to the positive homogeneity of the directional Lelong numbers, we can take $a_j\in\mathbf{Z}^n_+$. Then by (7), $\nu(u_j,0,a)$ is equal to the multiplicity of the function $f_j^{(a)}(z)=f_j(z^a)$.

We would also like to mention that (32) gives a lower bound for the Milnor number $\mu(F,0)$ of a singular point 0 of a holomorphic function F (i.e. for the multiplicity of the isolated zero of the mapping $f=\operatorname{grad}F$

at 0) in terms of the indices $I(F,0,a)$ (7) of F. Since $I(\partial F/\partial z_k,0,a) \ge I(F,0,a) - a_k$,

$$\mu(F,0) \ge \prod_{1\le k\le n} \left(\frac{I(F,0,a)}{a_k} - 1\right).$$

Finally, as follows from (7), the set $\overline{\mathbf{R}_+^n \setminus \Theta_{u,0}}$ is the Newton polyhedron for the system $(f_1,\ldots,f_n)$ at 0 (see Introduction). Therefore, $n!\,V(\Theta_{u,0})$ is the Newton number of $(f_1,\ldots,f_n)$ at 0, and (33) becomes the bound for m_f due to A.G. Kouchnirenko (see [1], Th. 22.8). So, for any plurisubharmonic function u, we will call the value $N(u,x)$ *the Newton number of* u *at* x.

5 Indicators as logarithmic tangents

Let $u \in PSH(\Omega,0)$, $u(0) = -\infty$. We will consider the following problem: under what conditions on u, its residual measure equals its Newton number?

Of course, the relation

$$\exists \lim_{z\to 0} \frac{u(z)}{\Psi_{u,0}(z)} = 1 \tag{34}$$

is sufficient, however it seems to be too restrictive. On the other hand, as the example $u(z) = \log(|z_1+z_2|^2 + |z_2|^4)$ shows, the condition

$$\lim_{\lambda\to 0} \frac{u(\lambda z)}{\Psi_{u,0}(\lambda z)} = 1 \quad \forall z \in \mathbf{C}^n \setminus \{0\}$$

does not guarantee the equality $\tau(u,0) = N(u,0)$.

To weaken (34) we first give another description for the local indicators. In [6], a compact family of plurisubharmonic functions

$$u_r(z) = u(rz) - \sup\{u(y) :\ |y| < r\}_{r>0}$$

was considered and the limit sets, as $r \to 0$, of such families were described. In particular, the limit set need not consist of a single function, so a plurisubharmonic function can have several (and thus infinitely many) tangents. Here we consider another family generated by a plurisubharmonic function u.

Given $m \in \mathbf{N}$ and $z \in \mathbf{C}^n$, denote $z^m = (z_1^m,\ldots,z_n^m)$ and set

$$T_m u(z) = m^{-1} u(z^m).$$

Clearly, $T_m u \in PSH(\Omega \cap D)$ and $T_m u \in PSH_-(\overline{D_r})$ for any $r \in \mathbf{R}_+^n \cap D^*$ (i.e. $0 < r_k < 1$) for all $m \ge m_0(r)$.

Proposition 8 *The family* $\{T_m u\}_{m\ge m_0(r)}$ *is compact in* $L^1_{loc}(D_r)$.

Proof. Let $M(v,\rho)$ denote the mean value of a function v over the set $\{z:\ |z_k|=\rho_k,\ 1\le k\le n\}$, $0<\rho_k\le r_k$, then $M(T_m u,\rho)=m^{-1}M(u,\rho^m)$. The relation

$$m^{-1}M(u,\rho^m)\nearrow \Psi_{u,0}(\rho) \text{ as } m\to\infty \tag{35}$$

implies $M(T_m u,\rho)\ge M(T_{m_0}u,\rho)$. Since $T_m u\le 0$ in D_r, it proves the compactness.

Theorem 8 *(a)* $T_m u\to\Psi_{u,0}$ *in* $L^1_{loc}(D)$*;*

(b) if $u\in PSH(\Omega,0)$ *then* $(dd^cT_m u)^n\to\tau(u,0)\,\delta(0)$.

