Multidimensional residue theory and applications

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Abstract. This paper is a sample prepared to illustrate the use of the American Mathematical Society's LaTeX document class `amsbook` and publication-specific variants of that class.

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APPENDIX A

Currents on complex manifolds

A.1. Differential forms, currents, positive currents

A.1.1. Positive or strongly positive differential forms over a $\mathbb{C}$- vector space V. All $\mathbb{C}$-vector spaces V admit a canonical orientation, the rule being that the form

$$\frac{1}{(2i)^n}\bigwedge_{j=1} d\overline{\zeta}_j \wedge d\zeta_j = \bigwedge_{k=1}^{n} (d\xi_k \wedge d\eta_k)$$

$(\zeta_j = \xi_j + i\eta_j,\ j = 1, ..., n)$ is the *volume form.* This canonical orientation is preserved by any holomorphic change of coordinates. It implies (in particular, when the $\mathbb{C}$-vector space V is the complex tangent space $T_z(\mathcal{X})$ at the point z of a finite dimensional, complex analytic manifold $\mathcal{X}$) an orientation over $\mathcal{X}$, which is equivalent to the fact that there exists a positive, non-vanishing (n, n)-form over the whole complex manifold $\mathcal{X}$. The above notion of positivity[1], which is often materialized through the interplay between holomorphic coordinates systems (i.e. $(\zeta_1, ..., \zeta_n)$) and anti-holomorphic ones (i.e. $(\overline{\zeta}_1, ..., \overline{\zeta}_n)$) will be of great importance to us; note that the methods of real differential geometry miss such a useful tool.

Definition A.1 (positive differential forms). If V is complex vector space equipped with canonical orientation, a (p, p)-form $\varphi \in \bigwedge^{p,p} V^* := \bigwedge^p V^* \otimes \overline{\bigwedge^p V^*}$ is said to be *positive*[2] if and only if $\varphi \wedge \bigwedge_{j=1}^{n-p} (i\alpha_j \wedge \overline{\alpha}_j)$ is a positive (n, n)-form for arbitrary elements $\alpha_1, ..., \alpha_{n-p}$ in the dual V^*. The form ω is said to be *strongly positive* if it can be expressed as a linear combination with positive coefficients of forms of type $\bigwedge_{j=1}^{p} (i\beta_j \wedge \overline{\beta}_j)$, where $\beta_1, ..., \beta_p \in V^*$. The form φ is said to be negative (*resp.* strongly negative) if and only if $-\varphi$ is positive (*resp.* strongly positive).

Remark A.1. Any *strongly positive* form (as also, by duality, any *positive* form) is real, i.e. satisfies $\varphi = \overline{\varphi}$. A $(1, 1)$-form $i \sum_{j,k} h_{jk} d\zeta_j \wedge d\overline{\zeta}_k$ over the $\mathbb{C}$-vector space V is positive if and only if the sesquilinear form

$$\sum_{j=1}^{n}\sum_{k=1}^{n} h_{jk}\, d\zeta_j \otimes \overline{d\zeta_k} \ : u = (u_1, ..., u_n) \mapsto \sum_{j,k} h_{jk} u_j \overline{u}_k$$

induces a hermitian, semi-positive metric over V.

[1] A real differentiable manifold $\mathcal{X}$ of dimension N is called *orientable* if and only if the determinant bundle $\bigwedge^N T^*(\mathcal{X})$ admits a trivialization, that is, there exists a non-vanishing global section section of this bundle to act as a *volume form.* The existence of such a *volume form* allows the use of *integration theory* over the variety $\mathcal{X}$.

[2] One has to be careful here: we understand *positive* as *non-negative* ; dealing with other concepts of positivity, in particular for bundles (see for example Appendix B, Section B.3.4), *positive* will carry a stronger meaning, i.e. *positive definite.*

We note that any wedge product of forms, all of which are strongly positive, is strongly positive. If all of them, but one, are strongly positive, then the wedge product remains positive.

Remark A.2. The notions of *positivity* and *strong positivity* for differential (p,p)-forms differ when $2 \leq p \leq n-2$: a positive (p,p)-forme $i^{p^2}\beta \wedge \overline{\beta}$ is strongly positive if and only if β splits as the wedge product of p elements of V^*, which is not always the case as soon as $2 \leq p \leq n-2$.

A.1.2. Currents on complex analytic manifolds (positive currents). In the present section, we will consider a complex analytic manifold $\mathcal{X}$ of dimension n. We will also need to deal with a locally trivial complex vector bundle $E \to \mathcal{X}$ (not necessarily holomorphic) with rank m. For every (p,q), $0 \leq p,q \leq n$, denote as $\mathcal{D}^{(n-p,n-q)}(\mathcal{X},E)$ the $\mathbb{C}$-vector space of global, smooth (C^∞) sections with support compact of the $\mathbb{C}$-vector bundle $(T^{n-p,n-q}_{\mathcal{X},\mathbb{R}})^* \otimes_{\mathbb{R}} E$ (for the vocabulary about vector complex bundles, we refer to Appendix B).

Definition A.2 (notion of current). The dual space of the $\mathbb{C}$-vector space $\mathcal{D}^{(n-p,n-q)}(\mathcal{X},E)$ is, by definition, the $\mathbb{C}$-space of *currents of bi-degree* (p,q) (which are also called (p,q)-currents, or currents with bi-dimension $(n-p,n-q)$) over $\mathcal{X}$, with values in the dual bundle $E^* \to \mathcal{X}$.

The above dual is denoted all along the monography as ${}'\mathcal{D}^{(p,q)}(\mathcal{X},E^*)$. When $E = \mathcal{X}\times\mathbb{C}$, we use the abridged notations $\mathcal{D}^{(n-p,n-q)}(\mathcal{X},\mathcal{X}\times\mathbb{C}) = \mathcal{D}^{(n-p,n-q)}(\mathcal{X})$ and ${}'\mathcal{D}^{(p,q)}(\mathcal{X},\mathcal{X}\times\mathbb{C}) = {}'\mathcal{D}^{(p,q)}(\mathcal{X})$. In case the vector bundle $E \to \mathcal{X}$ is the trivial bundle $\mathcal{X}\times\mathbb{C} \to \mathbb{C}$, it happens useful to identify currents of bi-degree (p,q) with differential (p,q)-forms, whose coefficients are distributions. If $(e_1,...,e_m)$ is a local frame for the $\mathbb{C}$-vector bundle $E \to \mathcal{X}$ over the open chart (U,τ) and if $\varphi \in \mathcal{D}^{(n-p,n-q)}(U,E)$, then one can express φ in local coordinates $(\zeta_1,...,\zeta_n)$ in $\tau(U) \subset \mathbb{C}^n$ in the form $\varphi = \sum_{j=1}^m \varphi_j \otimes e_j$, where

$$\varphi_j := \sum_{\substack{1\leq j_1<\cdots<j_{n-p}\leq n\\ 1\leq k_1<\cdots<k_{n-q}\leq n}} \varphi_{j,J,K}\, d\zeta_I \wedge d\overline{\zeta}_K\,, \quad \varphi_{j,J,K} \in \mathcal{D}(\tau(U),\mathbb{C})\,,$$

with the convention J and K to be respectively $(n-p)$-uplets et $(n-q)$-uplets with strictly increasing ordering of indices $\{j_1,...,j_{n-p}\}$, $\{j_1,...,j_{n-q}\}$ and where

$$d\zeta_J := \bigwedge_{l=1}^{n-p} d\zeta_{j_l}\,,\ d\overline{\zeta}_K := \bigwedge_{l=1}^{n-q} d\overline{\zeta}_{k_l}.$$

An element $T \in {}'\mathcal{D}^{(p,q)}(\mathcal{X},E^*)$ is thus represented in the open neighborhood U (in local coordinates in $\tau(U)$) as $T = \sum_{j=1}^m T^j \otimes e_j^*$, where each T^j can be developped as

$$T^j := \sum_{\substack{1\leq j_1'<\cdots<j_p'\leq n\\ 1\leq k_1'<\cdots<k_q'\leq n}} T^{j,J',K'} d\zeta_{J'} \wedge d\overline{\zeta}_{K'}\,, \quad T^{j,J',K'} \in \mathcal{D}'(\tau(U),\mathbb{C})\,.$$

Therefore, it is suitable to introduce the duality bracket $\langle T,\varphi\rangle$:

$$\langle T,\varphi\rangle = (-2i)^n \sum_{j=1}^m \sum_{\substack{|J|=n-p\\ |K|=n-q}} \epsilon_{J,K}\, T^{j,J^c,K^c}(\varphi_{j,J,K})\,,$$

where $J^c := \{1,...,n\} \setminus J$, $K^c := \{1,...,n\} \setminus K$ (ordered in a strictly increasing manner) and $\epsilon_{J,K} = \pm 1$ is defined by

$$i^n d\zeta_{J^c} \wedge d\overline{\zeta}_{K^c} \wedge d\zeta_J \wedge d\overline{\zeta}_K = 2^n \epsilon_{J,K} \bigwedge_{k=1}^{n} (d\xi_k \wedge d\eta_k).$$

When $E \to \mathcal{X}$ is the trivial vector bundle $\mathcal{X} \times \mathbb{C} \to \mathcal{X}$, the notion of *positive current* plays a crucial role.

Definition A.3 (positivity concepts for currents). A current $T \in {}'\mathcal{D}^{(p,p)}(\mathcal{X}, \mathbb{C})$ is said to be *positive*[3] if and only if $\langle T, \varphi \rangle \geq 0$ for every $(n-p, n-p)$ form $\varphi \in \mathcal{D}^{(n-p,n-p)}(\mathcal{X})$ such that, for every z in $\mathcal{X}$, $\varphi(z)$ is *strongly positive* form in $[(T^{n-p,n-q}_{\mathcal{X},\mathbb{R}})^* \otimes_{\mathbb{R}} \mathbb{C}]_{\mathcal{X},z}$. The current is said to be *strongly positive* if this holds for all forms $\varphi \in \mathcal{D}^{(n-p,n-p)}(\mathcal{X})$ which are just *positive* at every point z of $\mathcal{X}$.

Remark A.3. If $T = \sum_{J,K} T^{J,K} dz_J \wedge d\overline{z}_K$ is a positive, (p,p)-current, then the measures $i^{p^2} T^{J,J}$ are positive measures (it suffices to test T on strongly positive forms $i^{(n-p)^2} dz_{J^c} \wedge d\overline{z}_{J^c}$). The complex measures $T^{J,K}$ satisfy $T^{K,J} = \overline{T^{J,K}}$ (by duality argument, since T is real on positive forms) and, if $\lambda_1, ..., \lambda_n$ are arbitrary positive or zero coefficients, the very important inequality between measures :

$$\text{(A.1)}\quad \lambda_J \lambda_K |T^{J,K}| \leq 2^{n-p} \sum_{K\cap J \subset M \subset K \cup J} \lambda_M^2 T^{M,M}\,, \quad \lambda_L := \prod_{l \in L} \lambda_l \text{ si } L \subset \{1,...,n\}$$

(for such inqualities, we refer for example to [**De0**] (chapter 3)).

The fact that any positive, closed, (p,p)-current T on complex analytic manifold has necessarily measure coefficients (see Remark A.3) implies a useful observation: it has order 0. If, in addition, it is closed ($dT = 0$), then it is identically equal to zero as soon as its support[4] is carried by a closed analytic subset of $\mathcal{X}$ of codimension strictly greater than p; this can be seen when testing the action of the current in a neighborhood of a regular point of its support (which we assume here to be of dimension strictly less then $n-p$). Every (p,p)-current of order 0 (i.e. with measure coefficients) over $\mathcal{X}$ such that dT is also of order 0 is called *normal* ; any (p,p) normal current supported by a closed analytic set with codimension strictly greater than p is necesseraly equal to 0.

Example A.1 (plurisubharmonic functions). If u is a locally integrable and locally *plurisubharmonic* function[5] in $\mathcal{X}$, then the current $dd^c u := (i/2\pi)\partial\overline{\partial} u$ is positive and closed. Furthermore, every positive, closed $(1,1)$-courant T over $\mathcal{X}$ can be expressed locally as $dd^c u$, where u is plurisubharmonic and locally integrable.

Example A.2. Positive, closed, $(1,1)$-currents (which are locally of the form $dd^c u$, with u being plurisubharmonic function, see the example A.1), can be multiplied following a procedure of the type *integration by parts*, always under the condition that the involved plurisubharmonic functions u_j, $j = 1, ..., p$ are locally bounded :

$$\text{(A.2)}\qquad dd^c u_p \wedge \cdots \wedge dd^c u_1 = dd^c \Big[u_p dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1 \Big].$$

[3]The remark in the definition A.1 ; *positive* means always here *non-negative*.

[4]The *support* of a current is the complement of the largest open over which the action of the currents is equal to zero.

[5]This means that the function u is upper semi-continuous, has values in $[-\infty, +\infty[$, and is subharmonic in the intersection of any complex line with its open domain of definition.

Note that the term in the right hand side of (A.2) makes sense (as a distribution); moreover, if T is a positive, closed current, then the same is true for the current $dd^c u \wedge T := dd^c[uT]$ when u is a locally bounded, plurisubharmonic function; this can be proved using regularization of u by convolution, which allows to replace u by a plurisubharmonic function $u^{(k)} = u * \rho_k$, where $(\rho_k)_k$ is some C^∞- approximation of the Dirac mass in $\mathbb{C}^n$; weak convergence of $dd^c[u^{(k)}T]$ to $dd^c[uT]$ shows that $dd^c[uT]$ is indeed a positive current. Thus, induction (A.2) allows to define positive, closed currents

$$dd^c u_p \wedge \cdots \wedge dd^c u_1$$

when the functions u_j, $j = 1, ..., p$, are locally bounded, plurisubharmonic functions.

Example A.3. [a geometric construction] It is important to get rid of the restriction for the plurisubharmonic functions u_j to be locally integrable in the inductive construction (A.2). One particularly important example is the case when $\mathcal{X}$ is the domain $U \subset \mathbb{C}^n$ and $T = dd^c[\log(|f_1|^2 + \cdots + |f_m|^2)]$, the functions $f_1, \ldots, f_m$ being holomorphic in U. The function $\log \|f\|^2 = \log(|f_1|^2 + \cdots + |f_m|^2)$ is almost everywere well defined in U; it is a plurisubharmonic function, which is also locally integrable. In fact, the singularities are of $\log \|f\|^2$ are logarithmic, hence integrable. For a justification of local integrability for $\log \|f\|^2$, we can invoke the *Lojasiewicz inequality*: if $f_1, ..., f_m$ are m holomorphic functions in a neighborhood of the origin in $\mathbb{C}^n$, defining the closed analytic subset A in this neighborhood, then there exists (see for example [**BoR, Lo, Tou**]) a minimal exponent $\alpha \in \mathbb{Q}^+$ such that, for every $\epsilon > 0$, the inequality

$$\sup_{j=1,\ldots,m} |f_j(\zeta)| \geq \kappa_\epsilon [d(\zeta, A)]^{\alpha+\epsilon}\,, \quad \kappa_\epsilon > 0\,, \tag{A.3}$$

holds in a neighborhood of the origin; combined with the local description of analytic subsets, as given in the *Nœther preparation theorem* (see Appendix D, Section D.1.4 and in particular Proposition D.1), this result implies the local integrability of the function $\log \|f\|$. We take the opportunity here to emphasize the role of *Lojasiewicz inequalities* in complex analytic geometry. This implies that $T = dd^c \log \|f\|^2$ is an example of a $(1,1)$ closed, positive current (see Example A.1). However, the method presented in Example A.2 for the construction of the successive wedge powers $T, T \wedge T, ...,$ of the current T cannot be carried out here, since $\log \|f\|^2$ is not anymore a locally bounded function. Nevertheless, it remains possible to proceed with integration by parts, as in Example A.2, in order to define $dd^c u_k \wedge \cdots \wedge dd^c u_1$ for $k = 1, ..., p$, as soon as the sets $\mathrm{Sing}(u_1), ..., \mathrm{Sing}(u_p)$ (that is the complements of the sets about which the functions u_j are locally bounded) are such that $\mathrm{Sing}(u_j) \subset A_j$, where A_j is a closed, analytic subset of $\mathcal{X}$, and

$$\forall\, k = 1, ..., p\,,\ \forall\, 1 \leq j_1 < \cdots < j_k \leq p\,,\ \mathrm{codim}\,(A_{j_1} \cap \cdots \cap A_{j_k}) \leq k\,. \tag{A.4}$$

For a proof of this difficult result (based for example on *Chern-Nirenberg inequalities*), we refer the reader to Chapter 4 in [**De0**]. If $T = dd^c \log[\|f\|^2]$, $f_1, ..., f_m$ being m holomorphic functions in a domain U of $\mathbb{C}^n$, the inductive method described in Example A.2 can be carried out (of one admits the mentioned result here) to define wedge powers of T, up to the order $p = \mathrm{codim}\,\{f_1 = \cdots = f_m = 0\}$; in fact, condition (A.4) remains then satisfied, since $\mathrm{Sing}(\log \|f\|^2) = \{f_1 = \cdots = f_m = 0\}$ and all sets A_j, $j = 1, ..., p$ are equal to $\{f_1 = \cdots = f_m = 0\}$ in this particular case. Note however that condition (A.4) cannot be invoked anymore for the construction of $T^{\wedge k}$ when $k > p = \mathrm{codim}\,(\{f_1 = \cdots = f_m = 0\})$.

A.1.3. Trivial extension of positive, closed currents. This section is devoted to the *theorem of extension of positive closed currents* due to H. El Mir [**ElM**].

Definition A.4. A subset of complex analytic manifold $\mathcal{X}$ is called *pluripolar* if it can be locally defined by $u^{-1}(\{-\infty\})$, where u is locally integrable, plurisubharmonic function.

The following example is among the most common to illustrate the above definition.

Example A.4. Every closed analytic subset A of a complex analytic manifold $\mathcal{X}$, defined locally as a set of common zeroes of m holomorphic functions $f_1, ..., f_m$, is pluripolar since the function

$$z \mapsto \log |f| := \frac{\log(|f_1|^2 + \cdots + |f_m|^2)}{2}$$

is locally integrable and plurisubharmonic (see Example A.3 above).

The important extension theorem by H. El Mir [**ElM**] asserts that pluripolar sets do not constitute an obstacle for the (trivial) extension of positive, closed, currents.

Theorem A.1 (El Mir extension theorem). *Let $\mathcal{X}$ be a complex analytic manifold of dimension n, E be a closed, pluripolar subset, T be a positive (hence , with measure coefficients), closed, (p,p)-current in the open set $\mathcal{X} \setminus E$. If T has a finite mass in a neighborhood of every point of a E (i.e. the sum of total masses of measure coefficients of $i^{p^2} T$ is locally finite about any point in E), then the positive current obtained by trivial extension of the measure coefficients of T (i.e. taking them with no mass on E), remains a positive, closed current.*

Remark A.4. If T is a positive, closed current over $\mathcal{X}$ and if $E \subset \mathcal{X}$ is a pluripolar, closed subset, then the current obtained by the trivial extension (defining it to be equal to 0) is the restriction of T to the open set $\mathcal{X} \setminus E$ and is denoted by $T \cdot \mathbf{1}_{\mathcal{X}\setminus E}$. The positive, closed current $T \cdot \mathbf{1}_E := T - T \cdot \mathbf{1}_{\mathcal{X}\setminus E}$ is thus a current supported by the pluripolar set E. The restriction operation of a positive, closed current over the manifold $\mathcal{X}$ to some closed pluripolar subset E, namely $T \mapsto T \cdot \mathbf{1}_E$, is of particular importance for integration currents as well as residual currents (which are neither positive nor in general closed anymore).

A.2. Lelong numbers of positive, closed, (p,p)-currents

Let $\mathcal{X}$ be a *Stein manifold*[6] . This amounts to assume that $\mathcal{X}$ can be written as an increasing union of sets $\{u_0 < N\}$, $N = 1, 2, ...$, which are relatively compact in X, u_0 being a strictly plurisubharmonic function. Since we will essentially deal here with local results and every complex manifold is locally Stein, such an hypothesis on $\mathcal{X}$ ($\mathcal{X}$ Stein) will not be restrictive for our purpose.

Let T be a positive, closed, (p,p)-current over $\mathcal{X}$ and $u : \mathcal{X} \longrightarrow [-\infty, \infty[$ be a plurisubharmonic, continuous function, which is semi-exhaustion over the support of T, i.e. there exists $R > 0$ such that

$$\operatorname{Supp} T \cap \{\zeta \in \mathcal{X}\, ;\, u(\zeta) < R\} \subset\subset \mathcal{X}. \tag{A.5}$$

[6] For a definition of Stein manifolds and a recap of fundamental properties of ideal sheaves in this setting (Cartan's theorems A and B), we refer to Appendix D.

PROPOSITION A.1. *If T and u below are fixed and R is such that (A.5) is valid, then the function*

$$r \in]-\infty, R[\longmapsto \nu(T,u,r) := \int_{\{z\in\mathcal{X}\,;\,u(z)<r\}} T \wedge (dd^c u)^{n-p}$$

is a positive, increasing function of r over $]-\infty, R[$. The limit of this function whenever r tends to $-\infty$ exists and is equal to

$$\nu(T,u) \ := \int_{\{u=-\infty\}} T \wedge (dd^c u)^{n-p}.$$

This positive limit is called the generalized Lelong number of T associated to the semi-exhaustion function u.

An interesting and typical example here is the following.

EXAMPLE A.5 (ordinary Lelong number). If x is a point of support of the current T, $\mathcal{X}$ a local Stein chart about x, $u(\zeta) := \log|\zeta - x|$ (in local coordinates about x), the Lelong number such defined is called *the Lelong number of the current T at the point x* and is denoted by $\nu_x(T)$. This is a strictly positive number(except in the case when T is the zero current in the neighborhood of est x). The Lelong number $\nu_x(T)$ at the point x is also realized as the limit, whenever r decreases to 0, of the quotient of *mass* of T inside the euclidian ball (in $\mathbb{C}^n_\zeta$) with center at x and of radius r

$$\frac{\pi^{n-p}}{(n-p)!}\int_{X\cap\{|\zeta-x|\le r\}} T\wedge (dd^c|\zeta-x|^2)^{n-p}$$

by the $(n-p)$-dimensional volume of this euclidian ball

$$\frac{\pi^{n-p}}{(n-p)!}r^{2(n-p)}.$$

That is,

$$\nu_x(T) := \lim_{r\to 0^+}\left(\frac{1}{r^{2(n-p)}}\int_{\mathcal{X}\cap\{|\zeta-x|\le r\}} T\wedge (dd^c|\zeta-x|^2)^{n-p}\right). \tag{A.6}$$

Another useful approach of the Lelong number $\nu_x(T)$ of a (p,p) positive closed current T over an n dimensional complex manifold $\mathcal{X}$ is the following:

$$\nu_x(T) = \lim_{\lambda\to 0^+}\Big(\overline{\partial}|\zeta-x|^{2\lambda}\wedge\frac{\partial\log|\zeta-x|^2}{2i\pi}\wedge(dd^c\log|\zeta-x|^2)^{p-1}\wedge T\Big). \tag{A.7}$$

The right-hand side of (A.7) is interpreted as the limit at $\lambda = 0$ of a current valued function of the complex parameter λ which happens to be holomorphic in $\{\operatorname{Re}\lambda > 0\}$. Formula (A.7) is obtained thanks to integration by parts: for any test function φ in a neighborhood of x, one has (the product operations being defined as in Example A.3)

$$
\begin{aligned}
&\nu_x(T) = \Big\langle T \wedge (dd^c \log|\zeta - x|^2)^{n-p}\,,\,\varphi\Big\rangle = \\
= \;&\lim_{\lambda\to 0^+} \Big\langle \frac{|\zeta|^{2\lambda}-1}{\lambda}\,T \wedge (dd^c \log|\zeta - x|^2)^{n-p-1}\,,\, dd^c\varphi\Big\rangle \\
= \;&\lim_{\lambda\to 0^+} \Big\langle \overline{\partial}|\zeta - x|^{2\lambda} \wedge \frac{\partial \log|\zeta - x|^2}{2i\pi} \wedge (dd^c \log|\zeta - x|^2)^{n-p-1} \wedge T\,,\,\varphi\Big\rangle \\
&\quad + \lim_{\lambda\to 0^+} \Big\langle |\zeta - x|^{2\lambda}\, T \wedge (dd^c \log|\zeta - x|^2)^{n-p}\,,\,\varphi\Big\rangle \\
= \;&\lim_{\lambda\to 0^+} \Big\langle \overline{\partial}|\zeta - x|^{2\lambda} \wedge \frac{\partial \log|\zeta - x|^2}{2i\pi} \wedge (dd^c \log|\zeta - x|^2)^{n-p-1} \wedge T\,,\,\varphi\Big\rangle \\
&\quad + \Big\langle \mathbf{1}_{\mathbb{C}^n\setminus\{x\}} \cdot [T \wedge (dd^c \log|\zeta - x|^2)^{n-p}]\,,\,\varphi\Big\rangle \\
= \;&\lim_{\lambda\to 0^+} \Big\langle \overline{\partial}|\zeta - x|^{2\lambda} \wedge \frac{\partial \log|\zeta - x|^2}{2i\pi} \wedge (dd^c \log|\zeta - x|^2)^{n-p-1} \wedge T\,,\,\varphi\Big\rangle + 0\,.
\end{aligned}
$$

A.3. Integration currents on analytic subsets

A.3.1. Construction and elementary properties. If A is an irreducible analytic subset of dimension $n-p$ ($1 \leq p \leq n$) of a n-dimensional complex manifold $\mathcal{X}$, it follows from the local "*à la Nœther*" representation of analytic subsets (see Appendix D, Section D.1.4, in particular Proposition D.1), that the positive (p,p) current defined over $X \setminus A_{\rm sing}$ by

$$
\langle [A]_{\rm red}, \varphi\rangle := \int_A \varphi = \int_{A_{\rm reg}} \varphi\,,\ \varphi \in \mathcal{D}^{(n-p,n-p)}(\mathcal{X} \setminus A_{\rm sing})
$$

has a finite mass in a neighborhood of every point of $A_{\rm sing}$. This current (defined in the open set $\mathcal{X} \setminus A_{\rm sing}$) is closed over this open set and the analytic subset $A_{\rm sing}$ is pluripolar (example A.4).

Definition A.5 (integration current on an a closed analytic subset). If A denotes an irreducible analytic subset of dimension $n-p$ of an analytic complex manifold of dimension n, the *integration current* $[A]$ (or $[A]_{\rm red}$, in order to emphasize the fact that it depends on the irreducible, hence reduced, cycle $[A]$) on the set A is the (p,p) current defined through the trivial extension (in the sense of the theorem A.1) of the closed positive (p,p) current on $\mathcal{X} \setminus A$:

$$
\varphi \in \mathcal{D}^{(n-p,n-p)}(\mathcal{X} \setminus A) \mapsto \int_{\mathcal{X}\setminus A_{\rm sing}} \varphi.
$$

The integration current $[A]$ is positive and closed[7] over $\mathcal{X}$, with support the irreducible analytic subset A.

If C denotes the $(n-p,n-p)$-cycle

$$
C = \sum_\gamma m_\gamma C_\gamma
$$

[7]It is the same here to say ∂ and $\overline{\partial}$-closed because the coefficients of $i^{(n-p)^2}T$ verify $T^{J,K} = \overline{T^{K,J}}$ (see Remark A.3).

(where all C_γ are irreducible cycles), the *current of integration* $[C]$ *over the cycle* C is by definition the closed (p,p)-current (not positive anymore, but always with measures coefficients) defined by

$$\langle [C], \varphi\rangle := \sum_\gamma m_\gamma [C_\gamma].$$

Note here that if A is an analytic subset of $\mathcal{X}$, the current $[C]\cdot \mathbf{1}_A$ (see Remark A.4) is the current

$$[C^A] = [C]\cdot \mathbf{1}_A = \sum_{\{\gamma\,;\,C_\gamma\subset A\}} m_\gamma [C_\gamma].$$

Similarly

$$[C]\cdot \mathbf{1}_{\mathcal{X}\setminus A} = \sum_{\{\gamma\,;\,C_\gamma\not\subset A\}} m_\gamma [C_\gamma].$$

We remark here that this operation of *cutting out* is not such easy to describe from the algebraic point of view. If C is an effective cycle, $\mathcal{I}(C)$ being the associated coherent sheaf of ideals, and if $\mathcal{A}$ is a coherent sheaf of ideals such that the quotient sheaf $\mathcal{O}_\mathcal{X}/\mathcal{A}$ has A for support, the coherent sheaf $\mathcal{I}(C^{\mathcal{X}\setminus A})$ attached to the cycle

$$C^{\mathcal{X}\setminus A} := \sum_{\{\gamma\,;\,C_\gamma\not\subset A\}} m_\gamma C_\gamma$$

is

$$\mathcal{I}(C^{\mathcal{X}\setminus A}) = \bigcup_{k=1}^{\infty} [\mathcal{I}(C) \ : \mathcal{A}^k]\,, \tag{A.8}$$

where $[(I(C))_x : \mathcal{A}_x^k]$ denotes, for every $x\in\mathcal{X}$, the ideal $\mathcal{O}_{\mathcal{X},x}$ (called *carrier* of $\mathcal{A}_x^k$ inside $(\mathcal{I}(C))_x$) consists of elements $h\in\mathcal{O}_{\mathcal{X},x}$ such that $h\mathcal{A}_x^k\subset (I(C))_x$. Note that (A.8) involves an asymptotic formulation. This *cutting out* operation appeals to the important notion of *gap sheaf* (see e.g. [**Mass**], Chapter 1) in algebraic geometry.

Remark A.5 (currents on reduced analytic spaces). The definition A.5 fits naturally in the setting where $\mathcal{X}$ is replaced with a reduced, complex, analytic space (see Appendix D, Section D.4) We may assume that $\mathcal{X}$ is irreducible with complex dimension n. Locally (in a neighborhood of a point $x\in\mathcal{X}$), we can consider $\mathcal{X}$ to be an analytic subset A_x of dimension n of an open set U_x in $\mathbb{C}^{N_x}$ $(N_x\geq n)$. If Y is an irreducible closed analytic subset of $\mathcal{X}$ of dimension $n-p$ and $y\in Y$, the subset $Y\cap U_y$ can be considered as a closed analytic subset of A_y, thus also a closed analytic subset (with complex dimension $n-p$) of the open set U_y in $\mathbb{C}^{N_y}$. Hence, one can define in the open set U_y (and therefore globally) the integration current $[Y]=[Y]_{\rm red}$ over the analytic subset $Y\cap U_y$. A $(n-p,n-p)$- smooth differential form on $\mathcal{X}$ (expressed in the open chart U_x) is (by definition) an element of the quotient of the space of $(n-p,n-p)$-forms defined in the neighborhood of A_x inside U_x by the subspace of $(n-p,n-p)$-forms defined in a neighborhood of A_x inside U_x and identically zero over the complex analytic manifold $A_{x,\rm reg}$ (one should speak rigorously about germs of differential forms on A). Thus, the integration current $[Y]=[Y]_{\rm red}$ over $Y\subset\mathcal{X}$ introduced above defines an element of the dual space of the space of $(n-p,n-p)$-forms over $\mathcal{X}$, with compact support in $\mathcal{X}$. It is, in some sense which is easy to make precise, a closed positive current on $\mathcal{X}$. More generally, one may define (p,q)-currents on the complex (reduced) analytic space $(V,\mathcal{O}_V)$ (see Appendix D, Definition D.10) as elements of the dual of the space of

$(n-p, n-q)$ smooth differential forms with compact support on $(V, \mathcal{O}_V)$, in the sense precised above (in the particular case $p=q$).

The integration current is the fundamental building block for positive closed currents. This will become profound in the last subsection A.4 together with the formulation of the important *stratification theorem* of Y.T. Siu ([**Siu**], see Theorem A.5 in this Appendix), which we will comment and exploit. At this point we will introduce some first results in this direction. Recall that (this is a theorem due to Thie) that the Lelong number of the integration current $[A]$ over an analytic, closed, irreducible set A coincides with the number μ of *branches* in the Nœther representation of the closed, analytic subset A (see Appendix D, Section D.1.4): it is precisely the number μ which is approached (in the case $T=[A]$) as the limit of the quotients (A.6). Then, for $x \in A$, the Lelong number $\nu_x([A])$ is always a positive integer. When $p=1$ and $A=\{f=0\}$ with f being irreducible in $\mathcal{O}_{\mathcal{X},x}$, this number is equal to the multiplicity $\mu_x(f)$, that is to the valuation at $\zeta=0$ of $\zeta \mapsto f(x+\zeta)$.

