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The tropical Grassmannian

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Abstract. In tropical algebraic geometry, the solution sets of polynomial equations are piecewise-linear. We introduce the tropical variety of a polynomial ideal, and we identify it with a polyhedral subcomplex of the Gröbner fan. The tropical Grassmannian arises in this manner from the ideal of quadratic Plücker relations. It parametrizes all tropical linear spaces. Lines in tropical projective space are trees, and their tropical Grassmannian $\mathcal{G}_{2,n}$ equals the space of phylogenetic trees studied by Billera, Holmes and Vogtmann. Higher Grassmannians offer a natural generalization of the space of trees. Their faces correspond to monomial-free initial ideals of the Plücker ideal. The tropical Grassmannian $\mathcal{G}_{3,6}$ is a simplicial complex glued from 1035 tetrahedra.

1 Introduction

The *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \min, +)$ is the set of real numbers augmented by infinity with the *tropical addition*, which is taking the minimum of two numbers, and the *tropical multiplication* which is the ordinary addition [10]. These operations satisfy the familiar axioms of arithmetic, e.g. distributivity, with ∞ and 0 being the two neutral elements. *Tropical monomials* $x_1^{a_1} \dots x_n^{a_n}$ represent ordinary linear forms $\sum_i a_i x_i$, and *tropical polynomials*

$$F(x_1, x_2, \dots, x_n) = \sum_{a \in \mathcal{A}} C_a x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}, \quad \text{with } \mathcal{A} \subset \mathbb{N}^n,\ C_a \in \mathbb{R}, \tag{1}$$

represent piecewise-linear convex functions $F : \mathbb{R}^n \to \mathbb{R}$. To compute $F(x)$, we take the minimum of the affine-linear forms $C_a + \sum_{i=1}^n a_i x_i$ for $a \in \mathcal{A}$. We define the *tropical hypersurface* $\mathcal{T}(F)$ as the set of all points x in $\mathbb{R}^n$ for which this minimum is attained at least twice, as a runs over $\mathcal{A}$. Equivalently, $\mathcal{T}(F)$ is the set of all points $x \in \mathbb{R}^n$ at which F is not differentiable. Thus a tropical hypersurface is an $(n-1)$-dimensional polyhedral complex in $\mathbb{R}^n$.

The rationale behind this definition will become clear in Section 2, which gives

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a self-contained development of the basic theory of tropical varieties. For further background and pictures see [14, §9]. Every tropical variety is a finite intersection of tropical hypersurfaces (Corollary 2.3). But not every intersection of tropical hypersurfaces is a tropical variety (Proposition 6.3). Tropical varieties are also known as *logarithmic limit sets* [2], *Bieri–Groves sets* [4], or *non-archimedean amoebas* [7]. Tropical curves are the key ingredient in Mikhalkin's formula [9] for planar Gromov–Witten invariants.

The object of study in this paper is the tropical Grassmannian $\mathscr{G}_{d,n}$ which is a polyhedral fan in $\mathbb{R}^{\binom{n}{d}}$ defined by the ideal of quadratic Plücker relations. All of our main results regarding $\mathscr{G}_{d,n}$ are stated in Section 3. The proofs appear in the subsequent sections. In Section 4 we prove Theorem 3.4 which identifies $\mathscr{G}_{2,n}$ with the space of phylogenetic trees in [5]. A detailed study of the fan $\mathscr{G}_{3,6} \subset \mathbb{R}^{20}$ is presented in Section 5. In Section 6 we introduce tropical linear spaces and we prove that they are parametrized by the tropical Grassmannian (Theorem 3.6). In Section 7 we show that the tropical Grassmannian $\mathscr{G}_{3,7}$ depends on the characteristic of the ground field.

2 The tropical variety of a polynomial ideal

Let K be an algebraically closed field with a valuation into the reals, denoted $\deg : K^* \to \mathbb{R}$. We assume that 1 lies in the image of deg and we fix $t \in K^*$ with $\deg(t) = 1$. The corresponding local ring and its maximal ideal are

$$R_K = \{c \in K : \deg(c) \geqslant 0\} \quad \text{and} \quad M_K = \{c \in K : \deg(c) > 0\}.$$

The residue field $k = R_K/M_K$ is algebraically closed. Given any ideal

$$I \subset K[x] = K[x_1, x_2, \dots, x_n],$$

we consider its affine variety in the *n-dimensional algebraic torus* over K,

$$V(I) = \{u \in (K^*)^n : f(u) = 0 \text{ for all } f \in I\}.$$

Here $K^* = K \backslash \{0\}$. In all our examples, K is the algebraic closure of the rational function field $\mathbb{C}(t)$ and "deg" is the standard valuation at the origin. Then $k = \mathbb{C}$, and if $c \in \mathbb{C}[t]$ then $\deg(c)$ is the order of vanishing of c at 0.

Every polynomial in $K[x]$ maps to a tropical polynomial as follows. If

$$f(x_1, \dots, x_n) = \sum_{a \in \mathscr{A}} c_a x_1^{a_1} \dots x_n^{a_n} \quad \text{with } c_a \in K^* \text{ for } a \in \mathscr{A}. \tag{2}$$

and $C_a = \deg(c_a)$, then $\operatorname{trop}(f)$ denotes the tropical polynomial F in (1).

The following definitions are a variation on Gröbner basis theory [13]. Fix $w \in \mathbb{R}^n$. The *w-weight* of a term $c_a \cdot x_1^{a_1} \dots x_n^{a_n}$ in (2) is $\deg(c_a) + a_1 w_1 + \dots + a_n w_n$. The *initial form* $\operatorname{in}_w(f)$ of a polynomial f is defined as follows. Set $\tilde{f}(x_1, \dots, x_n) = f(t^{w_1} x_1, \dots, t^{w_n} x_n)$. Let v be the smallest weight of any term of f, so that $t^{-v} \tilde{f}$ is a non-zero element in $R_K[x]$. Define $\mathrm{in}_w(f)$ as the image of $t^{-v}\tilde{f}$ in $k[x]$. We set $\mathrm{in}_w(0) = 0$. For $K = \overline{\mathbb{C}(t)}$ and $k = \mathbb{C}$ this means that the initial form $\mathrm{in}_w(f)$ is a polynomial in $\mathbb{C}[x]$.

Given any ideal $I \subset K[x]$ its *initial ideal* is defined to be

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle \subset k[x].$$

Theorem 2.1. *For an ideal $I \subset K[x]$ the following subsets of $\mathbb{R}^n$ coincide*:

(a) *The closure of the set* $\{(\deg(u_1), \ldots, \deg(u_n)) : (u_1, \ldots, u_n) \in V(I)\}$;

(b) *The intersection of the tropical hypersurfaces* $\mathcal{T}(\mathrm{trop}(f))$ *where* $f \in I$;

(c) *The set of all vectors $w \in \mathbb{R}^n$ such that* $\mathrm{in}_w(I)$ *contains no monomial.*

The set defined by the three conditions in Theorem 2.1 is denoted $\mathcal{T}(I)$ and is called the *tropical variety* of the ideal I. Variants of this theorem already appeared in [14, Theorem 9.17] and in [7, Theorem 6.1], without and with proof respectively. Here we present a short proof which is self-contained.

Proof. First we show that (b) contains (a). As (b) is clearly closed, it is enough to consider any point $w = (\deg(u_1), \ldots, \deg(u_n))$ in the set (a) and show it lies in (b). For any $f \in I$ we have $f(u_1, \ldots, u_n) = 0$ and this implies that the minimum in the definition of $F = \mathrm{trop}(f)$ is attained at least twice at w. This condition is equivalent to $\mathrm{in}_w(f)$ not being a monomial. This shows that (a) is contained in (b), and (b) is contained in (c). It remains to prove that (c) is contained in (a). Consider any vector w in (c) such that $w = (\deg(v_1), \ldots, \deg(v_n))$ for some $v \in (K^*)^n$. Since the image of the valuation is dense in $\mathbb{R}$ and the set defined in (a) is closed, it suffices to prove that $w = (\deg(u_1), \ldots, \deg(u_n))$ for some $u \in V(I)$. By making the change of coordinates $x_i = x_i \cdot v_i^{-1}$, we may assume that $w = (0, 0, \ldots, 0)$.

Since $\mathrm{in}_w(I)$ contains no monomial and since k is algebraically closed, by the Nullstellensatz there exists a point $\bar{u} \in V(\mathrm{in}_w(I)) \subset (k^*)^n$. Let $\bar{m}$ denote the maximal ideal in $k[x]$ corresponding to $\bar{u}$. Let S be the set of polynomials f in $R_K[x]$ whose reduction modulo M_K is not in $\bar{m}$. Then S is a multiplicative set in $R_K[x]$ disjoint from I. Consider the induced map

$$\varphi : R_K \to S^{-1}R_K[x]/S^{-1}(I \cap R_K[x]).$$

We claim that φ is injective. Suppose not, and pick $c \in R_K \backslash \{0\}$ with $\varphi(c) = 0$, so we can find $f \in S$ such that $cf \in I$. Since c^{-1} exists in K, this implies $f \in I$ which is a contradiction.