Proof. Let g be a partial limit of the sequence $T_m u$, that is $T_{m_s}u\to g$ as $s\to\infty$ for some sequence m_s. For the function $v(z)=\sup\{u(y):\ |y_k|\le |z_k|,\ 1\le k\le n\}$ and any $r\in\mathbf{R}^n_+\cap D^*$ we have by (30)

$$T_m u(z)\le (T_m v)\,(z)\le \Psi_{u,0}(r^{-1}\cdot z)$$

and thus

$$g(z)\le\Psi_{u,0}(z)\ \forall z\in D. \tag{36}$$

On the other hand, the convergence of $T_{m_s}u$ to g in L^1 implies $M(T_{m_s}u,r)\to M(g,r)$ ([3], Prop. 4.1.10). By (35), $M(T_{m_s}u,r)\to\Psi_{u,0}(r)$, so $M(g,r)=\Psi_{u,0}(r)$ for every $r\in\mathbf{R}^n_+\cap D^*$. Being compared with (36) it gives us $g\equiv\Psi_{u,0}$, and the statement *(a)* follows.

To prove *(b)* we observe that for each $\alpha\in(0,1)$

$$\int_{\alpha D}(dd^cT_m u)^n=\int_{\alpha^m D}(dd^cu)^n\to\tau(u,0)$$

as $m\to\infty$, and for $0<\alpha<\beta<1$

$$\lim_{m\to\infty}\int_{\beta D\setminus\alpha D}(dd^cT_m u)^n=\lim_{m\to\infty}\left[\int_{\beta^m D}(dd^cu)^n-\int_{\alpha^m D}(dd^cu)^n\right]=0.$$

The theorem is proved.

So, Theorem 8 shows us that $\tau(u,0)=N(u,0)$ if an only if $(dd^cT_m u)^n\to(dd^c\Psi_{u,0})^n$. And now we are going to find conditions for this convergence.

Recall the definition of the inner $\mathcal{C}_{n-1}$-capacity introduced in [17]: for any Borel subset E of a domain ω,

$$\mathcal{C}_{n-1}(E,\omega)=\sup\{\int_E(dd^cv)^{n-1}\wedge\beta_1:\ v\in PSH(\omega),\ 0<v<1\}.$$

It was shown in [17] that convergence of uniformly bounded plurisubharmonic functions v_j to v in $\mathcal{C}_{n-1}$-capacity imples $(dd^c v_j)^n \to (dd^c v)^n$. In our situation, neither $T_m u$ nor $\Psi_{u,0}$ are bounded, so we will modify the construction from [17].

Set

$$E(u,m,\delta) = \{z \in D \setminus \{0\} : \frac{T_m u(z)}{\Psi_{u,0}(z)} > 1+\delta\}, \quad m \in \mathbf{N},\ \delta > 0.$$

Theorem 9 *Let $u \in PSH(\Omega,0), \rho \in (0,1/4), N > 0$, and a sequence $m_s \in \mathbf{N}$ be such that*

1) $u(z) > -Nm_s$ on a neighbourhood of the sphere $\partial B_{\rho^{m_s}}$, $\forall s$;

2) $\lim_{s\to\infty} \mathcal{C}_{n-1}(B_\rho \cap E(u,m_s,\delta), D) = 0 \quad \forall \delta > 0.$

Then $(dd^c T_m u)^n \to (dd^c \Psi_{u,0})^n$ on D.

Proof. Without loss of generality we can take $u \in PSH_-(D,0)$. Consider the functions $v_s(z) = \max\{T_{m_s}u(z), -N\}$ and $v = \max\{\Psi_{u,0}(z), -N\}$. We have $v_s = T_{m_s}u$ and $v = \Psi_{u,0}$ on a neighbourhood of ∂B_ρ, $v_s = v = -N$ on a neighbourhood of 0, $v_s \le v$ on B_ρ, and $v_s \ge (1+\delta)v$ on $B_\rho \setminus E(u,m_s,\delta)$.

We will prove the relations

$$(dd^c v_s)^k \wedge (dd^c v)^l \to (dd^c v)^{k+l} \tag{37}$$

for $k = 1,\ldots,n,\ l = 0,\ldots,n-k$. As a consequence, it will give us the statement of the theorem. Indeed, by Theorem 8,

$$\int_{B_\rho} (dd^c v_s)^n = \int_{B_\rho} (dd^c T_{m_s} u)^n \to \tau(u,0)$$

while

$$\int_{B_\rho} (dd^c v)^n = \int_{B_\rho} (dd^c \Psi_{u,0})^n = N(u,0),$$

and (37) with $k = n$ provides the coincidence of the right-hand sides of these relations and thus the convergence of $(dd^c T_m u)^n$ to $(dd^c \Psi_{u,0})^n$.