If T is a positive, closed, (p,p)-current over X and A is an analytic subset of X, then one can put

$$\nu_A(T) := \inf\{\nu_x(T)\,;\, x \in A\}. \tag{A.9}$$

The following proposition will be of use to us, because we will study the integration currents through their approximations (see Section A.3.2 below).

Proposition A.2. *Let T be a positive, closed, (p,p)-current and $A \subset X$ be an irreducible, analytic subset of dimension $n-p$. Then, one has the equality*

$$T \cdot \mathbf{1}_A = \nu_A(T)[A] \tag{A.10}$$

and, as a consequence, the inequality $T \geq \nu_A(T)[A]$ holds. In particular, if the support of T is an analytic set of pure dimension $n-p$ and the irreducible components of A are denoted as A_γ, then

$$T = \sum_\gamma \nu_{A_\gamma}(T)\,[A_\gamma]. \tag{A.11}$$

A.3.2. Approximations of integration currents over analytic subsets. A closed analytic subset A of a complex analytic manifold is locally defined as the set of common solutions of a system of analytic equations $s_j=0$. In fact, this description of analytic subsets is more *algebraic* than purely *geometric* (since it involves the s_j). Therefore, it is very important to be able to express the integration current on such a closed analytic set A, that is when $A=V(s)$ is the zero set of a holomorphic section s of some holomorphic vector bundle of rank m, $E \longrightarrow \mathcal{X}$.

The algebraic information about s is then carried by the coherent sheaf of ideals $\mathcal{J}[s]$ defined as follows. Let

$$\rfloor_s\ : E^* = \bigwedge^1 E^* \longmapsto \bigwedge^0 E := X \times \mathbb{C}$$

be the interior product induced by s, that is, when expressed in a holomorphic frame $(e_1, ..., e_m)$ for $E \to \mathcal{X}$ (above some local chart U),

$$\rfloor_s[(\xi_1 e_1^*(\zeta) + \cdots + \xi_m e_m^*(\zeta)] := \sum_{j=1}^m \xi_j \sigma_j(\zeta)$$

whenever

$$s(z) := \sum_{j=1}^{m} \sigma_j(\zeta) e_j(\zeta)\,,\ \forall\, \zeta \in U\,.$$

The ideal $(\mathcal{J}[s])_x$ is defined as

$$(\mathcal{J}(s))_x = \rfloor_s \Big[\mathcal{O}_{\mathcal{X},x}(\mathcal{X}, E^*)\Big]\,. \tag{A.12}$$

Thus $V(s)$ is the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{J}[s]$, though $\mathcal{J}(s)$ does not necesseraly define $V(s)$ as a reduced closed analytic subset of $\mathcal{X}$. Additionally, we equip the vector bundle $E \to \mathcal{X}$ with an hermitian metric $|\ |$.

Above the open set $\mathcal{X} \setminus V(s)$, we denote as $E_{\mathcal{X}\setminus V(s)} \to \mathcal{X} \setminus V(s)$ the restriction vector bundle (equipped with the hermitian structure induced by the hermitian structure $|\ |$) and as $L_s \to \mathcal{X} \setminus V(s)$ the holomorphic line bundle whose fiber above the point $x \in \mathcal{X}\setminus V(s)$ is the complex line in E_x generated by $s(x)$. This line bundle $L_s \to \mathcal{X} \setminus V(s)$ inherits also the hermitian structure $|\ |$ and so does the quotient bundle

$$\frac{E_{|\mathcal{X}\setminus V(s)}}{L_s} \longrightarrow \mathcal{X} \setminus V(s).$$

We also introduce the total Chern form (see the definition in Appendix B)

$$C\Big(E_{|\mathcal{X}\setminus V(s)}/L_s, |\ |\Big) = \sum_{k\geq 0} c_k\Big(E_{|\mathcal{X}\setminus V(s)}/L_s, |\ |\Big)$$

of the quotient bundle

$$\frac{E_{|\mathcal{X}\setminus V(s)}}{L_s} \longrightarrow \mathcal{X} \setminus V(s)$$

for the Chern connection induced by the hermitian structure $|\ |$. This Chern form is a C^∞ closed differential form in the open set $\mathcal{X} \setminus V(s)$, which extends[8] to the whole of $\mathcal{X}$ as a closed differentiable form with locally integrable coefficients still denoted as $C\Big(E_{|\mathcal{X}\setminus V(s)}/L_s, |\ |\Big)$.

We state in this geometric context a useful result obtained by J.R. King in 1970 ([**King, Meo2, And3**]). Such a result implies the holonomy properties that will presented in the next subsection A.3.3.

Theorem A.2 (approximation of the integration current via analytic continuation). *Let s be a holomorphic section of the holomorphic hermitian vector bundle $(E \to \mathcal{X}, |\ |)$ of rank m above a complex analytic manifold $\mathcal{X}$ and $\mathcal{J}[s]$ be the corresponding $\mathcal{O}_X$-coherent sheaf (defined as* (A.12)*). Let $(V_\gamma(s))_\gamma$ the irreducible components of $V(s) = s^{-1}(0)$ and $\mu_{s,\gamma}$ the Hilbert-Samuel multiplicity* [9] *of $(\mathcal{J}[s])_{x_\gamma}$ at a generic point x_γ on $V(s)$. Let $C(\mathcal{J}[s]) = \sum_\gamma \mu_{s,\gamma}[V_\gamma]_{\rm red}$ be the cycle associated to $\mathcal{J}[s]$. If p denotes the codimension of $V(s)$, then the function*

$$\lambda \in \{\mathrm{Re}\,\lambda >> 0\} \longmapsto \overline{\partial}|s|^{2\lambda} \wedge \frac{\partial \log |s|^2}{2i\pi} \wedge c_{p-1}\Big(E_{|\mathcal{X}\setminus V(s)}/L_s, |\ |\Big) \in \mathcal{D}^{'(p,p)}(\mathcal{X}) \tag{A.13}$$

has a meromophic continuation over the whole complex plane. This continuation is holomorphic in a half plane $\{\mathrm{Re}\,\lambda > -\eta\}$ for some $\eta > 0$ and its value at $\lambda = 0$ is exactly the integration current $[C(\mathcal{J}[s])_{n-p}]$ attached to the purely $n-p$ dimensional

[8] This fact follows from the proof of theorem A.2 below.

[9] See Section A.3.4 below in this Appendix A.

cycle $C(\mathcal{J}[s])_{n-p}$ *corresponding to the component of dimension* $n-p$ *of the cycle* $C(\mathcal{J}[s])$. *Furthermore, if* φ *is any* $(n-p, n-p)$ *test form, then*
(A.14)
$$\Big\langle [C(\mathcal{J}[s])_{n-p}], \varphi \Big\rangle = \lim_{\epsilon \to 0^+} \left(\frac{\epsilon\, p}{2i\pi} \int_{\mathcal{X}} \frac{\overline{\partial}|s|^2 \wedge \partial |s|^2}{|s|^2(|s|^2+\epsilon)^{p+1}} \wedge c_{p-1}\Big(E_{|\mathcal{X}\setminus V(s)}/L_s, |\ |\Big) \wedge \varphi \right).$$

Remark A.6. When $E \to \mathcal{X}$ is the trivial bundle $\mathcal{X} \times \mathbb{C}^m \to \mathcal{X}$, one can replace

$$c_{p-1}\Big(E_{|\mathcal{X}\setminus V(s)}/L_s, |\ |\Big)$$

with $(dd^c[\log |s|^2])^{p-1}$. This was in fact the way the result was formulated (in this particular setting) by J. R. King in [**King**] : one has

$$[C(\mathcal{J}[s])_{n-p}] = \mathbf{1}_{V(s)} \cdot (dd^c \log |s|^2)^p\,, \tag{A.15}$$

where the wedge product on the right-hand side is defined thanks to the iterative procedure introduced in Example A.3 ; the multiplication by $\mathbf{1}_{V(s)}$ was defined in Section A.1.3 (see Remark A.4).

A.3.3. Some operational consequences of regular holonomicity. One (indirect) important consequence of Theorem A.2 is a result, due J.E. Björk [**Bj3, Bj4**], concerning *distribution coefficients* (here positive measures) of the integration current over an analytic set: they are *regular holonomic*, in some sense precisely introduced in [**Bj3, Bj4**]. We will need here one operational aspect of the property inherited from *regular holonomy* and mention it here, without entering the theory of $\mathcal{D}_{\mathcal{X}}$-modules ; it is indeed a fact of crucial importance (see [**Bj2**]).

Proposition A.3 (holonomy of the distribution coefficients for the integration current). *Let* A *be an analytic irreducible subset of codimension* p *in an open set* U *of* $\mathbb{C}^n$ *and* τ_A *be one of he measure coefficients of the integration current over the analytic set* A. *Let* $(h_1, ..., h_l)$ *be a tuple of holomorphic functions in* U, *not all of them identically equal to zero on the support of* τ_A, *and* $|h|^2 := |h_1|^2 + \cdots + |h_l|^2$. *For every* $z \in U$, *the exists a functional equation (called a Bernstein-Sato equation)*

$$b_z(\lambda)[|h|^{2\lambda} \otimes \tau_A] = \mathcal{Q}_z(\lambda, \zeta, \overline{\zeta}, \partial/\partial\zeta, \partial/\partial\overline{\zeta})[|h|^{2(\lambda+1)} \otimes \tau_A]\,, \tag{A.16}$$

where b_z *denotes one polynomial, all of whose roots are strictly negative, rational numbers, and* $\mathcal{Q}_z$ *is a germ of differential operator with analytic coefficients, depending polynomially on a parameter* λ. *The polynomial* b_z, *which can be chosen to be minimal, is called a Bernstein-Sato polynomial.*

Remark A.7. If $l = 1$, instead of (A.16), one has a formal *holomorphic identity*

$$b_z(\lambda)[h^\lambda \otimes \tau_A] = \mathcal{Q}_z(\lambda, \zeta, \partial/\partial\zeta)[h^{\lambda+1} \otimes \tau_A]\,, \tag{A.17}$$

or even a true identity (in the sense of distributions)

$$b_z(\lambda)\Big[\frac{|h|^{2\lambda}}{h} \otimes \tau_A\Big] = \overline{\mathcal{Q}_z}(\lambda, \overline{\zeta}, \partial/\partial\overline{\zeta})\Big[|h|^{2\lambda}\frac{\overline{h}}{h} \otimes \tau_A\Big]. \tag{A.18}$$

Here b_z is a monic polynomial with strictly negative rational roots. Such identities reveal to be very useful to perform the integration by parts (for example, when extending Proposition A.4 below to the restricted case, see the end of this subsection).

As a consequence of Proposition A.3, we uncover the following result : if $[A]$ is an integration current over an irreducible, analytic subset A of $\mathcal{X}$ and if W is an analytic subset of A, whose dimension is strictly smaller than the dimension of A, then one can define the currents

$$\begin{aligned} \mathbf{1}_W \cdot [A] &= \Big[(1-|s|^{2\lambda})\,[A]\Big]_{|\lambda=0} \\ \mathbf{1}_{\mathcal{X}\setminus W} \cdot [A] &= \Big[|s|^{2\lambda}\,[A]\Big]_{|\lambda=0}, \end{aligned}$$

where s is a section of hermitian vector bundle $E \to \mathcal{X}$ such that $V(s) = W$. We remark that the current $\mathbf{1}_W \cdot [A]$ is always the zero current, which really means that the integration current $[A]$ over an irreducible, analytic subset A has the *standard extension property* (S. E. P.) *respect to its support* A. This property simply states the fact (which is evident here) that the current $[A]$ does not split its mass over none of the proper analytic subset of its support (for example: the subset $A_{\rm sing}$). This can be rephrased as follows: $[A]$ is the *standard extension* of its restriction over each open set $\mathcal{X} \setminus W$, where W is a proper analytic subset of A.

The holonomy property of distribution coefficients of an integration current over some analytic subset (Proposition A.3) allows to extend the following result.

Proposition A.4. *Let $\mathcal{X}$ be an analytic,complex manifold of dimension n, $H \subset \mathcal{X}$ be a closed hypersurface, f be an element of $\mathcal{M}_{\mathcal{X}}(\mathcal{X})$, whose polar set is included in the hypersurface H. The holomorphic function $f_{|\mathcal{X}\setminus H} \in \mathcal{O}_{\mathcal{X}}(\mathcal{X} \setminus H)$ extends as a distribution to the whole complex manifold $\mathcal{X}$. Conversely, if H is an hypersurface of an analytic, complex manifold of dimension n, and f is an element of $\mathcal{O}_{\mathcal{X}}(\mathcal{X} \setminus H)$ which extends as a distribution over the whole manifold $\mathcal{X}$, then f extends as a global section of the sheaf $\mathcal{M}_{\mathcal{X}}$ over $\mathcal{X}$ with polar set included in H.*

Remark A.8 (meromorphic forms and currents). Proposition A.4 extends to the restricted case as follows. If A is an irreducible, analytic subset of $\mathcal{X}$ of codimension p, and ω is a $(k,0)$-meromorphic differential form on A, $(k \le n-p)$, which is holomorphic in $A \setminus H_A$, where H_A is a hypersurface of A containing the set $A_{\rm sing}$, then the $(p+k,p)$ current $(\omega \wedge [A])_{|\mathcal{X}\setminus H_A}$ (which is $\overline{\partial}$-closed in $\mathcal{X} \setminus H_A$) has a standard continuation as a current in the ambient manifold $\mathcal{X}$. If $H_A := \{h_A = 0\}$, this $(p+k,p)$-courant $\omega \wedge [A]$ over the whole manifold $\mathcal{X}$ is the weak limit, when ϵ tends to 0^+, of $(\chi_{\{|h_A|\ge\epsilon\}}\omega \wedge [A])_{\epsilon>0}$. This follows from the regular holonomy of the distribution coefficients τ_A of the integration current $[A]$ (see Remark A.7). When the standard extension $\omega \wedge [A]$ satisfies $\overline{\partial}[\omega \wedge [A]] = 0$ in the whole ambient manifold $\mathcal{X}$, ω is called a *holomorphic form on A in the sense of Barlet* [**Ba1**]. We can also formulate the converse assertion : if ω is a holomorphic $(k,0)$-form on the complex manifold $A \setminus H_A$, such that $\omega \wedge [A]$ (defined as a $\overline{\partial}$-closed current on the open set $\mathcal{X} \setminus H_A$) extends as a current to the whole manifold $\mathcal{X}$, then the form ω extends as a meromorphic $(k,0)$-form on A (i.e. a meromorphic form in a neighborhood of A in the ambient manifold $\mathcal{X}$).

A.3.4. Hilbert-Samuel multiplicities and integration currents. Let $\mathcal{X}$ be a complex, analytic manifold of dimension n and $\mathcal{I}$ be a coherent sheaf of ideals of the sheaf of the structure sheaf $\mathcal{O}_{\mathcal{X}}$. If x is a point of the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$, let $\mathfrak{M}_{\mathcal{I},x}$ be the maximal ideal of the local ring $\mathbf{R}_{\mathcal{I},x} = \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_x$. Given such a point

x, one can consider the graded algebra

$$\mathrm{Grad}_{\mathfrak{M}_{\mathcal{I},x}}(\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_x) := \bigoplus_{k=0}^{\infty} \frac{\mathfrak{M}^k_{\mathcal{I},x}}{\mathfrak{M}^{k+1}_{\mathcal{I},x}}.$$

Assume that the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ is of pure dimension $n-p$ in a neighborhood of the point z. Then $n-p$ is also *the Krull dimension* of the local ring $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_x$.

Definition A.6 (Hilbert function). The Hilbert function of $(\mathbf{R}_{\mathcal{I},x},\mathfrak{M}_{\mathcal{I},x})$ is by definition the function from $\mathbb{N}$ into $\mathbb{N}$, which associates to d the integer $\dim_{\mathbf{R}_{\mathcal{I},x}/\mathfrak{M}_{\mathcal{I},x}}(\mathbf{R}_{\mathcal{I},x}/\mathfrak{M}^d_{\mathcal{I},x})$.

An argument from commutative algebra (for example see [**Ha1**], chapter 1 or [**JP**]) ensures that for d sufficiently large, the Hilbert function of $(\mathbf{R}_{\mathcal{I},x},\mathfrak{M}_{\mathcal{I},x})$ takes the same value in d as does a polynomial with rational coefficients of degree $n-p\le n$ (called *the Hilbert-Samuel polynomial* of $(\mathbf{R}_{\mathcal{I},x},\mathfrak{M}_{\mathcal{I},x})$) :

$$\Xi \mapsto \mathrm{HP}_{(\mathbf{R}_{\mathcal{I},x},\mathfrak{M}_{\mathcal{I},x})}(\Xi) = \sum_{k=0}^{n-p-1} h_k \Xi^k + \frac{\mu_x(\mathcal{I})}{(n-p)!}\Xi^{n-p}\,,$$

where $\mu_x(\mathcal{I})$ is a positive integer, called *Hilbert-Samuel multiplicity* of $\mathcal{I}$ at the point x where $\mathbf{R}_{\mathcal{I},x}$ is of dimension $n-p$. Note that for $d\ge 1$,

$$\dim_{\mathbf{R}_{\mathcal{I},x}/\mathfrak{M}_{\mathcal{I},x}}(\mathbf{R}_{\mathcal{I},x}/\mathfrak{M}^d_{\mathcal{I},x}) = \sum_{k=0}^{d-1} \dim_{\mathbf{R}_{\mathcal{I},x}/\mathfrak{M}_{\mathcal{I},x}}(\mathfrak{M}^k_{\mathcal{I},x}/\mathfrak{M}^{k+1}_{\mathcal{I},x}).$$

The key result concerning the Hilbert-Samuel multiplicity at the point x of the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ is that, by construction, it does not depend really depend on $\mathcal{I}$, but on the *integral closure* $\overline{\mathcal{I}}_x$ (see [**LeT**], proposition 1.18) of $\mathcal{I}_x$ in $\mathcal{O}_{\mathcal{X},x}$ (for the notion of integral closure of a coherent ideal sheaf, see Appendix D).

The *Lelong number* $\nu_x([A])$ of the integration current $[A]=[A]_{\mathrm{red}}$ over an irreducible analytic subset is interpreted as the *Hilbert-Samuel multiplicity* $\mu_x(\mathcal{I}_A)$ at the point x of $\mathcal{I}_{A,x}$, where $\mathcal{I}_A$ denotes the sheaf of ideals attached to the analytic set A. This relates a rather simple geometric interpretation (the Lelong number being understood as a limit of quotients of volumes, see (A.6)) to an algebraic definition which involves also some asymptotic formulation (that of the Hilbert-Samuel multiplicity above). The essence for this ubiquity is the key formula in $\mathbb{C}^n$:

$$dd^c \log\|\zeta\|^2 = \int_{\alpha\in\mathbb{P}^n(\mathbb{C})} [\langle \alpha,\zeta\rangle = 0]\, d\sigma(\alpha)\,, \tag{A.19}$$

where $d\sigma$ denotes the normalized Fubini-Study metric $dd^c(\log[|z_0|^2+\cdots+|z_n|^2])$ in $\mathbb{P}^n(\mathbb{C})$. It reflects in the following proposition (which is an application of (A.7) above to the particular case $T=[A]$).

Proposition A.5. *Let A be an irreducible, analytic set of dimension $n-p$ of $\mathcal{X}$ and $\zeta=(\zeta_1,\dots,\zeta_n)$ be the generators of the maximal ideal $\mathfrak{M}_{\mathcal{X},z}$ in $\mathcal{O}_{\mathcal{X},x}$ in the neighborhood U_x of the point x. The function*

$$\lambda\in\{\mathrm{Re}\,\lambda >> 1\} \longmapsto \overline{\partial}|\zeta|^{2\lambda}\wedge\frac{\partial[\log|\zeta|^2]}{2i\pi}\wedge(dd^c[\log|\zeta|^2])^{n-p-1}\wedge[A]_{\mathrm{red}} \in {}'\mathcal{D}^{(n,n)}(U_x)$$

extends as a meromorphic function in $\mathbb{C}$, which is holomorphic in a neighborhood of the origin and with value at $\lambda = 0$ given by

$$\begin{aligned}&\left[\overline{\partial}|\zeta|^{2\lambda} \wedge \frac{\partial[\log|\zeta|^2]}{2i\pi} \wedge (dd^c[\log|\zeta|^2])^{n-p-1} \wedge [A]_{\mathrm{red}}\right]_{\lambda=0}\\ = \;& \mathbf{1}_A \cdot \int_{\alpha\in(\mathbb{P}^n)^{n-p}} [\langle\alpha_1,\zeta\rangle = 0] \wedge \cdots \wedge [\langle\alpha_{n-p},\zeta\rangle = 0]\, d\sigma(\alpha_1) \otimes \cdots \otimes d\sigma(\alpha_{n-p})\\ = \;& \nu_x([A]) \times [x] = \mu_x(\mathcal{I}_A) \times [x].\end{aligned}$$

A.4. Siu stratification

Integration currents are the primary tools in order to express positive, closed currents. The following major result is due to Y.T.Siu [**Siu**].

Theorem A.3 (Siu analyticity theorem). *Let T be a positive, closed (p,p)-current over an analytic, complex manifold $\mathcal{X}$. The level subsets*

$$E_c(T) := \{x \in \mathcal{X}\,;\, \nu_x(T) \geq c\}\,,\ c > 0,$$

are analytic subsets of $\mathcal{X}$ and have a codimension at least equal to p.

The above theorem, has been generalized by J.P. Demailly in 1987 [**De1**] :

Theorem A.4. *Let $\mathcal{X}$ be a Stein manifold , $\mathcal{Y}$ be a complex analytic manifold, $u\ : \mathcal{X} \times \mathcal{Y} \longmapsto [-\infty,+\infty[$ be a plurisubharmonic, continuous function, which in addition is locally Hölder in y with exponent in $]0,1]$. If T is a positive, closed current over $\mathcal{X}$ such that u be semi-exhaustive*[10] *over the support of T, the level subsets*

$$E_c(T,u) := \{y \in \mathcal{Y}\,;\, \nu(T,u(\cdot,y)) \geq c\}\,,\ c > 0,$$

are analytic subsets of $\mathcal{Y}$.

Theorem A.3, combined with Proposition A.2, implies the following result, which could be called the *stratification theorem for positive, closed currents.*

Theorem A.5 (Siu's stratification theorem). *Let T be a positive, closed (p,p)-current over a complex, analytic manifold $\mathcal{X}$. Then, the current T is expressed as a weak limit*

$$\lim_{M\to+\infty}\Big(\sum_{j=1}^{M}\lambda_j[A_j]\Big) + N\,, \tag{A.20}$$

where the A_j, $j \in \mathbb{N}^*$, are irreducible, analytic subsets of $\mathcal{X}$ of dimension $n-p$, the constants λ_j, $j \in \mathbb{N}^*$, are strictly positive numbers, and N is a positive, closed current, which can be considered as negligible, since for every $c > 0$, the level subsets $E_c(N)$ are of codimension strictly greater that p. Furthermore, any such stratification is unique (up to the labelling of the A_j). The part $T - N$ of the decomposition is called singular part of the positive current T.

[10] Recall here the terminology used in Section A.2: the fact that u is *semi-exhaustive* over the support of T means that for every compact K of $\mathcal{Y}$, there exists $R(K)$ such that

$$\{(x,y) \in \operatorname{Supp} T \times K\,;\, \varphi(x,y) \leq R(K)\}$$

is relatively compact in $\mathcal{X} \times \mathcal{Y}$.

APPENDIX B

Hermitian bundles

In the present appendix we will present the necessary material on Hermitian Geometry, whose aim is to make our monograph as self-contained as possible. The main reference (and inspiration) is Chapter 5 in the book by J.P. Demailly [**De0**]. An alternative, but equally valuable source is the book by R.O. Wells [**We0**].

B.1. Differentiable manifolds and real or complex vector bundles

B.1.1. Smooth real differentiable manifolds. A smooth *real differentiable manifold* of real dimension N consists in the following data :

(1) a separable topological space $\mathcal{X}$, which is countable at infinity (i.e. it is an increasing countable union of compact sets) ;
(2) an atlas (U_α, τ_α) of open *charts* U_α in $\mathcal{X}$, which cover $\mathcal{X}$, and where τ_α is, for every α, an homeomorphism between U_α and an open set V_α in $\mathbb{R}^N$, such that, for every pair of indices (α, β), $\tau_{\alpha\beta} = \tau_\alpha \circ (\tau_\beta)^{-1}$ is a C^∞-diffeomorphism between $\tau_\beta(V_\alpha \cap V_\beta)$ and $\tau_\alpha(V_\alpha \cap V_\beta)$.

The *real tangent space* $T_{\mathbb{R},x}(\mathcal{X})$ can be interpreted from three different points of view ; we will see later on how this reflects on the interpretation of real vector fields.

A) The first model is a *geometric model* : if U_α is an open chart containing x, one introduces the *germs of* C^1 *curves* $t \in I =]-\epsilon, \epsilon[\mapsto \gamma(t) \in U_\alpha$ lying in $\mathcal{X}$ in a neighborhood of x, passing through x ($\gamma(0) = x$ and $\tau_\alpha \circ \gamma$ is of class C^1) ; elements in $T_{\mathbb{R},x}(\mathcal{X})$ are then *tangent classes* at x of such germs (the *tangent class* of the class of (I, γ) being realized as $d_0[\tau_\alpha \circ \gamma](1)$).
B) The second model is an *algebraic model* ; it consists in interpreting the $\mathbb{R}$-vector space $T_{\mathbb{R},x}(\mathcal{X})$ as the $\mathbb{R}$-vector space of *real derivations* of the $\mathbb{R}$-algebra $\mathcal{E}_{\mathcal{X},x}$ of germs of smooth real valued functions at the point x, a *real derivation* being a real $\mathbb{R}$-linear map from $\mathcal{E}_{\mathcal{X},x}$ into itself that satisfies the Leibniz rule $D[fg] = f(a)D[g] + g(a)D[f]$; in this frame-work, every element of $T_{\mathbb{R},x}(\mathcal{X})$ is described in terms of its action as a real derivation.
C) The third point of view is an even more *algebraic model* : first we still introduce the $\mathbb{R}$-algebra $\mathcal{E}_{\mathcal{X},x}$ (viewed this time as a local ring) and its maximal ideal $\mathfrak{M}_{\mathcal{X},x}$; then one interprets the elements of $T_{\mathbb{R},x}(\mathcal{X})$ as the elements of the dual space $(\mathfrak{M}_{\mathcal{X},x}/(\mathfrak{M}_{\mathcal{X},x})^2)^*$; an element in $T_{\mathbb{R},x}(\mathcal{X})$ is then considered through its action on the $\mathbb{R}$-vector space $\mathfrak{M}_{\mathcal{X},x}/(\mathfrak{M}_{\mathcal{X},x})^2$ as a $\mathbb{R}$-linear form.

B.1.2. Real or complex, locally trivial vector bundle of finite rank. Let $\mathcal{X}$ be a smooth real manifold of dimension n. A *real, locally trivial, vector bundle of rank* m over $\mathcal{X}$ (we will frequently use the notation $E \to \mathcal{X}$) is a smooth real manifold E of dimension $n + m$, together with :

(1) a C^∞-map $\pi : E \longrightarrow \mathcal{X}$, called *projection*, the set $E_x = \pi^{-1}(\{x\})$ being the *fiber* above x ;

(2) a $\mathbb{R}$-vector space structure of dimension m on each fiber $E_x = \pi^{-1}(x)$ such that the vector space structure is locally trivial, which means there exists an open covering $\{U_\alpha\}_{\alpha\in I}$ of $\mathcal{X}$, together with C^∞ diffeomorphisms, known as *local trivializations*, $\theta_\alpha\ : \pi^{-1}(U_\alpha) \leftrightarrow U_\alpha \times \mathbb{R}^m$ such that

$$\pi_{\pi^{-1}(U_\alpha)}(e) = \mathrm{pr}_{U_\alpha}(\theta_\alpha(e)), \quad \forall\, e \in \pi^{-1}(U_\alpha), \tag{B.1}$$

and, for every $x \in U_\alpha$, the map θ_α realizes a $\mathbb{R}$-linear isomorphism between $E_x = \pi^{-1}(\{x\})$ and $\mathbb{R}^m$.

It follows from condition (2) above that, for every $\alpha, \beta \in I$, the map

$$\theta_{\alpha,\beta} = \theta_\alpha \circ \theta_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^m \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^m$$

is a linear automorphism on each fiber $\{x\} \times \mathbb{R}^m$. Hence one has

$$\theta_{\alpha,\beta}(x,\eta) = (x, g_{\alpha,\beta}(x)\cdot\eta),\ (x,\eta) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^m,$$

where $(g_{\alpha,\beta})_{(\alpha,\beta)\in I\times I}$ is a collection of C^∞-$\mathrm{GL}(m,\mathbb{R})$-valued maps on $\mathcal{X}$, which satisfies the 1-*cocycle relation*

$$g_{\alpha,\beta} \circ g_{\beta,\gamma} = g_{\alpha,\gamma} \text{ on } \mathrm{U}_\alpha \cap \mathrm{U}_\beta \cap \mathrm{U}_\gamma\,,\ \alpha,\beta,\gamma \in \mathrm{I}. \tag{B.2}$$

A (smooth) section of the bundle $E \to \mathcal{X}$ over some open set $U \subset \mathcal{X}$ is a C^∞-map s from U to E (both equipped with their structure of real differentiable manifolds) such that, for any $x \in U$, $s(x) \in E_x$.

Example B.1 (the tangent real vector bundle $T_{\mathbb{R}}(\mathcal{X})$). One defines a structure of $2N$-real manifold on $\bigcup_{x\in\mathcal{X}} T_{\mathbb{R},x}(\mathcal{X})$ by defining the local charts

$$\widetilde{\tau}_{\pi^{-1}(U_\alpha)}(x,\dot\gamma) = (\tau_\alpha(x), d_0(\tau_\alpha\circ\gamma)(1))\ ;$$

we thus construct a structure of real, locally trivial, C^∞-vector bundle of rank N over $\mathcal{X}$. This is the *real tangent vector bundle* $T_{\mathbb{R}}(\mathcal{X}) \to \mathcal{X}$. The sections of $T_{\mathbb{R},x}(\mathcal{X})$ over the open set U in $\mathcal{X}$ of this bundle are called C^∞-*real vector fields* over the open set U. In the local chart $U_\alpha = \tau_\alpha^{-1}(V_\alpha)$ about some point $x \in \mathcal{X}$, we represent the vector field in the form $\sum_{j=1}^N a_j(x)\frac{\partial}{\partial x_j}$, where $a_1, ..., a_N$ are C^∞-functions in $V_\alpha = \tau(U_\alpha)$.

A more abstract approach to the notion of the real, locally trivial, C^∞-vector bundle is the following: any such bundle is given (up to isomorphim of locally trivial bundles of the same rank[1]) by an open covering $(U_\alpha)_{\alpha\in I}$ of $\mathcal{X}$, together with a 1-*cocycle* in the sense of Čech, that is, for every $(\alpha,\beta) \in I\times I$, a C^∞-map

$$g_{\alpha,\beta}\ : U_\alpha \cap U_\beta \to \mathrm{GL}(m,\mathbb{R})$$

such that the 1-cocycle relations (B.2) hold. The structure of the locally trivial, C^∞-vector bundle of rank m associated to this 1-cocycle $(g_{\alpha,\beta})_{(\alpha,\beta)\in I\times I}$ is obtained by equipping the disjoint union of $\{U_\alpha\} \times \mathbb{R}^m$ with the quotient topology, identifying the pairs (x,v) et $(x, g_{\alpha,\beta}(x).v)$, where α,β are arbitrary indices in I and $x \in U_\alpha \cap U_\beta$.

[1]Two $\mathbb{R}$-vector bundles $E_1 \overset{\pi_1}{\to} \mathcal{X}$ and $E_2 \overset{\pi_2}{\to} \mathcal{X}$ locally trivial and of the same rank are *isomorphic* if and only if there exists a C^∞ diffeomorphism $f\ : \mathcal{X} \to \mathcal{X}$, a C^∞-diffeomorphism $F\ : E_1 \to E_2$ such that $f\circ\pi_1 = \pi_2 \circ F$; the same definition holds for $\mathbb{C}$-vector bundles of the same rank.