The injectivity of φ implies that there is some minimal prime P of the ring on the right hand side such that $P \otimes_{R_K} K$ is a proper ideal in $K[x]/I$. There exists a maximal ideal of $K[x]/I$ containing $P \otimes_{R_K} K$, and since K is algebraically closed, this maximal ideal has the form $\langle x_1 - u_1, \ldots, x_n - u_n \rangle$ for some $u \in V(I) \subset (K^*)^n$. We claim that $u_i \in R_K$ and $u_i \cong \bar{u}_i \bmod M_K$. This will imply $\deg(u_1) = \cdots = \deg(u_n) = 0$ and hence complete the proof.

Consider any $x_i - u_i \in I$. By clearing denominators, we get $a_i x - b_i \in I \cap R_K[x]$

with $b_i/a_i = u_i$, and not both a_i and b_i lie in M_K. If $a_i \in M_K$, then $a_i x - b_i \cong -b_i \bmod M_K$. Hence $\mathrm{in}_w(I)$ contains the reduction of b_i modulo M_K, which is a unit of K^* and hence equals the unit ideal. This is a contradiction. If $a_i \notin M_K$ and $-b_i/a_i \not\cong \bar{u}_i \bmod M_K$ then the reduction of $a_i x - b_i$ modulo M_K does not lie in $\bar{m}$. This means that $a_i x - b_i \in S$ and is a unit of $S^{-1}R_K[x]$, so P is the unit ideal. But then P is not prime, also a contradiction. This completes the proof. □

The key point in the previous proof can be summarized as follows:

Corollary 2.2. *Every zero over k of the initial ideal* $\mathrm{in}_w(I)$ *lifts to a zero over K of I.*

By a zero of an ideal I in $K[x]$ we mean a point on its variety in $(K^*)^n$. The notion of (reduced) Gröbner bases is well-defined for ideals $I \subset K[x]$ and (generic) weight vectors w, and, by adapting the methods of [13, §3] to our situation, we can compute a *universal Gröbner basis* $\mathrm{UGB}(I)$. This is a finite subset of I which contains a Gröbner basis for I with respect to any weight vector $w \in \mathbb{R}^n$. From Part (c) of Theorem 2.1 we derive:

Corollary 2.3 (Finiteness in Tropical Geometry). *The tropical variety* $\mathcal{T}(I)$ *is the intersection of the tropical hypersurfaces* $\mathcal{T}(\mathrm{trop}(f))$ *where* $f \in \mathrm{UGB}(I)$.

The following result is due to Bieri and Groves [4]. An alternative proof using Gröbner basis methods appears in [14, Theorem 9.6].

Theorem 2.4 (Bieri–Groves Theorem). *If I is a prime ideal and* $K[x]/I$ *has Krull dimension r, then* $\mathcal{T}(I)$ *is a pure polyhedral complex of dimension r.*

We shall be primarily interested in the case when $k = \mathbb{C}$ and $K = \overline{\mathbb{C}(t)}$. Under this hypothesis, the ideal I is said to have *constant coefficients* if the coefficients c_a of the generators f of I lie in the ground field $\mathbb{C}$. This implies $C_a = \deg(c_a) = 0$ in (1), where $F = \mathrm{trop}(f)$. Our problem is now to solve a system of tropical equations all of whose coefficients are identically zero:

$$F(x_1, x_2, \ldots, x_n) = \sum_{a \in \mathcal{A}} 0 \cdot x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}. \tag{3}$$

Here the tropical variety is a subfan of the Gröbner fan of an ideal in $\mathbb{C}[x]$.

Corollary 2.5. *If I has constant coefficients then* $\mathcal{T}(I)$ *is a fan in* $\mathbb{R}^n$.

3 Results on the tropical Grassmannian

We fix a polynomial ring in $\binom{n}{d}$ variables with integer coefficients:

$$\mathbb{Z}[p] = \mathbb{Z}[p_{i_1 i_2 \ldots i_d} : 1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n].$$

The *Plücker ideal* $I_{d,n}$ is the homogeneous prime ideal in $\mathbb{Z}[p]$ consisting of the algebraic relations among the $d \times d$-subdeterminants of any $d \times n$-matrix with entries in any commutative ring. The ideal $I_{d,n}$ is generated by quadrics, and it has a well-known quadratic Gröbner basis (see e.g. [12, Theorem 3.1.7]). The projective variety of $I_{d,n}$ is the *Grassmannian* $G_{d,n}$ which parametrizes all d-dimensional linear subspaces of an n-dimensional vector space.

The *tropical Grassmannian* $\mathscr{G}_{d,n}$ is the tropical variety $\mathscr{T}(I_{d,n})$ of the Plücker ideal $I_{d,n}$, over a field K as in Section 2. Theorem 2.1 (c) implies

$$\mathscr{G}_{d,n} = \{w \in \mathbb{R}^{\binom{n}{d}} : \mathrm{in}_w(I_{d,n}) \text{ contains no monomial}\}.$$

The ring $(\mathbb{Z}[p]/I_{d,n}) \otimes K$ is known to have Krull dimension $(n-d)d+1$. Therefore Theorem 2.4 and Corollary 2.5 imply the following statement.

Corollary 3.1. *The tropical Grassmannian $\mathscr{G}_{d,n}$ is a polyhedral fan in $\mathbb{R}^{\binom{n}{d}}$. Each of its maximal cones has the same dimension, namely $(n-d)d+1$.*

We show in Section 7 that the fan $\mathscr{G}_{d,n}$ depends on the characteristic of K if $d = 3$ and $n \geqslant 7$. All results in Sections 2–6 are valid over any field K.

It is convenient to reduce the dimension of the tropical Grassmannian. This can be done in three possible ways. Let φ denote the linear map from $\mathbb{R}^n$ into $\mathbb{R}^{\binom{n}{d}}$ which sends an n-vector $(a_1, a_2, \ldots, a_n)$ to the $\binom{n}{d}$-vector whose $(i_1, \ldots, i_d)$-coordinate is $a_{i_1} + \cdots + a_{i_d}$. The map φ is injective, and its image is the common intersection of all cones in the tropical Grassmannian $\mathscr{G}_{d,n}$. Note that the vector $(1, \ldots, 1)$ of length $\binom{n}{d}$ lies in image(φ). We conclude:

- The image of $\mathscr{G}_{d,n}$ in $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}(1, \ldots, 1)$ is a fan $\mathscr{G}'_{d,n}$ of dimension $d(n-d)$.
- The image of $\mathscr{G}_{d,n}$ or $\mathscr{G}'_{d,n}$ in $\mathbb{R}^{\binom{n}{d}}/\mathrm{image}(\varphi)$ is a fan $\mathscr{G}''_{d,n}$ of dimension $(d-1) \cdot (n-d-1)$. No cone in this fan contains a non-zero linear space.
- Intersecting $\mathscr{G}''_{d,n}$ with the unit sphere yields a polyhedral complex $\mathscr{G}'''_{d,n}$. Each maximal face of $\mathscr{G}'''_{d,n}$ is a polytope of dimension $nd - n - d^2$.

We shall distinguish the four objects $\mathscr{G}_{d,n}, \mathscr{G}'_{d,n}, \mathscr{G}''_{d,n}$ and $\mathscr{G}'''_{d,n}$ when stating our theorems below. In subsequent sections less precision is needed, and we sometimes identify $\mathscr{G}_{d,n}, \mathscr{G}'_{d,n}, \mathscr{G}''_{d,n}$ and $\mathscr{G}'''_{d,n}$ if there is no danger of confusion.

Example 3.2. $(d = 2, n = 4)$ The smallest non-zero Plücker ideal is the principal ideal $I_{2,4} = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \rangle$. Here $\mathscr{G}_{2,4}$ is a fan with three five-dimensional cones $\mathbb{R}^4 \times \mathbb{R}_{\geqslant 0}$ glued along $\mathbb{R}^4 = \mathrm{image}(\varphi)$. The fan $\mathscr{G}''_{2,4}$ consists of three half rays emanating from the origin (the picture of a tropical line). The zero-dimensional simplicial complex $\mathscr{G}'''_{2,4}$ consists of *three points*.

Example 3.3. $(d = 2, n = 5)$ The tropical Grassmannian $\mathscr{G}'''_{2,5}$ is the *Petersen graph* with 10 vertices and 15 edges. This was shown in [14, Example 9.10].

The following theorem generalizes both of these examples. It concerns the case $d = 2$, that is, the tropical Grassmannian of lines in $(n-1)$-space.

Theorem 3.4. *The tropical Grassmannian $\mathscr{G}'''_{2,n}$ is a simplicial complex known as the space of phylogenetic trees. It has $2^{n-1} - n - 1$ vertices, $1 \cdot 3 \ldots (2n-5)$ facets, and its homotopy type is a bouquet of $(n-2)!$ spheres of dimension $n-4$.*

A detailed description of $\mathscr{G}_{2,n}$ and the proof of this theorem will be given in Section 4. Metric properties of the *space of phylogenetic trees* were studied by Billera, Holmes and Vogtmann in [5] (our n corresponds to Billera, Holmes and Vogtmann's $n+1$.) The abstract simplicial complex and its homotopy type had been found earlier by Vogtmann [16] and by Robinson and Whitehouse [11]. The description has the following corollary. Recall that a simplicial complex is a *flag complex* if the minimal non-faces are pairs of vertices. This property is crucial for the existence of unique geodesics in [5].