We prove (37) by induction in k. Let $k = 1, 0 \le l \le n-1, \delta > 0$. For any test form $\phi \in \mathcal{D}_{n-l-1,n-l-1}(B_\rho)$,

$$\left| \int dd^c v_s \wedge (dd^c v)^l \wedge \phi - \int (dd^c v)^{l+1} \wedge \phi \right| = \left| \int (v - v_s)(dd^c v)^l \wedge dd^c \phi \right|$$

$$
\begin{aligned}
&\le C_\phi \int_{B_\rho} (v-v_s)(dd^c v)^l \wedge \beta_{n-l} \\
&= C_\phi \left[\int_{B_\rho \setminus E_{s,\delta}} + \int_{B_\rho \cap E_{s,\delta}} \right] (v-v_s)(dd^c v)^l \wedge \beta_{n-l} \\
&= C_\phi \quad [I_1(s,\delta) + I_2(s,\delta)],
\end{aligned}
$$

where, for brevity, $E_{s,\delta} = E(u, m_s, \delta)$.

We have

$$
I_1(s,\delta) \le \delta \int_{B_\rho} |v| (dd^c v)^l \wedge \beta_{n-l} \le C\delta
$$

with a constant C independent of s, and

$$
\begin{aligned}
I_2(s,\delta) &\le N \int_{B_\rho \cap E_{s,\delta}} (dd^c v)^l \wedge \beta_{n-l} \\
&\le C(N,\rho,l) \cdot \mathcal{C}_{n-1}(B_\rho \cap E_{s,\delta}, D) \longrightarrow 0.
\end{aligned}
$$

Since $\delta > 0$ is arbitrary, it proves (37) for $k = 1$.

Let us now have got (37) for $k = j$ and $0 \le l \le n-j$. For $\phi \in \mathcal{D}_{n-l-j,n-l}(B_\rho)$,

$$
\int (dd^c v_s)^{j+1} \wedge (dd^c v)^l \wedge \phi = \int (dd^c v_s)^j \wedge (dd^c v)^{l+1} \wedge \phi
$$

$$
+ \int \left[(dd^c v_s)^{j+1} \wedge (dd^c v)^l - (dd^c v_s)^j \wedge (dd^c v)^{l+1} \right] \wedge \phi.
$$

The first integral in the right-hand side converges to $\int (dd^c v)^{l+j+1} \wedge \phi$ by the induction assumption. The second integral can be estimated similarly to the case $k = 1$:

$$
\left| \int \left[(dd^c v_s)^{j+1} \wedge (dd^c v)^l - (dd^c v_s)^j \wedge (dd^c v)^{l+1} \right] \wedge \phi \right|
$$

$$
\begin{aligned}
&\le C_\phi \left[\int_{B_\rho \setminus E_{s,\delta}} + \int_{B_\rho \cap E_{s,\delta}} \right] (v-v_s)(dd^c v_s)^j (dd^c v)^l \wedge \beta_{n-j-l} \\
&= C_\phi \quad [I_3(s,\delta) + I_4(s,\delta)].
\end{aligned}
$$

Since $(dd^c v_s)^j \wedge (dd^c v)^l \to (dd^c v)^{j+l}$,

$$
\int (dd^c v_s)^j (dd^c v)^l \wedge \beta_{n-j-l} \le C \quad \forall s
$$

and

$$
I_3(s,\delta) \le \delta \int_{B_\rho} |v| (dd^c v_s)^j (dd^c v)^l \wedge \beta_{n-j-l} \le CN\delta.
$$

Similarly,

$$
\begin{aligned}
I_4(s,\delta) &\le N \int_{B_\rho \cap E_{s,\delta}} (dd^c v_s)^j (dd^c v)^l \wedge \beta_{n-j-l} \\
&\le C(N,\rho,j,l) \cdot \mathcal{C}_{n-1}(B_\rho \cap E_{s,\delta}, D) \longrightarrow 0,
\end{aligned}
$$

and (37) is proved.

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