B.1.3. Operations on isomorphism classes of vector bundles. The next important feature about vector bundles is that one can define *operations* between locally trivial vector bundles (up to isomorphisms of $\mathbb{R}$-bundles of the same rank), for example the addition $E_1 \oplus E_2$ of two vector bundles $E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$ with respective ranks m_1 and m_2 (that is a vector bundle of the rank m_1+m_2, which fiber above x is identified with the direct sum $E_{1,x} \oplus_{\mathbb{R}} E_{2,x}$), the exterior powers (until the power m, known also *determinant bundle*) of a locally trivial vector bundle $E \to \mathcal{X}$ of rank m. The p-th exterior power is a vector bundle of rank $m!/(p!(m-p)!)$, if $p = 0, ..., m$; the power of order 0 is defined to be, by convention, the trivial vector bundle $\mathcal{X} \times \mathbb{R}$). One can also consider the *dual vector bundle* $E^* \to \mathcal{X}$ of a locally trivial, real vector bundle $E \to \mathcal{X}$ of rank m, or the tensor product $E_1 \otimes E_2$ of two locally trivial, real vector bundles $E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$ with respective ranks m_1 and m_2 (this is a bundle of'rank $m_1 \times m_2$, which fiber above x is identified with $E_{1,x} \otimes_{\mathbb{R}} E_{2,x}$), etc. To perform such constructions, we work in any case with an open covering of $\mathcal{X}$ which is the refinement of the two open coverings used to define the 1-cocycles that determine the bundles $E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$; then we define operations on the attached 1-cocycles such as direct sum, tensorial product between elements in $\mathrm{GL}(m_1, \mathbb{R})$ and $\mathrm{GL}(m_2, \mathbb{R})$, wedge powers of an element in $\mathrm{GL}(m, \mathbb{R})$, dual of such an element, etc.

The set of isomorphism classes of locally trivial real vector bundles of rank 1 (such bundles are also called *real line bundles*) can be equipped with a structure of a commutative group (the addition being the tensorial product).

Furthermore, given locally trivial, real vector bundles $E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$ with corresponding ranks m_1 and m_2, one can define, up to isomorphism of locally trivial vector bundles of the same rank, the locally trivial vector bundle $\mathrm{Hom}_{\mathbb{R}}(E_1, E_2)$, whose fiber above the point x is $\mathrm{Hom}_{\mathbb{R}}(E_{1,x}, E_{2,x})$ (it is a locally trivial vector bundle of rank $m_1 \times m_2$ which is isomorphic to $E_2 \otimes E_1^*$).

Exactly as above, we can also define the notion of *locally trivial complex vector bundle* of (complex) rank m over a real differentiable manifold $\mathcal{X}$. The fibers E_x are now $\mathbb{C}$- vector spaces and θ_α realizes a linear isomorphism between $\pi^{-1}(\{x\})$ and $\{x\} \times \mathbb{C}^m$ for any x in the trivialization open chart U_α such that $\pi^{-1}(U_\alpha) \leftrightarrow U_\alpha \times \mathbb{C}^m$ via θ_α. If one adopts the point of view of 1-cocycles, then equivalence classes of locally trivial, complex vector bundles of rank m are constructed starting with 1-cocycles with values in $\mathrm{GL}(m, \mathbb{C})$. In order to construct, as an example, such a complex vector bundle of rank N, one complexifies, for every $x \in \mathcal{X}$, the $\mathbb{R}$-vector space $T_{\mathbb{R},x}(\mathcal{X})$. This will be seen in the forthcoming section B.3, dealing with, as $\mathcal{X}$, a complex analytic manifold of complex dimension n (that is with underlying real structure with dimension $2n$).

Example B.2 (the cotangent vector bundle $T^*_{\mathbb{R}}(\mathcal{X})$). The *real cotangent vector bundle* $T^*_{\mathbb{R}}(\mathcal{X})$ is the dual of the tangent real vector bundle. Smooth sections of this bundle over some open set U are the differential 1-forms with C^∞-real valued coefficients over U. These forms are expressed locally in the open chart U_α as $\omega = \sum_{j=1}^N \omega_j dx_j$, where $\omega_1, ..., \omega_N$ are C^∞-functions in the open set $V_\alpha = \tau_\alpha(U_\alpha) \subset \mathbb{R}^N$. Smooth complex 1-differential forms are sections of the $\mathbb{C}$-vector bundle $T^*_{\mathbb{R}}(\mathcal{X}) \otimes_{\mathbb{R}} \mathbb{C}$.

Given a locally trivial vector bundle $E \to \mathcal{X}$ (real or complex) with rank m, we introduce the vector bundles $\bigwedge^p T^*_{\mathbb{R}}(\mathcal{X}) \otimes_{\mathbb{R}} E$. A (smooth) section of this vector bundle over the open set U is called a *C^∞-real differential E-valued p-form* over U.

The space of such E-valued p-forms over U will be denoted by

$$C_p^\infty\Big(U, \bigwedge^p T^*_{\mathbb{R}}(\mathcal{X}) \otimes_{\mathbb{R}} E\Big) = C_p^\infty(U, E).$$

A (smooth) section over U of the bundle $\bigwedge^p T^*_{\mathbb{R}}(\mathcal{X})$ (that is of $\bigwedge^p T^*_{\mathbb{R}}(\mathcal{X}) \otimes \mathbb{R}$) is simply called C^∞ *real differential p-form* over U; similarly, a (smooth) section over U of the complex bundle $\bigwedge^p T^*_{\mathbb{R}}(\mathcal{X}) \otimes_{\mathbb{R}} \mathbb{C}$ is called a C^∞ *complex differential p-form* over U. We call p the *degree* of the differential form.

B.1.4. Connection on a locally trivial $\mathbb{K}$-vector bundle.

Definition B.1 (connection on a vector bundle). Let $\mathcal{X}$ be a differentiable manifold of dimension N and let $E \to \mathcal{X}$ be a locally trivial, real (*resp.* complex) vector bundle of rank m above $\mathcal{X}$. A C^∞ connection D on E is a differential, $\mathbb{R}$-linear operator of order 1

$$D \ : \bigoplus_{p=0}^{N} C_p^\infty(\mathcal{X}, E) \longmapsto \bigoplus_{p=0}^{N} C_p^\infty(\mathcal{X}, E)$$

acting from $C_p^\infty(\mathcal{X}, E)$ into $C_{p+1}^\infty(\mathcal{X}, E)$, for every $p = 0, ..., N$, which satisfies the Leibniz rule:

$$\text{(B.3)} \quad D[f \wedge \omega] = df \wedge \omega + (-1)^p f \wedge D[\omega] \qquad \forall\, f \in C_p^\infty(\mathcal{X}, \mathbb{K})\,,\ \forall\, \omega \in C_q^\infty(\mathcal{X}, E),$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (whether the bundle E is real or complex). Note that the linearity which is imposed to D remains in any case $\mathbb{R}$-linearity. Connections will always for us be C^∞.

It is very convenient to describe the action of the connection locally, in a open neighborhood U, over which the vector bundle E is trivial. Let $\theta = \theta_\alpha$ be the trivialization morphism, as defined in (B.1). Compose with θ^{-1} the mappings $x \mapsto (x, \epsilon_j)$, where $(\epsilon_1, ..., \epsilon_m)$ is the canonical base of the vector space $\mathbb{K}^m$. One obtains thus a system $(e_1, ..., e_m)$ of sections of E over U, which is called a *frame* . An element s of $C_p^\infty(U, E)$ (*i.e.* a smooth, differential p-form over U, with values in E) can be expressed with respect to this frame in the form

$$s = \sum_{j=1}^{m} \sigma_j \otimes e_j,$$

where $\sigma_j \in C_p^\infty(U, \mathbb{K})$. If one writes

$$D[e_j] = \sum_{k=1}^{m} a_{kj} \otimes e_k\,,\ j = 1, ..., m\,,$$

where the $a_{kj} \in C_1^\infty(U, \mathbb{K})$ are smooth 1-forms (with values in $\mathbb{K}$) over U, then (B.3) is expressed (in short) by

$$\text{(B.4)} \qquad D\Big[\sum_{j=1}^{m} \sigma_j \otimes e_j\Big] = \sum_{j=1}^{m} \Big(d\sigma_j + \sum_{k=1}^{m} a_{jk} \wedge \sigma_k\Big) \otimes e_j\,.$$

This means that, once the trivialization θ is fixed, the action of D on $C_\bullet^\infty(U, E)$ is given by

$$\text{(B.5)} \qquad D[s] =_\theta d\sigma + A \wedge \sigma \qquad \text{if } s = \sum_{j=1}^{m} \sigma_j \otimes e_j\,,$$

where $\sigma = (\sigma_1, ..., \sigma_m)$ denotes the coordinates of the $\mathbb{K}^m$-valued function representing s over U with respect to the frame provided by the trivialization θ, d represents the *de Rham connection* over U for the trivial bundle $U \times \mathbb{K}^m \to U$, and A (which contributes to the correcting term $A \wedge \sigma$) is a matrix of real smooth differential 1-forms in U.

Proposition B.1 (changing of frame). *If $(e_1, ..., e_m)$ and $(\widetilde{e}_1, ..., \widetilde{e}_m)$ are two frames corresponding to two different trivializations θ and $\tilde{\theta}$ over the same open set U of $\mathcal{X}$ and s is the section of E over U such that*

$$s = \sum_{j=1}^{m} \sigma_j \otimes e_j = \sum_{j=1}^{m} \widetilde{\sigma}_j \otimes \widetilde{e}_j$$

with $\widetilde{\sigma} = g \cdot \sigma$, then the matrices A and $\widetilde{A}$ which represent (as in (B.5)*) the action of a given connection D on E within respectively the frame e and the frame $\widetilde{e}$ are related by the so-called gauge transformation law*

$$A = g^{-1} \cdot \widetilde{A} \cdot g + g^{-1} \cdot dg. \tag{B.6}$$

B.1.5. Curvature tensor of a connection.

Definition B.2 (curvature of a connection). Let $E \to \mathcal{X}$ be a real or complex, locally trivial, vector bundle (with rank m) above a real differentiable manifold $\mathcal{X}$, and D be a connection on this bundle, as in Definition B.1. The operator

$$D^2 \ : C^\infty_\bullet(\mathcal{X}, E) \to C^\infty_{\bullet+2}(\mathcal{X}, E)$$

is called the *curvature operator* of the connection D. The differential two-form $\Theta(D) \in C^\infty_2(\mathcal{X}, \mathrm{Hom}_{\mathbb{K}}(E, E))$ such that $D^2[s] = \Theta(D) \wedge s$ is called the *curvature tensor* of the connection D.

If $(e_1, ..., e_m)$ denotes the frame induced by the trivialization θ of E over the open set U of the manifold $\mathcal{X}$, and A denotes the matrix of 1-forms describing the action of D relatively to the decomposition of a section with respect to the frame (as in (B.4)), then applying the rule (B.5), one shows that the curvature operator is expressed in terms of coordinates with respect to the above frame in the form :

$$D^2\Big[\sum_{j=1}^{m} \sigma_j \otimes e_j\Big] = \sum_{j=1}^{m} \tau_j \otimes e_j \qquad \text{where} \qquad \tau = (dA + A \wedge A) \wedge \sigma \,. \tag{B.7}$$

Here $\sigma_1, ..., \sigma_m$ are smooth q-forms with values in $\mathbb{K}$ over the open set U. The matrix of 2-forms

$$\Theta = dA + A \wedge A \tag{B.8}$$

is called the *curvature matrix* of the connection D over U, expressed with respect to the frame $(e_1, ..., e_m)$. This is also the matrix of the curvature tensor $\Theta(D)$ with respect to the frame.

B.1.6. Operations on bundles and connections. We present here briefly how operations on bundles described in susection B.1.3 reflect in the constructions of adapted connections and how to express their curvature tensors. Let $E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$ be two locally trivial vector bundles (real or complex) with respective ranks m_1 and m_2, over the real differentiable manifold $\mathcal{X}$. Assume that both

$E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$ are equipped with connections (respectively D_{E_1} and D_{E_2}). Then

$$D_{E_1 \oplus E_2} := D_{E_1} \oplus D_{E_2}$$

defines a connection on $E_1 \oplus E_2$, whose tensor of curvature is $\Theta(D_{E_1}) \oplus \Theta(D_{E_2})$. There is also a unique connection $D_{E_1 \otimes E_2}$ on $E_1 \otimes E_2$ such that

$$D_{E_1 \otimes E_2}(s_1 \wedge s_2) = D_{E_1}(s_1) \wedge s_2 + (-1)^{\deg s_1} s_1 \wedge D_{E_2}(s_2)$$

for every differential form s_1 of $\mathcal{X}$ with values in E_1, and for every differential form s_2 of $\mathcal{X}$ with values in E_2. Furthermore, the following identity is true

$$\Theta(D_{E_1 \otimes E_2}) = \Theta(D_{E_1}) \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes \Theta(D_{E_2}). \tag{B.9}$$

If D is a connection on E, then

$$u \in C^\infty_\bullet(\mathcal{X}, E^*) \longmapsto D_{E^*}(u) \ : s \longmapsto d(u \cdot s) - (-1)^{\deg u} u \cdot D_E(s) \tag{B.10}$$

is a connection on the dual vector bundle E^* with $\Theta(D_{E^*}) = -[\Theta(D_E)]^t$, (^{t} denotes here the transposition operator from $\mathrm{Hom}_{\mathbb{K}}(E, E)$ into $\mathrm{Hom}_{\mathbb{K}}(E^*, E^*)$). One can construct a connection on $\mathrm{Hom}_{\mathbb{K}}(E_1, E_2)$ from the connections D_{E_1} and D_{E_2} on the vector bundles $E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$ setting

$$D_{\mathrm{Hom}_{\mathbb{K}}(E_1,E_2)}(v) \ : s \longmapsto D_{E_2}(v \cdot s) - (-1)^{\deg v} v \cdot D_{E_1}(s) \tag{B.11}$$

forall all $v \in C^\infty_\bullet(\mathcal{X}, \mathrm{Hom}_{\mathbb{K}}(E_1, E_2))$. For this connection, one has

$$\Theta(D_{\mathrm{Hom}_{\mathbb{K}}(E_1,E_2)}) = \Theta(D_{E_2}) \otimes \mathrm{Id}_{E_1^*} - \mathrm{Id}_{E_2} \otimes [\Theta(D_{E_1})]^t. \tag{B.12}$$

Such identity follows from (B.9) since there exists a natural $\mathbb{K}$-isomorphism between $\mathrm{Hom}_{\mathbb{K}}(E_1, E_2)$ and $E_2 \otimes E_1^*$. In particular, one has the *Bianchi identity*

$$D_{\mathrm{Hom}_{\mathbb{K}}(E,E)}(\Theta(D_E)) = D_E(\Theta(D_E)(\cdot)) - \Theta(D_E)[D_E(\cdot)] = D_E^3(\cdot) - D_E^3(\cdot) = 0. \tag{B.13}$$

Finally, respect to wedge powers, given $E \to \mathcal{X}$ equipped with a connection D, there is a unique connection $D_{\wedge p}$ on the vector bundle $\bigwedge^p E$ (that can be defined inductively), such that

$$D_{\wedge p}(s_1 \wedge \cdots \wedge s_p) = \sum_{j=1}^p (-1)^{\deg s_1 + \ldots + \deg s_{j-1}} s_1 \wedge \cdots \wedge s_{j-1} \wedge D(s_j) \wedge s_{j+1} \wedge \cdots \wedge s_p$$

for any $s_1, ..., s_p \in C^\infty_\bullet(\mathcal{X}, E)$. Its tensor curvature is given as

$$\Theta(D_{\wedge p}) = \sum_{j=1}^p s_1 \wedge \cdots \wedge s_{j-1} \wedge [\Theta(D) \cdot s_j] \wedge s_{j+1} \wedge \cdots \wedge s_p. \tag{B.14}$$

In the particular case $p = m$, the curvature tensor $\Theta(D_{\wedge m})$ can be viewed as a scalar (the determinant vector bundle has rank 1). This scalar is the trace of $\Theta(D)$, considered as an element of $C^\infty(\mathcal{X}, \mathrm{Hom}_{\mathbb{K}}(E, E))$.

B.2. Hermitian structure on a complex vector bundle

Let $\mathcal{X}$ be a differentiable manifold, $E \to \mathcal{X}$ be a locally trivial, complex vector bundle on $\mathcal{X}$ of complex rank m. We can equip any such bundle with an *hermitian structure* . This means that for every fiber E_x, $x \in \mathcal{X}$, we can define a positive metric

$$\xi \longmapsto |\xi|_x^2 \,,$$

in order that the map $E \longmapsto [0, \infty[$ associating to $(x, \xi) \in E_x$ the positive number $|\xi|_x^2$ is C^∞.

Let $E \to \mathcal{X}$ be a locally trivial, complex vector bundle of rank m over a differentiable manifold $\mathcal{X}$ with real dimension N. Assume that the vector bundle E is equipped with a hermitian metric. Denote the metric induced on the fiber E_x as $|\ |_x$ and the corresponding scalar product $\langle\ ,\ \rangle_x$. For every $0 \leq p, q \leq N$, the scalar product induces a sesqui-linear mapping

$$\langle\ ,\ \rangle\ : C_p^\infty(\mathcal{X}, E) \times C_q^\infty(\mathcal{X}, E) \longmapsto C_{p+q}^\infty(\mathcal{X}). \tag{B.15}$$

One can express this mapping using trivializations. Actually, a local trivialization above an open subset U allows to express the forms with respect to a frame $(e_1, ..., e_m)$ over U. Then, the sesqui-linear mapping (B.15) is defined as

$$\Big\langle \sum_{j=1}^m \sigma_j(\zeta) \otimes e_j(\zeta)\, ,\, \sum_{k=1}^m \tau_k(\zeta) \otimes e_k(\zeta) \Big\rangle := \sum_{j=1}^m \sum_{k=1}^m (\sigma_j(\zeta) \wedge \overline{\tau}_k(\zeta))\, \langle e_j(\zeta), e_k(\zeta)\rangle_\zeta$$

where $\sigma_1, ..., \sigma_m$ are smooth p-forms in U, and $\tau_1, ..., \tau_m$ are smooth q-forms in U. This bracket operation makes sense globally since the metric is globally defined over E.

Definition B.3 (compatibility of a connection with a metric). Let $E \to \mathcal{X}$ be a complex vector bundle of rank m equipped with a hermitian metric $|\ |$, inducing the bracket operation (B.15) between smooth E-valued p-forms and smooth E-valued q forms. A connection D on E is *compatible with the hermitian structure* $|\ |$ *on the bundle* E if and only if

$$d[\langle s, t\rangle] = \langle D(s), t\rangle + (-1)^p \langle s, D(t)\rangle\ ,\ \forall\, s \in C_p^\infty(\mathcal{X}, E),\ \forall\, t \in C_q^\infty(\mathcal{X}, E). \tag{B.16}$$

If the frame $(e_1, ..., e_m)$ is orthonormal (respect to the hermitien metric $|\ |$) and if $s = \sum_j \sigma_j \otimes e_j$, $t = \sum_k \tau_k \otimes e_k$, then

$$\langle s, t\rangle = \sum_{j=1}^n \sigma_j \wedge \tau_j.$$

Applying the operator d, one finds then

$$d[\langle s, t\rangle] = \langle d\sigma, \tau\rangle + (-1)^p \langle \sigma, d\tau\rangle. \tag{B.17}$$

But $D[s] =_\theta d\sigma + A \wedge \sigma$ and $D[t] =_\theta d\tau + A \wedge \tau$, where A represents the matrix of the connection D respect to this frame (see (B.4) or (B.5)). Comparing (B.17) to (B.16), one observes that

$$A^* = -A \tag{B.18}$$

where A is the connection matrix. Note also that iA is a 1-form with values in the $\mathbb{C}$-vector space Herm$\,(C_1^\infty(\mathcal{X}, \mathbb{C}))$ of hermitian matrices with 1-complex differential forms as coefficients. Using the fact that $d \circ d = 0$, one checks that

$$\langle \Theta(D) \wedge s, t\rangle = -\langle s, \Theta(D) \wedge t\rangle\,,$$

and hence $i\Theta(D) \in C_2^\infty(\mathrm{Herm}\,(E, E))$.

In the case when $m = 1$ (the vector bundle is then called a *line bundle*), the compatibility of D with the metric means just that the matrix A is a 1-form taking real values. In this case $i\Theta(D) \in C_2^\infty(\mathcal{X}, \mathbb{R})$.

B.3. Holomorphic hermitian bundles over complex manifolds

B.3.1. Complex manifolds. A *complex manifold* $\mathcal{X}$ of dimension n consists in the following data:

(1) a topological vector space $\mathcal{X}$ which is countable at infinity;
(2) a collection of charts (U_α, τ_α) on $\mathcal{X}$, where the $(U_\alpha)_\alpha$ realize a covering of $\mathcal{X}$, and, for any α, $\tau_\alpha\ : U_\alpha \leftrightarrow V_\alpha \subset \mathbb{C}^n$ is an homeomorphism, such that, for every pair of indices (α, β), the map $\tau_{\alpha\beta} = \tau_\alpha \circ (\tau_\beta)^{-1}$ is a biholomorphism between $\tau_\beta(V_\alpha \cap V_\beta)$ and $\tau_\alpha(V_\alpha \cap V_\beta)$.

Example B.3 (the example of the projective space). Besides $\mathbb{C}^n$, the most common example of complex manifold is the projective space $\mathbb{P}^n(\mathbb{C})$, which is realized as the geometric quotient of $\mathbb{C}^{n+1}\setminus\{(0,...,0)\}$ by the equivalence relation of co-linearity between vectors in $\mathbb{C}^{n+1} \setminus \{(0,...,0)\}$. The homogeneous coordinates $[z_0 : \cdots : z_n]$ represent points of the projective space. Open sets $U_j := \{[z_0 : \cdots : z_n]\,;\, z_j \neq 0\}$, $j = 0,...,n$ are the charts for this complex manifold with transition functions $\phi_{i,j} = z_j/z_i$ on $U_i \cap U_j$. It is important to think here of $\mathbb{P}^n(\mathbb{C})$ as a hyperplane at *infinity* $\mathcal{U}_0 := \{[0 : z_0 : \cdots : z_n]\}$ in $\mathbb{P}^{n+1}(\mathbb{C})$ (the homogeneous coordinates being $(\tau, z_0, ..., z_n)$, in the following sense : if $(z_0, ..., z_n)$ is a point in $\mathbb{C}^{n+1}\setminus\{(0,...,0)\}$, the complex line $\{(0, \lambda z_0, ..., \lambda z_n)\,;\, \lambda \in \mathbb{C}\}$ intersects the hyperplane at infinity of the projective space $\mathbb{P}^{n+1}(\mathbb{C})$ exactly at the point identified with $[z_0 : \cdots : z_n] \in \mathbb{P}^n(\mathbb{C})$. We remark here that one of the most important $(n+1, n)$-kernels of the multidimensional complex analysis, namely the *Bochner-Martinelli kernel* in $\mathbb{C}^{n+1}\setminus\{(0,...,0)\}$, can by deduced from the positive volume form $(dd^c \log \|z\|^2)^n$ on $\mathbb{P}^n(\mathbb{C})$ (called the *Fubini-Study volume form*) by multiplication with dt/t, and then averaging along the orbits issued from points in $\mathbb{P}^n(\mathbb{C})$ viewed as points at infinity of $\mathbb{P}^{n+1}(\mathbb{C})$. We conclude this example by pointing out, since they play an important role in this monography, that complex manifolds obtained (as $\mathbb{P}^n(\mathbb{C})$) by gluing together copies of $\mathbb{C}^n$ *via* monomial transition functions, are known as smooth *toric varieties* (see [**Elh, Dan, Ew**] for such constructions).

Consider now the underlying real differentiable structure $\mathcal{X}_\mathbb{R}$ on $\mathcal{X}$, and, for every point $x \in \mathcal{X}$, the real tangent subspace $T_{\mathbb{R},x}(\mathcal{X}_\mathbb{R})$ ($\mathbb{R}$-vector space of dimension $2n$), equipped with its *almost complex structure*, that is with the linear involution $J(x)$ of $T_{\mathbb{R},x}(\mathcal{X}_\mathbb{R})$ whose action, described in local coordinates $(\xi_1, \eta_1, ..., \xi_n, \eta_n)$ (where $\zeta_k = \xi_k + i\eta_k$ for $k = 1, ..., n$) is given by

$$J(\partial/\partial\xi_k) = \partial/\partial\eta_k\ ,\ J(\partial/\partial\eta_k) = -\partial/\partial\xi_k\ ,\ k = 1, ..., n.$$

It induces a suitable decomposition of $T_{\mathbb{R},x}(\mathcal{X}_\mathbb{R})$ into two proper subspaces. It also allows to equip $T_{\mathbb{R},x}(\mathcal{X}_\mathbb{R})$ with a structure of $\mathbb{C}$-vector space (with complex dimension n). Namely the operator J will correspond to multiplication by i in what will be the *complex tangent space* $T_x(\mathcal{X}) = T_{\mathbb{R},x}(\mathcal{X}_\mathbb{R})$.

The three equivalent ways to think of $T_{\mathbb{R},x}(\mathcal{X}_\mathbb{R})$ as a complex vector space (it will then be called the *holomorphic tangent vector space* and denoted as $T(\mathcal{X})$) are the following ones.

A) The first model is a *geometric model*. If U_α is an open chart containing z, one introduces the germs of analytic disks $\gamma\ : \xi + i\eta \in D(0, \epsilon) \mapsto \gamma(\xi + i\eta) \in U_\alpha$ at x, lying in $\mathcal{X}$ in a neighborhood of x and passing through x ($\gamma(0) = x$ and $\tau_\alpha \circ \gamma$ is holomorphic in V_α). Elements in $T_x(\mathcal{X})$ are *tangent classes at* x of such germs (the equivalence class of (D,γ) is realized by $d_0[\tau_\alpha \circ \gamma](1)$), the differentiation here being the complex differentiation.

B) the second model, is an algebraic model. It consists in interpreting the $\mathbb{C}$-vector space $T_x(\mathcal{X})$ as the $\mathbb{C}$-vector space of *complex derivations* of the nœtherian $\mathbb{C}$-algebra of germs of holomorphic functions $\mathcal{O}_{\mathcal{X},x}$ at the point x. That is, $\mathbb{C}$-linear applications from $\mathcal{O}_{\mathcal{X},x}$ to $\mathbb{C}$ satisfying the Leibniz rule : $D[fg] = f(a)D[g] + g(a)D[f]$. This model describes an element of $T_x(\mathcal{X})$ in terms of its action (as a derivation of $\mathcal{O}_{\mathcal{X},x}$).

C) the third model is also an algebraic model. One still introduce the $\mathbb{C}$-algebra $\mathcal{O}_{\mathcal{X},x}$, together with its maximal ideal $\mathfrak{M}_{\mathcal{X},x}$. The $\mathbb{C}$-vector space $T_x(\mathcal{X})$ is identified as the dual of the $\mathbb{C}$-vector space $\mathfrak{M}_{\mathcal{X},x}/(\mathfrak{M}_{\mathcal{X},x})^2$.

For every $x \in \mathcal{X}$, we have the following decomposition of the complexification of $T_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}}) = T_x(\mathcal{X})$:

$$\begin{aligned}\mathbb{C} \otimes_{\mathbb{R}} T_x(\mathcal{X}) &= \mathbb{C} \otimes_{\mathbb{R}} T_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}}) = T_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}}) \oplus iT_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}})\\ &= T_x^{1,0}(\mathcal{X}) \oplus T_x^{0,1}(\mathcal{X}) \simeq T_x(\mathcal{X}) \oplus \overline{T_x(\mathcal{X})}\,,\end{aligned}$$

where $T_x(\mathcal{X})$ denotes the complex tangent subspace at the point x (equipped with the structure of $\mathbb{C}$-vector space of dimension n thanks to J) and $\overline{T_x(\mathcal{X})}$ denotes its conjugate ($T_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}})$ equipped with the conjugate complex structure $-J$), the corresponding isomorphisms above being realized by

$$\begin{aligned}\xi \in T_x(\mathcal{X}) &\leftrightarrow \frac{\xi - iJ(\xi)}{2} \in T_x^{1,0}(\mathcal{X})\\ \xi \in \overline{T_x(\mathcal{X})} &\leftrightarrow \frac{\xi + iJ(\xi)}{2} \in T_x^{0,1}(\mathcal{X})\end{aligned}$$

(recall that J denotes the operator of multiplication by i on the fibers $T_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}})$).

Over the complex manifold $\mathcal{X}$ we then have at our disposal *holomorphic tangent bundle* $T^{1,0}_{\mathcal{X}} \simeq T(\mathcal{X})$, whose sections over the open set U are the *holomorphic vector fields*, that is, the vector fields ξ which are written locally as

$$\sum_{j=1}^{n} a_j(\zeta) \frac{\partial}{\partial \zeta_j}\,,$$

where $a_1, ..., a_n$ are C^∞-complex valued functions in $V_\alpha = \tau_\alpha(U \cap U_\alpha)$. The fiber over x (modulo the $\mathbb{C}$-isomorphism mentioned in (B.19)) is the complex tangent space $T_{\mathbb{C},x}(\mathcal{X}) = T_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}})$, equipped this time with its complex structure.

Similarly, we introduce the *antiholomorphic tangent bundle* $T^{0,1}_{\mathcal{X}}$ whose sections over the open set U are the *antiholomorphic vector fields* , that is, the vector fields ξ locally expressed as

$$\sum_{j=1}^{n} a_j(\zeta) \frac{\partial}{\partial \overline{\zeta}_j}\,,$$

where $a_1, ..., a_n$ are C^∞-complex valued functions in $V_\alpha = \tau_\alpha(U \cap U_\alpha)$. The fiber over x (modulo the $\mathbb{C}$-isomorphism mentioned in (B.19)) is the *conjugate* complex tangent bundle is the complex tangent, that is $T_{\mathbb{R},x}(\mathcal{X}_{\mathbb{R}})$ equipped this time with the complex structure associated to $-J$.

Given a locally trivial complex vector bundle $E \to \mathcal{X}$, one can define the complex vector bundles

$$\Big[\bigwedge^p (T_{\mathcal{X}}^{1,0})^* \otimes_{\mathbb{C}} \bigwedge^q (T_{\mathcal{X}}^{0,1})^*\Big] \otimes_{\mathbb{C}} E = (T_{\mathcal{X}}^{p,q})^* \otimes_{\mathbb{C}} E\,,\ p,q \in \mathbb{N}\,,\ p+q \leq 2n.$$

Smooth sections of this bundle over the open set are C^∞ *complex (p,q)-forms in U, with values in E*. We denote as $C^\infty_{p,q}(U,E)$ the $\mathbb{C}$-vector space of these sections.

Remark B.1. Let $E \to \mathcal{X}$ be a $\mathbb{C}$-vector bundle over a complex manifold $\mathcal{X}$ and $D\ : C^\infty_\bullet(\mathcal{X},E) \to C^\infty_{\bullet+1}(\mathcal{X},E)$ be a connection on $E \to \mathcal{X}$. Because of the $\mathbb{R}$-linearity of D, it splits as $D = D_C + D''_C$, where $D'_C\ : C^\infty_{\bullet,q}(\mathcal{X},E) \to C^\infty_{\bullet+1,q}(\mathcal{X},E)$ for any q and $D''_C\ : C^\infty_{p,\bullet}(\mathcal{X},E) \to C^\infty_{p,\bullet+1}(\mathcal{X},E)$ for any p.

B.3.2. Holomorphic bundles and Chern connection.

Definition B.4. [holomorphic bundle] A locally trivial complex vector bundle $E \to \mathcal{X}$ with rank m over a n-dimensional complex manifold $\mathcal{X}$ is called *holomorphic* if and only if E is equipped with a structure of complex manifold of dimension $n+m$ such that:

- the projection $\pi\ : (x,\xi) \mapsto z$ holomorphic from E into $\mathcal{X}$;
- there exists a covering $(U_\alpha)_\alpha$ of $\mathcal{X}$ such that for each α, E is trivializable over the open set U_α and the trivialization morphism θ_α is an holomorphic map from $\pi^{-1}(U_\alpha)$ in $U_\alpha \times \mathbb{C}^m$.

Example B.4 (holomorphic line bundles and Picard group). As seen in Appendix C, one can associate to a Cartier divisor (U_α, s_α) on $\mathcal{X}$ the isomorphism class of line bundle corresponding the 1-cocycle $(s_\alpha/s_\beta)_{\alpha,\beta} \in Z^1(\mathcal{X},(U_\alpha)_\alpha,\mathcal{O}^*_{\mathcal{X}})$. This map is a surjective homomorphism from the group of Cartier divisors on $\mathcal{X}$ onto the group of isomorphism classes of holomorphic line bundles. Its kernel coincides with the subgroup of principal Cartier divisors. The *Picard group* of $\mathcal{X}$ (see Appendix C), that is the quotient of the group of Cartier divisors by the subgroup of principal ones, is then isomorphic to the group of isomorphism classes of holomorphic line bundles, or also to the Čech cohomology group $\check{H}^1(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})$ (since two 1-cocycles define the same isomorphism class if and only if they differ from an exact one).