Corollary 3.5. *The simplicial complex $\mathscr{G}'''_{2,n}$ is a flag complex.*

We do not have a complete description of the tropical Grassmannian in the general case $d \geqslant 3$ and $n - d \geqslant 3$. We did succeed, however, in computing all monomial-free initial ideals $\mathrm{in}_w(I_{d,n})$ for $d = 3$ and $n = 6$:

Theorem 3.6. *The tropical Grassmannian $\mathscr{G}'''_{3,6}$ is a 3-dimensional simplicial complex with 65 vertices, 550 edges, 1395 triangles and 1035 tetrahedra.*

The proof and a complete description of $\mathscr{G}_{3,6}$ will be presented in Section 5. We shall see that $\mathscr{G}_{3,6}$ differs in various ways from the tree space $\mathscr{G}_{2,n}$. Here is one instance of this, which follows from Theorem 5.4. Another one is Corollary 4.4 versus Proposition 5.5.

Corollary 3.7. *The tropical Grassmannian $\mathscr{G}'''_{3,6}$ is not a flag complex.*

If X is a d-dimensional linear subspace of the vector space K^n, then (the topological closure of) its image $\deg(X)$ under the degree map is a polyhedral complex in $\mathbb{R}^n$. Such a polyhedral complex arising from a d-plane in K^n is called a *tropical d-plane in n-space*. Since X is invariant under scaling, every cone in $\deg(X)$ contains the line spanned by $(1, 1, \ldots, 1)$, so we can identify $\deg(X)$ with its image in $\mathbb{R}^n / \mathbb{R}(1, 1, \ldots, 1) \simeq \mathbb{R}^{n-1}$. Thus $\deg(X)$ becomes a $(d-1)$-dimensional polyhedral complex in $\mathbb{R}^{n-1}$. For $d = 2$, we get a tree.

The classical Grassmannian $G_{d,n}$ is the projective variety in $\mathbb{P}^{\binom{n}{d}-1}$ defined by the Plücker ideal $I_{d,n}$. There is a canonical bijection between $G_{d,n}$ and the set of d-planes through the origin in K^n. The analogous bijection for the tropical Grassmannian $\mathscr{G}'_{d,n}$ is the content of the next theorem.

Theorem 3.8. *The bijection between the classical Grassmannian $G_{d,n}$ and the set of d-*

planes in K^n induces a unique bijection $w \mapsto L_w$ between the tropical Grassmannian $\mathcal{G}'_{d,n}$ and the set of tropical d-planes in n-space.

Theorems 3.4, 3.6 and 3.8 are proved in Sections 4, 5 and 6 respectively.

4 The space of phylogenetic trees

In this section we prove Theorem 3.4 which asserts that the *tropical Grassmannian of lines* $\mathcal{G}_{2,n}$ coincides with the *space of phylogenetic trees* [5]. We begin by reviewing the simplicial complex $\mathbf{T}_n$ underlying this space.

The vertex set $\mathrm{Vert}(\mathbf{T}_n)$ consists of all unordered pairs $\{A, B\}$, where A and B are disjoint subsets of $[n] := \{1, 2, \ldots, n\}$ having cardinality at least two, and $A \cup B = [n]$. The cardinality of $\mathrm{Vert}(\mathbf{T}_n)$ is $2^{n-1} - n - 1$. Two vertices $\{A, B\}$ and $\{A', B'\}$ are connected by an edge in $\mathbf{T}_n$ if and only if

$$A \subset A' \quad \text{or} \quad A \subset B' \quad \text{or} \quad B \subset A' \quad \text{or} \quad B \subset B'. \tag{4}$$

We now define $\mathbf{T}_n$ as the flag complex with this graph. Equivalently, a subset $\sigma \subseteq \mathrm{Vert}(\mathbf{T}_n)$ is a face of $\mathbf{T}_n$ if any pair $\{\{A, B\}, \{A', B'\}\} \subseteq \sigma$ satisfies (4).

The simplicial complex $\mathbf{T}_n$ was first introduced by Buneman (see [1, §5.1.4]) and was studied more recently by Robinson–Whitehouse [11] and Vogtmann [16]. These authors obtained the following results. Each face σ of $\mathbf{T}_n$ corresponds to a semi-labeled tree with leaves $1, 2, \ldots, n$. Here each internal node is unlabeled and has at least three neighbors. Each internal edge of such a tree defines a partition $\{A, B\}$ of the set of leaves $\{1, 2, \ldots, n\}$, and we encode the tree by the set of partitions representing its internal edges. The facets (= maximal faces) of $\mathbf{T}_n$ correspond to *trivalent trees*, that is, semi-labeled trees whose internal nodes all have three neighbors. All facets of $\mathbf{T}_n$ have the same cardinality $n - 3$, the number of internal edges of any trivalent tree. Hence $\mathbf{T}_n$ is pure of dimension $n - 4$. The number of facets (i.e. trivalent semi-labeled trees on $\{1, 2, \ldots, n\}$) is the *Schröder number*

$$(2n-5)!! = (2n-5) \times (2n-7) \times \cdots \times 5 \times 3 \times 1. \tag{5}$$

It is proved in [11] and [16] that $\mathbf{T}_n$ has the homotopy type of a bouquet of $(n-2)!$ spheres of dimension $n - 4$. The two smallest cases $n = 4$ and $n = 5$ are discussed in Examples 3.2 and 3.3. Here is a description of the next case:

Example 4.1. ($n = 6$) The two-dimensional simplicial complex $\mathbf{T}_6$ has 25 vertices, 105 edges and 105 triangles, each coming in two symmetry classes:

$$15 \text{ vertices like } \{12, 3456\}, \quad 10 \text{ vertices like } \{123, 456\},$$

$$60 \text{ edges like } \{\{12, 3456\}, \{123, 456\}\}, \quad 45 \text{ edges like } \{\{12, 3456\}, \{1234, 56\}\},$$

$$90 \text{ triangles like } \{\{12, 3456\}, \{123, 456\}, \{1234, 56\}\},$$

$$15 \text{ triangles like } \{\{12, 3456\}, \{34, 1256\}, \{56, 1234\}\}.$$

Each edge lies in three triangles, corresponding to restructuring subtrees.

We next describe an embedding of $\mathbf{T}_n$ as a simplicial fan into the $\frac{1}{2}n(n-3)$-dimensional vector space $\mathbb{R}^{\binom{n}{2}}/\mathrm{image}(\varphi)$. For each trivalent tree σ we first define a cone B_σ in $\mathbb{R}^{\binom{n}{2}}$ as follows. By a *realization* of a semi-labeled tree σ we mean a one-dimensional cell complex in some Euclidean space whose underlying graph is a tree isomorphic to σ. Such a realization of σ is a metric space on $\{1, 2, \dots, n\}$. The *distance* between i and j is the length of the unique path between leaf i and leaf j in that realization. Then we set

$$B_\sigma = \{(w_{12}, w_{13}, \dots, w_{n-1,n}) \in \mathbb{R}^{\binom{n}{2}} : -w_{ij} \text{ is the distance from leaf } i \text{ to leaf } j \text{ in some realization of } \sigma\} + \mathrm{image}(\varphi).$$

Let C_σ denote the image of B_σ in the quotient space $\mathbb{R}^{\binom{n}{2}}/\mathrm{image}(\varphi)$. Passing to this quotient has the geometric meaning that two trees are identified if their only difference is in the lengths of the n edges adjacent to the leaves.

Theorem 4.2. *The closure $\bar{C}_\sigma$ is a simplicial cone of dimension $|\sigma|$ with relative interior C_σ. The collection of all cones $\bar{C}_\sigma$, as σ runs over $\mathbf{T}_n$, is a simplicial fan. It is isometric to the Billera–Holmes–Vogtmann space of trees.*

Proof. Realizations of semi-labeled trees are characterized by the *four point condition* (e.g. [1, Theorem 2.1], [6]). This condition states that for any quadruple of leaves i, j, k, l there exists a unique relabeling such that

$$w_{ij} + w_{kl} = w_{ik} + w_{jl} \leqslant w_{il} + w_{jk}. \tag{6}$$

Given any tree σ, this gives a system of $\binom{n}{4}$ linear equations and $\binom{n}{4}$ linear inequalities. The solution set of this linear system is precisely the closure $\bar{B}_\sigma$ of the cone B_σ in $\mathbb{R}^{\binom{n}{2}}$. This follows from the *Additive Linkage Algorithm* [6] which reconstructs the combinatorial tree σ from any point w in B_σ.

All of our cones share a common linear subspace, namely,

$$\bar{B}_\sigma \cap -\bar{B}_\sigma = \mathrm{image}(\varphi). \tag{7}$$

This is seen by replacing the inequalities in (6) by equalities. The cone $\bar{B}_\sigma$ is the direct sum (8) of this linear space with a $|\sigma|$-dimensional simplicial cone. Let $\{e_{ij} : 1 \leqslant i < j \leqslant n\}$ denote the standard basis of $\mathbb{R}^{\binom{n}{2}}$. Adopting the convention $e_{ji} = e_{ij}$, for any partition $\{A, B\}$ of $\{1, 2, \dots, n\}$ we define

$$E_{A,B} = \sum_{i \in A} \sum_{j \in B} e_{ij}.$$

These vectors give the generators of our cone as follows:

$$\bar{B}_\sigma = \mathrm{image}(\varphi) + \mathbb{R}_{\geqslant 0}\{E_{A,B} : \{A, B\} \in \sigma\}. \tag{8}$$

From the two presentations (6) and (8) it follows that

$$\bar{B}_\sigma \cap \bar{B}_\tau = \bar{B}_{\sigma\cap\tau} \quad \text{for all } \sigma, \tau \in \mathbf{T}_n. \tag{9}$$

Therefore the cones B_σ form a fan in $\mathbb{R}^{\binom{n}{2}}$, and this fan has face poset $\mathbf{T}_n$. It follows from (8) that the quotient $\bar{C}_\sigma = \bar{B}_\sigma/\mathrm{image}(\varphi)$ is a pointed cone.