Example B.5 (some basic holomorphic line bundles on $\mathbb{P}^n(\mathbb{C})$). On $\mathbb{P}^n(\mathbb{C})$, the Cartier divisor (U_j,f_j), $j=0,...,n$, where $U_j := \{[z_0 : \cdots : z_n]\,;\, z_j \neq 0\}$ and $f_j([z_0 : \cdots : z_n]) = z_j/z_0$ induces the Weil divisor $-[z_0=0]$ (see also Example C.1 in Appendix C) . The corresponding isomorphism class in the Picard group is denoted as $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$. Holomorphic sections of $(\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1))^{\otimes N} := \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(N)$ can be expressed in homogeneous coordinates as homogeneous polynomials with total degree N. The dual bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-1) := (\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1))^*$ corresponds to the effective Weil divisor $[z_0 = 0]$. Global holomorphic sections of the bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-N)$ for $N > 0$ extend naturally to homogeneous functions with degree $-N$ on $\mathbb{C}^{n+1}$, therefore are all trivial and equal to the zero section. The holomorphic line bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-1) = (\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1))^*$ is isomorphic to the *tautological line bundle* on $\mathbb{P}^n(\mathbb{C})$, that is the subbundle of the trivial bundle $\mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}$ which fiber above $[z_0 : \cdots : z_n]$ is the complex line $\mathbb{C}^{n+1}$ with $(z_0,...,z_n)$ as direction. The Picard group of $\mathbb{P}^n(\mathbb{C})$ is isomorphic to $\mathbb{Z}$.

If $E \to \mathcal{X}$ is a complex vector bundle of rank m and if $\mathcal{O}_{\mathcal{X}}$ denotes the sheaf of germs of holomorphic functions over $\mathcal{X}$, then we note that the sheaf of holomorphic sections of E, denoted as $\mathcal{O}_{\mathcal{X}}(E)$, is a locally free sheaf of rank m over the sheaf $\mathcal{O}_{\mathcal{X}}$. That is, for every point x of $\mathcal{X}$ there exists a neighborhood U_x of x such that $\mathcal{O}_{\mathcal{X}}(E)_{|U_x}$ is isomorphic to $(\mathcal{O}_{\mathcal{X}})_{|U_x})^m$. For those readers who are interested to know more about *sheaf theory*, they can consult the books by R. Godement [**God**], by H. Grauert and R. Remmert [**GrR**] (see also [**GRo**]). We will present some of this material on sheaf theory in the introduction of Section D.2 in Appendix C.

All operations involving complex vector bundles, such as described in Section B.1.6, preserve the class of holomorphic vector bundles. In particular, when $E \to \mathcal{X}$ is an holomorphic vector bundle, such is the case of course for $E^* \to \mathcal{X}$.

Assume that $E \to \mathcal{X}$ is a holomorphic vector bundle of rank m over a complex manifold of dimension n and that $s \in \mathbb{C}^\infty_{p,q}(\mathcal{X}, E)$ ($p, q \in \mathbb{N}$ with $p+q \leq 2n$). Let U_α and U_β be two open charts of $\mathcal{X}$ with nonempty intersection and over which exist holomorphic trivializations (respectively θ_α and θ_β). Then, if $\sigma_\alpha[s]$ (*resp.* $\sigma_\beta[s]$) denotes the section of the trivial bundle $U_\alpha \times \mathbb{C}^m \to U_\alpha$ (*resp.* $U_\beta \times \mathbb{C}^m \to U_\beta$) obtained by composition $\theta_\alpha \circ s$ (*resp.* $\theta_\beta \circ s$) with the projections over $\mathbb{C}^m$, one has

$$\sigma_\alpha[s](z) = g_{\alpha,\beta}(\zeta) \cdot \sigma_\beta[s](\zeta) \qquad \forall\, \zeta \in U_\alpha \cap U_\beta.$$

As usual, $(g_{\alpha,\beta})_{\alpha,\beta}$ denotes here the 1-cocycle (assumed to be here holomorphic and not only C^∞). The holomorphicity of this cocycle leads to

$$\overline{\partial}[\sigma_\alpha[s]] \equiv g_{\alpha,\beta} \cdot \overline{\partial}[\sigma_\beta[s]] \quad \text{in} \quad U_\alpha \cap U_\beta.$$

Hence, the collection of forms locally represented as $(\overline{\partial}\sigma_\alpha[s])_\alpha$ fit together as a global element $D''[s] \in C^\infty_{p,q+1}(\mathcal{X}, E)$. One defines thus a connection D'' of type $(0,1)$ (that is from $C^\infty_{p,\bullet}(\mathcal{X}, E)$ to $C^\infty_{p,\bullet+1}(\mathcal{X}, E)$, but, remember, $\mathbb{R}$-linear) which is intrinsically attached to the complex structure.

Definition B.5 (the canonical connection attached to an holomorphic vector bundle). The connection D'' associated with the holomorphic vector bundle $E \to \mathcal{X}$ equipped with its complex structure is called the $(0,1)$*canonical connection* of the holomorphic bundle $E \to \mathcal{X}$. It depends only on the complex structure of $E \to \mathcal{X}$.

This connection D'' induces the definition of the *Dolbeault complex*

$$C^\infty_{p,0}(\mathcal{X}, E) \xrightarrow{D''} \cdots \xrightarrow{D''} C^\infty_{p,q}(\mathcal{X}, E) \xrightarrow{D''} C^\infty_{p,q+1}(\mathcal{X}, E) \xrightarrow{D''} \cdots,$$

which will be in this monograph a tool of extreme importance.

Definition B.6 (compatibility of a connection with a complex structure). Let $E \to \mathcal{X}$ an holomorphic vector bundle with rang m over some complex manifold $\mathcal{X}$. A connection D on $E \to \mathcal{X}$ is compatible with the complex structure of $E \to \mathcal{X}$ if its $(0,1)$ component D''_C in the splitting $D = D'_C + D''_C$ (see Remark B.1) equals the canonical $(0,1)$-connection D'' attached to the complex structure of $E \to \mathcal{X}$.

Let us now equip holomorphic vector bundles with hermitian metrics (besides their complex structure).

Note first a point that will be important to us respect to duality. Let $(E \to \mathcal{X}, |\ |)$ an holomorphic vector bundle equiped with an hermitian metric. This metric induces a linear isometry (called *complex conjugaison*) between $E \to \mathcal{X}$ and $E^* \to \mathcal{X}$: if s is an holomorphic section of $E \to \mathcal{X}$, the conjugate section $s^* \ : \mathcal{X} \to E^*$ of $E^* \to \mathcal{X}$ is defined by

(B.19) $$s^*(x)(\xi) = \langle \xi, s(x) \rangle_x\,,\ \forall\, x \in \mathcal{X}\,,\ \forall\, \xi \in E_x.$$

This isometry extends naturally to an isometry between spaces of differential forms with values in E and corresponding spaces of differential with values in E^*.

The other crucial point is the notion of *Chern connection*.

PROPOSITION B.2 (Chern connection). *Let $(E \to \mathcal{X}, |\ |)$ be a holomorphic bundle equipped with its complex structure (D'' being the $(0,1)$-canonical connection), together with a hermitian metric $|\ |$. There exists a unique connection $D = D_{E,|\ |}$ over E which is compatible at the same time with both the complex structure (see Definition B.6) and the hermitian metric (see Definition B.3). This unique connection is called the Chern connection of the holomorphic hermitian vector bundle $(E \to \mathcal{X}, |\ |)$. Its curvature tensor Θ is called the Chern curvature tensor of $(E \to \mathcal{X}, |\ |)$.*

REMARK B.2. If D denotes the Chern connection associated to the holomorphic hermitian vector bundle $(E \to \mathcal{X}, |\ |)$ (with the canonical connection D''), then $D^2 = D'_C \circ D'' + D'' \circ D'_C$, since $(D'')^2 = (D_C)^2 = 0$. The curvature tensor $\Theta_{E,|\ |}$ of this Chern connection $D = D_{E,|\ |}$ is then a section of the bundle

$$((T^{1,0}_{\mathcal{X}})^* \otimes (T^{0,1}_{\mathcal{X}})^*) \otimes \mathrm{Hom}_{\mathbb{C}}(E,E) \simeq ((T^{1,0}_{\mathcal{X}})^* \otimes (T^{0,1}_{\mathcal{X}})^*) \otimes (E \otimes E^*)$$

which can be represented, with respect to a frame, as a matrix of $(1,1)$-forms. Furthermore, $i\Theta_{E,|\ |}$ has the hermitian symmetry (see Section B.2), which means it can be expressed with respect to an *orthonormal* frame as

$$i\Theta_{E,|\ |} = \sum_{j=1}^{m}\sum_{k=1}^{m}\Big(\sum_{l,p} u_{jk;lp}\, d\zeta_l \wedge \overline{d\zeta_p}\Big)\, e_j \otimes e_k^*$$

with $u_{jk;pl} = \overline{u_{jk;lp}}$. Be careful the frame needs to be *orthonormal* (which is generally impossible to realize with an holomorphic frame for an arbitrary non trivial metric)!

We observe also that if $E \to \mathcal{X}$ is a complex holomorphic bundle of rank m equipped with a metric $|\ |$ and if $H(\zeta)$ denotes the Gram matrix

$$H(\zeta) := \Big[\langle e_j(\zeta)\,,\, e_k(\zeta)\rangle_\zeta\Big]_{1\le j,k\le m}$$

associated to the metric expressed in the *holomorphic* frame $(e_1, ..., e_m)$, then the matrix of the Chern connection $D_{E,|\ |}$, expressed with respect to the same frame, is the matrix of $(1,0)$-forms

$$A(\zeta) = [H(\zeta)]^{-1} \cdot \partial[H(\zeta)].$$

Furthermore, a direct computation shows that the curvature matrix, expressed with respect to this holomorphic frame $(e_1, ..., e_m)$, is

$$\Theta_{E,|\ |}(\zeta) = \overline{\partial}\Big([H(\zeta)]^{-1} \cdot \partial[H(\zeta)]\Big).$$

We conclude this section with the following proposition, related to the specific case of connections on the holomorphic tangent bundle $T(\mathcal{X})$.

PROPOSITION B.3 (torsion of a metric on the holomorphic tangent bundle). *Let $T(\mathcal{X}) \to \mathcal{X}$ be the holomorphic tangent bundle of a complex n dimensional manifold* *and* $|\ | = h$ *be a hermitian metric on* $T(\mathcal{X}) \to \mathcal{X}$ *diagonalized in a smooth unitary local frame* $(\xi_1^*, ..., \xi_n^*)$ *for* $(T(\mathcal{X}))^*$ *as*

$$h = \sum_{1\leq j,k\leq n} \xi_j^* \otimes \overline{\xi_k^*}\,.$$

There exists a unique $n \times n$ *matrix* A *of* 1*-complex differential forms satisfying* $A^* := {}^t\overline{A} = -A$ *and such that*

$$\begin{bmatrix} d\xi_1^* \\ \vdots \\ d\xi_n^* \end{bmatrix} = A \wedge \begin{bmatrix} \xi_1^* \\ \vdots \\ \xi_n^* \end{bmatrix} + \tau_h[\xi^*]\,, \tag{B.20}$$

where $\tau_h[\xi^*]$ *denotes a column-matrix of* $(2,0)$*-complex differential forms called the torsion matrix of the metric* h *(expressed with respect to the unitary frame* $\xi^* = (\xi_1^*, ..., \xi_n^*)$ *for the vector bundle* $(T(\mathcal{X}))^*$ *equipped with the metric induced by* h*). The matrix of the Chern connection* $D_{T(\mathcal{X}),|\ |}$ *with respect to the basis* $(\xi_1, ..., \xi_n)$*, dual to the basis* $(\xi_1^*, ..., \xi_n^*)$*, is equal to* $-{}^tA$*.*

B.3.3. Chern forms of a holomorphic bundle. Characteristic classes. Let $E \to \mathcal{X}$ be a holomorphic vector bundle of rank m over the complex manifold $\mathcal{X}$ of complex dimension n. Suppose that $E \to \mathcal{X}$ is equipped with a hermitian metric $|\ |$ and denote by D the Chern connection constructed for the holomorphic vector bundle (Proposition B.2). It is the unique connection over $E \to \mathcal{X}$ which is compatible both with the complex structure and the hermitian metric. Let us denote $\Theta_{E,|\ |} \in C_2^\infty(\mathcal{X}, \mathrm{Hom}_{\mathbb{C}}(E,E))$ the curvature tensor of this connection, or equivalently, the Chern curvature tensor of the holomorphic vector bundle $E \to \mathcal{X}$ equipped with the metric $|\ |$. If $(e_1, ..., e_m)$ is a local frame, this curvature tensor is expressed locally with respect to it as

$$\Theta_{E,|\ |} = \sum_{j=1}^m \sum_{k=1}^m a_{jk}\, e_j \otimes e_k^*.$$

One uses here the isomorphism from $\mathrm{Hom}_{\mathbb{C}}(E,E)$ into $E \otimes E^*$, where $e_j(\zeta) \otimes e_k^*(\zeta)$ corresponds to the $\mathbb{C}$-endomorphism of E_ζ expressed in the basis $(e_1(\zeta), ..., e_m(\zeta))$ by the matrix $[\delta_{jk,j'k'}]_{1\leq j',k'\leq m}$. The $a_{j,k}$ are elements of $C_2^\infty(\mathcal{X}, \mathbb{C})$ (*i.e.* complex differential 2-forms over $\mathcal{X}$). Using the same *frame*, we can write:

$$\begin{aligned} \det\Big(\frac{i}{2\pi}\Theta_{E,|\ |} + \mathrm{Id}_E\Big) &= \det\left(\frac{i}{2\pi}\Big(\sum_{j=1}^m\sum_{k=1}^m a_{jk} e_j \otimes e_k^*\Big) + \sum_{j=1}^m e_j \otimes e_j^*\right) \\ &= 1 + \sum_{0<p\leq \inf(m,n)} c_p(E, |\ |) \in \bigoplus_{p\leq \inf(m,n)} C_{2p}^\infty(\mathcal{X}, \mathbb{C})\,, \end{aligned} \tag{B.21}$$

where $c_p(E, |\ |) \in C_{2p}^\infty(\mathcal{X}, \mathbb{C})$, $p = 1, 2, ...$ is a $2p$-form (in fact a (p,p)-differential form) over $\mathcal{X}$.

Definition B.7 (Chern forms). Let $(E \to \mathcal{X}, |\ |)$ be a holomorphic vector bundle of rank m over the complex manifold $\mathcal{X}$, equipped with a hermitian metric $|\ |$, with Chern curvature tensor $\Theta_{E,|\ |}$. The differential forms $c_1(E, |\ |), c_2(E, |\ |), \ldots$ defined by (B.21) are called the *Chern forms* of the holomorphic vector bundle $E \to \mathcal{X}$ equipped with the hermitian metric $|\ |$.

Explicit computations show that the *total Chern form* defined as the sum of all Chern forms $c_p(E, |\ |)$, $0 < 2p \leq m$, i.e.

$$C(E, |\ |) := 1 + \sum_{p \leq \inf(m,n)} c_p(E, |\ |)$$

is a closed form. We now turn to an essentially different interpretation, eventually more convenient for us, of Chern forms (see [**And3**], Section 2). Our presentation follows here the presentation in [**And3**].

We introduce for that purpose the exterior algebra over $\mathcal{X}$

$$\Lambda = \bigwedge \Big((T^{1,0}_{\mathcal{X}})^* \oplus (T^{0,1}_{\mathcal{X}})^* \oplus E \oplus E^* \Big)$$

(we consider it as a bundle $\Lambda \to \mathcal{X}$). Any differential form s with values in E, which is locally expressed in a local frame as

$$s = \sum_{j=1}^{m} \sigma_j \otimes e_j,$$

will be identified with a unique section of $\Lambda \to \mathcal{X}$ through the correspondence

$$s = \sum_{j=1}^{m} \sigma_j \otimes e_j \longleftrightarrow \widetilde{s} := \sum_{j=1}^{m} \sigma_j \wedge e_j.$$

In the same way, any differential form S with values in $\mathrm{Hom}_{\mathbb{C}}(E, E)$, expressed locally as

$$S = \sum_{j=1}^{m} \sum_{k=1}^{m} \Sigma_{j,k} \otimes e_j \otimes e_k^*,$$

will be identified with a unique section of $\Lambda \to \mathcal{X}$ through the correspondence

$$S = \sum_{j=1}^{m} \sum_{k=1}^{m} \Sigma_{j,k} \otimes e_j \otimes e_k^* \longleftrightarrow \widetilde{S} := \sum_{j=1}^{m} \sum_{k=1}^{m} \Sigma_{k,j} \wedge e_j \wedge e_k^*.$$

Any connection D over E becomes then a $\mathbb{R}$-linear mapping from the space $C^\infty(\mathcal{X}, \Lambda)$ (of smooth sections of $\Lambda \to \mathcal{X}$) into itself. This map, denoted by $\widetilde{D}$, is viewed as an anti-derivation with respect to the exterior product[2]. The action of the $\widetilde{D}$ is the following :

- it acts as D on the e_j ;
- it acts as D^* (dual connection of D, defined in (B.10)), on the e_k^* ;
- it acts as the de Rham operator d over the factors from $(T^{1,0}_{\mathcal{X}})^* \oplus (T^{0,1}_{\mathcal{X}})^*$.

We remark immediately that the action of $\widetilde{D}$ described as above leads to

$$\widetilde{D_{\mathrm{Hom}_{\mathbb{C}}(E,E)}[S]} = \widetilde{D}[\widetilde{S}] \qquad \forall\, S \in C^\infty_\bullet(\mathcal{X}, \mathrm{Hom}_{\mathbb{C}}(E, E)). \tag{B.22}$$

Bianchi identity (B.13) implies then, following (B.22), that $\widetilde{D}[\widetilde{\Theta_{E,|\ |}}] = 0$ provided that $\Theta_{E,|\ |}$ is the Chern curvature tensor of $(E \to \mathcal{X}, |\ |)$. Again, following (B.22), one has also that $\widetilde{D}[\widetilde{I}] = 0$ if

$$\widetilde{I} := \sum_{j=1}^{m} e_j \wedge e_j^*$$

[2]In this case the Leibniz rule is $\widetilde{D}[\widetilde{s} \wedge \widetilde{t}] = D[\widetilde{s}] \wedge \widetilde{t} + (-1)^{\deg \widetilde{s}}\, \widetilde{s} \wedge \widetilde{D}[\widetilde{t}]$.

is the section of $\bigwedge \to \mathcal{X}$ identified with Id_E. A section ω of the bundle $\bigwedge \to \mathcal{X}$ can be expressed in a unique way with respect to the frame $(e_1, ..., e_m)$ in the form

$$\omega = \omega' \wedge \frac{\widetilde{I}^m}{m!} + \omega'' ,$$

where ω'' has total degree strictly inferior to m in all entries e_j^*, e_k, $1 \leq j, k \leq m$. It is convenient to set

$$\int_e \omega := \omega',$$

in order to rewrite the total Chern form as

$$C(E, |\ |) = \int_e \Big(\frac{i}{2\pi}\widetilde{\Theta}_{E,|\ |} + \widetilde{I}\Big)^m = \int_e \exp\Big(\frac{i}{2\pi}\widetilde{\Theta}_{E,|\ |} + \widetilde{I}\Big). \tag{B.23}$$

REMARK B.3. It follows from the computation above that every Chern form $c_p(E, |\ |)$, $p = 1, 2, ...,$ can be considered as an element of $C^\infty_{p,p}(\mathcal{X}, E \otimes E^*)$ with hermitian symmetry, i.e. that can be expressed in a local orthonormal frame as

$$c_p(E, |\ |) = \sum_{j=1}^m \sum_{k=1}^m \Big(\sum_{|L|=|P|=p} u_{jk;LP}\, d\zeta_L \wedge \overline{d\zeta_P}\Big)\, e_j \otimes e_k^*,$$

where $u_{jk;PL} = \overline{u_{jk;LP}}$.

We conclude by stating the following proposition (which also follows from the reinterpretation (B.23) (see [**And3**], Section 2, for more details) :

PROPOSITION B.4. *Let $(E \to \mathcal{X}, |\ |)$ is a holomorphic vector bundle of rank m over a complex manifold $\mathcal{X}$ with dimension n. Assume that $E \to \mathcal{X}$ is equipped with the hermitian metric $|\ |$. Then the total Chern class $C(E, |\ |)$, therefore all the Chern forms $c_p(E, |\ |)$ for $0 < p \leq \inf(m, n)$, are d-closed forms. In addition, the cohomology class of $C(E, |\ |)$ in the de Rham cohomology $H^\bullet(\mathcal{X}, \mathbb{C})$, therefore also all the cohomology classes in $H^\bullet(\mathcal{X}, \mathbb{C})$ of the Chern forms $c_p(E, |\ |)$ for $0 < p \leq \inf(m, n)$, are independent of the hermitian metric.*

DEFINITION B.8 (Chern characteristic classes). If $E \to \mathcal{X}$ is a holomorphic vector bundle over the complex manifold $\mathcal{X}$, the de Rham cohomology class of the total Chern form $C(E, |\ |)$, where $|\ |$ is a hermitian metric over the bundle $E \to \mathcal{X}$, is called the *characteristic class* of the holomorphic bundle $E \to \mathcal{X}$. The cohomology classes of the Chern forms $c_p(E, |\ |)$, $p \leq \inf(m, n)$, are called the *Chern characteristic classes* of the holomorphic vector bundle $E \to \mathcal{X}$. The total characteristic class, as the characteristic classes, depend only on the holomorphic bundle $E \to \mathcal{X}$ and not on the hermitian metric.

B.3.4. Positivity of holomorphic hermitian bundles. We summarize here (and compare) various concepts of *positivity* for holomorphic hermitian vector bundles $(E \to \mathcal{X}, |\ |)$ over a n-dimensional complex manifold $\mathcal{X}$.

Let $(E \to \mathcal{X}, |\ |)$ be such an holomorphic vector bundle with rank m, equipped with an hermitian metric $|\ |$. It follows from Remark B.3 that the first Chern form $c_1(E, |\ |)$, which can be expressed in an orthonormal local frame as

$$i \sum_{j=1}^m \sum_{k=1}^m \Big(\sum_{l,p} u_{jk;lp}\, d\zeta_l \wedge \overline{d\zeta_p}\Big)\, e_j \otimes e_k^* = i \sum_{j=1}^m \sum_{k=1}^m \Big(\sum_{l,p} u_{jk;lp}\, d\zeta_l \wedge \overline{d\zeta_p}\Big)\, (e_j^*)^* \otimes e_k^* ,$$

where $u_{jk;pl} = \overline{u_{kj;lp}}$. Thus, it induces an hermitian form

$$\theta_E = \sum_{1\leq j,k\leq m} \sum_{1\leq l,p\leq n} u_{jk;lp} (d\zeta_l \otimes e_j^*) \otimes (\overline{d\zeta_p \otimes e_k^*})$$

on $T(\mathcal{X}) \otimes E$ if one sets

$$\text{(B.24)} \qquad \theta_E(\xi \otimes e_j, \eta \otimes e_k) = \Big(\sum_{l,p} u_{jk;lp} d\zeta_l \wedge \overline{d\zeta_p}\Big)(\xi, \overline{\eta}), \quad 1 \leq j,k \leq m.$$

That is

$$\theta_E\Big(\sum_{j=1}^m \sum_{l=1}^n \xi_{jl} \frac{\partial}{\partial \zeta_l} \otimes e_j, \sum_{j=1}^m \sum_{l=1}^n \xi_{jl} \frac{\partial}{\partial \zeta_l} \otimes e_j\Big) = \sum_{\substack{1\leq j,k\leq m\\ 1\leq l,p\leq n}} u_{jk;lp} \xi_{jl} \overline{\xi}_{kp}\,.$$

Remark B.4. When $E \to \mathcal{X}$ is an holomorphic line bundle, i.e. $m = 1$, the fact that θ_E defines an hermitian semi-positive form on $T(\mathcal{X})$ is equivalent (see Remark A.1 in Appendix A) to the fact that $c_1(E, |\ |)$ is a positive form, or, which is equivalent, to the fact that $c_1(E^*, |\ |)$ (with the metric induced by $|\ |$ by duality) is a negative form (its opposite is positive).

Remark B.5. The first Chern form $c_1(E, |\ |) = (i/2\pi)\Theta_{E,|\ |}$ induces also an hermitian form θ_E^* on $T^*(\mathcal{X}) \otimes E^*$, if one sets

$$\text{(B.25)} \qquad \theta_E^*(\xi \otimes e_j^*, \eta \otimes e_k^*) = \Big(\sum_{l,p} u_{jk;lp} d\zeta_l \wedge \overline{d\zeta_p}\Big)(\xi, \overline{\eta}), \quad 1 \leq j,k \leq m.$$

Since $\Theta_{E^*,|\ |^*} = -\Theta_{E,|\ |}$, the form θ_E^* is hermitian positive (*resp.* semi-positive) if and only if θ_{E^*} is hermitian negative (*resp.* semi-negative).

Definition B.9 (three concepts of positivity). For an hermitian holomorphic bundle $(E \to \mathcal{X}, |\ |)$, one may introduce the three following concepts of positivity. These three concepts coincide for line bundles.

a) The holomorphic hermitian bundle $(E \to \mathcal{X}, |\ |)$ is said to be *Nakano positive*, i.e. $E >_{\mathrm{N}} 0$ (*resp. Nakano semi-positive*, i.e. $E \geq_{\mathrm{N}} 0$) if and only if the hermitian form θ_E associated to $c_1(E, |\ |)$ as in (B.24) defines a scalar product (*resp.* a positive form) on each fiber of the bundle $T(\mathcal{X}) \otimes E \to \mathcal{X}$. The holomorphic hermitian bundle $(E \to \mathcal{X}, |\ |)$ is said to be *Nakano negative*, i.e. $E <_{\mathrm{N}} 0$ (*resp. Nakano semi-negative*, i.e. $E \leq_{\mathrm{N}} 0$) if and only if $-\theta_E$ defines a scalar product (*resp.* a positive form) on each fiber of the bundle $T(\mathcal{X}) \otimes E \to \mathcal{X}$.

b) The holomorphic hermitian bundle $(E \to \mathcal{X}, |\ |)$ is said to be *Griffiths positive* , i.e. $E >_{\mathrm{G}} 0$ (*resp. Griffiths semi-positive*, i.e. $E \geq_{\mathrm{G}} 0$) if and only if

$$\forall\, \xi \neq 0 \in T_x(\mathcal{X}),\ \forall\, e_x \neq 0 \in E_x,\ \theta_E(\xi \otimes e_x, \xi \otimes e_x) > 0$$
$$(\textit{resp.}\ \forall\, \xi \in T_x(\mathcal{X}),\ \forall\, e_x \in E_x,\ \theta_E(\xi \otimes e_x, \xi \otimes e_x) \geq 0).$$

The holomorphic hermitian bundle $(E \to \mathcal{X}, |\ |)$ is said to be *Griffiths negative*, i.e. $E <_{\mathrm{G}} 0$ (*resp. Griffiths semi-negative*, i.e. $E \leq_{\mathrm{G}} 0$) if and only if

$$\forall\, \xi \in T_x(\mathcal{X}),\ \xi \neq 0,\ \forall\, e_x \in E_x, e_x \neq 0,\ \theta_E(\xi \otimes e_x, \xi \otimes e_x) < 0$$
$$(\textit{resp.}\ \forall \xi \in T_x(\mathcal{X}),\ \forall\, e_x \in E_x,\ \theta_E(\xi \otimes e_x, \xi \otimes e_x) \leq 0).$$

c) The holomorphic hermitian bundle $(E \to \mathcal{X}, |\ |)$ is said to be *Bott-Chern positive*, i.e. $E >_{\mathrm{BC}} 0$ (resp. *Bott-Chern semi-positive*, i.e. $E \geq_{\mathrm{BC}} 0$) if and only if the hermitian form θ_{E^*} associated to $c_1(E, |\ |)$ as in (B.25) defines a scalar product (*resp.* a positive form) on $T^*(\mathcal{X}) \otimes E^*$.

REMARK B.6. Since $i\Theta_{E^*,|\ |^*} = -i^t\Theta_{E,|\ |}$ and

$$\theta_{E^*}(\xi\otimes\langle\cdot,e\rangle,\xi'\otimes\langle\cdot,e'\rangle) = -\theta_E(\xi\otimes e,\xi'\otimes e')$$

and for any local sections ξ,ξ' of $T(\mathcal{X})$, for any local sections e,e' of E, the conditions $E\geq_{\mathrm{G}} 0$ and $E^*\leq_{\mathrm{G}} 0$ are equivalent.

REMARK B.7. The fact that $E\geq_{\mathrm{BC}} 0$ is also equivalent (see [**And3**]) to the fact that there exist $\Phi_1,...,\Phi_M$ in $T^*(\mathcal{X})\otimes E$ such that

$$c_1(E,|\ |) = i\sum_{l=1}^{M}\Phi_j\otimes\Phi_j^*\,,$$

where $\Phi\mapsto\Phi^*$ denotes the conjugaison isometry introduced in (B.19). It then follows from the method leading to (B.23) that, as soon as $E\geq_{\mathrm{BC}} 0$, all Chern forms $c_p(R,|\ |)$, $p=1,2,...$ are positive.

The three positivity notions introduced in Definition B.9 coincide for line bundles ($m=1$). Note that this is the only case where it happens in general. However Nakano positivity (*resp.* Nakano semi-negativity) always implies Griffiths positivity (*resp.* Griffiths semi-negativity).

EXAMPLE B.6. On $\mathbb{P}^n(\mathbb{C})$, the line bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$, equipped with the usual metric

$$|\xi|_z = \frac{|\xi|}{\|z\|}\,,\ z=[z_0:\cdots:z_n]\,,$$

is positive: its first Chern form is $dd^c[\log\|z\|^2]$, which is the Kähler form on $\mathbb{P}^n(\mathbb{C})$. The tautological line bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(-1)$, equipped with the metric induced by that on $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$ is negative, since its first Chern form is $-dd^c[\log\|z\|^2]$.

B.3.5. Positivity and algebricity.

DEFINITION B.10 (Projective algebraic manifolds). A compact n-dimensional analytic manifold $\mathcal{X}$ is said to be *algebraic projective* if and only if there exists an holomorphic embedding $\mathcal{X}\xrightarrow{\iota}\mathbb{P}^N(\mathbb{C})$ for some $N\in\mathbb{N}$.

We recall here the important theorem of Chow, which can be viewed as a consequence of Remmert-Stein Theorem : if A is a purely $n+1$-dimensional closed analytic subset of $\mathbb{C}^{N+1}\setminus\{(0,...,0)\}$ with $n\geq 0$, its closure $\overline{A}$ in $\mathbb{C}^{N+1}$ is a closed analytic subset of $\mathbb{C}^{N+1}$. Chow's theorem reflects the G. A. G. A. principle formulated by J.P. Serre [**Ser1**].

THEOREM B.1 (Chow's theorem). *If A is a closed analytic subset of $\mathbb{P}^N(\mathbb{C})$, A is the zero set of an homogeneous polynomial ideal in $\mathbb{C}[X_0,...,X_N]$. In particular, if $\mathcal{X}$ is a n-dimensional submanifold in $\mathbb{P}^N(\mathbb{C})$, one can find homogeneous polynomials $P_1,...,P_M$ in $\mathbb{C}[X_0,...,X_N]$ such that*

$$\mathcal{X} = \{[z]\in\mathbb{P}^N(\mathbb{C})\,;\,P_1(z)=\cdots=P_M(z)=0\},$$
$$\Big\{[z]\in\mathbb{P}^N(\mathbb{C})\,;\,P_1(z)=\cdots=P_M(z)=\Delta_{\iota_1}(z)=\cdots=\Delta_{\iota_\mu}(z)=0\Big\}=\emptyset\,,$$

where $\Delta_{\iota_1},...,\Delta_{\iota_\mu}$ denote the minors with rank $N-n$ of the jacobian matrix

$$\frac{\partial(P_1,...,P_M)}{\partial(z_0,...,z_N)} = \Big[\frac{\partial P_j}{\partial z_k}\Big]_{\substack{1\leq j\leq M\\ 0\leq k\leq N}}.$$

We conclude this Appendix with the theorem of Kodaira [**Kod1**], which provides a caracterization of projective algebraic varieties based on the concept of positivity for line bundles.

THEOREM B.2 (Kodaira theorem). *A compact analytic manifold $\mathcal{X}$ is projective algebraic if and only if $\mathcal{X}$ carries a positive hermitian holomorphic line bundle $(E \to \mathcal{X}, |\ |)$.*

APPENDIX C

Divisors and Chow groups

C.1. Cartier divisors and Čech cohomology on complex manifolds

The two notions of Cartier and Weil divisor coincide within the frame of a smooth complex manifold. However, in a reduced complex analytic space (see Appendix D, Definition D.10), these two notions differ. The first one is related to the functional point of view (hence it is more analytic) and thus, intrinsically, brings into the picture the notion of sheaf, together with related algebraic ideas and methods. On the opposite side, the second one is related to the geometric point of view. The two points of view merge in the context of complex geometry.

C.1.1. Cartier divisors and line bundles.

Definition C.1 (Cartier divisor). We call *Cartier divisor* in a complex manifold $\mathcal{X}$ (assumed to be connected) the collection $\{(U_\alpha, f_\alpha)_\alpha\}$, where

(1) $\{(U_\alpha)_\alpha\}$ realizes a covering of $\mathcal{X}$ with open subsets ;
(2) for every α, the function f_α is meromorphic and not identically equal to 0 in U_α, such that for every pair of indices (α,β), the function f_α/f_β has a holomorphic extension into $U_\alpha \cap U_\beta$ with values in $\mathbb{C}^*$.

Defining a Cartier divisor in $\mathcal{X}$ amounts to define a global section of the quotient sheaf $\mathcal{M}^*_{\mathcal{X}}/\mathcal{O}^*_{\mathcal{X}}$. Here, $\mathcal{M}^*_{\mathcal{X}}$ denotes the multiplicative sheaf of non-zero sections of $\mathcal{M}_{\mathcal{X}}$. The sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{M}_{\mathcal{X}}$ (sometimes called sheaf of regular functions on $\mathcal{X}$, besides the sheaf $\mathcal{O}_{\mathcal{X}}$ of holomorphic functions on $\mathcal{X}$) is the sheaf of meromorphic functions in $\mathcal{X}$, that is, its stalk $\mathcal{M}_{\mathcal{X},x}$ at x is defined as the quotient field of the integral domain $\mathcal{O}_{\mathcal{X},x}$ of germs of holomorphic functions at the point x in $\mathcal{X}$. In terms of multidimensional complex analysis, when $\mathcal{X} = U$ is an open set in $\mathbb{C}^n$, defining a Cartier divisor in $\mathcal{X} = U$ amounts to define a set of data for the second Cousin problem in U.

To any Cartier divisor d in $\mathcal{X}$ corresponds an *holomorphic line bundle* $L \to \mathcal{X}$, called the associated line bundle, and denoted by $\mathcal{O}(d)$ (or $[-d]$, as it will be justified later on, see Remark C.2). Its transition functions, defined as

$$x \in U_\alpha \cap U_\beta \mapsto g_{\alpha,\beta}(x) := f_\alpha(x)/f_\beta(x),$$

determine a 1-cocycle $(U_\alpha \cap U_\beta, (g_{\alpha,\beta})_{\alpha,\beta})$ (with $g_{\alpha,\beta} \in \mathcal{O}^*_{\mathcal{X}}(U_\alpha \cap U_\beta)$). The collection of functions $(f_\alpha)_\alpha$ corresponds then to the global meromorphic section of the bundle $\mathcal{O}(d)$. The sheaf of holomorphic sections of $\mathcal{O}(d) = [-d]$ is identified with the locally free sheaf $\mathcal{K}(d)$, where

$$\mathcal{K}(d)_{|U_\alpha} = \mathcal{K}(d)(U_\alpha) := \frac{1}{f_\alpha}\mathcal{O}_{\mathcal{X}}(U_\alpha). \tag{C.1}$$

Its trivialization over the open set U_α is the mapping

$$\sigma \in \mathcal{K}(d)_{|U_\alpha} = \mathcal{K}(d)(U_\alpha) \longmapsto f_\alpha \sigma \in \mathcal{O}_{\mathcal{X}}(U_\alpha).$$

Conversely, the choice of a global meromorphic section $(f_\alpha)_\alpha$ of a line bundle (such a section corresponds to a Cartier divisor in $\mathcal{X}$) induces the construction of 1-cocycle $(g_{\alpha,\beta})_{\alpha,\beta}$ ($g_{\alpha,\beta} = f_\alpha/f_\beta$ in $U_\alpha \cap U_\beta$), thus determining the line bundle itself.

We can therefore identify the set of Cartier divisors on $\mathcal{X}$ either with the set of holomorphic line bundles $L \to \mathcal{X}$, or with the set of locally free sheafs of $\mathcal{O}_{\mathcal{X}}$-modules with rank 1. To any holomorphic line bundle $L \to \mathcal{X}$, one can attach the sheaf $\mathcal{O}(L)$ of its holomorphic sections. If $L \to X$ corresponds to a Cartier divisor d, then this sheaf is identified with the locally free sheaf $\mathcal{K}(d)$ introduced in (C.1).

The set of Cartier divisors in $\mathcal{X}$ is equipped with natural addition : the sum of two Cartier divisors corresponding to the given divisors $(U_{1,\alpha}, f_{1,\alpha})_\alpha$ and $(U_{2,\alpha}, f_{2,\alpha})_\alpha$ is the Cartier divisor described by the collection $(U_{1,\alpha} \cap U_{2,\alpha}, f_{1,\alpha} f_{2,\alpha})_\alpha$. The set of the Cartier divisors in $\mathcal{X}$, equipped with this addition, inherits a group structure and becomes the *group of divisors* of $\mathcal{X}$, denoted by $\mathrm{Div}\,(\mathcal{X})$. One very important subgroup of $\mathrm{Div}\,(\mathcal{X})$ is the subgroup of *principal divisors*. This is the subgroup which consisting of Cartier divisors which are defined by a global non zero section of the sheaf $\mathcal{M}_{\mathcal{X}}$ (in $\mathcal{X}$). The principal divisor $\mathrm{div}(f)$ attached to F is the Cartier divisor associated to the open cover $(U_\alpha, F_{|U_\alpha})$, where $(U_\alpha)_\alpha$ is an atlas of charts for $\mathcal{X}$. Two Cartier divisors d_1 and d_2 in $\mathcal{X}$ are called equivalent if and only if the divisor $d_1 - d_2$ is a principal divisor. We denote by $\mathrm{Pr}(\mathcal{X})$ or by $\mathrm{Div}_0(\mathcal{X})$ the subgroup of principal Cartier divisors.

C.1.2. Čech cohomology (a brief review). Before stating the important Proposition C.1, which specifies the algebraic correspondance between isomorphism classes of holomorphic line bundles and Cartier divisors, we need to review basic facts about Čech cohomology.

Čech cohomology is defined over a topological space $\mathcal{X}$ relatively to a sheaf of abelian groups, though we will use here frequently sheaves of commutative rings or $\mathcal{O}_{\mathcal{X}}$-modules [1]. The examples of sheaves of abelian groups which we use here are those of the constant sheaves $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, ($\mathcal{F}(U)$, $U \subset \mathcal{X}$, being one of these additive discrete or continuous groups) and the sheaves of rings $\mathcal{E}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}$ (considered as sheaves of additive groups) or the sheaves $\mathcal{E}_X^*, \mathcal{O}_X^*$ (considered as sheaves of multiplicative groups).

Given an open covering $(U_\alpha)_\alpha$ of $\mathcal{X}$, a Čech k-*cochain* is by definition a mapping which associates to any intersection of $k+1$ open sets $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ an element $h_{\alpha_0,\dots,\alpha_k} \in \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$, i.e. a section of the sheaf $\mathcal{F}$ over the open set $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$. Due to the group structure of $\mathcal{F}(U)$, one can define a structure of additive group on the set of k-cochains and thus obtain the group of k-cochains $\check{\mathrm{C}}^k(\mathcal{X}, \mathcal{U}, \mathcal{F})$ subordinated to the covering $\mathcal{U} = (U_\alpha)_\alpha$.

Furthermore, the boundary morphism $\delta = \delta_k$ from the group $\check{\mathrm{C}}^k(\mathcal{X}, \mathcal{U}, \mathcal{F})$ into the group $\check{\mathrm{C}}^{k+1}(\mathcal{X}, \mathcal{U}, \mathcal{F})$ is the group homomorphism defined as follows. For $k = 0$, take

$$(\delta_0 h)_{\alpha,\beta} := \delta^0 h(U_\alpha \cap U_\beta) = h(U_\beta)_{|U_\alpha \cap U_\beta} - h(U_\alpha)_{|U_\alpha \cap U_\beta}.$$

[1] With respect to the notions of *sheaf of abelian groups*, *sheaf of commutative rings*, *sheaf of $\mathcal{O}_{\mathcal{X}}$-modules* , one can found more material in Appendix D, Subsection D.2.1.

For $k=1$, take

$$(\delta^1 h)_{\alpha,\beta,\gamma} := \delta^1 h(U_\alpha \cap U_\beta \cap U_\gamma) = \Big[h(U_\beta \cap U_\gamma) - h(U_\gamma \cap U_\alpha) + h(U_\alpha \cap U_\beta)\Big]_{|U\alpha\cap U_\beta \cap U_\gamma},$$

and continue in the same vein for $k \geq 2$. It is easy to see that $\delta_{k+1} \circ \delta_k = \delta \circ \delta = 0$ and that, for $k \geq 1$, the image $\check{\mathrm{B}}^k(\mathcal{X},\mathcal{U},\mathcal{F})$ of $\check{\mathrm{C}}^{k-1}(\mathcal{X},\mathcal{U},\mathcal{F})$ by $\delta = \delta_{k-1}$ is a subgroup (called *the k-th Čech coboundary subgroup*) of the kernel $\check{\mathrm{Z}}^k(\mathcal{X},\mathcal{U},\mathcal{F})$ of $\delta = \delta_k$ (called *the subgroup of Čech k-cocycles*). The quotient group

$$\check{\mathrm{H}}^k(\mathcal{X},\mathcal{U},\mathcal{F}) := \frac{\check{\mathrm{Z}}^k(\mathcal{X},\mathcal{U},\mathcal{F})}{\check{\mathrm{B}}^k(\mathcal{X},\mathcal{U},\mathcal{F})}$$

materializes the obstruction for a k-cocycle to be a k-coboundary and is called k-th *Čech cohomology group* of $\mathcal{X}$, with values in $\mathcal{F}$, subordinated to the open covering $\mathcal{U}$. To define the groups $\check{\mathrm{H}}^k(\mathcal{X},\mathcal{F})$ independently of the covering $\mathcal{U}$, one has to take the inductive limit with respect to all possible coverings $\mathcal{U}$ of $\mathcal{X}$ (more and more refined).

C.1.3. Isomorphism classes of holomorphic line bundles; the Picard group. Let us recall from Appendix B (Section B.1.2, transposed here to the holomorphic context) the notion of isomorphism between holomorphic line bundles. We say that two holomorphic line bundles $L_1 \stackrel{\pi_1}{\to} \mathcal{X}$ and $L_2 \stackrel{\pi_2}{\to} \mathcal{X}$ over a complex manifold $\mathcal{X}$ are *isomorphic* if and only if there exist a biholomorphic function $f\ : \mathcal{X} \to \mathcal{X}$, a biholomorphic mapping $F\ : L_1 \to L_2$ such that $f \circ \pi_1 = \pi_2 \circ F$. This definition extends naturally to the notion of isomorphism between two holomorphic vector bundles of the same rank $E_1 \to \mathcal{X}$ and $E_2 \to \mathcal{X}$. Thus, the set of isomorphism classes of line bundles over $\mathcal{X}$, can be equipped with the operation which consists in making the tensorial product of representatives $L_1 \to \mathcal{X}$ and $L_2 \to \mathcal{X}$ of two isomorphism classes. More specifically, it can be easily seen that the tensor product $(L_1 \otimes L_2) \to \mathcal{X}$ gives a holomorphic vector bundle which is a representative of the class obtained by tensoring any representant in the isomorphism class of $L_1 \to \mathcal{X}$ with any representant in the isomorphism class of $L_2 \to \mathcal{X}$. This induces a group structure on the set of equivalence classes of holomorphic line bundles on $\mathcal{X}$.

PROPOSITION C.1 (Picard group). *The group of isomorphism classes of holomorphic line bundles over $\mathcal{X}$ is isomorphic to the quotient of the group of the Cartier divisors* $\mathrm{Div}(\mathcal{X})$ *by the subgroup of the Cartier principal divisors* $\mathrm{Pr}(\mathcal{X})$*, or to the cohomology group $\check{H}^1(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})$ for the* Č*ech cohomology, that is, it is isomorphic to the quotient*

$$\check{\mathrm{H}}^1(\mathcal{X},\mathcal{O}^*_{\mathcal{X}}) = \lim_{\overrightarrow{\mathcal{U}}} \frac{\check{\mathrm{Z}}^1(\mathcal{X},\mathcal{U},\mathcal{O}^*_{\mathcal{X}})}{\check{\mathrm{B}}^1(\mathcal{X},\mathcal{U},\mathcal{O}^*_{\mathcal{X}})}$$

obtained as the inductive limit of the quotients of groups of 1*-cocycles* $\check{\mathrm{Z}}^1(\mathcal{X},\mathcal{U},\mathcal{O}^*_{\mathcal{X}})$*, attached to an open covering $\mathcal{U}$ of $\mathcal{X}$, by their* 1*-coboundary subgroups* $\check{\mathrm{B}}^1(\mathcal{X},\mathcal{U},\mathcal{O}^*_{\mathcal{X}})$*. The group of isomorphism classes of holomorphic vector bundles is called the Picard group of the complex manifold $\mathcal{X}$ (and denoted as* $\mathrm{Pic}(\mathcal{X})$*).*

C.1.4. The first Chern class of a C^∞ line bundle. We take the opportunity here to define the *first Chern class of a line bundle* $L \to \mathcal{X}$ (non necessarily holomorphic, but here just C^∞) over the complex manifold $\mathcal{X}$. The case of holomorphic vector bundles will be a particular case.

Exactly as in Proposition C.1 (in the holomorphic context), one can show (in the C^∞ context) that there exists an isomorphism between the group of isomorphism classes of C^∞ line bundles $L \to \mathcal{X}$ over $\mathcal{X}$ and the Čech cohomology groups $\check{H}^1(\mathcal{X}, \mathcal{E}^*_{\mathcal{X}})$, where $\mathcal{E}_{\mathcal{X}}$ (here instead of $\mathcal{O}_{\mathcal{X}}$ used previously in the holomorphic case, see Proposition C.1) denotes the sheaf of germs of C^∞ functions over $\mathcal{X}$: to the isomorphism class of $L \to \mathcal{X}$, one associates the class in $\check{\mathrm{H}}^1(\mathcal{X}, \mathcal{E}^1_{\mathcal{X}})$ of the 1-cocycle $(U_\alpha \cap U_\beta, g_{\alpha,\beta})_{\alpha,\beta}$ corresponding to it. On the other hand, one has the long exact sequence of coherent sheafs

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E}_X \stackrel{\exp(2i\pi(\cdot))}{\longrightarrow} \mathcal{E}^*_X \to 1. \tag{C.2}$$

Furthermore, the Čech cohomology groups $\check{\mathrm{H}}^1(\mathcal{X}, \mathcal{E}_{\mathcal{X}})$ and $\check{\mathrm{H}}^2(\mathcal{X}, \mathcal{E}_{\mathcal{X}})$ are zero because of the flatness of the sheaf $E_{\mathcal{X}}$. Taking from (C.2) the long cohomology sequence, it follows that $\check{\mathrm{H}}^1(\mathcal{X}, \mathcal{E}^*_{\mathcal{X}})$ and $\check{\mathrm{H}}^2(\mathcal{X}, \mathbb{Z})$ are isomorphic (as additive groups). Thus, it is natural to set the following definition.

Definition C.2 (the first Chern class of a C^∞ line bundle). If $L \to \mathcal{X}$ is a C^∞ $\mathbb{C}$-line bundle over a complex manifold $\mathcal{X}$, then we call *first Chern class* of the line bundle $L \to \mathcal{X}$ the image in $\check{\mathrm{H}}^2(\mathcal{X}, \mathbb{Z})$ of the class in $\check{\mathrm{H}}^1(\mathcal{X}, \mathcal{E}^*_{\mathcal{X}})$ of the 1-cocycle (with respect to the sheaf $\mathcal{E}^*_{\mathcal{X}}$) attached to the C^∞ line bundle $L \to \mathcal{X}$.

Remark C.1. When $L \to \mathcal{X}$ was an holomorphic line bundle, we introduced in Appendix B, Definition B.8, the notion of Chern characteristic class $c_1(L)$ of $L \to \mathcal{X}$, as an element in $H^2_{\mathrm{DR}}(\mathcal{X}, \mathbb{R})$. In fact, the two groups $H^2_{\mathrm{DR}}(\mathcal{X}, \mathbb{R})$ and $\check{\mathrm{H}}^2(\mathcal{X}, \mathbb{R})$ are isomorphic and the element in $\check{\mathrm{H}}^2(\mathcal{X}, \mathbb{R})$ which corresponds to $c_1(E)$ via this isomorphism happens to define an element in $\check{\mathrm{H}}^2(\mathcal{X}, \mathbb{Z})$, which is precisely the first Chern class of $L \to \mathcal{X}$ in Definition C.2.

C.2. Weil divisors on a complex manifold

A closed analytic hypersurface in a complex manifold $\mathcal{X}$ is, by definition, a closed subset of $\mathcal{X}$ defined in a local chart about each of its points x by a local equation $f_x = 0$, where $f_x \in \mathcal{O}_{\mathcal{X},x}$. It is called irreducible if it cannot be written as a union of two hypersurfaces H_1 and H_2 such that $H_1 \neq H$ and $H_2 \neq H$.

Definition C.3 (Weil divisor). A *Weil divisor* in a n-dimensional complex manifold $\mathcal{X}$ (assumed here to be connected) is a locally finite[2], linear combination with integer coefficients of closed analytic irreducible hypersurfaces $(H_\gamma)_\gamma$:

$$D = \sum_\gamma m_\gamma H_\gamma. \tag{C.3}$$

When all m_γ are positive or zero, the Weil divisor D is called *effective* . The set of Weil divisors inherits the structure of additive commutative group, called also *the group of* $(n-1)$*-cycles in* $\mathcal{X}$.

To any Cartier $d = (U_\alpha, f_\alpha)_\alpha$ on $\mathcal{X}$, one can naturally associate a Weil divisor

$$D = \sum_{H \text{ hypersurface of } \mathcal{X}} \mathrm{order}_H(d) \times H\,,$$

where the order of $d = (f_\alpha)_\alpha$ along the hypersurface H is defined in a local chart U_α intersecting H as follows : use an arbitrary regular point x in $U_\alpha \cap H$ such

[2]Locally finite means that, given an arbitrary compact subset $K \subset\subset \mathcal{X}$, there are only finitely many hypersurfaces H_γ intersecting K and such that $m_\gamma \neq 0$ in the formal development (C.3).

that one can assume that $U_\alpha \cap H = \{\zeta_n = 0\}$ in local coordinates about this point. The order $\mu := \mathrm{order}_H(d)$ is then defined as the exponent $\mu = \mu_{H,x} \in \mathbb{Z}$ such that, in a neighborhood of x, one has in local coordinates $f_\alpha(\zeta) = \zeta_n^\mu u_\alpha(\zeta)$, u_α being a meromorphic function which is not identically equal to zero on $\{\zeta_n = 0\}$. This exponent does not depend on the regular point x in H.

Furthermore, one can associate a Weil divisor (denoted also by $\mathrm{div}(F)$, as that was done for Cartier divisors in Section C.1) to a meromorphic function F which is not identically zero in $\mathcal{X}$, namely

$$\mathrm{div}(F) = \sum_H \mathrm{order}_H(F)\, H.$$

One thus obtains so-called *principal Weil divisors* . The group of Weil divisors quotiented by the subgroup of principal Weil divisors is denoted as $A_{n-1}(\mathcal{X})$ and called the *Chow group* of order $n-1$ of the complex manifold $\mathcal{X}$.

On a complex manifold, it is possible, given a Weil divisor $D = \sum_\gamma m_\gamma H_\gamma$, to associate to it a Cartier divisor. To this end, it is enough, given any irreducible closed hypersurface H_γ and a point $x \in \mathcal{X}$, to construct a local chart $U_{\gamma,x}$ about x and an irreducible equation $\{h_{\gamma,x} = 0\}$ defining (in a reduced way) H_γ in local coordinates in $U_{\gamma,x}$ (one may have $h_{\gamma,x} \equiv 1$ in case $x \notin H_\gamma$). Given a point x in $\mathcal{X}$, there are finitely many γ such that $m_\gamma \neq 0$. Take then U_x as the intersection of all such H_γ and

$$f_x = \prod_H h_{\gamma,x}^{m_\gamma}.$$

Then $(U_x, f_x)_{x\in\mathcal{X}}$ defines a Cartier divisor.

REMARK C.2. Note that, if s is an holomorphic section of the Cartier divisor $d = (U_x, f_x)_x$ associated to the Weil divisor $D = \sum_\gamma m_\gamma H_\gamma$, then s induces a Weil divisor

$$\mathrm{div}(s) = \sum_H \mathrm{order}_H(s)\, H$$

such that $\mathrm{div}(s) + D$ is effective. This explains why the line bundle $\mathcal{O}(d)$ associated to a Cartier divisor as in subsection C.1.1 is also denoted as $[-d]$.

Therefore, on an analytic complex n-dimensional manifold $\mathcal{X}$, the notions of Cartier and Weil divisors coincide. It follows then from Proposition C.1 that the Picard group $\mathrm{Pic}(\mathcal{X})$ and the $(n-1)$-Chow group $A_{n-1}(\mathcal{X})$ coincide in this case. In the framework of reduced complex analytic spaces, introduced in Appendix D, Definition D.10, one can still define the two notions, but they to not coincide anymore, even though $\mathcal{X}$ is normal: one can always attach to a Cartier divisor a Weil divisor (as before, once one uses the normalization introduced in Appendix D, Theorem D.4), but, with respect to the converse, there are in general obstructions: one can attach to a Weil divisor D a Cartier divisor if and only if D can be expressed locally as a principal Weil divisor.

EXAMPLE C.1 (the projective space $\mathbb{P}^n(\mathbb{C})$). In the complex manifold $\mathbb{P}^n(\mathbb{C})$, the Cartier divisor, corresonding to the finite family (U_j, f_j), $j = 0, ..., n$, where $U_j := \{[z_0 : \cdots : z_n]\,;\, z_j \neq 0\}$ and $f_j([z_0 : \cdots : z_n]) = z_j/z_0$ induces the Weil divisor $-[z_0 = 0]$. The isomorphism class of the line bundle corresponding to the Picard group $\mathrm{Pic}(\mathbb{P}^n(\mathbb{C}))$ is denoted by $\mathcal{O}(1)$. It generates the Picard group of $\mathbb{P}^n(\mathbb{C})$, which is isomorphic to $\mathbb{Z}$. The Weil divisor $[z_0 = 0] = \mathrm{div}(z_0)$ is a generator for the $(n-1)$ Chow ring $A_{n-1}(\mathbb{P}^n(\mathbb{C}))$ (see also Example B.5 in Appendix B).

EXAMPLE C.2 (homogeneous coordinates on smooth complete toric varieties). One important class of examples, extending naturally that of the projective space $\mathbb{P}^n(\mathbb{C})$, is the class of smooth, complete toric varieties, obtained by gluing copies of the affine space $\mathbb{C}^n$, through identifications induced by monomial transformations. The construction of the toric variety lies on *a complete simple fan* Σ, which is a finite family of cones, that satisfy the following conditions:

- each cone σ is strictly rational, that is, it is generated by elements of $\mathbb{Z}^n$ and does not contain any subspace of the vector space $\mathbb{R}^n$, except the trivial subspace 0.
- the fan Σ partitions $\mathbb{R}^n$ in such a way that any face $\tau \in \tau(\sigma)$ of any cone $\sigma \in \Sigma$ still belongs to Σ, and the intersection of two cones $\sigma_1, \sigma_2 \in \Sigma$ is a face of both cones σ_1 and σ_2.
- it is assumed that the cones σ of dimension n in the family are generated by a base of $\mathbb{Z}^n$. The $n \times n$ matrix formed by these base vectors is assumed to have its determinant equal to ± 1.

In order to realize a compact complex manifold gluing together copies of $\mathbb{C}^n$ (in correspondance with n-dimensional cones in the simple fan) with respect to the choice of the fan, one makes the identifications through the following monomial maps : to such a n-dimensional cone generated by primitive vectors (a vector with integer coordinates is called primitive if its coordinates are relatively prime)

$$\eta_j = (\eta_{j1}, ..., \eta_{jn}) \,,\ j = 1, ..., n$$

one associates the monomial transformation

$$(\zeta_1, ..., \zeta_n) \longmapsto \Big(\prod_{j=1}^{n} \zeta_j^{\eta_{j1}}, ..., \prod_{j=1}^{n} \zeta_j^{\eta_{jn}} \Big)$$

(see [**Dan, Elh, Ew**]). The $d = n + r$ cones $\xi_1, ..., \xi_d$ of dimension 1 of the fan (also called rays) are put into correspondence ([**Co1**]) with *homogeneous coordinates* $z_1, ..., z_d$ in a such a manner that the complex manifold $\mathcal{X}$ of dimension n thus constructed is materialized as the geometric quotient

$$\mathcal{X} \simeq \frac{\mathbb{C}^d \setminus \Big\{ z \in \mathbb{C}^d \text{ s.t. } \prod_{\xi_j \notin \tau(\sigma)} z_j = 0 \,;\, \sigma \in \Sigma(n) \Big\}}{\Big\{ (t_1, ..., t_d) \in \mathbb{C}^d \,;\, \prod_{j=1}^{d} t_j^{\xi_{jk}} = 1 \,,\ k = 1, ..., n \Big\}} \tag{C.4}$$

of

$$\mathbb{C}^d \setminus \Big\{ z \in \mathbb{C}^d \,\text{s.t.} \prod_{\xi_j \notin \tau(\sigma)} z_j = 0 \,;\, \sigma \in \Sigma(n) \Big\}$$

by the subgroup G (isomorphic to $(\mathbb{C}^*)^r$, $r = d - n$)

$$G = \Big\{ (t_1, ..., t_d) \in \mathbb{C}^d \,;\, \prod_{j=1}^{d} t_j^{\xi_{jk}} = 1 \,,\ k = 1, ..., n \Big\} \,,$$

where $\xi_j = (\xi_{j1}, ..., \xi_{jn})$ is a primitive vector generating the ray ξ_j. A simple illustration of the main ideas of the above construction is the realization of the projective space $\mathbb{P}^n(\mathbb{C})$ as a geometric quotient in the case $r = d - n = 1$. The Picard group of such a complex manifold is isomorphic to $\mathbb{Z}^r$. If in addition one assumes that the vectors $\xi_1, ..., \xi_n$ generate one of the n-dimensional cones $\sigma \in \Sigma$, then the classes of the Cartier divisors related to the Weil divisors $\{z_{n+1} = 0\}$,...,$\{z_{n+r} = 0\}$

form a basis for the additive group $\mathrm{Pic}(\mathcal{X})$. The classes of the Weil divisors $[z_j = 0]$, $j = n+1, ..., n+r$ in $A_{n-1}(\mathcal{X})$ generate $A_{n-1}(\mathcal{X})$ as a free $\mathbb{Z}$-module with rank r. Remark also that the open subset of $\mathbb{C}^{n+r}$

$$\mathbb{C}^{n+r} \setminus \Big\{ z \in \mathbb{C}^{n+r} \text{ s.t. } \prod_{\xi_j \text{ non face de } \sigma} z_j = 0\,;\, \sigma \in \Sigma\,,\ \dim(\sigma) = n \Big\}$$

(which is the complement of the zero set of a specific monomial ideal, called *irrelevant ideal* $\mathcal{I}_{\mathrm{irr}}(\mathcal{X})$) appears as a "bundle" with complex tori $(\mathbb{C}^*)^r$ as fibers (instead of vectorial spaces as in the case of tautological sheaf over $\mathbb{P}^n(\mathbb{C})$, see Example B.5 in Appendix B) over the basis $\mathcal{X}$.

C.3. Chow groups on a reduced analytic space

In this section, $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ denotes a n-dimensional reduced analytic space ($\mathcal{O}_{\mathcal{X}}$ being the structure sheaf) and $\mathcal{M}_{\mathcal{X}}$ denotes the sheaf of meromorphic functions on $\mathcal{X}$, that is the stalk $\mathcal{M}_{\mathcal{X},x}$ is the quotient field of the integral domain $\mathcal{O}_{\mathcal{X},x}$. The above setting allows to deal with singularities, think for example of $\mathcal{X}$ being properly embedded as a purely dimensional analytic subset $\mathcal{X}$ in some ambiant complex manifold $\widetilde{\mathcal{X}}$ of dimension $N > n$).

The notion of Weil divisor on $\mathcal{X}$ is defined exactly as it was defined in the case where $\mathcal{X}$ was non singular. Definition C.3 remains valid. A Weil divisor is a locally finite linear combination with integer coefficients

$$D = \sum_{\gamma} m_\gamma H_\gamma\,,$$

where the H_γ are irreducible analytic hypersurfaces. A closed analytic hypersurface $H \subset \mathcal{X}$ is a closed subset of $\mathcal{X}$ that can be defined about each of its point x as $H = \{f_x = 0\}$, where $f_x \in \mathcal{O}_{\mathcal{X},x}$. The set of Weil divisors inherits naturally a structure of abelian additive group.

Let us assume that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is normal (see Appendix D, Definition D.11. Then, $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *regular in codimension one* at each point, which implies that, given any meromorphic function $F \in \mathcal{M}_{\mathcal{X}}(\mathcal{X})$ and any irreducible closed hypersurface H in $\mathcal{X}$, one can define the order $\mathrm{order}_H(F)$ as the valuation of F along H at a generic point of H. Therefore, as in Section C.2, one can associate to any global section F of $\mathcal{M}_{\mathcal{X}}$ a Weil divisor

$$\mathrm{div}(F) = \sum_H \mathrm{order}_H(F)\, H\,.$$

Such Weil divisors of the form $\mathrm{div}(F)$ are called *principal*. Let us come to the notion of rational equivalence between cycles (see for example [**Ha1**], Appendix 1, [**Fu**]), or [**Fu2**], Chapter 5, for the specific example of toric varieties which will be of particular interest for us in this monograph).

Definition C.4 (the Chow group $A_{n-1}(\mathcal{X})$). Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ a reduced complex analytic space and $\overline{\mathcal{X}} \xrightarrow{\pi} \mathcal{X}$ be its normalization (see Appendix D, Theorem D.4). Two Weil divisors D_1 and D_2 on $\mathcal{X}$ are said to be *rationally equivalent* as codimension 1-cycles on $\mathcal{X}$ if and only if $D_1 - D_2 = \pi_*(\mathrm{div}(F))$ for some $F \in \mathcal{M}_{\overline{\mathcal{X}}}(\overline{\mathcal{X}})$, which means (in terms of integration currents on the complex analytic spaces $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and

$(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}})$[3]) that $[D_1 - D_2] = \pi_*[\mathrm{div}(F)]$. The group of Weil divisors modulo rational equivalence is the Chow group $A_{n-1}(\mathcal{X})$.

When A is any irreducible closed analytic subset of a reduced complex analytic set $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, one can view $(A, (\mathcal{O}_{\mathcal{X}})_{|A})$ as a reduced analytic space and therefore, in view of Definition C.4, introduce the notion of principal Weil divisor on A. Then, one can also consider the notion of rational equivalence between Weil divisors on $(A, (\mathcal{O}_{\mathcal{X}})_{|A}))$. Let $\overline{A} \xrightarrow{\pi_A} A$ be the normalization of the reduced analytic space $(A, (\mathcal{O}_{\mathcal{X}})_{|A})$. A Weil divisor D_A on A is called principal (with respect to A) if and only if there is element $F_A \in \mathcal{M}_{\overline{A}}(\overline{A})$ such that $[D_A] = (\pi_A)_*[\mathrm{div}(F_A)]$. If ι_A denotes the embedding of A in $\mathcal{X}$, then the principal cycle $D_A = \pi_*[\mathrm{div}(F_A)]$ induces a codimension (codim$A + 1$)-cycle on $\mathcal{X}$, namely $(\iota_A)_*(D_A) = (\iota_A \circ \pi_A)_*[\mathrm{div}(F_A)]$. We are then led to the following definition.

Definition C.5 (the Chow groups $A_r(\mathcal{X})$, $r = 0, ..., n-1$). Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ an irreducible n-dimensional complex analytic space and $0 \leq r \leq n-1$. Two r-analytic cycles on $\mathcal{X}$ are said to be rationally equivalent if and only they differ by a locally finite linear combination of cycles of the form $(\iota_{A_\gamma})_*(D_{A_\gamma}) = (\iota_{A_\gamma} \circ \pi_{A_\gamma})_*[\mathrm{div}(F_{A_\gamma})]$ (as introduced above), the A_γ being closed irreducible analytic subsets of $\mathcal{X}$ with dimension $r+1$. The additive abelian group of r-analytic cycles on $\mathcal{X}$ modulo rational equivalence is called the Chow group $A_r(\mathcal{X})$.