We get the desired conclusion for the cones $\bar{C}_\sigma$ by taking quotients modulo the common linear subspace (7). The resulting fan in $\mathbb{R}^{\binom{n}{2}}/\mathrm{image}(\varphi)$ is simplicial of pure dimension $n-3$ and has face poset $\mathbf{T}_n$. It is isometric to the Billera–Holmes–Vogtmann space in [5] because their metric is flat on each cone $\bar{C}_\sigma \simeq \mathbb{R}^{|\sigma|}_{\geqslant 0}$ and extended by the gluing relations $\bar{C}_\sigma \cap \bar{C}_\tau = \bar{C}_{\sigma\cap\tau}$. □

We now turn to the tropical Grassmannian and prove our first main result. We shall identify the simplicial complex $\mathbf{T}_n$ with the fan in Theorem 4.2.

Proof of Theorem 3.4. The Plücker ideal $I_{2,n}$ is generated by the $\binom{n}{4}$ quadrics

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \quad \text{for } 1 \leqslant i < j < k < l \leqslant n.$$

The tropicalization of this polynomial is the disjunction of linear systems

$$\begin{aligned} w_{ij} + w_{kl} &= w_{ik} + w_{jl} \leqslant w_{il} + w_{jk} \quad \text{or} \\ w_{ij} + w_{kl} &= w_{il} + w_{jk} \leqslant w_{ik} + w_{jl} \quad \text{or} \\ w_{ik} + w_{jl} &= w_{il} + w_{jk} \leqslant w_{ij} + w_{kl}. \end{aligned}$$

Every point w on the tropical Grassmannian $\mathscr{G}_{2,n}$ satisfies this for all quadruples i, j, k, l, that is, it satisfies the four point Condition (6). The Additive Linkage Algorithm reconstructs the unique semi-labeled tree σ with $w \in C_\sigma$. This proves that every relatively open cone of $\mathscr{G}_{2,n}$ lies in the relative interior of a unique cone C_σ of the fan $\mathbf{T}_n$ in Theorem 4.2.

We need to prove that the fans $\mathbf{T}_n$ and $\mathscr{G}_{2,n}$ are equal. Equivalently, every cone C_σ is actually a cone in the Gröbner fan. This will be accomplished by analyzing the corresponding initial ideal. In view of (9), it suffices to consider maximal faces σ of $\mathbf{T}_n$. Fix a trivalent tree σ and a weight vector $w \in C_\sigma$. Then, for every quadruple i, j, k, l, the inequality in (6) is strict. This means combinatorially that $\{\{i,l\},\{j,k\}\}$ is a four-leaf subtree of σ.

Let J_σ denote the ideal generated by the quadratic binomials $p_{ij}p_{kl} - p_{ik}p_{jl}$ corresponding to all four-leaf subtrees of σ. Our discussion shows that $J_\sigma \subseteq \mathrm{in}_w(I_{2,n})$. The proof will be completed by showing that the two ideals agree:

$$J_\sigma = \mathrm{in}_w(I_{2,n}). \tag{10}$$

This identity will be proved by showing that the two ideals have a common initial monomial ideal, generated by square-free quadratic monomials.

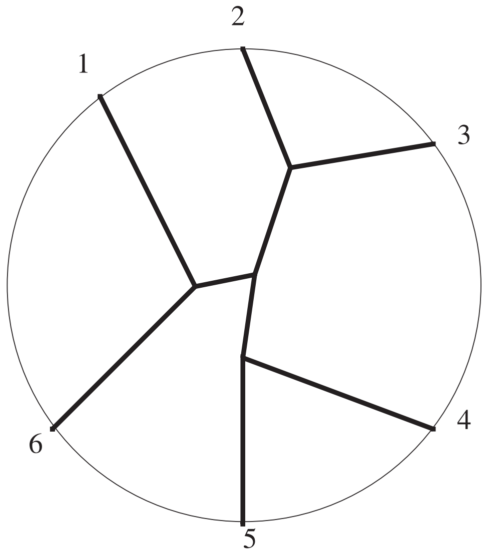

Figure 1. A Circular Labeling of a Tree with Six Leaves

We may assume, without loss of generality, that $-w$ is a strictly positive vector, corresponding to a planar realization of the tree σ in which the leaves $1, 2, \ldots, n$ are arranged in circular order to form a convex n-gon (Figure 1).

Let M be the ideal generated by the monomials $p_{ik}p_{jl}$ for $1 \leqslant i < j < k < l \leqslant n$. These are the crossing pairs of edges in the n-gon. By a classical construction of invariant theory, known as *Kempe's circular straightening law* (see [12, Theorem 3.7.3]), there exists a term order $\prec_{\text{circ}}$ on $\mathbb{Z}[p]$ such that

$$M = \text{in}_{\prec_{\text{circ}}}(I_{2,n}). \tag{11}$$

Now, by our circular choice w of realization of the tree σ, the crossing monomials $p_{ik}p_{jl}$ appear as terms in the binomial generators of J_σ. Moreover, the term order $\prec_{\text{circ}}$ on $\mathbb{Z}[p]$ refines the weight vector w. This implies

$$\text{in}_{\prec_{\text{circ}}}(\text{in}_w(I_{2,n})) = \text{in}_{\prec_{\text{circ}}}(I_{2,n}) = M \subseteq \text{in}_{\prec_{\text{circ}}}(J_\sigma). \tag{12}$$

Using $J_\sigma \subseteq \text{in}_w(I_{2,n})$ we conclude that equality holds in (12) and in (10). □

The simplicial complex $\Delta(M)$ represented by the squarefree monomial ideal M is an iterated cone over the boundary of the polar dual of the *associahedron*; see [12, page 132]. The facets of $\Delta(M)$ are the triangulations of the n-gon. Their number is the common degree of the ideals $I_{2,n}$, J_σ and M:

$$\text{the } (n-2)^{\text{nd}} \text{ Catalan number} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

The reduced Gröbner basis of (11) has come to recent prominence as a key example in the Fomin–Zelevinsky theory of *cluster algebras* [8]. Note also:

Corollary 4.3. *There exists a maximal cone in the Gröbner fan of the Plücker ideal $I_{2,n}$ which contains, up to symmetry, all cones of $\mathcal{G}_{2,n}$.*

Proof. The cone corresponding to the initial ideal (11) has this property. □

Corollary 4.4. *Every initial binomial ideal of $I_{2,n}$ is a prime ideal.*

Proof. If $\mathrm{in}_w(I_{2,n})$ is a binomial ideal then w must satisfy the four point Condition (6) with strict inequalities. Hence $\mathrm{in}_w(I_{2,n}) = J_\sigma$ for some semi-labeled trivalent tree σ. The ideal J_σ is radical and equidimensional because its initial ideal $M = \mathrm{in}_{\prec_{\mathrm{circ}}}(J_\sigma)$ is radical and equidimensional (unmixed).

To show that J_σ is prime, we proceed as follows. For each edge e of the tree σ we introduce an indeterminate y_e. Consider the polynomial ring

$$\mathbb{Z}[y] = \mathbb{Z}[y_e : e \text{ edge of } \sigma].$$

Let ψ denote the homomorphism $\mathbb{Z}[p] \to \mathbb{Z}[y]$ which sends p_{ij} to the product of all indeterminates y_e corresponding to edges on the unique path between leaf i and leaf j. We claim that $\mathrm{kernel}(\psi) = J_\sigma$.

A direct combinatorial argument shows that the convex polytope corresponding to the toric ideal $\mathrm{kernel}(\psi)$ has a canonical triangulation into $\frac{1}{n-1}\binom{2n-4}{n-2}$ unit simplices (namely, $\Delta(M)$). Hence $\mathrm{kernel}(\psi)$ and J_σ are both unmixed of the same dimension and the same degree. Since $\mathrm{kernel}(\psi)$ is obviously contained in J_σ, it follows that the two ideals are equal. □

Corollary 4.5. *The tropical Grassmannian $\mathcal{G}_{2,n}$ is characteristic-free.*

This means that we can consider the Plücker ideal $I_{2,n}$ in the polynomial ring $K[p]$ over any ground field K when computing its tropical variety. All generators $p_{ij}p_{kl} - p_{ik}p_{jl}$ of the initial binomial ideals J_σ have coefficients $+1$ and -1, so $J_\sigma \otimes k$ contains no monomial in $k[p]$, even if $\mathrm{char}(k) > 0$.