C.4. Chow groups in the algebraic context

Add the references [**HaLT**] *and* [**Groth**] *(XII, Theorem 4.4), where it is mentionned (or proved) the equality of between the algebraic and analytic Picard group for a complete integral (i.e irreducible and reduced) algebraic scheme. Is it also true for Chow groups ?*

Example C.3 (simplicial complete toric varieties and orbifolds). Keep the two first items for the definition of the complete simple rational fan Σ as in Example C.2 but, instead of the last one, assume that any cone in Σ is rational simplicial, that is, it is generated by a set of primitive vectors which can be completed as a basis of $\mathbb{R}^n$ (not anymore of $\mathbb{Z}^n$). Such a rational fan is now called *simplicial* [4]. The object realized as the geometric quotient (C.4) inherits a structure of reduced analytic set; it is a singular algebraic variety, called a *toric complete simplicial variety.* The fact that the transition maps are monomial makes indeed this structure more precise from the algebraic point of view: it is called an *orbifold* structure. If $\xi_1, ..., \xi_n$ are n rays in the fan which generate one of the n-dimensional cones, and $\xi_{n+1}, ..., \xi_{n+r}$ being the remaining ones, the corresponding classes of the Weil divisors $[z_{n+j} = 0]$, $j = 1, ..., r$ (expressed in homogeneous coordinates $[z_1 : \cdots : z_{n+r}]$ on $\mathcal{X}$, one coordinate being associated to each ray, see Example C.2) generate the Chow group $A_{n-1}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. In this case $A_{n-1}(\mathcal{X})$ has rank r, is not in general free, but can be decomposed as $A_{n-1}(\mathcal{X})_{\mathrm{tor}} \oplus L$, where L denotes a free group with rank r

[3]See Appendix A, Section A.3, in particular Remark A.5, for the notion of integration current on a Weil divisor on a reduced analytic space.

[4]Note that any rational simplicial fan can be algorithmically refined (introducing additional cones) in order to become simple. In fact, one can forget completely about the simpliciality hypothesis (that is the third item among properties which are imposed to the fan) since any rational fan can be refined (thanks to subdivisions of cones) in order to become simplicial (for a description of the algorithmic procedure, see [**MumK**], or also [**Fu2**], Section 2.6).

and $A_{n-1}(\mathcal{X})_{\rm tor}$ is a finite abelian group which corresponds to the torsion part of $A_{n-1}(\mathcal{X})$.

C.5. Ampleness for holomorphic line bundles

The notions of *semi-ampleness* and *ampleness* for an holomorphic vector bundle $E \to \mathcal{X}$ over a n-dimensional complex manifold are both notions of algebraic nature. We restrict mainly ourselves in this section to the case of line bundles, in correspondence (as seen from subsection C.1.1 above) with Cartier divisors.

Definition C.6 (various notions of ampleness for holomorphic line bundles). Let $L \to \mathcal{X}$ be a holomorphic line bundle above the n-dimensional complex manifold $\mathcal{X}$. Let $H^0(\mathcal{X}, L)$ be the $\mathbb{C}$-vector space of global holomorphic sections of $L \to \mathcal{X}$.

(1) The line bundle $L \to \mathcal{X}$ is said to be *globally generated* if and only if, for any $x \in \mathcal{X}$, the evaluation map
$$s \in H^0(\mathcal{X}, L) \longmapsto s(x) \in L_x$$
is surjective, i.e. any local holomorphic section of $L \to \mathcal{X}$ can be expressed as a linear combination of global ones.

(2) The holomorphic line bundle $L \to \mathcal{X}$ is said to be *semi-ample* if and only if, for any $p \in \mathbb{N}$ large enough, the line bundle $L^{\otimes^p} \to \mathcal{X}$ is globally generated.

(3) The holomorphic line bundle $L \to \mathcal{X}$ is said to be *very ample* if and only if the evaluation maps
$$\begin{aligned} s \in H^0(\mathcal{X}, L) \quad &\longmapsto \quad [j^1(s)]_x := \Big(\dot{s}_x \text{ modulo } \mathfrak{M}^2_{\mathcal{X},x}\mathcal{O}_{\mathcal{X},x}(L)\Big) \in \frac{\mathcal{O}_{\mathcal{X},x}(L)}{\mathfrak{M}^2_{\mathcal{X},x}\mathcal{O}_{\mathcal{X},x}(L)} \\ s \in H^0(\mathcal{X}, L) \quad &\longmapsto \quad s(x) + s(x') \in L_x \oplus L'_x \end{aligned}$$
are both surjective[5] for any x, x' in $\mathcal{X}$ such that $x \neq x'$. Here $\mathfrak{M}_{\mathcal{X},x}$ denotes the maximal ideal in the local ring $\mathcal{O}_{\mathcal{X},x}$.

(4) The holomorphic line bundle $L \to \mathcal{X}$ is said to be *ample* if and only if $L^{\otimes^p} \to \mathcal{X}$ is very ample for $p \in \mathbb{N}$ large enough.

Remark C.3. Similar definitions hold for holomorphic vector bundles of rank $m > 1$. One needs just to replace the tensorial product $L^{\otimes^p} \to \mathcal{X}$ in the second and fourth items by $S^p\, E \to \mathcal{X}$, where $S^p\, E \to \mathcal{X}$ denotes the p-symmetric power[6] of $E \to \mathcal{X}$.

The relation between such algebraic notions and the metric notions of *positivity* in the Griffiths or Nakano sense (see Appendix D, Section B.3.4, Definition B.9) are illustrated by the following proposition.

Proposition C.2 (ampleness and positivity). *Let $E \to \mathcal{X}$ an holomorphic vector bundle with rank m over a complex n-dimensional analytic manifold $\mathcal{X}$.*

[5]One calls $[j^1(s)]_x$ the 1-jet of s at the point x ; if one replaces $\mathfrak{M}^2_{\mathcal{X},x}$ by $\mathfrak{M}^{k+1}_{\mathcal{X},x}$, one gets the k-jet of s at x, denoted as $[j^k(s)]_x$.

[6]When V is a $\mathbb{C}$-vector space of dimension m, the symmetric power $S^p(V)$ is the quotient of $V^{\otimes^p}$ under the natural action of the symmetric group $\mathcal{S}_p$: $\mathfrak{s} \cdot (v_1 \otimes \cdots \otimes v_p) = v_{\mathfrak{s}(1)} \otimes \cdots \otimes v_{\mathfrak{s}(p)}$ for any arbitrary permutation $\mathfrak{s}$ of $\{1, ..., p\}$.

- *If $E \to \mathcal{X}$ is globally generated, one may equip $E \to \mathcal{X}$ with an hermitian metric $|\ |$ such that the hermitian holomorphic vector bundle $(E \to \mathcal{X}, |\ |)$ is semi-positive in the sense of Griffiths (or such that the dual hermitian bundle $(E^* \to \mathcal{X}, |\ |)$ is semi-negative in the sense of Nakano).*
- *In particular, if $L \to \mathcal{X}$ is a semi-ample holomorphic line bundle, one may equip $L \to \mathcal{X}$ with an hermitian metric $|\ |$ such that the holomorphic hermitian line bundle $(L \to \mathcal{X}, |\ |)$ is semi-positive.*

EXAMPLE C.4 (ampleness on $\mathbb{P}^n(\mathbb{C})$). Over $\mathbb{P}^n(\mathbb{C})$, the line bundle $\mathcal{O}(1) \to \mathbb{P}^n(\mathbb{C})$, (see C.1) associated to the Weil divisor $-[z_0 = 0]$, is the prototype of a very ample bundle. If $\mathcal{X}$ is a projective algebraic manifold (i.e. is embedded via ι in some $\mathbb{P}^N(\mathbb{C})$ for N large enough), the line bundle $\iota^*\mathcal{O}(1) \to \mathcal{X}$ (that is, if one considers $\mathcal{X} \subset \mathbb{P}^N(\mathbb{C})$, the line bundle $\mathcal{O}(1)_{|\mathcal{X}} \to \mathcal{X}$ is a very ample holomorphic bundle over $\mathcal{X}$.

EXAMPLE C.5 (ampleness on smooth complete toric varieties). We illustrate here the various notions of ampleness in the context of n-dimensional complete toric manifolds, such as introduced in Example C.2 above. The presentation of such a manifold as the geometric quotient (C.4) allows to associate to any homogeneous coordinate $z_1, ..., z_{n+r}$ (that is to any ray ξ_j of the simple rational fan[7] the effective Weil divisor expressed in homogeneous coordinates as $D_j = [z_j = 0]$ (see [**Co1**] or [**CLO**]). Given the Weil divisor

$$D = a_1 D_1 + \cdots + a_{n+r} D_{n+r}, , \qquad a_1, ..., a_{n+r} \in \mathbb{Z}\,,$$

one can associate to it the convex polyedron P_D defined in the dual space $\mathbb{R}^n_{x_1^*,...,x_n^*}$ as

$$P_D := \{x^* \in (\mathbb{R}^n)^*\ ; \langle x^*, \xi_j\rangle + a_j \geq 0\,,\ j = 1, ..., n+r\}. \tag{C.5}$$

If $f(x_1, ..., x_n)$ is a generic Laurent polynomial with support P_D in which one substitutes

$$x_k := \prod_{j=1}^{n+s} z_l^{\xi_{jk}}\,,\ k = 1, ..., n,$$

one gets a rational function in $(z_1, ..., z_{n+r})$ which denominator M_D is precisely $z_1^{a_1} \cdots z_{n+r}^{a_{n+r}}$; denote as F the numerator of this rational function. Such a rational function F/M_D can be interpreted as a global holomorphic section of the line bundle $\mathcal{O}(D) = [-D]$, and the Weil divisor $\mathrm{div}(f) + \sum_1^{n+r} D_j$ (that is the Cartier divisor $\mathrm{div}(F)$) is effective, see Remark C.2 above. Let Δ_D the convex polyedron in $(\mathbb{R}^+)^{n+r}_{t_1,...,t_{n+r}}$ defined as the support of F for f generic. If $(\xi_1, ..., \xi_n)$ generate one of the n-dimensional cones σ of the simple fan Σ, let $\Delta_{D,\sigma}$ be the projection of Δ_D on the space $\mathbb{R}^n_{t_1,...,t_n}$. To any such n dimensional cone σ in the simple fan Σ, one can associate in such a way a convex polyedron $\Delta_{D,\sigma}$ in $(\mathbb{R}^+)^n$. The line bundle $\mathcal{O}(D) = [-D]$ is semi-ample if and only if, for any n-dimensional cone σ in Σ, the origin is a vertex of $\Delta_{D,\sigma}$, which means (assuming for the sake of simplicity that σ is generated by the rays $\xi_1, ..., \xi_n$) that the vector

$$u^*_{D,\sigma} := -\sum_{j=1}^{n} a_j \xi_j$$

[7] For the sake of simplicity, we also denote as ξ_j the primitive vector in $\mathbb{Z}^n$ that generates the ray ξ_j.

is, for any such σ, a vertex of P_D, or that the Weil divisor

$$D_\sigma := \sum_{j=1}^{r} (a_{n+j} + \langle u^*_{D,\sigma}, \xi_{n+j} \rangle)\, D_{n+j}$$

is effective. The line bundle $\mathcal{O}(D)$ is ample if and only if, for any n-dimensional cone in Σ, the Weil divisor D_σ is strictly effective, i.e. $a_{n+j} + \langle u^*_{D,\sigma}, \xi_{n+j} \rangle > 0$ for any $j = 1, ..., r$. Note that it follows from Kodaira's Theorem [**Kod1**] that such a toric complete manifold $\mathcal{X}(\Sigma)$ is *projective* (see Example C.4) if and only if it admits an ample Weil divisor. The line bundle $\mathcal{O}(D)$ is very ample if additionally the extremities $(\delta_{jk})_{1 \leq k \leq n}$, $j = 1, ..., n$, of the vectors in the canonical basis of $\mathbb{R}^n_{t_1, \ldots, t_n}$ belong to the polyedron $\Delta_{D,\Sigma}$ for any n-dimensional cone σ in the fan Σ. The various notions of ampleness can also be interpreted in another way. Any point x^* in the affine space $\mathbb{R}^n_{x_1^*, \ldots, x_n^*}$ can be expressed in a unique way $\sum_{j \in J} x^*_{j_l} \xi_{j_l}$, where $J \subset \{1, ..., n+r\}$, the cone generated by the ξ_{j_l} being the smallest cone in Σ that contains x^* (the uniqueness of such decomposition follows. One can associate to the Weil divisor D the function $\Psi_D \; : \mathbb{R}^n_{x_1^*, \ldots, x_n^*} \mapsto \mathbb{R}$ defined as

$$\Psi_D \Big(\sum_{j \in J} x^*_{j_l} \xi_{j_l} \Big) = - \sum_{l} x^*_{j_l} a_{j_l}.$$

The line bundle $\mathcal{O}(D)$ is semi-ample if and only if this support function Ψ_D is convex, that is $\Psi_D(u^* + v^*) \geq \Psi_D(u^*) + \Psi_D(v^*)$ for any u^*, v^* in $\mathbb{R}^n_{x_1^*, \ldots, x_n^*}$. The line bundle $\mathcal{O}(D)$ is ample if and only if the function Ψ_D is strictly convex, that is

$$\Psi_D(u^* + v^*) \geq \Psi_D(u^*) + \Psi_D(v^*)\,,$$

the equality being satisfied if and only if u^* and v^* belong to the same cone in the fan Σ. When the line bundle $\mathcal{O}(D)$ is ample, the convex polyedron P_D defined in (C.5) is an n-dimensional integral polytope in $(\mathbb{R}^n)^*$ which is combinatorially dual to Σ.

APPENDIX D

Analytic sets, Normalization and Log resolutions

D.1. Analytic sets

D.1.1. Analytic sets and corresponding ideal sheaves.

Definition D.1 (closed analytic subset). Let $\mathcal{X}$ be a complex manifold of dimension n, with $\mathcal{O}_{\mathcal{X}}$ as structural sheaf. A subset $A \subset \mathcal{X}$ is a *closed analytic subset* of $\mathcal{X}$ if A is a closed subset of $\mathcal{X}$ which is expressed locally (in an open chart U about each of its points, with respect to local coordinates in this chart) as the set of common zeroes of a finite family of elements in $\mathcal{O}_{\mathcal{X}}(U)$. An analytic subset $A \subset \mathcal{X}$ is said to be *irreducible* if it cannot be decomposed into a union of two closed analytic subsets A_1 and A_2 such that $A_1 \neq A$ and $A_2 \neq A$.

If $A \subset \mathcal{X}$ is an analytic subset, then one can associate with it a sheaf of ideals $\mathcal{I}_A \subset \mathcal{O}_{\mathcal{X}}$, where

$$\mathcal{I}_{A,x} := \{f \in \mathcal{O}_{\mathcal{X},x}\,;\, f = 0 \text{ on } A_x\}\,, \tag{D.1}$$

where A_x denotes the germ of the analytic set A at the current point x. For any point $x \in \mathcal{X}$, the ideal $\mathcal{I}_{A,x} \subset \mathcal{O}_{\mathcal{X},x}$ is a radical ideal (i.e. it is equal to its own radical). From the geometric point of view, the germ A_x of the support of $\mathcal{O}_x/\mathcal{I}_{A,x}$ is the union of its isolated components, which are germs at x of irreducible closed analytic sets in a convenient neighborhood of x, which correspond to prime ideals $\mathfrak{P}_{x,\iota}$ such that $\mathcal{I}_{A,x} = \bigcap_\iota \mathfrak{P}_{A,x,\iota}$.

On the other hand, any ideal $\mathcal{I}_x$ in the nœtherian local ring $\mathcal{O}_{\mathcal{X},x}$ admits a finite primary decomposition (not necessarily unique)

$$\mathcal{I}_x = \bigcap_\iota \mathfrak{Q}_{x,\iota}\,, \tag{D.2}$$

where the ideals $\mathfrak{Q}_{x,\iota}$ are primary ideals. While the above decomposition (D.2) is not unique, it leads to the finite list of distinct prime ideals $\mathfrak{P}_{x,\iota} := \sqrt{\mathfrak{Q}_{x,\iota}}$, called *associated ideals* of the $\mathcal{O}_{\mathcal{X},x}$-module $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_x$. This list is independent of the primary decomposition (D.2). Among these associated primes, one distinguishes the ones which (considered within the list) are minimal with respect to inclusion. Denote them as $\mathfrak{P}^{\rm isol}_{x,j}$, $j = 1, ..., M^{\rm isol}_x$. The corresponding germs at x of the closed analytic sets

$$V(\mathfrak{P}^{\rm isol}_{x,j}) = \{\zeta\,;\, f(\zeta) = 0 \quad \forall f \in \mathfrak{P}^{\rm isol}_{x,j}\}\,, \quad j = 1, ..., M^{\rm isol}_x,$$

are called *isolated components* of the germ of analytic set $A_x = \mathrm{Supp}\,(\mathcal{O}_x/\mathcal{I}_x)$. The other prime ideals among the list of associated primes correspond, when taking into consideration the germs at x of their zero sets, to germs of closed irreducible analytic subsets, called *embedded components* of A_x at the point x. Geometrically speaking, these embedded components cannot be seen, since each of them is properly included as a germ of subanalytic subset about x in one of the irreducible isolated components of the support A_x of $\mathcal{O}_x/\mathcal{I}_x$. This really means that purely geometric methods are unable to restitute the whole algebraic information carried by the ideal $\mathcal{I}_x$, that is, by the germ of closed analytic set A_x (viewed through the radical ideal that defines it, or equivalently, through its set of analytic defining equations $\{f_\iota = 0\}_\iota$). On the contrary, methods inspired by multidimensional complex analysis help to fill (at least partially) that bridge between the rough geometric information provided by the isolated primes (together with their corresponding Hilbert-Samuel multiplicities recovered as Lelong numbers, see Appendix A, Section A.3.4) and the (sometimes too much!) precise algebraic information carried by $\mathcal{I}_{A,x}$, which definitely needs some knowledge about the invisible embedded components of A_x.

For every isolated component of the germ of the set A at the point x (corresponding to the isolated prime $\mathfrak{P}^{\rm isol}_{A_x,j}$), one defines the *dimension of the component at the point* x as the *Krull dimension* of the quotient ring $\mathcal{O}_{\mathcal{X},x}/\mathfrak{P}^{\rm isol}_{A_x,j}$, that is, the largest length $\max(\kappa)$ of the strictly increasing chains of distinct prime ideals

$$\mathfrak{P}^{\rm isol}_{A_x,j} \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_\kappa \subsetneq \mathcal{O}_{\mathcal{X},x}.$$

The *local dimension of* A *at the point* x, denoted as $\dim_x A$, is defined to be the maximum of the dimensions of the isolated components of A_x at this point, that is, the maximum of Krull dimensions of the rings $\mathcal{O}_{\mathcal{X},x}/\mathfrak{P}^{\rm isol}_{A_x,j}$, $j = 1, ..., M^{\rm isol}_{A_x}$.

If $f_1, ..., f_M$ generate a prime ideal $\mathfrak{P}$ in the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions at the origin, and $\dim \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{P} = n - p$, i.e. $\dim_0\{f_1 = \cdots = f_M = 0\} = n - p$, then all minors of rank p of the Jacobian matrix

$$\Big(\frac{\partial f_j}{\partial \zeta_k}\Big)_{1\leq j\leq M, 1\leq k\leq n}$$

cannot vanish identically over the germ at the origin of $\{f_1 = \cdots = f_M\}$ (see for example [**Ha1, GRo**]). It follows that, if A is a closed irreducible analytic subset of $\mathcal{X}$, then the set $A_{\rm sing}$ of *singular points* of A, that is, the set of points in A about which A cannot be described as a submanifold, is a proper analytic subset of A such that, for any $x \in A_{\rm sing}$, $\dim(A_{\rm sing})_x < \dim A_x$. It implies that $A \setminus A_{\rm sing} = A_{\rm reg}$ is dense in A. Non singular points of the set A are called *regular points* of A . Since A is irreducible, the function $x \mapsto \dim A_x$ is constant on $A_{\rm reg}$. Its constant value on $A_{\rm reg}$ is defined to be the *dimension* of the irreducible closed analytic set A. It takes values between 0 and $n - 1 = \dim \mathcal{X} - 1$. A closed analytic subset $A \subset \mathcal{X}$ is said to be of *pure dimension* (or *purely dimensional*) if all its irreducible components are of the same dimension.

To conclude the present section,let us mention the classical *box principle*. Suppose $\mathfrak{Q}_1, ..., \mathfrak{Q}_M$ are M ideals in $\mathcal{O}_{\mathcal{X},x}$ and $h_1, ..., h_N$ be N elements in the same local ring such that, for any $j = 1, ..., M$, there exists at least one h_k so that $h_k \notin \mathfrak{Q}_j$. Then, for generic complex coefficients $\lambda_1, ..., \lambda_N$, one has $\sum_{l=1}^N \lambda_l h_l \notin \mathfrak{Q}_j$. The same result holds in the global setting (polynomial ideals in $\mathbb{K}[X_1, ..., X_n]$ or homogeneous polynomial ideals in $\mathbb{K}[X_0, ..., X_n]$ provided $\mathbb{K}$ is an infinite commutative field).

D.1.2. Complete intersection. Among configurations of analytic subsets of pure dimension equal to $n - p$ in the complex manifold $\mathcal{X}$, the most interesting for us is that of *complete intersection*. It is pertinent here to distinguish the local point of view from the global one.

Definition D.2 (complete intersection : local and global notions). Let A be a closed analytic subset with pure dimension $n-m$ ($1 \le m \le n$) in an n-dimensional complex manifold $\mathcal{X}$.

- The analytic subset A is said to be *locally complete intersection* in $\mathcal{X}$ if and only if it can be locally defined about each of its points x as the intersection of exactly m hypersurfaces intersecting at x.
- The analytic subset A is said to be *globally complete intersection* in $\mathcal{X}$ if and only if A is the set of common zeros of a global holomorphic section $s \in H^0(\mathcal{X}, E)$ of an holomorphic vector bundle $E \to \mathcal{X}$ of rank exactly m over $\mathcal{X}$.

Remark D.1. Local and global aspects differ : in the first case, given two open charts U_α and U_β with non-empty intersection, it is possible for the set A to be defined in U_α as $\{f_{\alpha 1} = \cdots = f_{\alpha m} = 0\}$, and in U_β as $\{f_{\beta 1} = \cdots = f_{\beta m} = 0\}$, with $f_\alpha = \Phi(f_\beta)$ in $U_\alpha \cap U_\beta$. Here Φ is a (m,m) matrix of holomorphic functions in $U_\alpha \cap U_\beta$, whose determinant is not necessarily an invertible element of $\mathcal{O}_{\mathcal{X}}(U_\alpha \cap U_\beta)$. In case A is a globally complete intersection with U_α and U_β being local charts over which $E \to \mathcal{X}$ can be trivialized, then one knows that it is possible to choose f_α and f_β so that they correspond to coordinates in the trivialisations above U_α and U_β of the same holomorphic section $s \in H^0(\mathcal{X}, E)$. One has then $f_\alpha = g_{\alpha\beta}(f_\beta)$, where $g_{\alpha\beta} \in H^0(\mathcal{X}, \mathrm{GL}(m, \mathbb{C}))$ (see Appendix B, Section B.3.2).

Remark D.2. It is important to distinguish the notion of complete intersection to that of regular sequence[1]! The fact that a $(n-m)$-purely dimensional closed analytic set A is globally defined as a complete intersection, that is, as the zero set of an holomorphic section s of a m-holomorphic vector bundle $E \to \mathcal{X}$, does not imply that $(\sigma_{1,x}, ..., \sigma_{m,x}))$ (where $s = \sigma_1 \otimes e_1 + \cdots + \sigma_m \otimes e_m$ when expressed in an holomorphic frame about x) is a regular sequence in $\mathcal{O}_{\mathcal{X},x}$, unless $x \in s^{-1}(0)$. Take for example as E the trivial bundle over $\mathbb{C}^3$ and $A = s^{-1}(0) = \{(0,0,0)\}$, where $s(\zeta) = (\zeta_1(\zeta_3+1), \zeta_2(\zeta_3+1), \zeta_3)$. The sequence $(s_{1,x}, s_{2,x}, s_{3,x})$ is not regular (in this order) about any point in the complex plane $\{\zeta_3 + 1 = 0\}$.

Example D.1. Any closed analytic set in $\mathbb{P}^n(\mathbb{C})$ is algebraic projective, that is defined as the zero set of a finite number of homogeneous polynomials in $z_0, ..., z_n$ (see Appendix B, Theorem B.1). If $P_1, ..., P_m$ are m homogeneous polynomials in $z_0, ..., z_n$ with respective total degrees $D_1, ..., D_m$, then the section $(P_1, ..., P_m)$ of the m-vector bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(D_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(D_m)$ defines a globally complete intersection if and only if $\{[z] \in \mathbb{P}^n(\mathbb{C})\,;\, P_1(z) = \cdots = P_m(z) = 0\}$ is purely $(n-m)$ dimensional.

We conclude this section with an heuristic remark. It is easier to describe the notion of complete intersection in geometric terms than algebraically, as the following equivalence shows. If $f_1, ..., f_m$ are m holomorphic functions in an open pseudoconvex domain of $\mathbb{C}^n$ (or more generally, m holomorphic functions over a *Stein manifold* $\mathcal{X}$, see the definition in Section D.2 below), then $f_1, ..., f_m$ either define a $(n-m)$- dimensional globally complete intersection in $\mathcal{X}$ or have no common

[1]A sequence $(s_1, ..., s_m)$ in a commutative ring $\mathbf{R}$ is said to be *regular* in $\mathbf{R}$ if and only if $s_1 \neq 0$ and, for any $j = 1, ..., m-1$, s_{j+1} is not a zero divisor in the quotient ring $\mathbf{R}/(s_1, ..., s_j)$.

zero in $\mathcal{X}$ if and only if, for every $k \in \mathbb{N}^*$, for every homogeneous relation

$$\sum_{i_1+\cdots+i_m=k} a_{i_1,\dots,i_m}(\zeta) f_1^{i_1}(\zeta) f_2^{i_2}(\zeta)\dots f_m^{i_m}(\zeta) \in \Big((f_1,\dots,f_m)\,\mathcal{O}_{\mathcal{X}}(\mathcal{X})\Big)^{k+1},$$

where $a_{i_1,\dots,i_m}$ are holomorphic functions in $\mathcal{X}$, one has that

$$\forall\, i_1,\dots,i_m\,,\ a_{i_1,\dots,i_m} \in (f_1,\dots,f_m)\mathcal{O}_{\mathcal{X}}(\mathcal{X})\,.$$

Thus one can see that the rather simple geometric concept of complete intersection (here global) corresponds, expressed in algebraic terms, to a more involved asymptotic characterization ("for every $k \in \mathbb{N}^*$, for every relation, etc."). Compare also to the two approaches (geometric or algebraic) to the notion of gap sheaf, where the same phenomenon occurs (Appendix A, Section A.3.1) : a simple geometric formulation on one side, an asymptotic more involved algebraic formulation on the other side.

D.1.3. Local presentation of hypersurfaces. We give here a convenient geometric "presentation" for a closed analytic hypersurface A in an n-dimensional manifold $\mathcal{X}$ about one of its points. It is based on *Weierstrass preparation theorem.* Without loss of generality, this local study reduces to the case when $\mathcal{X} = \Omega$, Ω being a neighborhood of the origin in $\mathbb{C}^n$.

Assume that $A = \{f = 0\}$, where f is an holomorphic function in a neighborhood of the origin in $\mathbb{C}^n$, which is not identically zero there, such that $f(0) = 0$ and reduced, that is $\overline{A \setminus \{df = 0\}} = A$ in a neighborhood of 0. Denote the coordinates as $z_1,\dots,z_{n-1},w$. One can express this function in a neighborhood of the origin as

$$f(z_1,\dots,z_{n-1},w) = f_\mu(z_1,\dots,z_{n-1},w) + o(|(z_1,\dots,z_{n-1},w)|^\mu),$$

where f_μ is a homogeneous polynomial of degree $\mu \in \mathbb{N}^*$. After a generic linear change of coordinates, one may assume that in a disc $\overline{\delta_w(0,\epsilon)}$, with $0 < \epsilon < 1$ being sufficiently small,

$$f(0,0,\dots,0,w) = w^\mu h(w)\,,\ |h(w)| \geq \delta > 0\,,\ \forall\, w \in \overline{\delta_w(0,\epsilon)}\,.$$

It follows from Rouché's thoeorem, that for $\|z\| = \|(z_1,\dots,z_{n-1})\|$ sufficiently small (depending on ϵ, i.e. in a small poly-disc Δ_z), the function $w \mapsto f(z,w)$ has exactly μ zeros $w_j(z)$, $j = 1,\dots,\mu$, counted with their multiplicities, in the closed disc $\overline{\delta_w(0,\epsilon)}$, all zeroes being in the interior of the open disc. For every $k = 1,\dots,\mu$, the residue formula implies that

$$\sum_{j=1}^{\mu} [w_j(z)]^k = \frac{1}{2i\pi}\int_{|\zeta|=\epsilon} \zeta^k\,\frac{\dfrac{\partial f}{\partial z_n}(z,\zeta)}{f(z,\zeta)}\,d\zeta\,,$$

showing that the Newton sums

$$z \mapsto \Phi_k(z) := \sum_{j=1}^{\mu} [w_j(z)]^k\,,\quad k = 1,\dots,\mu\,,$$

of the roots $w_j(z)$, $j = 1,\dots,\mu$ are holomorphic functions of z in a neighborhood of the origin $z = 0$. It follows from Newton's formulae[2] that the same is true for the

[2]Note here that the fact to work over $\mathbb{C}$, i.e. over a field with characteristic 0, is essential for such formulae to hold.

symmetric functions $z \mapsto \mathfrak{s}_1(z)$,..., $z \mapsto \mathfrak{s}_{\tilde{\mu}}(z)$ of the roots of $w \to f(z,w)$. Consider then the function

$$(z,w) \mapsto P(z,w) := w^\mu - \varphi_1(z) w^{\mu-1} + \cdots + (-1)^\mu \varphi_\mu(z).$$

The function

$$(z,w) \mapsto \frac{f(z,w)}{P(z,w)}$$

is holomorphic and does not vanish in a neighborhood of the origin in $\mathbb{C}^n$. Describing the zero set of f in a neighborhood of the origin is equivalent to describe the zero set of the function $(z,w) \mapsto P(z,w)$. The set $A \cap (\Delta_z \times \delta_w(0,\epsilon))$ corresponds then to the family of leaves of a μ-to-1 covering $\pi : A \cap (\Delta_z \times \delta_w(0,\epsilon)) \to \Delta_z$. If $\sigma(z)$ denotes the discriminant of $X \to P(X,z)$, then the set $A \cap (\Delta_z \times \delta_w(0,\epsilon))$ is a smooth manifold of dimension $n-1$ over the set $\{z\,;\,\sigma(z) \neq 0\}$. Since f is reduced, $\sigma \not\equiv 0$ in Δ_z. The set $\{\sigma(z) = 0\}$ over which the covering is ramified (at least two leaves cross) is called the *discriminant locus* ; its inverse image by π is the *ramification locus* on A. The number μ of leaves equals the *multiplicity* $\mu_0(f)$ of f at the origin. That is, μ is equal to the degree of the homogeneous component of lowest degree (or also the *valuation*) of $(z,w) \mapsto f(z,w)$ at the origin. This multiplicity is also equal to the Lelong number $\nu_0([A]_{\rm red})$ of the integration current $[A]_{\rm red}$ at the origin (Section A.3.1 in Appendix A).

D.1.4. E. Nœther local presentation of closed analytic sets. In this section, we extend the local presentation for closed hypersurfaces described in Section D.1.3 to the case of purely dimensional closed analytic subsets. E. Nœther's normalization lemma will be the counterpart for Weierstrass preparation theorem.

The following proposition ([**GrR**]) plays an important role in what follows :

Proposition D.1 (presentation of closed analytic sets). *Let $A \subset \Omega$ be a closed, analytic subset of pure dimension $n-m \in \{0,...,n-1\}$ in the open neighborhood Ω of the origin in $\mathbb{C}^n$ such that $0 \in A$. There exists a neighborhood Ω' of the origin in Ω, m holomorphic functions $f_1,...,f_m$ in Ω', defining in Ω' a complete intersection*

$$\widetilde{A} = \{f_1 = \cdots = f_m = 0\}\,,$$

in a such way that $A \cap \Omega'$ is a union a finite number $\mu = \nu_0([A]_{\rm red})$ of irreducible components $\widetilde{A}_\iota$ of the analytic subset $\widetilde{A}$, satisfying in addition $df_1 \wedge \cdots \wedge df_m \not\equiv 0$ over the set $\widetilde{A}_\iota$ for every ι.