5 The Grassmannian of 3-planes in 6-space

In this section we study the case $d = 3$ and $n = 6$. The Plücker ideal $I_{3,6}$ is minimally generated by 35 quadrics in the polynomial ring in 20 variables,

$$\mathbb{Z}[p] = \mathbb{Z}[p_{123}, p_{124}, \ldots, p_{456}].$$

We are interested in the 10-dimensional fan $\mathcal{G}_{3,6}$ which consists of all vectors $w \in \mathbb{R}^{20}$ such that $\mathrm{in}_w(I_{3,6})$ is monomial-free. The four-dimensional quotient fan $\mathcal{G}''_{3,6}$ sits in $\mathbb{R}^{20}/\mathrm{image}(\varphi) \simeq \mathbb{R}^{14}$ and is a fan over the three-dimensional polyhedral complex

$\mathscr{G}'''_{3,6}$. Our aim is to prove Theorem 3.6, which states that $\mathscr{G}'''_{3,6}$ consists of 65 vertices, 550 edges, 1395 triangles and 1035 tetrahedra.

We begin by listing the vertices. Let E denote the set of 20 standard basis vectors e_{ijk} in $\mathbb{R}^{\binom{6}{3}}$. For each 4-subset $\{i,j,k,l\}$ of $\{1,2,\ldots,6\}$ we set

$$f_{ijkl} = e_{ijk} + e_{ijl} + e_{ikl} + e_{jkl}.$$

Let F denote the set of these 15 vectors. Finally consider any of the 15 *tripartitions* $\{\{i_1,i_2\},\{i_3,i_4\},\{i_5,i_6\}\}$ of $\{1,2,\ldots,6\}$ and define the vectors

$$g_{i_1i_2i_3i_4i_5i_6} := f_{i_1i_2i_3i_4} + e_{i_3i_4i_5} + e_{i_3i_4i_6} \quad \text{and}$$
$$g_{i_1i_2i_5i_6i_3i_4} := f_{i_1i_2i_5i_6} + e_{i_3i_5i_6} + e_{i_4i_5i_6}.$$

This gives us another set G of 30 vectors. All 65 vectors in $E \cup F \cup G$ are regarded as elements of the quotient space $\mathbb{R}^{\binom{6}{3}}/\mathrm{image}(\varphi) \simeq \mathbb{R}^{14}$. Note that

$$g_{i_1i_2i_3i_4i_5i_6} = g_{i_3i_4i_5i_6i_1i_2} = g_{i_5i_6i_1i_2i_3i_4}.$$

Later on, the following identity will turn out to be important in the proof of Theorem 5.4:

$$g_{i_1i_2i_3i_4i_5i_6} + g_{i_1i_2i_5i_6i_3i_4} = f_{i_1i_2i_3i_4} + f_{i_1i_2i_5i_6} + f_{i_3i_4i_5i_6}. \tag{13}$$

Lemma 5.1 and other results in this section were found by computation.

Lemma 5.1. *The set of vertices of $\mathscr{G}_{3,6}$ equals $E \cup F \cup G$.*

We next describe all the 550 edges of the tropical Grassmannian $\mathscr{G}_{3,6}$.

(EE) There are 90 edges like $\{e_{123}, e_{145}\}$ and 10 edges like $\{e_{123}, e_{456}\}$, for a total of 100 edges connecting pairs of vertices both of which are in E. (By the word "like", we will always mean "in the S_6 orbit of", where S_6 permutes the indices $\{1,2,\ldots,6\}$.)

(FF) This class consists of 45 edges like $\{f_{1234}, f_{1256}\}$.

(GG) Each of the 15 tripartitions gives exactly one edge, like $\{g_{123456}, g_{125634}\}$.

(EF) There are 60 edges like $\{e_{123}, f_{1234}\}$ and 60 edges like $\{e_{123}, f_{1456}\}$, for a total of 120 edges connecting a vertex in E to a vertex in F.

(EG) This class consists of 180 edges like $\{e_{123}, g_{123456}\}$. The intersections of the index triple of the e vertex with the three index pairs of the g vertex must have cardinalities $(2,1,0)$ *in this cyclic order*.

(FG) This class consists of 90 edges like $\{f_{1234}, g_{123456}\}$.

Lemma 5.2. *The 1-skeleton of $\mathscr{G}'''_{3,6}$ is the graph with the 550 edges above.*

Let Δ denote the *flag complex* specified by the graph in the previous lemma. Thus Δ is the simplicial complex on $E \cup F \cup G$ whose faces are subsets σ with the property that each 2-element subset of σ is one of the 550 edges. We will see that $\mathscr{G}_{3,6}$ is a subcomplex homotopy equivalent to Δ.

Lemma 5.3. *The flag complex Δ has* 1,410 *triangles,* 1,065 *tetrahedra,* 15 *four-dimensional simplices, and it has no faces of dimension five or more.*

The facets of Δ are grouped into seven symmetry classes:

Facet FFFGG: There are 15 four-dimensional simplices, one for each partition of $\{1,\dots,6\}$ into three pairs. An example of such a tripartition is $\{\{1,2\},\{3,4\},\{5,6\}\}$. It gives the facet $\{f_{1234}, f_{1256}, f_{3456}, g_{123456}, g_{125634}\}$. The 75 tetrahedra contained in these 15 four-simplices are not facets of Δ.

The remaining 990 tetrahedra in Δ are facets and they come in six classes:

Facet EEEE: There are 30 tetrahedra like $\{e_{123}, e_{145}, e_{246}, e_{356}\}$.

*Facet EEFF*1: There are 90 tetrahedra like $\{e_{123}, e_{456}, f_{1234}, f_{3456}\}$.

*Facet EEFF*2: There are 90 tetrahedra like $\{e_{125}, e_{345}, f_{3456}, f_{1256}\}$.

Facet EFFG: There are 180 tetrahedra like $\{e_{345}, f_{1256}, f_{3456}, g_{123456}\}$.

Facet EEEG: There are 240 tetrahedra like $\{e_{126}, e_{134}, e_{356}, g_{125634}\}$.

Facet EEFG: There are 360 tetrahedra like $\{e_{234}, e_{125}, f_{1256}, g_{125634}\}$.

While Δ is an abstract simplicial complex on the vertices of $\mathscr{G}'''_{3,6}$, it is not embedded as a simplicial complex because relation (13) shows that the five vertices of the four dimensional simplices only span a three dimensional space. Specifically, they form a bipyramid with the F-vertices as the base and the G-vertices as the two cone points.

We now modify the flag complex Δ to a new simplicial complex Δ' which has pure dimension three and reflects the situation described in the last paragraph. The complex Δ' is obtained from Δ by removing the 15 FFF-triangles $\{f_{1234}, f_{1256}, f_{3456}\}$, along with the 30 tetrahedra FFFG and the 15 four-dimensional facets FFFGG containing the FFF-triangles. In Δ', the bipyramids are each divided into three tetrahedra arranged around the GG-edges. The following theorem implies both Theorem 3.6 and Corollary 3.7.

Theorem 5.4. *The tropical Grassmannian $\mathscr{G}'''_{3,6}$ equals the simplicial complex Δ'. It is not a flag complex because of the* 15 *missing FFF-triangles. The homology of $\mathscr{G}'''_{3,6}$ is concentrated in* (*top*) *dimension* 3; $H_3(\mathscr{G}'''_{3,6}, \mathbb{Z}) = \mathbb{Z}^{126}$.

The integral homology groups were computed independently by Michael Joswig and Volkmar Welker. We are grateful for their help.

This theorem is proved by an explicit computation. The correctness of the result can be verified by the following method. One first checks that the seven types of cones described above are indeed Gröbner cones of $I_{3,6}$ whose initial ideals are monomial-free. Next one checks that the list is complete. This relies on a result in [7] which guarantees that $\mathcal{G}_{3,6}$ is connected in codimension 1. The completeness check is done by computing the link of each of the known classes of triangles. Algebraically, this amounts to computing the (truly zero-dimensional) tropical variety of $\mathrm{in}_w(I_{3,6})$ where w is any point in the relative interior of the triangular cone in question. For all but one class of triangles the link consists of three points, and each neighboring 3-cell is found to be already among our seven classes. The links of the triangles are as follows:

Triangle EEE: The link of $\{e_{146}, e_{256}, e_{345}\}$ consists of $e_{123}, g_{163425}, g_{142635}$.

Triangle EEF: The link of $\{e_{256}, e_{346}, f_{1346}\}$ consists of $f_{1256}, g_{132546}, g_{142536}$.

Triangle EEG: The link of $\{e_{156}, e_{236}, g_{142356}\}$ consists of $e_{124}, e_{134}, f_{1456}$.

Triangle EFF: The link of $\{e_{135}, f_{1345}, f_{2346}\}$ consists of $e_{236}, e_{246}, g_{153426}$.

Triangle EFG: The link of $\{e_{235}, f_{2356}, g_{143526}\}$ consists of $e_{145}, f_{1246}, e_{134}$.

Triangle FFG: The link of $\{f_{1236}, f_{1345}, g_{134526}\}$ consists of $e_{126}, e_{236}, g_{132645}$.

Triangle FGG: The link of $\{f_{1456}, g_{142356}, g_{145623}\}$ consists of f_{2356} and f_{1234}.

The FGG triangle lies in the interior of our bipyramid FFFGG and is incident to two of the three FFGG tetrahedra which make up the triangulation of that bipyramid. It is not contained in any other facet of $\mathcal{G}'''_{3,6}$.

The 15 bipyramids are responsible for various counterexamples regarding $\mathcal{G}_{3,6}$. This includes the failure of Corollaries 3.5 and 4.4 to hold for $d \geqslant 3$.

Proposition 5.5. *Not every initial binomial ideal of $I_{3,6}$ is prime. More precisely, if w is any vector in the relative interior of an FFGG cone, then* $\mathrm{in}_w(I_{3,6})$ *is the intersection of two distinct codimension* 10 *primes in* $\mathbb{Z}[p]$.

Proof. We may assume that $w = f_{1256} + f_{3456} + g_{123456} + g_{125634}$. Explicit computation (using [13, Corollary 1.9]) reveals that $\mathrm{in}_w(I_{3,6})$ is generated by

$$
\begin{array}{lll}
p_{124}p_{135} - p_{123}p_{145}, & p_{123}p_{146} - p_{124}p_{136}, & p_{125}p_{136} - p_{126}p_{135}, \\
p_{125}p_{146} - p_{126}p_{145}, & p_{135}p_{146} - p_{136}p_{145}, & p_{123}p_{245} - p_{124}p_{235}, \\
p_{123}p_{246} - p_{124}p_{236}, & p_{126}p_{235} - p_{125}p_{236}, & p_{125}p_{246} - p_{126}p_{245}, \\
p_{134}p_{235} - p_{135}p_{234}, & p_{136}p_{234} - p_{134}p_{236}, & p_{136}p_{235} - p_{135}p_{236}, \\
p_{134}p_{245} - p_{145}p_{234}, & p_{134}p_{246} - p_{146}p_{234}, & p_{146}p_{245} - p_{145}p_{246}, \\
p_{135}p_{346} - p_{136}p_{345}, & p_{146}p_{345} - p_{145}p_{346}, & p_{135}p_{245} - p_{145}p_{235}, \\
p_{135}p_{256} - p_{156}p_{235}, & p_{156}p_{245} - p_{145}p_{256}, & p_{135}p_{456} - p_{145}p_{356}, \\
p_{136}p_{246} - p_{146}p_{236}, & p_{136}p_{256} - p_{156}p_{236}, & p_{146}p_{256} - p_{156}p_{246},
\end{array}
$$

$$p_{136}p_{456} - p_{146}p_{356}, \quad p_{235}p_{246} - p_{236}p_{245}, \quad p_{235}p_{346} - p_{236}p_{345},$$
$$p_{245}p_{346} - p_{246}p_{345}, \quad p_{235}p_{456} - p_{245}p_{356}, \quad p_{246}p_{356} - p_{236}p_{456},$$
$$p_{136}p_{245} - p_{135}p_{246}, \quad p_{145}p_{236} - p_{135}p_{246}, \quad p_{146}p_{235} - p_{135}p_{246},$$
$$p_{123}p_{456} - p_{124}p_{356}, \quad p_{134}p_{256} - p_{156}p_{234}.$$

The ideal $\mathrm{in}_w(I_{3,6})$ is the intersection of the two codimension 10 primes

$$P = \mathrm{in}_w(I_{3,6}) + \langle p_{125}p_{346} - p_{126}p_{345}\rangle \quad \text{and}$$
$$Q = \mathrm{in}_w(I_{3,6}) + \langle p_{135}, p_{136}, p_{145}, p_{146}, p_{235}, p_{236}, p_{245}, p_{246}\rangle.$$

The degrees of the ideals P, Q and $I_{3,6}$ are 38, 4 and 42 respectively. □

We close this section with one more counterexample arising from the triangulated bipyramid in $\mathscr{G}'''_{3,6}$. It was proved in [3] that the $d \times d$-minors of a generic $d \times n$-matrix form a *universal Gröbner basis*. A question left open in that paper is whether the maximal minors also form a *universal sagbi basis*. It is well-known that they form a sagbi basis for a specific term order. See [13, Theorem 11.8] and the discussion in Section 6 below. The question was whether the sagbi basis property holds for all other term orders. We show that the answer is "no": the maximal minors are not a universal sagbi basis.

Corollary 5.6. *There exists a term order on* 18 *variables such that the* 3×3*-minors of a generic* 3×6*-matrix are not a sagbi basis in this term order.*

Proof. Consider the 3×6-matrix in [15, Example 1.8 and Proposition 3.13]:

$$W = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}$$

Let $w \in \mathbb{R}^{\binom{6}{3}}$ be its vector of *tropical* 3×3*-minors*. The coordinates of w are

$$w_{ijk} = \min\{W_{1i} + W_{2j} + W_{3k}, W_{1i} + W_{3j} + W_{2k}, W_{2i} + W_{1j} + W_{3k},$$
$$W_{2i} + W_{3j} + W_{1k}, W_{3i} + W_{1j} + W_{2k}, W_{3i} + W_{2j} + W_{1k}\}.$$

This vector represents the centroid of our bipyramid: $w = g_{123456} + g_{125634}$. We consider the 3×3-minors of the following matrix of indeterminates:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{pmatrix} \tag{14}$$

The initial forms of its 3×3-minors with respect to the weights W are

$$\begin{array}{llll}
p_{123} = z_1 x_2 y_3, & p_{124} = z_1 x_2 y_4, & p_{125} = y_1 z_2 x_5, & p_{126} = y_1 z_2 x_6, \\
p_{134} = -z_1 y_3 x_4, & p_{135} = -z_1 y_3 x_5, & p_{136} = -z_1 y_3 x_6, & p_{145} = -z_1 y_4 x_5, \\
p_{146} = -z_1 y_4 x_6, & p_{156} = z_1 x_5 y_6, & p_{234} = -z_2 y_3 x_4, & p_{235} = -z_2 y_3 x_5, \\
p_{236} = -z_2 y_3 x_6, & p_{245} = -z_2 y_4 x_5, & p_{246} = -z_2 y_4 x_6, & p_{256} = z_2 x_5 y_6, \\
p_{345} = -z_3 y_4 x_5, & p_{346} = -z_3 y_4 x_6, & p_{356} = y_3 z_5 x_6, & p_{456} = y_4 z_5 x_6.
\end{array}$$

Fix any term order $\prec$ which refines W. The criterion in [13, §11] will show that the 3×3-minors are not a sagbi basis with respect to $\prec$. The toric ideal of algebraic relations on the twenty monomials above is precisely the prime P in the proof of Proposition 5.5. The ideal P strictly contains $\mathrm{in}_w(I_{3,6})$. Both have codimension 10 but their degrees differ by 4. Using [13, Theorem 11.4] we conclude that the 3×3-minors are not a sagbi basis for $\prec$. $\square$

6 Tropical planes

The Grassmannian $G_{d,n}$ is the parameter space for all d-dimensional linear planes in K^n. We now prove the analogous statement in tropical geometry (Theorem 3.8). But there are also crucial differences between the classical planes and tropical planes. For instance, most tropical planes are not complete intersections of tropical hyperplanes (see Example 6.2 and Proposition 6.3). Our combinatorial theory of tropical d-planes is a direct generalization of the Buneman representation of trees (the $d = 2$ case) and thus offers mathematical tools for possible future applications in phylogenetic analysis.

Proof of Theorem 3.8. The tropical Grassmannian $\mathcal{G}'_{d,n}$ is a fan of dimension $(n-d)d$ in $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}(1,1,\ldots,1) \simeq \mathbb{R}^{\binom{n}{d}-1}$. We begin by describing the map which takes a point w in $\mathcal{G}'_{d,n}$ to the associated tropical d-plane $L_w \subset \mathbb{R}^n$. Given w, we consider the tropical polynomials

$$F_J(x_1,\ldots,x_n) = \sum_{j \in J} w_{J\backslash\{j\}} \cdot x_j, \tag{15}$$

where J runs over all subsets of cardinality $d+1$ in $[n]$. We define L_w as the subset of $\mathbb{R}^n$ which is the intersection of the $\binom{n}{d+1}$ tropical hypersurfaces $\mathcal{T}(F_J)$. We claim that L_w is a tropical d-plane. Pick a point $\xi \in (K^*)^{\binom{n}{d}}$ which is a zero of $I_{d,n}$ and satisfies $w = \deg(\xi)$. The d-plane X represented by ξ is cut out by the $\binom{n}{d+1}$ linear equations derived from Cramer's rule:

$$f_J(x_1,\ldots,x_n) = \sum_{j \in J} \pm \xi_{J\backslash\{j\}} \cdot x_j = 0 \tag{16}$$

The tropicalization of this linear form is the tropical polynomial in (15), in symbols, $\operatorname{trop}(f_J) = F_J$. It is known that the f_J form a universal Gröbner basis for the ideal they generate [13, Proposition 1.6]. Therefore, Corollary 2.3 shows that L_w is indeed a tropical d-plane. In fact, we have

$$\deg(X) = L_w = L_{\deg(\xi)}.$$

This proves that the map $w \mapsto L_w$ surjects the tropical Grassmannian onto the set of all tropical d-planes, and it is the only such map which is compatible with the classical bijection between $G_{d,n}$ and the set of d-planes in K^n.

It remains to be shown that the map $w \mapsto L_w$ is injective. We do this by constructing the inverse map. Suppose we are given L_w as a subset of $\mathbb{R}^n$. We need to reconstruct the coordinates $w_{i_1\dots i_d}$ of w up to a global additive constant. Equivalently, for any $(d-1)$-subset I of $[n]$ and any pair $j,k \in [n]\backslash I$, we need to reconstruct the real number $w_{I\cup\{j\}} - w_{I\cup\{k\}}$.