Proposition D.1 provides a useful geometric presentation of A about the origin. If $f_1,...,f_m$ are m holomorphic functions in a neighborhood Ω of the origin defining there a complete intersection $\widetilde{A}$, then it follows from the normalization lemma of E. Nœther that there is a linear change of coordinates $\zeta = (z_1,...,z_{n-m},w_1,...,w_m)$ about the origin, such that if $\Delta_{z,w} \subset \Omega \subset \mathbb{C}_z \times \mathbb{C}_w$ is a convenient open polycylinder $\Delta_z \times \delta_w$ centered at $(0,0)$, the projection

$$\pi\ : (z,w) \in \widetilde{A} \cap (\Delta_z \times \delta_w) \longmapsto z \in \Delta_z$$

is a proper map. This linear change of coordinates may be chosen as generic, and the degree of the corresponding proper map π remains generally constant, equal to its minimal value, namely the Lelong number at the origin of $[\widetilde{A}]_{\rm red}$ (see Appendix A, Section A.3.1). The number of irreducible components of $\widetilde{A}$ in a sufficiently small neighborhood of $(0,0)$ is thus bounded from above by $\widetilde{\mu} = \nu_0([\widetilde{A}]_{\rm red})$. Since $df_1 \wedge \cdots \wedge df_m \not\equiv 0$ over each $\widetilde{A}_\iota$, one has that

$$J(z,w) := \frac{\partial(f_1,...,f_m)}{\partial(w_1,...,w_m)} \not\equiv 0$$

over $\widetilde{A}_\iota \cap (\Delta_z \times \delta_w)$ for every ι. Let $\{z \in \Delta_z \,:\, \widetilde{\sigma}(z) = 0\}$ be the discriminant locus of the projection π and the set $\{z \in \Delta_z \,:\, \sigma(z) = 0\} \subset \{z \in \Delta_z \,:\, \widetilde{\sigma}(z) = 0\}$ be that of $\pi_{|A}$. Note that $\{z \in \Delta_z \,:\, \sigma(z) = 0\}$ is the image on Δ_z by the proper projection π of the set

$$A \cap \{(z,w) \in \Delta_z \times \delta_w \,:\, J(z,w) = 0\}.$$

Observe that above the set $\{z \in \Delta_z \,:\, \widetilde{\sigma}(z) \neq 0\}$, $\widetilde{A} \cap (\Delta_z \cap \delta_w)$ consists in $\widetilde{\mu}$ disjoint leaves which are submanifolds of dimension $n-m$ parameterized by the coordinates z. The properness of π implies in fact that, for every $z \in \Delta_z$,

$$\pi^{-1}(z) \cap \widetilde{A} = \{(z, w^{(1)}(z)), ..., (z, w^{(\widetilde{\mu})}(z))\}$$
$$\pi^{-1}(z) \cap A = \{(z, w^{(1)}(z)), ..., (z, w^{(\mu)}(z))\}$$

where $\widetilde{\mu} = \nu_0([\widetilde{A}]_{\rm red})$ and $\mu = \nu_0([A]_{\rm red})$ are the Lelong numbers of the respective integration currents $[\widetilde{A}]_{\rm red}$ and $[A]_{\rm red}$ at the origin. Moreover, above the $\Delta_z \setminus \{\sigma(z) = 0\}$, where $\{\sigma = 0\} = \pi((A \cap (\Delta_z \times \delta_w)) \cap \{J(z,w) = 0\})$, $A \cap (\Delta_z \times \delta_w)$ consists in μ disjoint leaves which are submanifolds of dimension $n-m$ parameterized by the coordinates z.

D.1.5. Weakly holomorphic functions ; Oka universal denominator.

Definition D.3. Let $A \subset \mathcal{X}$ be a closed analytic subset in a n-dimensional complex manifold and h be a function from A to $\mathbb{C}$.

1. The function h is *weakly holomorphic function* on A if and only if $h_{|A_{\rm reg}}$ is holomorphic (as a function defined on $A_{\rm reg}$ equipped with its structure of $(n-1)$-complex manifold $(A_{\rm reg}, (\mathcal{O}_{\mathcal{X}})_{|A_{\rm reg}})$) and h is locally bounded about any point $x \in A$.
2. The function h is *c-holomorphic* ([**Whi, Chirk, Lo**]) on A if and only if $h_{|A_{\rm reg}}$ is holomorphic (see above) and h is continuous on A.
3. The function h is *strongly holomorphic* on A if and only if it is the restriction to A of a holomorphic function $\widetilde{h}$ in some open neighborhood of A in the ambient manifold $\mathcal{X}$.

While the two first notions are intrinsic (they do not depend on the embedding $\iota_A \,:\, A \to \mathcal{X}$), the third one is not.

One has the following useful characterizations for the two first concepts.

Proposition D.2. *Let $\mathcal{X}$ be a complex manifold, $A \subset \mathcal{X}$ be a closed analytic subset.*

- *A holomorphic map $h \,:\, A_{\rm reg} \to \mathbb{C}$ extends as a weakly holomorphic function to the whole of A if and only if the closure (in $\mathcal{X} \times \mathbb{C}$) of its graph*

$$\Gamma_{A_{\rm reg}}(h) = \{(x, h(x))\,;\, x \in A_{\rm reg}\}$$

over $A_{\rm reg}$ is a closed analytic subset in $\mathcal{X} \times \mathbb{C}$.

- *A function* $h : A \to \mathbb{C}$ *is c-holomorphic on* A *if and only if it is continuous on* A *and its graph* $\Gamma_A(h) = \{(x, h(x))\,;\, x \in A\}$ *is a closed analytic subset in* $\mathcal{X} \times \mathbb{C}$.

Weakly holomorphic functions (as well as c-holomorphic functions) on a closed analytic subset $A \subset \mathcal{X}$ are restrictions to A of meromorphic functions in a neighborhood of A in the ambient manifold $\mathcal{X}$. We have even a much more precise (and important) result.

Theorem D.1 (existence of the Oka universal denominator). *Let* $\mathcal{X}$ *be a complex manifold,* $A \subset \mathcal{X}$ *be a closed analytic subset,* h *be a weakly holomorphic function on* A. *Then, for any* $x \in A$, h *can be expressed about* x *as the restriction to* A *of a meromorphic function in a neighborhood of* x *in the ambient manifold* $\mathcal{X}$. *For* $x \in A$ *fixed, the denominator of this meromorphic extension can be chosen independent of* h.

Proof. Since the proof of this important result depends on the local Nœther presentation given in Section D.1.4, we give a sketch of it here ($\mathcal{X} = \Omega$ being a neighborhood of $x = 0$ in $\mathbb{C}^n$) based on the presentation described above. We start from this presentation and keep the previous notations. There exist holomorphic functions $h_{jk} : \Delta_z \times \delta_w \times \delta_w \to \mathbb{C}$ such that

$$f_j(z,u) - f_j(z,w) = \sum_{k=1}^{m} h_{jk}(z,u,w)(u_j - w_j),\ j = 1, ..., m.$$

This comes from Hefer division formulas for poly-cylinders (see below (D.4)) in Section D.2). If h is a holomorphic function in the submanifold $A_{\rm reg} \cap (\Delta_z \times \delta_w)$, then it can be extended holomorphically to a holomorphic function in the submanifold $\widetilde{A}_{\rm reg} \times (\Delta_z \times \delta_w)$ by setting $h = 0$ on $(\widetilde{A}_{\rm reg} \setminus A_{\rm reg}) \cap (\Delta_z \times \delta_w)$. One can see then that the function

$$\widetilde{H}\ : (z,w) \in \Delta_z \times \delta_w \longmapsto \sum_{\iota=1}^{\widetilde{\mu}} h(z, w^{(j)}(z)) \det[h_{jk}](z, w, w^{(\iota)})(z)$$

is holomorphic in both variables $(z, w) \in (\Delta_z \setminus \{\sigma = 0\}) \times \delta_w$. Its restriction to $A \cap ((\Delta_z \setminus \{\sigma = 0\}) \times \delta_w)$ equals $hJ_{|A}$. If h is bounded in $A_{\rm reg}$, Riemann's theorem shows that the function $\widetilde{H}$ has an analytic continuation to holomorphic function defined in $\Delta_z \times \delta_w$ and

$$\forall\, (z,w) \in A \cap (\Delta_z \cap \delta_w)\,,\ h(z,w) = \frac{\widetilde{H}(z,w)}{J(z,w)}\,.$$

The last equality proves that h coincides with the restriction to $A \cap (\Delta_z \times \delta_w)$ of a meromorphic function in $\Delta_z \cap \delta_w$, with J (independent of h) as denominator. □

D.2. Coherence and the theorems of Oka and Cartan

D.2.1. Sheaves of rings or $\mathcal{O}_{\mathcal{X}}$-modules ; coherence. Let us recall briefly here the basic notions about sheafs. In order to define a *pre-sheaf* $\mathcal{F}$ *of commutative rings* over a topological space $\mathcal{X}$, one needs :

(1) First to set a collection of commutative rings $\{\mathcal{F}(U)\}_{U \subset \mathcal{X}}$, U being an arbitrary open subset of $\mathcal{X}$. The ring $\mathcal{F}(U)$ is called *the ring of sections of the sheaf* $\mathcal{F}$ *over the open set* U.

(2) Then to associate to every pair of open sets U, V satisfying $U \subset V$, a *restriction* map $\rho_{U,V} \ : \mathcal{F}(V) \to \mathcal{F}(U)$ which is an homomorphism of rings, so that $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$ whenever $U \subset V \subset W$ and $\rho_{U,U} = \mathrm{Id}_E$.

The *pre-sheaf* $\mathcal{F}$ becomes a *sheaf of commutative rings* if it satisfies in addition the two following "gluing" axioms :

A) Let $(U_\alpha)_\alpha$ be a covering with open sets of an arbitrary open set U ; whenever $F, G \in \mathcal{F}(U)$ and $\rho_{U_\alpha,U}(F) = \rho_{U_\alpha,U}(G)$ for every α, then $F = G$.

B) Let $(U_\alpha)_\alpha$ be a covering with open sets of an arbitrary open set U ; if a collection $(F_\alpha)_\alpha$ with $F_\alpha \in \mathcal{F}(U_\alpha)$ satisfies

$$\rho_{U_\alpha \cap U_\beta, U_\alpha}(F_\alpha) = \rho_{U\alpha \cap U_\beta, U_\beta}(F_\beta) \quad \forall\, \alpha, \beta,$$

then there is an element $F \in \mathcal{F}(U)$ such that $\rho_{U_\alpha,U}(F) = F_\alpha$ for any index α.

For any $x \in \mathcal{X}$, one defines the stalk $\mathcal{F}_x$ of a sheaf of commutative rings at x as the direct limit $\lim\limits_{x \ni U} \mathcal{F}(U)$. This means that an element of $\mathcal{F}_x$ can be represented as an element in $\mathcal{F}(U)$ for some neighborhood U of x, taking into account that two such sections in $\mathcal{F}(U_1)$ and $\mathcal{F}(U_2)$ are identified if they coincide on a neighborhood U of x lying in $U_1 \cap U_2$.

Definition D.4 (support of a sheaf). The support of a sheaf of commutative rings on $\mathcal{X}$ is the set of points in $\mathcal{X}$ where $\mathcal{F}_x \neq 0$.

One may also consider sheaves of non-commutative rings, such as the sheaf $\mathcal{D}_\mathcal{X}$ of differential operators with holomorphic coefficients over a complex manifold $\mathcal{X}$ of dimension n. The commutators $[\partial_j, z_k]$ are equal to the Kronecker symbols δ_{jk}. One may also consider the sheaves of left $\mathcal{D}_\mathcal{X}$-modules. Basic references concerning the theory of $\mathcal{D}_\mathcal{X}$ modules are the books by J.E. Björk [**Bj1, Bj2**].

We introduce here the fundamental notion of *coherence* for a sheaf $\mathcal{F}$ of $\mathcal{O}_\mathcal{X}$-modules on a n-dimensional complex manifold $\mathcal{X}$.

Definition D.5 (coherence of a sheaf). A sheaf $\mathcal{F}$ of $\mathcal{O}_\mathcal{X}$-modules on a n-dimensional complex manifold $\mathcal{X}$ is said to be *coherent* (or $\mathcal{O}_\mathcal{X}$ coherent) if and only if it fulfills the two following conditions :

(1) for any x in $\mathcal{X}$, there exists an open neighborhood U_x of x and q_x elements $s_1, ..., s_{q_x}$ in $\mathcal{O}_\mathcal{X}(U_x)$ such that, for every $x' \in U_x$, the $\mathcal{O}_{\mathcal{X},x'}$-module $\mathcal{F}_{x'}$ is generated by the germs $s_{j,x'}$, $j = 1, ..., q_x$; one phrases this condition saying that $\mathcal{F}$ is *locally finitely generated* ;

(2) for any open subset U of $\mathcal{X}$, for any choice of sections $s_1, ..., s_q$ of $\mathcal{F}(U)$, the $(\mathcal{O}_\mathcal{X})_{|U}$-sub-sheaf of $(\mathcal{O}_\mathcal{X}^{\oplus q})_{|U}$ of relations $\mathcal{R}_U(s_1, ..., s_q)$, that is, the kernel of the sheaf homomorphism

$$(g_{1,\zeta}, ..., g_{q,\zeta}) \in \mathcal{O}_{\mathcal{X},\zeta}^{\oplus q} \longmapsto \sum_{j=1}^{q} g_{j,\zeta}\, s_{j,\zeta} \in \mathcal{F}_\zeta\,, \qquad \zeta \in U,$$

is also locally finitely generated.

The main examples, which we will meet throughout this monograph, are the following :

(1) The sheaf $\mathcal{O}_\mathcal{X}$ itself is a coherent sheaf on $\mathcal{X}$ (as a sheaf of $\mathcal{O}_\mathcal{X}$-modules over itself). This is *Oka coherence theorem* . The same is true for locally

free sheaves[3] of $\mathcal{O}_{\mathcal{X}}$-modules, in particular for the sheaf of holomorphic sections $\mathcal{O}_{\mathcal{X}}(E)$ of any holomorphic, locally trivial, vector bundle $E \to \mathcal{X}$. Locally free sheaves with rank 1 (one can take $m_x = 1$ for any x) are said to be *invertible* ; invertible sheaves are in correspondence with holomorphic line bundles through the operation that assigns to a line bundle its sheaf of holomorphic sections (see Appendix C).

(2) If A is a closed analytic subset of $\mathcal{X}$, the sheaf of ideals $\mathcal{I}_A \subset \mathcal{O}_{\mathcal{X}}$ defined in (D.1) is a coherent sheaf on $\mathcal{X}$ (this is a theorem of H. Cartan). Since one has the exact sequence of sheafs of $\mathcal{O}_{\mathcal{X}}$-modules

$$0 \to \mathcal{I}_A \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}/\mathcal{I}_A \to 0\,,$$

the same holds for the quotient sheaf $\mathcal{O}_{\mathcal{X}}/\mathcal{I}_A$.

(3) If $\pi\ : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a proper morphism between two complex manifolds and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_{\widetilde{\mathcal{X}}}$-modules on $\widetilde{\mathcal{X}}$, the sheaf $\pi_*[\mathcal{F}]$, which is the direct image of the sheaf $\mathcal{F}$ by π, is a coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}$ [**Grau**].

D.2.2. Stein manifolds ; Cartan Theorems A and B. We now turn to a natural class of complex manifolds (in a sense that they carry enough nonconstant holomorphic functions, which is not for example the case of a compact complex manifold such as $\mathbb{P}^n(\mathbb{C})$), that of *Stein manifolds.* A complex manifold X of dimension n, with structural sheaf $\mathcal{O}_{\mathcal{X}}$ is a *Stein manifold* if satisfies the following the three following properties :

(1) it is holomorphically convex, that is the *holomorphic convex hull*

$$\widehat{K}_{\mathcal{O}_{\mathcal{X}}} := \{x \in \mathcal{X}\,,\ |f(x)| \leq \sup_K |f|\,,\ \forall f \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})\}$$

of any compact subset $K \subset\subset \mathcal{X}$ remains compact;

(2) it carries enough holomorphic functions in order to separate points, that is, for any points $x, y \in \mathcal{X}$, $x \neq y$, there exists an element $f \in \mathcal{O}_{\mathcal{X}}$ such that $f(x) \neq f(y)$.

It is known that any Stein manifold $\mathcal{X}$ of dimension n can be holomorphically imbedded into $\mathbb{C}^{2n+1}$. One has also two extremely powerful results, due to H. Cartan.

Theorem D.2 (Cartan Theorem A). *Let $\mathcal{X}$ be a Stein manifold of dimension n. Any coherent analytic sheaf of $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}$ is spanned by its global sections, that is, for any open set $U \subset \mathcal{X}$, $\rho_{U,\mathcal{X}}[\mathcal{F}(\mathcal{X})]$ generates $\mathcal{F}(U)$ as a $\mathcal{O}_{\mathcal{X}}(U)$-module. In particular, for each $x \in \mathcal{X}$, one can find global sections $f_{1,x}, ..., f_{n,x}$ in $\mathcal{F}(\mathcal{X})$ such that $(f_{1,x}, ..., f_{n,x})$ defines a local system or coordinates about x.*

Theorem D.3 (Cartan Theorem B). *Let $\mathcal{X}$ be a Stein manifold of dimension n. If $\mathcal{F}$ is a coherent analytic sheaf of $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}$, then for each $q > 0$, $\check{H}^q(\mathcal{X}, \mathcal{F}) = 0$, where $\check{H}^q(\mathcal{X}, \mathcal{F})$ denotes the Čech cohomology group with values in $\mathcal{F}$.*

[3] A sheaf of $\mathcal{O}_{\mathcal{X}}$-modules is *locally free* if any point x in X admits a neighborhood U_x such that $\mathcal{F}(U_x)$ is isomorphic to the $\mathcal{O}(U_x)$-module $(\mathcal{O}(U_x))^{\oplus m_x}$ for some $m_x \in \mathbb{N}^*$. This means that there exist elements $s_1, ..., s_m$ in $\mathcal{F}(U_x)$ such that, for any $x' \in U_x$, the mapping

$$(\sigma_{1,x'}, ..., \sigma_{m_x,x'}) \in \mathcal{O}_{\mathcal{X},x'}^{\oplus m_x} \mapsto \sigma_{1,x'} s_{1,x'} + \cdots + \sigma_{m_x,x'} s_{m_x,x'}$$

is an isomorphism of $\mathcal{O}_{\mathcal{X},x'}$-modules.

One of the most remarkable consequences of Theorem A for us will be the following : if $F, f_1, ..., f_m$ are $m+1$ holomorphic functions on a Stein manifold $\mathcal{X}$, such that locally, at every point $x \in X$, $F_x \in (f_{1,x}, ..., f_{m,x})\, \mathcal{O}_{\mathcal{X},x}$ at the level of germs, then there exist holomorphic functions $a_1, ..., a_m$ on $\mathcal{X}$ such that $F \equiv \sum_{j=1}^m a_j f_j$ over all of $\mathcal{X}$. An important example occurs when $\mathcal{X} = \Omega \times \Omega \subset \mathbb{C}^n_z \times \mathbb{C}^n_w$, where Ω is a pseudo-convex domain of $\mathbb{C}^n_z$ (that is a Stein open subset of $\mathbb{C}^n$) and $f_j(z,w) := z_j - w_j$, $j = 1, ..., n$. One could as well replace Ω by a Stein manifold $\mathcal{X}$. If f is a holomorphic function in Ω, then there exist n holomorphic functions $g_1, ..., g_n$ on $\Omega \times \Omega$, such that

$$f(z) - f(w) = \sum_{j=1}^n (z_j - w_j) g_j(z,w)\ \forall\, (z,w) \in \Omega \times \Omega. \tag{D.3}$$

This formula is called *Hefer division formula* . Hefer formula can easily be obtained when Ω is convex, since one can use in this case Taylor formula with integral remainder, actually
(D.4)

$$f(z) - f(w) = \int_0^1 \frac{d}{dt}[f(tz + (1-t)w)]\, dt = \sum_{j=1}^n (z_j - w_j) \int_0^1 \frac{\partial f}{\partial z_j}(tz + (1-t)w)\, dt.$$

This simple argument does not go through in the case of a pseudo-convex domain Ω.

Theorem B implies that every coherent sheaf $\mathcal{F} = \mathcal{O}_{\mathcal{X}}$ of $\mathcal{O}_{\mathcal{X}}$-modules on a Stein manifold $\mathcal{X}$ admits in a neighborhood of any compact K a *free resolution* . That is, there exists an exact sequence of sheaf homomorphisms in an open neighborhood U of K :
(D.5)

$$0 \longrightarrow \mathcal{O}_U^{\oplus r_N} \xrightarrow{F_N} \mathcal{O}_U^{\oplus r_{N-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_U^{\oplus r_2} \xrightarrow{F_2} \mathcal{O}_U^{\oplus r_1} \xrightarrow{F_1} \mathcal{O}_U^{\oplus r_0} \longrightarrow \mathcal{F}_U \longrightarrow 0.$$

If $\mathcal{F} = \mathcal{O}_{\mathcal{X}}/\mathcal{I}$, where $\mathcal{I}$ is a coherent sheaf of ideals in $\mathcal{O}_{\mathcal{X}}$, one has $\mathrm{Im}\,(\mathcal{O}_U^{\oplus r_1} \to \mathcal{O}_U^{\oplus r_0}) = \mathcal{I}_U$. A free resolution of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ is sometimes called by extension free resolution of $\mathcal{I}$.

Since every complex manifold $\mathcal{X}$ is locally Stein, one can define at every point x of $\mathcal{X}$, if $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules, the *minimal length of a resolution* of $\mathcal{F}$ at the point x. This number is called the *depth* of the coherent sheaf $\mathcal{F}$ at the point x. It is bounded above by the codimension of the support of the sheaf $\mathcal{F}$ at the point x. This follows from the fact that the exactness of the complex (D.5) is equivalent to the following (Theorem 20.9, [**Eis1**]) :

$$\mathrm{codim}\,\{x \in \mathcal{X}\,;\, \mathrm{rank}(F_j(x)) < r_j - r_{j+1} + \cdots \pm r_N\} \geq j \quad \forall\, j \in \mathbb{N}^*. \tag{D.6}$$

If $\mathcal{F} = \mathcal{O}_{\mathcal{X}}/\mathcal{I}$, where $\mathcal{I}$ is a coherent ideal sheaf, then the depth of $\mathcal{F}$ at the point x is then bounded from above by the codimension of the set of common zeroes of the elements of $\mathcal{I}_x$. The search for a free resolution such as (D.5) for the sheaf $\mathcal{F} = \mathcal{O}_{\mathcal{X}}/\mathcal{I}$, where $\mathcal{I}$ is a coherent sheaf of ideals, is known as a the *determination of syzygies* problem (see [**Eis2**]) for the quotient $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ or, by extension, for the coherent sheaf of ideals $\mathcal{I}$. In general, there is no algorithm which allows to compute such a free resolution within polynomial time.

Definition D.6 (Cohen-Macaulay sheafs and analytic sets). A coherent sheaf on a complex manifold $\mathcal{X}$ is said to be *Cohen-Macaulay* if and only if for any x in the support of $\mathcal{F}$, the minimal length ν_x of a free resolution of the sheaf $\mathcal{F}$ at x equals the codimension of the support of $\mathcal{F}$ at this point. If $\mathcal{I}$ is a coherent sheaf of ideals such that $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ is Cohen-Macaulay, then we say that the coherent sheaf of ideals $\mathcal{I}$ is Cohen-Macaulay. The analytic space $(A, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$, where A denotes the zero set of the coherent sheaf ideal $\mathcal{I}$ is by extension said to be Cohen-Macaulay if $\mathcal{I}$ is.

Remark D.3. When the coherent sheaf of ideals $\mathcal{I}$ is Cohen-Macaulay at a point $x \in \mathrm{Supp}\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$, all associated primes of $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_x$ are isolated.

Example D.2 (complete intersections are Cohen-Macaulay). Any closed analytic set A which is locally or globally complete intersection (Definition D.2) is Cohen-Macaulay. If A is defined about x as $A = \{f_1 = \cdots = f_m = 0\}$, a minimal free resolution for $\mathcal{O}_{\mathcal{X}}/(f_1, ..., f_m)\mathcal{O}_{\mathcal{X}}$ at x is given then by the Koszul complex associated to the sequence $(f_1, ..., f_m)$.

D.3. Cycles and coherent analytic sheaves

Definition D.7 (analytic cycle). Let $\mathcal{X}$ be a n-dimensional complex manifold. A *k-analytic cycle* in $\mathcal{X}$ $(0 \leq k \leq n-1)$ is a formal, locally finite, linear combination

$$C = \sum_{\gamma} m_\gamma C_\gamma,\ m_\gamma \in \mathbb{Z}$$

of irreducible closed analytic subsets C_γ, all of dimension k. The set of k-cycles in $\mathcal{X}$ has the natural structure of an additive commutative group. A k-cycle is said to *effective* if all coefficients m_γ are positive integers.

Before stating the proposition which describes (imperfectly, but this will be sufficient for our purpose) the correspondence between the coherent ideals over a complex manifold $\mathcal{X}$ and the effective cycles on it, we recall some basic terminology.

The *support* of a section $s \in \mathcal{F}(U)$ of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{X}}$-modules over some open set U is the closure in U of the set of points where this section does not vanish. The support of $\mathcal{F}$ is the subset of $\mathcal{X}$ defined as the set of points where the stalk $\mathcal{F}_x$ is non zero (Definition D.4); it is closed when $\mathcal{F}$ is $\mathcal{O}_{\mathcal{X}}$-coherent since $\mathcal{O}_{X,x}$ is nœtherian. In particular the suppport of a section $s \in \mathcal{O}_X/\mathcal{J}(U)$, where $\mathcal{J}$ denotes a sheaf of coherent ideals of $\mathcal{O}_{\mathcal{X}}$, is a closed analytic subset of U contained in the locus of common zeroes of elements in $\mathcal{J}(U)$.

Recall also that the notion of *length* for $\mathbb{A}$-modules ($\mathbb{A}$ being a commutative ring) is the pendant of that of *dimension* for $\mathbb{K}$-vector spaces ($\mathbb{K}$ being a commutative field).

Proposition D.3 (cycles *versus* coherent sheaves of ideals [**Ko2**]). *Let $\mathcal{X}$ be a complex manifold. Let C be an effective cycle in $\mathcal{X}$ formally expressed as*

$$C = C_0 + C_1 + \cdots + C_{n-1} = \sum_{k=0}^{n-1} \sum_{\gamma} m_{k,\gamma} C_{k,\gamma}\,,$$

where C_k is a k-cycle, the $C_{k,\gamma}$ are irreducible, closed analytic subsets of $\mathcal{X}$ with dimension k, and $m_{k,\gamma}$ are positive integers (the sums being locally finite). One can associate to it the coherent sheaf of ideals

$$\mathcal{I}(C) := \prod_{k=0}^{n-1} \prod_{\gamma} (I_{C_{k,\gamma}})^{m_{k,\gamma}}. \tag{D.7}$$

Conversely, assume that we are given a coherent sheaf of ideals $\mathcal{J} \neq \mathcal{O}_{\mathcal{X}}$. Denote by $\mathcal{F}_k$, for $k = 0, ..., n-1$, the subsheaf of sections s of the sheaf $\mathcal{O}_{\mathcal{X}}/\mathcal{J}$ whose support is a closed analytic subset of dimension at least equal to k in $\mathcal{X}$. If $C_{k,\gamma}$ denotes the irreducible components of the support of the sheaf $\mathcal{F}_k/\mathcal{F}_{k+1}$ and $x_{k,\gamma}$ is a generic point in $(C_{k,\gamma})_{\mathrm{reg}}$, one can associate to the sheaf of ideals $\mathcal{J}$ the cycle in $\mathcal{X}$:

$$C(\mathcal{J}) := \sum_{k=0}^{n-1} \sum_{\gamma} \mathrm{length}\,(\mathcal{F}_{k,x_{k,\gamma}})\, C_{k,\gamma}. \tag{D.8}$$

Furthermore, in this (imperfect) correspondence $\mathcal{I}(C(\mathcal{J})) \subset \mathcal{J}$.

When C is an effective analytic k-cycle in X

$$C := \sum_{\gamma} m_\gamma C_\gamma,$$

the sheaf of coherent ideals defined by

$$\mathcal{I}(C) := \prod_{\gamma} (\mathcal{I}_{C_\gamma})^{m_\gamma}$$

is, in some sense, "too big" in order to reveal some information in relation with given sets of equations for the various components C_γ. This is the reason to introduce for every irreducible subset C_γ a sub-sheaf of $\mathcal{I}_{C_\gamma}$ called the *Chow sheaf of ideals* denoted by $\mathcal{I}^{\mathrm{chow}}_{C_\gamma}$. That is, for every point $x \in \mathcal{X}$, $\mathcal{I}^{\mathrm{chow}}_{C_\gamma,x} \subset \mathcal{I}_{C_\gamma,x})$, and the *Chow sheaf of ideals* is defined to be equal to

$$\mathcal{I}^{\mathrm{chow}}(C) := \prod_{\gamma} (\mathcal{I}^{\mathrm{chow}}_{C_\gamma})^{m_\gamma}.$$

It remains to define the sheafs $\mathcal{I}^{\mathrm{chow}}_{C_\gamma}$, more generally $\mathcal{I}^{\mathrm{chow}}_{A}$ when $A \subset \mathcal{X}$ is a closed irreducible analytic set of dimension $0 \leq k \leq n-1$. An introduction to this important concept (for us) can be found in [**Ko1**]. It is indeed an intermediate notion, half way between the geometric point of view (which is usually too naive, since only isolated components, and not embedded ones, can be captured) and the algebraic point of view (which on the opposite is too precise, the non-unicity of the primary decomposition (D.2) being for example a stumbling block). This notion is in fact reminiscent of the notion of *contour apparent* developped by G. Monge.

Definition D.8 (admissible projection). Let $\mathcal{X}$ be a complex n-dimensional manifold. Let $A \subset \mathcal{X}$ be a closed irreducible, analytic subset of $\mathcal{X}$ of dimension k and x a point in A. A linear projection $\pi_x : U_x \mapsto \mathbb{C}^{k+1}$ from a neighborhood U_x of x, with values in $\mathbb{C}^{k+1}$, such that $\pi_x(x) = 0$, is said to be *admissible* with respect to the point x and the analytic set A if the restriction of π_x to $U_x \cap A$ is a proper map from $U_x \cap A$ into $\mathbb{C}^{k+1}$.

Remark D.4. Here U_x is considered as a local chart, thus as a neighborhood of 0 in $\mathbb{C}^n$, which justifies the use of linear projections. Note that generically, a linear projection $\pi : U_x \to \mathbb{C}^{k+1}$ such that $\pi_x(x) = 0$ is admissible.

It follows from Remmert-Stein theorem (see e.g. [**GRo**]) that the image by an admissible projection π_x of the analytic set $U_x \cap A$ is a closed analytic subset of the open set $\pi_x(U_x)$ in $\mathbb{C}^{k+1}$. Restricting our-selves to a neighborhood of the points of A_{reg}, we see that the set $\pi_x(U_x \cap A)$ is an hypersurface in the open set

$\pi_z(U_x) \subset \mathbb{C}^{k+1}$. This hypersurface is locally defined about the origin by a reduced equation $\{\sigma_{\pi_x} = 0\}$.

DEFINITION D.9 (Chow ideal of an irreducible analytic set). Let A be a closed irreducible analytic subset of dimension k in a complex manifold. The coherent sheaf of ideals $\mathcal{I}_A^{\rm chow} \subset \mathcal{I}_A$ is defined as $\mathcal{I}_{A,x}^{\rm chow} = \mathcal{O}_{\mathcal{X},x}$ if $x \notin A$ and, whenever x is a point in A, as the ideal of $\mathcal{O}_{\mathcal{X},z}$ generated by all the germs at x of functions $\zeta \mapsto \sigma_{\pi_x}(\pi_x(\zeta))$, whenever π_x runs over the family of all linear projections $\pi\ :\ U_x \to \mathbb{C}^{k+1}$ which are admissible with respect to x and to the analytic subset A, U_x being an arbitrary small neighborhhood about x.