Fix a very large positive rational number M and consider the $(n-d+1)$-dimensional plane defined by $x_i = M$ for $i \in I$. The intersection of this plane with L_w contains at least one point $x \in \mathbb{R}^n$, and this point can be chosen to satisfy $x_j \ll M$ for all $j \in [n]\backslash I$. This can be seen by solving the $d-1$ equations $x_i = t^M$ on any d-plane $X \subset K^n$ which tropicalizes to L_w.

Now consider the tropical polynomial (15) with $J = I \cup \{j,k\}$. Since x lies in $\mathcal{T}(F_J)$, and since $\max(x_j, x_k) \ll M = x_i$ for all $i \in I$, we conclude

$$w_{J\backslash\{k\}} + x_k = w_{J\backslash\{j\}} + x_j.$$

This shows that the desired differences can be read off from the point x:

$$w_{I\cup\{j\}} - w_{I\cup\{k\}} = x_j - x_k. \tag{17}$$

We thus reconstruct $w \in \mathcal{G}_{d,n}$ by locating $\binom{n}{d-1}$ special points on L_w. $\square$

The above proof offers an (inefficient) algorithm for computing the map $w \mapsto L_w$, namely, by intersecting all $\binom{n}{d+1}$ tropical hypersurfaces $\mathcal{T}(F_J)$. Consider the case $d = 2$. Here the $\binom{n}{3}$ tropical polynomials F_J in (15) are

$$F_{ijk} = w_{ij} \cdot x_k + w_{ik} \cdot x_j + w_{jk} \cdot x_i.$$

The tropical hypersurface $\mathcal{T}(F_{ijk})$ is the solution set to the linear system

$$\begin{aligned} w_{ij} + x_k = w_{ik} + x_j \leqslant w_{jk} + x_i \quad &\text{or} \\ w_{ij} + x_k = w_{jk} + x_i \leqslant w_{ik} + x_j \quad &\text{or} \\ w_{ij} + x_k = w_{jk} + x_i \leqslant w_{ik} + x_j. & \end{aligned}$$

The conjunction of these $\binom{n}{3}$ linear systems can be solved efficiently by a variant of the tree reconstruction algorithm in [6]. If r and $s \in \mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ are vertices of this tree connected by an edge e, then $r = s + c\sum_{i\in S} e_i$ for some $c > o$ where $S \subset [n]$ is the set of leaves on the s side of e. We regard the tree as a metric space by assigning the length c to edge e. The length of each edge is measured in lattice distance, so we get the tree with metric $-2w$.

Corollary 6.1. *Let w be a point in $\mathcal{G}_{2,n}$ which lies in the cone C_σ for some tree σ. The image of L_w in $\mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ is a tree of combinatorial type σ.*

The bijection $w \mapsto L_w$ of Theorem 3.8 is a higher-dimensional generalization of recovering a phylogenetic tree from pairwise distances among n leaves. For instance, for $d = 3$ we can think of w as data giving a proximity measure for any triple among n "leaves". The image of L_w in $\mathbb{R}^n/\mathbb{R}(1,\ldots,1)$ is a "phylogenetic surface" which is a geometric representation of such data.

The tropical Grassmannians $\mathcal{G}_{d,n}$ and $\mathcal{G}_{n-d,n}$ are isomorphic because the ideals $I_{d,n}$ and $I_{n-d,n}$ are the same after signed complementation of Plücker coordinates. Theorem 3.8 allows us to define the *dual* $(n-d)$-plane L^* of a tropical d-plane L in $\mathbb{R}^n$. If $L = L_w$ then $L^* = L_{w^*}$ where w^* is the vector whose $([n]\backslash I)$-coordinate is the I-coordinate of w, for all d-subsets I of $[n]$. One can check that a tropical hyperplane $\sum a_i \cdot x_i = 0$ contains L^* iff $(a_i) \in L_w$ and that $(L^*)^* = L$.

Example 6.2. Let $w = e_{12} + e_{34} + e_{56}$ in $\mathbb{R}^{\binom{6}{2}}$. Then L_w is a tropical 2-plane in $\mathbb{R}^6$. Its image in $\mathbb{R}^6/\mathbb{R}(1,\ldots,1)$ is a tree as in Figure 1, of type $\sigma = \{\{12, 3456\}, \{34, 1256\}, \{56, 1234\}\}$. The Plücker vector dual to w is

$$w^* = e_{3456} + e_{1256} + e_{1234} \in \mathcal{G}_{4,6} \subset \mathbb{R}^{\binom{6}{4}}.$$

We shall compute the tropical 4-plane L_{w^*} by applying the algorithm in the proof of Theorem 3.8. There are 6 tropical polynomials F_J as in (15), namely,

$$\begin{aligned}
F_{12345} &= 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 + 1\cdot x_5\\
F_{12346} &= 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 + 1\cdot x_6\\
F_{12356} &= 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 0\cdot x_5 + 0\cdot x_6\\
F_{12456} &= 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_4 + 0\cdot x_5 + 0\cdot x_6\\
F_{13456} &= 1\cdot x_1 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 + 0\cdot x_6\\
F_{23456} &= 1\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 + 0\cdot x_6
\end{aligned}$$

The tropical 4-plane L_{w^*} is the intersection of these six tropical hyperplanes:

$$\mathcal{T}(F_{12345}) \cap \mathcal{T}(F_{12346}) \cap \mathcal{T}(F_{12356}) \cap \mathcal{T}(F_{12456}) \cap \mathcal{T}(F_{13456}) \cap \mathcal{T}(F_{23456}).$$

We claim that L_{w^*} *is not a complete intersection*, i.e., there do no exist two tropical linear forms F and F' such that $L_{w^*} = \mathcal{T}(F) \cap \mathcal{T}(F')$. A tropical linear form $F =$

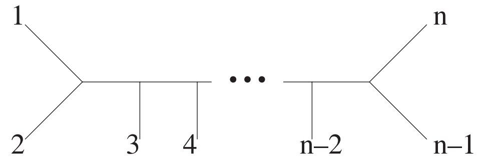

Figure 2. A Caterpillar Tree

$a_1x_1 + \cdots + a_6x_6$ vanishes on the dual 4-plane L_{w^*} if and only if the point $a = (a_1, \dots, a_6)$ lies in the 2-plane L_w. There are 9 types of such tropical linear forms F, one for each of the 9 edges of the tree L_w. For instance, the bounded edge $\{56, 1234\}$ represents the tropical forms

$$F = \alpha \cdot (x_1 + x_2 + x_3 + x_4) + \beta \cdot (x_5 + x_6) \quad \text{where } 0 < \alpha \leqslant \beta.$$

By checking all pairs of the 9 edges, we find that any conceivable intersection $\mathcal{T}(F) \cap \mathcal{T}(F')$ must contain a 5-dimensional cone like $\{x_1 + c = x_2 \ll x_3, x_4, x_5, x_6\}$, $\{x_3 + c = x_4 \ll x_1, x_2, x_5, x_6\}$ or $\{x_5 + c = x_6 \ll x_1, x_2, x_3, x_4\}$.

This example can be generalized as follows.

Proposition 6.3. *Let L_w be a tropical 2-plane in $\mathbb{R}^n$ whose tree is not combinatorially isomorphic to $\sigma = \{\{1, \dots, i\}, \{i+1, \dots, n\} : i = 2, 3, \dots, n-2\}$. Then the dual tropical $(n-2)$-plane L_{w^*} is not a complete intersection.*

The special tree σ in Proposition 6.3 is called the *caterpillar* in the phylogenetic literature (see Figure 2).

Proof. Suppose for contradiction that L_{w^*} is the intersection of the hyperplanes $\sum a_i \cdot x_i = 0$ and $\sum b_i \cdot x_i = 0$. The vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, regarded as elements of $\mathbb{R}^n/\mathbb{R}(1, \dots, 1)$, lie in the tree L_w. Denote by γ the path through L_w from a to b. Since L_w is not a caterpillar tree, the path γ goes through fewer than $n-1$ edges, so deleting those edges divides L into fewer than n connected components. Thus, there are two leaves of L, call them j and k, such that the none of the edges of γ separate j from k. Every edge of γ connects two points r and s with $s = r + c\sum_{i \in S} e_i$ where, in each case, either j and k both lie in S or neither do. Thus, $a_j - a_k = b_j - b_k$. Therefore, the intersection of the hyperplanes $\sum a_i \cdot x_i = 0$ and $\sum b_i \cdot x_i = 0$ contains every point (x_i) with $x_j + a_j = x_k + a_k$ and $x_i - x_j$ sufficiently positive for all $i \neq j, k$. But this is a codimension one subset of $\mathbb{R}^n/\mathbb{R}(1, \dots, 1)$ and we know that L_{w^*} is pure of codimension two. □

Our next goal is to give a combinatorial encoding of tropical planes. The basic object in our combinatorial encoding is a d-partition $\{A_1, \dots, A_d\}$. By a *d-partition* we mean an unordered partition of $[n]$ into d subsets A_i. Let L_w be a tropical d-plane and F a maximal cell of L_w. Thus F is a d-dimensional convex polyhedron in $\mathbb{R}^n$. The affine span of F is a d-dimensional affine space which is defined by equations of the special form

$$x_k - x_j = w_{J\backslash\{j\}} - w_{J\backslash\{k\}} \quad \text{(the right hand side is a constant)}$$

Such a system of equations defines a d-partition $\{A_1, \ldots, A_d\}$, namely, two indices j and k lie in the same block A_i if and only if the difference $x_k - x_j$ is constant on F. The number of blocks clearly equals d, the dimension of F.

Remark 6.4. A maximal face F of L_w is uniquely specified by its d-partition $\{A_1, \ldots, A_d\}$. It is a (bounded) polytope in $\mathbb{R}^n$ if and only if $|A_i| \geqslant 2$ for all i. Hence a tropical d-plane $L_w \subset \mathbb{R}^n$ has no bounded d-faces if $n \leqslant 2d - 1$.

We define the *type* of a tropical d-plane L, denoted type(L), to be the set of all d-partitions arising from the maximal faces of L. If $d = 2$ and $L = L_w$ with $w \in C_\sigma$, then type(L) is precisely the set σ together with the pairs $\{\{i\}, [n]\backslash\{i\}\}$ representing the unbounded edges of the tree L. This follows from Corollary 6.1. Thus type(L) generalizes the Buneman representation of semi-labeled trees (Section 4) to higher-dimensional tropical planes L.

The type of a tropical plane L_w is a strong combinatorial invariant, but it does not uniquely determine the cone of $\mathcal{G}_{d,n}$ which has w in its relative interior. We will see this phenomenon in the example below.

Example 6.5. We present three of the seven types in $\mathcal{G}_{3,6}$. In each case we display type(L_w) with the 15 obvious tripartitions $\{i, j, [6]\backslash\{i, j\}\}$ removed.

We begin with a type which we call the *sagbi type*:

$$\begin{aligned} &\{\{1,23,456\},\{1,56,234\},\{2,13,456\},\{2,56,134\},\\ EEFF1: \quad &\{3,12,456\},\{3,56,124\},\{4,12,356\},\{4,56,123\},\\ &\{5,12,346\},\{5,46,123\},\{6,12,345\},\{6,45,123\},\{12,34,56\}\} \end{aligned}$$

The next type is the *bipyramid type*. All three tetrahedra in a bipyramid *FFFGG* have the same type listed below. As the faces of $\mathcal{G}'''_{3,6}$ contain those w inducing different initial ideals $\mathrm{in}_w(I_{d,n})$, this example demonstrates that type(L_w) does not determine $\mathrm{in}_w(I_{d,n})$.

$$\begin{aligned} &\{\{1,34,256\},\{1,56,234\},\{2,34,156\},\{2,56,134\}\\ FFGG: \quad &\{3,12,456\},\{3,56,124\},\{4,12,356\},\{4,56,123\}\\ &\{5,12,346\},\{5,34,126\},\{6,12,345\},\{6,34,125\},\{12,34,56\}\} \end{aligned}$$

For all but one of the seven types in $\mathcal{G}_{3,6}$, the tropical plane L_w has 28 facets. The only exception is the type *EEEE*. Here the tropical plane L_w has only 27 facets, all of them unbounded.

$$EEEE: \quad \begin{array}{l}\{\{1,23,456\},\{1,234,56\},\{2,13,456\},\{2,135,46\}\\ \{3,12,456\},\{3,126,45\},\{4,26,135\},\{4,126,35\}\\ \{5,16,234\},\{5,126,34\},\{6,15,234\},\{6,135,24\}\}\end{array}$$

7 Dependence on the characteristic

Our definition of the tropical Grassmannian implicitly depended on the fields K and k. The ideal $I_{d,n}$ makes sense over any field and has the same generators (the classical Plücker relations). Nonetheless, the properties of the initial ideal $\mathrm{in}_w(I_{d,n})$ might depend on k, in particular, whether or not this ideal contains a monomial might depend on the characteristic of k. Hence, whether or not $w \in \mathcal{G}_{d,n}$ might depend on the characteristic of k.

In Corollary 4.5 we saw that this does not happen for $d = 2$, and it follows from the explicit computations in Section 5 that this does not happen for $\mathcal{G}_{3,6}$ either. In both of these cases, the tropical Grassmannian is characteristic-free. Another result that we observed in both of these nice cases is that it was enough to look at quadratic polynomials in $I_{d,n}$ to define the tropical Grassmannian. We shall see below that the same results do not hold for the next case $\mathcal{G}_{3,7}$. We summarize our result in the following theorem.

Theorem 7.1. *Let $d = 2$ or ($d = 3$ and $n = 6$). Then every monomial-free initial ideal of $I_{d,n}$ is generated by quadrics, and the tropical Grassmannian $\mathcal{G}_{d,n}$ is characteristic-free. Both of these properties fail if $d = 3$ and $n \geqslant 7$.*

Proof. It suffices to consider the case $d = 3$ and $n = 7$. An easy lifting argument will extend our example to the general case $d = 3$ and $n \geqslant 7$. The Plücker ideal $I_{3,7}$ is minimally generated by 140 quadrics in a polynomial ring $k[p_{123}, p_{124}, \ldots, p_{567}]$ in 35 unknowns over an arbitrary field k.

We fix the following zero-one vector. The appearing triples are gotten by a cyclic shift, and they correspond to the lines in the *Fano plane*:

$$w = e_{124} + e_{235} + e_{346} + e_{457} + e_{156} + e_{267} + e_{137} \in \mathbb{R}^{\binom{6}{3}}.$$

We next compute the initial ideal $\mathrm{in}_w(I_{3,7})$ under the assumption that the characteristic of k is zero. In a computer algebra system, this is done by computing the reduced Gröbner basis of $I_{3,7}$ over the field of rational numbers with respect to the (reverse lexicographically refined) weight order defined by $-w$. The reduced Gröbner basis is found to have precisely 196 elements, namely, 140 quadrics, 52 cubics, and 4 quartics. The initial ideal $\mathrm{in}_w(I_{3,7})$ is generated by the w-leading forms of the 196 elements in that Gröbner basis.

Among the 52 cubics in the Gröbner basis of $I_{3,7}$, we find the special cubic

$$\begin{aligned} f = 2 \cdot p_{123}p_{467}p_{567} &- p_{367}p_{567}\underline{p_{124}} - p_{167}p_{467}\underline{p_{235}} - p_{127}p_{567}\underline{p_{346}} \\ &- p_{126}p_{367}\underline{p_{457}} - p_{237}p_{467}\underline{p_{156}} + p_{134}p_{567}\underline{p_{267}} + p_{246}p_{567}\underline{p_{137}} + p_{136}\underline{p_{267}p_{457}}. \end{aligned}$$

Since $\mathrm{char}(k) \neq 2$, the leading form of this polynomial is the monomial

$$\mathrm{in}_w(f) = p_{123}p_{467}p_{567}.$$

This proves that w is not in the tropical Grassmannian $\mathcal{G}_{3,7}$.

On the other hand, suppose now that the characteristic of k equals two. In that case, the leading form of f is a polynomial with seven terms

$$\mathrm{in}_w(f) = -p_{367}p_{567}\underline{p_{124}} - p_{167}p_{467}\underline{p_{235}} - \cdots + p_{246}p_{567}\underline{p_{137}}.$$

This is not a monomial. In fact, none of the leading forms of the 196 Gröbner basis elements is a monomial. This proves that the initial ideal $\mathrm{in}_w(I_{3,7})$ contains no monomial, or equivalently, that w lies in the tropical Grassmannian $\mathcal{G}_{3,7}$ when $\mathrm{char}(k) = 2$. In fact, there is no inclusion in either direction between the tropical Grassmannians $\mathcal{G}_{3,7}$ in characteristic two and in characteristic zero. To see this, we modify our vector w as follows:

$$w' = w - e_{124} = e_{235} + e_{346} + e_{457} + e_{156} + e_{267} + e_{137} \in \mathbb{R}^{\binom{6}{3}}.$$

Then $\mathrm{in}_{w'}(f) = 2 \cdot p_{123}p_{467}p_{567} - p_{367}p_{567}\underline{p_{124}}$, which is not monomial if $\mathrm{char}(k) = 0$, but it is a monomial if $\mathrm{char}(k) = 2$. This shows that w' does not lie in $\mathcal{G}_{3,7}$ if the characteristic of k is two. By recomputing the Gröbner basis in characteristic zero, we find that the initial ideal $\mathrm{in}_{w'}(I_{3,7})$ contains no monomial, and hence does lie in $\mathcal{G}_{3,7}$ if the characteristic of k is zero.

The above argument also shows that, in any characteristic, either $\mathrm{in}_w(I_{3,7})$ or $\mathrm{in}_{w'}(I_{3,7})$ will be a monomial-free initial ideal which has a minimal generator of degree three. Quadrics do not suffice for $d = 3$ and $n \geqslant 7$. □

It is worth taking a moment to think about the intuitive geometry behind this argument. Let B be any subset of $\binom{[n]}{d}$; we can study the collection of points on the Grassmannian $G_{d,n}$ over the field k where the Plücker coordinate P_I is nonzero exactly for those $I \in B$. Such points exist exactly if B is the set of bases of a matroid of rank d on n points realizable over k.

Thus, when the characteristic of k is 2 there is a point $x \in G_{3,7}$ with $x_{ijk} = 0$ exactly when i, j and k are collinear in the Fano plane and no such point should exist in characteristic other than 2. Passing to the tropicalization, one would expect that in characteristic 2 there should be a point $y \in \mathcal{G}_{d,n}$ with $y_{ijk} = \infty$ for i, j and k collinear in the Fano plane and $y_{ijk} = 0$ otherwise. Intuitively, w is a perturbation of y so that w_{ijk} is 1 and not ∞.

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