D.4. Complex analytic spaces and normalization

Let $A \subset \Omega \subset \mathbb{C}^n$ and $B \subset \Omega' \subset \mathbb{C}^m$ be two closed, analytic subsets of the open sets Ω and Ω' respectively. A continuous function $f\ : A \to B$ is called a *morphism of analytic sets* from A into B if and only if, for every $x \in A$, there exists a neighborhood U_x of x in Ω, a holomorphic function F_x from U_x into $\mathbb{C}^m$, such that $(F_x)_{|A\cap\Omega} = f_{|A\cap\Omega}$. If such is the case, then for every $x \in A$, one can define the mapping

$$f_x^*\ : \mathcal{O}_{B,f(x)} := \frac{\mathcal{O}_{\Omega',f(x)}}{\mathcal{I}_{B,f(x)}} \longrightarrow \mathcal{O}_{A,x} := \frac{\mathcal{O}_{\Omega,x}}{\mathcal{I}_{A,x}}$$

as $f_x^*(g_{f(x)}) = (g \circ F_x)_x$. This map is called *comorphism* of f at the point x.

For an analytic subset $A \subset \Omega \subset \mathbb{C}^n$, denote by $\mathcal{O}_A$ the $\mathcal{O}_\Omega$-coherent sheaf (see Section D.2) $\mathcal{O}_A := \mathcal{O}_\Omega/\mathcal{I}_A$. Complex (reduced) analytic spaces are realized on the model of complex analytic manifolds (see Section B.3.1), except that one glues together closed analytic subsets $A_\alpha \subset \Omega_\alpha \subset \mathbb{C}^{n_\alpha}$ instead of copies of open sets of some $\mathbb{C}^n$.

DEFINITION D.10 (complex (reduced) analytic space). A *complex analytic space* (or, to be more precise, *reduced* [4] *complex analytic space*), $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ consists in the following data :

(1) a separable and locally compact topological space $\mathcal{X}$, which is countable at infinity ;
(2) a *structural sheaf* of rings of continuous functions $\mathcal{O}_\mathcal{X}$, together with a covering $(U_\alpha, \tau_\alpha)_\alpha$ of $\mathcal{X}$, where τ_α realizes an homeomorphism between U_α and some closed analytic subset $A_\alpha = \tau(U_\alpha) \subset \Omega_\alpha \subset \mathbb{C}^{n_\alpha}$ (for some $n_\alpha \in \mathbb{N}$), such that, for any index α, the comorphism

$$\tau_\alpha^*\ : g \in ((\mathcal{O}_{A_\alpha})_{|U_\alpha})_{\tau_\alpha(x)} = \frac{((\mathcal{O}_{\Omega_{\mathbb{C}^{n_\alpha}}})_{|\Omega_\alpha})_{\tau(x)}}{\mathcal{I}_{A_\alpha,\tau(x)}} \mapsto (g\circ\tau_\alpha)_x \in ((\mathcal{O}_\mathcal{X})_{|U_\alpha})_x\,,\ x \in U_\alpha$$

is an isomorphism between sheafs of rings.

If $\mathcal{X}$ is a complex analytic space, then the subset $\mathcal{X}_{\rm reg}$ of *regular points* of $\mathcal{X}$ (these are points about which $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a complex manifold with its structural sheaf of holomorphic functions) is dense in $\mathcal{X}$ (isolated points of $\mathcal{X}$ are considered as regular). One denotes by $\mathcal{X}_{\rm sing} = \mathcal{X} \setminus \mathcal{X}_{\rm reg}$ the set of *singular points* in $\mathcal{X}$. The closures of the connected components of $\mathcal{X}_{\rm reg}$ are called *irreducible components* of $\mathcal{X}$. The *dimension* of $\mathcal{X}$ at x, denoted as $\dim_x \mathcal{X}$, is by definition $\dim_{\tau_\alpha(x)}(A_\alpha)$, whenever (U_α, τ_α) is a local chart containing x . The function $x \mapsto \dim_z \mathcal{X}$ is

[4]Since one does not want here to enter the theory of schemes.

constant on the irreducible components of $\mathcal{X}$ and thus it is enough to evaluate it at regular points in $\mathcal{X}$. The complex space $\mathcal{X}$ is *of pure dimension* n if all its irreducible components are of dimension n.

Given a complex analytic space $\mathcal{X}$, one defines on $\mathcal{X}$ the sheaf $\widetilde{\mathcal{O}}_{\mathcal{X}}$ of functions $h : \mathcal{X} \to \mathbb{C}$ such that $h_{|\mathcal{X}_{\rm reg}}$ is holomorphic on $\mathcal{X}_{\rm reg}$ (considered as a complex manifold equipped with the structural sheaf $(\mathcal{O}_{\mathcal{X}})_{|\mathcal{X}_{\rm reg}}$) and h is locally bounded on $\mathcal{X}$, that is, for any $x \in \mathcal{X}$, one can find an open neighborhood U_x of x such that $|h(y)| \leq C(y)$ for any $y \in U_x \cap \mathcal{X}_{\rm reg}$. Such functions are also called *weakly holomorphic functions* on the complex space $\mathcal{X}$ (compare to Definition D.3). If $\mathcal{X}$ is irreducible (the set of regular points has then only one connected component), then the local rings $\mathcal{O}_{\mathcal{X},x}$ are integral domains and one may consider their fraction fields $\mathcal{M}_{\mathcal{X},x}$, $x \in \mathcal{X}$. The corresponding sheaf $\mathcal{M}_{\mathcal{X}}$ is then called *sheaf of meromorphic functions* (sometimes called *regular functions*) over $\mathcal{X}$. If $\mathcal{X}$ is not irreducible anymore, the sheaf $\mathcal{M}_{\mathcal{X}}$ is a sheaf of rings, $\mathcal{M}_{\mathcal{X},x}$ being the ring of fractions of $\mathcal{O}_{\mathcal{X},x}$, that is the quotient of $\mathcal{O}_{\mathcal{X},x}$ by the ideal of elements which are not zero-divisors. Theorem D.1 implies that $\widetilde{\mathcal{O}}_{\mathcal{X}} \subset \mathcal{M}_{\mathcal{X}}$. In fact, for $x \in \mathcal{X}$, $\widetilde{\mathcal{O}}_{\mathcal{X},x}$ is the *integral closure* of $\mathcal{O}_{\mathcal{X},z}$ in $\mathcal{M}_{\mathcal{X},x}$, that is, for every $x \in \mathcal{X}$, $\widetilde{\mathcal{O}}_{\mathcal{X},z}$ is the set of elements h_x in $\mathcal{M}_{\mathcal{X},x}$ satisfying a monic integral dependence relation :

$$h_x^M + o_1 h_x^{M-1} + \cdots + o_M, \text{ with } o_1, ..., o_M \in \mathcal{O}_{\mathcal{X},x}.$$

The notion of weakly holomorphic function on $\mathcal{X}$ appeals to *Riemann's analytic continuation theorem* : if $\mathcal{X}$ is a complex manifold of dimension n and A is an analytic subset of $\mathcal{X}$ with codimension 1, any function $f : \mathcal{X} \setminus A \to \mathbb{C}$ which is holomorphic in $\mathcal{X} \setminus A$ and localy bounded on $\mathcal{X}$, that is

$$\forall x \in \mathcal{X}\,,\ \exists U_x \ni x,\ \sup_{y \in U_x \setminus A} |f(y)| < +\infty\,, \tag{D.9}$$

extends holomorphically over the whole manifold $\mathcal{X}$. In case codim $A \geq 2$, the additional hypothesis (D.9) is redundant and the existence of the analytic continuation of f follows from Hartogs theorem (see e.g. [**GRo**]).

When $\mathcal{X}$ is a complex analytic space (equipped with its structure sheaf $\mathcal{O}_{\mathcal{X}}$ and supposed here to be irreducible with dimension n), Riemann's analytic continuation theorem fails to be true in general. In fact, one may have holomorphic functions in the dense subset $\mathcal{X}_{\rm reg}$ which are bounded in a neighborhood of a singular point $x \in \mathcal{X}_{\rm sing}$, but do not define elements of $\mathcal{O}_{\mathcal{X},x}$. Here is an example : let $\varphi : t \in D(0,\epsilon) \mapsto (t^3, t^2)$ be an injective (since 2 and 3 are coprime) holomorphic parametrization in a neighborhood of the origin in $\mathbb{C}^2$ of the analytic space $\mathcal{X}$ defined (as embedded in $\mathbb{C}^2$) by the equation $z_1^2 - z_2^3 = 0$; the function $h : x \mapsto \varphi^{-1}(x)$ is holomorphic in $\mathcal{X}_{\rm reg} = \mathcal{X} \setminus \{(0,0)\}$ and locally bounded in $\mathcal{X}$, but there is no holomorphic function $\widetilde{h}$ in a neighborhood of the origin in $\mathbb{C}^2$ such that $\widetilde{h}(t^3, t^2) = t$, which proves that φ^{-1} cannot be defined at the level of germs at $(0,0)$ as an element of $\mathcal{O}_{\mathcal{X},(0,0)}$.

The fact that Riemann's analytic continuation theorem fails in general on complex analytic spaces leads naturally to introduce, given a complex analytic space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, the concept of *normality at a given point* $x \in \mathcal{X}_{\rm sing}$ and of *normality* (that is normality at any point).

Definition D.11 (local and global notions of normality). A complex analytic space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is said to be *normal at a point* $x \in \mathcal{X}_{\rm sing}$ if and only if $\widetilde{\mathcal{O}}_{\mathcal{X},x} = \mathcal{O}_{\mathcal{X},x}$. The space $\mathcal{X}$ is said to be *normal* if and only if it is normal at any point x of $\mathcal{X}_{\rm sing}$.

Any complex analytic space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ can be *normalized* in the following sense.

THEOREM D.4 (Oka normalization theorem). *Let $\mathcal{X}$ be a complex analytic space. There exists a normal complex analytic space $(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}})$, together with a proper projection $\pi : \overline{\mathcal{X}} \to \mathcal{X}$ with $\#(\pi^{-1}(\{x\}) < +\infty$ for any x, such that $\overline{\mathcal{X}} \setminus \pi^{-1}(\mathcal{X}_{\rm sing})$ is dense in $\overline{\mathcal{X}}$ and π realizes an analytic isomorphism between $\overline{\mathcal{X}} \setminus \pi^{-1}(X_{\rm sing})$ and $\mathcal{X}_{\rm reg}$. Such a pair $((\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}), \pi)$ (or, in short, $(\overline{\mathcal{X}}, \pi)$), is said to be a normalization of the complex analytic space $\mathcal{X}$). It is unique in the following sense : given two such normalizations $(\overline{\mathcal{X}_1}, \pi_1)$ and $(\overline{\mathcal{X}_2}, \pi_2)$, $(\overline{\mathcal{X}_1}, \mathcal{O}_{\overline{\mathcal{X}_1}})$ and $(\overline{\mathcal{X}_2}, \mathcal{O}_{\overline{\mathcal{X}_2}})$ are isomorphic as complex analytic spaces.*

D.5. Blow-up and log resolutions

D.5.1. Normalized blow-up along a coherent sheaf. The geometric operation which consists in *blowing-up* plays an important role in analytic or algebraic geometry. Given a $\mathcal{O}_{\mathcal{X}}$-coherent sheaf of ideals $\mathcal{I}$ over a complex manifold $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, one will use in particular in this monograph the so-called *normalized blow-up of $\mathcal{X}$ along the sheaf of ideals $\mathcal{I}$* (or also *with center of the coherent ideal sheaf $\mathcal{I}$*)). This is, up to an isomorphism between (reduced) complex analytic spaces, an intrinsic object (depending on $\mathcal{X}$ and $\mathcal{I}$), see Proposition D.4 below. Here are the two operations in which is decomposed the normalized blow-up procedure, given $\mathcal{X}$ and the $\mathcal{O}_{\mathcal{X}}$-coherent sheaf $\mathcal{I}$.

(1) Blow-up the complex manifold $\mathcal{X}$ along the coherent sheaf $\mathcal{I}$ (or with as center the coherent sheaf $\mathcal{I}$), which leads, as decribed below, to a (reduced) complex analytic space $(\mathcal{X}_{\mathcal{I}}, \mathcal{O}_{\mathcal{X}_{\mathcal{I}}})$ (of the same dimension than $\mathcal{X}$), together with an holomorphic proper surjective projection $\pi : \mathcal{X}_{\mathcal{I}} \to \mathcal{X}$, such that the inverse image sheaf $\mathcal{I} \cdot \mathcal{O}_{\mathcal{X}_{\mathcal{I}}}$ of $\mathcal{I}$ by π is invertible (the support of $\mathcal{O}_{\mathcal{X}_{\mathcal{I}}}/\mathcal{I} \cdot \mathcal{O}_{\mathcal{X}_{\mathcal{I}}}$ is a closed analytic hypersurface $H_{\mathcal{I}}$ of $\mathcal{X}_{\mathcal{I}}$) and π realizes a biholomorphism between $\mathcal{X}_{\mathcal{I}} \setminus H_{\mathcal{I}}$ and $\mathcal{X} \setminus \mathrm{Supp}\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$.
(2) Find a normalization $\overline{\mathcal{X}_{\mathcal{I}}} \stackrel{\pi_N}{\to} \mathcal{X}_{\mathcal{I}}$ of the complex (reduced) analytic space $\mathcal{X}_{\mathcal{I}}$, such as described in Theorem D.4 above.

The *normalized blow-up* of $\mathcal{X}$ along $\mathcal{I}$ is then $\overline{\mathcal{X}_{\mathcal{I}}} \stackrel{\pi\circ\pi_N}{\to} \mathcal{X}$. The inverse image sheaf $\mathcal{I} \cdot \mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}}}$ *via* $\pi \circ \pi_N$ is also invertible, which means its support is a closed analytic hypersurface H in $\overline{\mathcal{X}_{\mathcal{I}}}$. Denote as $\pi_{\mathcal{I},N} = \pi \circ \pi_N$ and $(H_\gamma)_\gamma$ the irreducible components of the hypersurface $H = \mathrm{Supp}\,\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}}}/\mathcal{I} \cdot \mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}}}$. As $\overline{\mathcal{X}_{\mathcal{I}}}$ is normal, the rings $\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}},z}$ are regular in codimension 1 (see Appendix C). One can then associate a Weil divisor

$$D_{\mathcal{I}} := \sum_\gamma m_\gamma H_\gamma \tag{D.10}$$

to the coherent sheaf of ideals $\pi^*_{\mathcal{I},N}[\mathcal{I}]$, where

$$m_\gamma = \mathrm{length}\Big(\frac{\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}},x_\gamma}}{\mathcal{I} \cdot \mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}},x_\gamma}}\Big) \tag{D.11}$$

(here $\mathcal{I} \cdot \mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}},x_\gamma}$ denotes the inverse image sheaf of $\mathcal{I}$ *via* the mapping $\pi_{\mathcal{I},N}$ at x_γ, the generic point of the irreducible component H_γ). This divisor is called *exceptional divisor of the normalized blow-up* $\overline{X_{\mathcal{I}}} \stackrel{\pi_{\mathcal{I},N}}{\to} \mathcal{X}$. For a deeper understanding of these notions one can refer to [**Hir**] or [**Te1, Te2**].

The following important proposition shows the uniqueness, up to isomorphism, of the normalized blow-up (in terms of a universality property).

PROPOSITION D.4 (universality property of the normalized blow-up). *Let $\mathcal{X}$ be a complex manifold and $\mathcal{I}$ be a $\mathcal{O}_{\mathcal{X}}$-coherent sheaf of ideals on $\mathcal{X}$. The normalized blow-up of $\mathcal{X}$, $\overline{\mathcal{X}_{\mathcal{I}}} \stackrel{\pi_{\tilde{I},N}}{\to} \mathcal{X}$ along $\mathcal{I}$ has the following universality property : if $\overline{\mathcal{X}} \stackrel{\tau}{\to} \mathcal{X}$ is any morphism of complex (reduced) analytic spaces such that $\overline{\mathcal{X}}$ is normal and the inverse image sheaf $\mathcal{I}\cdot\mathcal{O}_{\overline{\mathcal{X}}}$ is invertible, then τ factorizes in a unique manner as $\tau = \pi_{\mathcal{I},N}\circ\tau_{\mathcal{I}}$, where $\tau_{\mathcal{I}}$ is a morphism of complex (reduced) analytic spaces from $\overline{\mathcal{X}}$ into the normalized blow-up $\overline{\mathcal{X}_{\mathcal{I}}}$.*

It follows from the universality property of the normalized blow-up that the following definition defines instrinsic analytic subsets contained in Supp$\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$.

DEFINITION D.12 (distinguished components of Supp$\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$ [**Fu, Fu1**]). Let $\mathcal{X}$ be a complex n-dimensional manifold and $\mathcal{I}$ a $\mathcal{O}_{\mathcal{X}}$-coherent sheaf of ideals. Let $\overline{\mathcal{X}_{\mathcal{I}}} \stackrel{\pi_N\circ\pi}{\to} \mathcal{X}$ be the normalized blow-up of $\mathcal{X}$ along the coherent sheaf $\mathcal{I}$ and let $(H_\gamma)_\gamma$ be the list of the irreducible components of its exceptional divisor (D.10). It follows from Remmert-Stein theorem that the images $(\pi_N\circ\pi)(H_\gamma)$ are closed irreducible subsets Z_γ of Supp$\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$. The irreducible closed analytic subsets Z^k_ι among the (in general highly redundant) list $(Z_\gamma)_\gamma$ which have codimension k, where codim$\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I}) \leq k \leq n$, are called *distinguished irreducible components of codimension* k of Supp$\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$.

Let us make explicit the blow-up of a complex analytic manifold $\mathcal{X}$ along a coherent sheaf of ideals $\mathcal{I}$ first in the case $\mathcal{I}=\mathcal{I}_{\mathcal{Y}}$, where $\mathcal{Y}$ is a closed submanifold in $\mathcal{X}$, then in the general case of a coherent $\mathcal{O}_{\mathcal{X}}$-sheaf $\mathcal{I}$ on $\mathcal{X}$ (see e.g. [**Smi1**]).

a) *Suppose $\mathcal{I}=\mathcal{I}_{\mathcal{Y}}$, where $\mathcal{Y}$ is a closed submanifold in $\mathcal{X}$.* The blow-up of $\mathcal{X}$ along $\mathcal{I}_{\mathcal{Y}}$ is an holomorphic vector bundle with rank codim$\,\mathcal{Y}$ above $\mathcal{Y}$. Its fiber above $y\in\mathcal{Y}$ is the *projectivization* $\mathbb{P}(N_y)$ of the space N_y of all directions orthogonal to $\mathcal{Y}$ at the point y, i.e. the $\mathbb{C}$-vector space obtained as the quotient of $N_y\setminus\{0\}$ by the co-linearity relation. The vector bundle such constructed above $\mathcal{Y}$ is the quotient bundle $T(\mathcal{X})_{|\mathcal{Y}}/T(\mathcal{Y})$, where $T(\mathcal{X})\to\mathcal{X}$ is the holomorphic complex tangent bundle to $\mathcal{X}$ (that one restricts above $\mathcal{Y}$ as a bundle above $\mathcal{Y}$) and $T(\mathcal{Y})\to\mathcal{Y}$ the holomorphic tangent complex to $\mathcal{Y}$ (considered as a sub-bundle of $T(\mathcal{X})_{|\mathcal{Y}}\to\mathcal{Y}$. Suppose the submanifold $\mathcal{Y}$ (here with dimension $n-m$) is defined in a local chart U about one of its points y as
$$\mathcal{Y} := \{\zeta\in\mathcal{X}\,;\, f_1(\zeta)=\cdots=f_m(\zeta)=0\}\,,$$
where $f_1,...,f_m$ are holomorphic functions in U, satisfying $df_1\wedge\cdots\wedge df_m\neq 0$ over $\mathcal{Y}\cap U$. The blow-up of U along $\mathcal{I}_{\mathcal{Y}\cap U}$ can be expressed, up to an isomorphism of complex manifolds, as the closure in $U\times\mathbb{P}^{m-1}(\mathbb{C})$ of the graph of the mapping
$$\zeta\in U\setminus\mathcal{Y}\mapsto [f_1(\zeta):\cdots:f_m(\zeta)].$$
This graph is defined as a set of points $(\zeta,[w_1:\cdots:w_m])$ in the manifold $U\times\mathbb{P}^{m-1}(\mathbb{C})$ satisfying
$$w_j f_{j+1}(\zeta) - w_{j+1} f_j(\zeta) = 0\,,\ j=1,...,m-1.$$
Then the projection $\pi_U\ : U_{\mathcal{I}_{|U}}\to U$ (which is then proper since $\mathbb{P}^{m-1}(\mathbb{C})$ is compact) is given by the projection
$$(\zeta,[w_1:\cdots:w_m])\mapsto\zeta\in U.$$

The inverse image $\pi_U^{-1}(\mathcal{Y})$ is a smooth hypersurface $H_{\mathcal{Y},U} := (\mathcal{Y}\cap U)\times \mathbb{P}^{m-1}(\mathbb{C})$ and π_U realizes a biholomorphism between $\mathcal{X}_{\mathcal{I}_{|U}} \setminus H_{\mathcal{Y},U}$ and $U \setminus \mathcal{Y}$.

b) *Consider the case where $\mathcal{I}$ is an arbitrary $\mathcal{O}_{\mathcal{X}}$-coherent sheaf on $\mathcal{X}$.* It is enough to blow-up (along $\mathcal{I}_{|U}$) a local chart U about a point y in the support $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$. Let $(f_1, ..., f_m)$ be m elements in $\mathcal{O}(U)$, such that $\mathcal{I}_x = ((f_1)_x, ..., (f_m)_x)$ for every $x \in U$. The blow-up $U_{\mathcal{I}_{|U}}$ of U along $\mathcal{I}_{|U}$ can be expressed (up to isomorphism of analytic spaces) as the closure in $U \times \mathbb{P}^{m-1}(\mathbb{C})$ of the graph of the map

$$\zeta \in U \setminus \mathrm{Supp}\,(\mathcal{O}_U/\mathcal{I}_{|U}) \mapsto [f_1(\zeta) : \cdots : f_m(\zeta)].$$

The projection $\pi\ : U_{\mathcal{I}_{|U}} \to U$ is given by the projection map

$$(\zeta, [w_1 : \cdots : w_m]) \mapsto \zeta \in U.$$

The complex analytic space $\mathcal{I}_{|U}$ depends only (up to isomorphisms of complex analytic spaces) on the coherent sheaf $\mathcal{I}_{|U}$, not on the system of generators $(f_1, ..., f_m)$ chosen in $\mathcal{O}_{\mathcal{X}}(U)$. Hartogs theorem implies that the inverse image by π of the support of the quotient sheaf $\mathcal{O}_U/\mathcal{I}_U$ (i.e. here the locus of the common zeroes of f_j, $j = 1, ..., m$, in U) cannot have components with codimension larger of equal to 2. Therefore it is an hypersurface $H_{\mathcal{I},U}$ in $U \times \mathbb{P}^{m-1}(\mathbb{C})$ and π realizes a biholomorphism between $U_{\mathcal{I}_{|U}} \setminus H_{\mathcal{I},U}$ and $U \setminus \bigcap_1^m f_j^{-1}(0) = U \setminus \mathrm{Supp}\,(\mathcal{O}_{\mathcal{X}})_{|U}/\mathcal{I}_U)$. That geometric description of the blow-up $U_{\mathcal{I}_{|U}}$ of U along the coherent sheaf $\mathcal{I}_{|U}$ will be enough for our needs in the core of the monograph.

The notion of *integral closure* of an ideal in a commutative ring is closely related to the *normalized blow-up*. We recall this algebraic definition.

Definition D.13 (integral closure of an ideal in a commutative ring). Let $\mathbb{A}$ be a commutative ring and $\mathfrak{a}$ be an ideal of $\mathbb{A}$. We call *integral closure* of $\mathfrak{a}$ in $\mathbb{A}$ the ideal $\overline{\mathfrak{a}}$ of $\mathbb{A}$ consisting of elements h of $\mathbb{A}$ satisfying the *homogeneous* integral relation :

$$h^N + u_1 h^{N-1} + \cdots + h_N = 0,$$

with $u_k \in \mathfrak{a}^k$, $k = 1, 2, ..., N$. An equivalent formulation is to say that hT, considered as an element of the graded algebra $\mathbb{A}[T]$, is integral over the Rees algebra

$$\widetilde{\mathfrak{a}} := \mathbb{A} \oplus \mathfrak{a}T \oplus \mathfrak{a}^2T^2 \oplus ... = \bigoplus_{k=0}^{\infty} \mathfrak{a}^k T^k \subset \mathbb{A}[T],$$

i.e. it satisfies a monic integral dependence relation

$$(hT)^M + \sum_{j=1}^{M} (hT)^{M-j} \widetilde{u}_j(T)\,,\ u_1(T), ..., u_M(T) \in \widetilde{\mathfrak{a}}.$$

If $\mathcal{X}$ is a complex analytic variety, $\mathcal{I}$ a $\mathcal{O}_{\mathcal{X}}$ coherent sheaf of ideals, one can consider the coherent sheaf $\overline{\mathcal{I}}$, where, for every $x \in \mathcal{X}$, $\overline{\mathcal{I}}_x$ denotes the integral closure of $\mathcal{I}_x$ in $\mathcal{O}_{\mathcal{X},x}$. The coherence of the sheaf of ideals $\overline{\mathcal{I}}$ follows from the theorem of Grauert on the transport of coherence through direct image [**Grau**] and Proposition D.5 below.

Proposition D.5. *Let U be an open set in a complex manifold $\mathcal{X}$ and f be a global section of the ideal $\overline{\mathcal{I}}$ in U. If $\pi\ : \overline{\mathcal{X}_{\mathcal{I}}} \to \mathcal{X}$ is the normalized blow-up of $\mathcal{X}$*

along $\mathcal{I}$ *and* $D_{\mathcal{I}}$ *its exceptional divisor* (D.10). *Then the principal Cartier divisor* $\mathrm{div}(f\circ\pi)$ *is such that*

$$\mathrm{div}(f\circ\pi)\geq D_{\mathcal{I}}.$$

This implies also that for any component H_γ *of* $\pi^{-1}(\mathrm{Supp}\,(\mathcal{O}_{\mathcal{X}}/\mathcal{I}))$ *intersecting* $\pi^{-1}(U)$*, one has*

$$\mathrm{length}\,\Big(\frac{(\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}},x_\gamma})_{|\pi^{-1}(U)}}{f\cdot(\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}},x_\gamma})_{|\pi^{-1}(U)}}\Big)\geq m_\gamma \tag{D.12}$$

at a generic point x_γ *of* H_γ*, the multiplicities* m_γ *being defined by* (D.11). *This claim is equivalent to the inclusion*

$$f\,(\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}}})_{|\pi^{-1}(U)}\subset\mathcal{I}\cdot(\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}}})_{|\pi^{-1}(U)}. \tag{D.13}$$

Conversely, if either condition (D.12) *or condition* (D.13) *is fulfilled, then* f *is a global section in* U *of the sheaf of the ideals* $\overline{\mathcal{I}}$*.*

Remark D.5. One can as well introduce the blow-up of a complex analytic space $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ (instead of a complex manifold) along a coherent $\mathcal{O}_{\mathcal{X}}$-sheaf of ideals $\mathcal{I}_X$, and then, normalize it in order to construct the normalized blow-up of $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ along $\mathcal{I}_{\mathcal{X}}$. Results such as Propositions D.4 and D.5 still hold. From the point of view of complex analytic scheme theory (see [**Ha1**]), one should point out that the blow-up $(\mathcal{X}_{\mathcal{I}},\mathcal{O}_{\mathcal{X}_{\mathcal{I}}})$ of $(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ along the $\mathcal{O}_{\mathcal{X}}$-coherent sheaf $\mathcal{I}$ corresponds to the complex analytic scheme

$$(X_{\mathcal{I}},\mathcal{O}_{\mathcal{X}_{\mathcal{I}}})=\mathrm{Proj}\,\Big(\bigoplus_{k=0}^{\infty}\mathcal{I}^k\Big).$$

The normalized blow-up $(\overline{\mathcal{X}_{\mathcal{I}}},\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}}})$ corresponds to the complex analytic scheme

$$(\overline{\mathcal{X}_{\mathcal{I}}},\mathcal{O}_{\overline{\mathcal{X}_{\mathcal{I}}}})=\mathrm{Proj}\,\Big(\overline{\bigoplus_{k=0}^{\infty}\mathcal{I}^k}\Big)=\mathrm{Proj}\,\Big(\bigoplus_{k=0}^{\infty}\overline{\mathcal{I}^k}\Big)$$

(refer for example to [**LeT**]).

D.5.2. Log resolutions, Hironaka theorem. Among closed non smooth hypersurfaces $\mathcal{H}$ in a n-dimensional complex manifold $\mathcal{X}$, *hypersurfaces with local normal crossings* play a major role, since their singularities can be explicitely described.

Definition D.14 (Hypersurface with normal crossings). Let $\mathcal{X}$ be a n-dimensional manifold and $\mathcal{H}$ a closed hypersurface in $\mathcal{X}$. The hypersurface $\mathcal{H}$ is *locally with normal crossings* if and only if, for any point $x\in\mathcal{H}$, there is a centered system of local coordinates in some open chart U about x such that

$$\mathcal{H}\cap U=\{\zeta\in U\,;\,\zeta_{i_1}\cdots\zeta_{i_{l_x}}=0\,,\ \{i_1,...,i_{l_x}\}\subset\{1,...,n\}\},$$

i.e. $\mathcal{H}$ can be locally described in centered local coordinates ζ about any of its points as the zero set of a monomial in ζ.

Definition D.15 (log resolution). Let $(\mathcal{X},\mathcal{O}_{\mathcal{X}}))$ a n-dimensional complex irreducible analytic space and $\mathcal{Y}$ a closed analytic subset in $\mathcal{X}$. A *log resolution* $\widetilde{\mathcal{X}}\overset{\pi}{\to}\mathcal{X}$ for $\mathcal{Y}$ consists :

(1) a complex manifold $(\widetilde{\mathcal{X}},\mathcal{O}_{\widetilde{\mathcal{X}}})$ where $\dim\widetilde{\mathcal{X}}=n$;

(2) a proper surjective holomorphic map $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$, such that the analytic subset $\pi^{-1}(\mathcal{Y}) \cup \mathrm{crit}(\pi)$ is a closed analytic hypersurface with normal crossings in $\widetilde{\mathcal{X}}$, where $\mathrm{crit}(\pi)$ denotes the set of critical points of π, that is points in $\widetilde{\mathcal{X}}$ about which π does not realize a local biholomorphism.

One of the main tools in developing multidimensional residue theory is the theorem on resolution of singularities due to H. Hironaka [**Hir**]. It justifies the existence of a log resolution for any analytic subset $\mathcal{Y}$ in an irreducible analytic complex space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

THEOREM D.5 (resolution of singularities). *Let $\mathcal{X}$ be an irreducible complex analytic space of dimension n and $\mathcal{Y}$ be a complex analytic subset of $\mathcal{X}$ containing $\mathcal{X}_{\mathrm{sing}}$. There exists a complex manifold $\widetilde{\mathcal{X}}_{\mathcal{Y}}$ of dimension n and a proper holomorphic mapping $\pi_{\mathcal{Y}} : \widetilde{\mathcal{X}}_{\mathcal{Y}} \to \mathcal{X}$, such that, if $\widetilde{\mathcal{H}} = \pi_{\mathcal{Y}}^{-1}(\mathcal{Y})$:*

- *the mapping $\pi_{\mathcal{Y}}$ realizes a biholomorphism between $\widetilde{\mathcal{X}}_{\mathcal{Y}} \setminus \widetilde{\mathcal{H}}$ and $\mathcal{X} \setminus \mathcal{Y}$;*
- *the sheaf of ideals $\mathcal{I}_{\mathcal{Y}} \cdot \mathcal{O}_{\widetilde{\mathcal{X}}_{\mathcal{Y}}}$ (the inverse image by $\pi_{\mathcal{Y}}$ of the sheaf of ideals $\mathcal{I}_{\mathcal{Y}}$ on $\mathcal{X}$) is an invertible sheaf whose support is an hypersurface which has locally normal crossings.*

REMARK D.6. The universality property of the normalized blow-up (Proposition D.4) implies the following : if $\mathcal{I}$ is a coherent sheaf of ideals on a complex manifold $\mathcal{X}$ (or even a complex analytic space, see Remark D.5 above) and $\mathcal{Y}$ denotes the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$, then any resolution of singularities $\widetilde{\mathcal{X}}_{\mathcal{Y}} \overset{\widetilde{\pi}_{\mathcal{Y}}}{\to} \mathcal{X}$ as in Theorem D.5 factorizes in a unique way as $\widetilde{\pi}_{\mathcal{Y}} = \pi_{\mathcal{I},N} \circ \theta_{\widetilde{\mathcal{X}}_{\mathcal{Y}}}$, where $\theta_{\widetilde{\mathcal{X}}_{\mathcal{Y}}}$ is a morphism of analytic spaces from $\widetilde{\mathcal{X}}$ into the normalized blow-up $\overline{\mathcal{X}_{\mathcal{I}}}$.

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