

WEIGHTED ADMISSIBILITY AND WELLPOSEDNESS OF LINEAR SYSTEMS IN BANACH SPACES

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ABSTRACT. We study linear control systems in infinite-dimensional Banach spaces governed by analytic semigroups. For $p \in [1, \infty]$ and $\alpha \in \mathbb{R}$ we introduce the notion of L^p -admissibility of type α for unbounded observation and control operators. Generalising earlier work by Le Merdy [20] and the first named author and Le Merdy [12] we give conditions under which L^p -admissibility of type α is characterised by boundedness conditions which are similar to those in the well-known Weiss conjecture. We also study L^p -wellposedness of type α for the full system. Here we use recent ideas due to Pruess and Simonett. Our results are illustrated by a controlled heat equation with boundary control and boundary observation where we take Lebesgue and Besov spaces as state space. This extends the considerations in [4] to non-Hilbertian settings and to $p \neq 2$.

1. INTRODUCTION AND MAIN THEOREMS

We are concerned with linear control systems of the following form

$$\begin{cases} x'(t) + Ax(t) &= Bu(t), & (t > 0) \\ x(0) &= x_0, \\ y(t) &= Cx(t), & (t > 0), \end{cases} \quad (1)$$

where $-A$ is the generator of a strongly continuous semigroup $(T(\cdot))$ in a Banach space X . The function $x(\cdot)$ takes values in X , the functions $u(\cdot)$ and $y(\cdot)$ take values in Banach spaces U and Y , respectively. The control operator B is an unbounded operator from U to X , and the observation operator C is an unbounded operator from X to Y . We refer e.g. to [25, 27, 32, 31]. A commonly used minimal assumption on B and C is that C is bounded $X_1(A) \rightarrow Y$ and B is bounded $U \rightarrow X_{-1}(A)$ where $X_1(A)$ denotes the domain $D(A)$ of A equipped with the graph norm, and $X_{-1}(A)$ denotes the completion of $(X, \|R(\lambda_0, A) \cdot\|_X)$ with λ_0 in the resolvent set $\rho(A)$ of A (cf., e.g., [9, Sect. II.5]). Note that, for $\lambda_0 \in \rho(A)$, the norm $\|(\lambda_0 - A) \cdot\|_X$ on $X_1(A)$ is equivalent to the graph norm of A . The semigroup $T(\cdot)$ has an extension to a strongly continuous semigroup $T_{-1}(\cdot)$ on $X_{-1} := X_{-1}(A)$ whose generator A_{-1} is an extension of A (cf. [9, Sect. II.5]). These extensions are needed in order to give a precise meaning to compositions involving the operator B .

Now let $\mathcal{X} := C([0, \infty), X)$ and, for each $\tau > 0$, $\mathcal{X}_\tau = C([0, \tau], X)$. Suppose that we are given spaces \mathcal{Y} of functions $\mathbb{R}_+ \rightarrow Y$ and \mathcal{U} of functions $\mathbb{R}_+ \rightarrow U$ with restrictions $\mathcal{Y}_\tau, \mathcal{U}_\tau$ to $[0, \tau]$, $\tau > 0$, respectively. Then the system (1) is called *wellposed* if, for each $\tau > 0$, state and output of the system (1) depend continuously on initial state and input, i.e., if the mapping

$$X \times \mathcal{U}_\tau \rightarrow X \times \mathcal{Y}_\tau, \quad (x_0, u(\cdot)) \mapsto (x(\tau), y(\cdot))$$

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is continuous. Since the solution to (1) has the (formal) representation

$$(x(t), y(t)) = (T(t)x_0 + \int_0^t T(s)Bu(t-s) ds, CT(t)x_0 + C \int_0^t T(s)Bu(t-s) ds), \quad t > 0,$$

and $T(\cdot)$ is strongly continuous, this means continuity of the three maps

$$\Psi_\tau : \begin{cases} X \rightarrow \mathcal{Y}_\tau \\ x_0 \mapsto CT(\cdot)x_0 \end{cases}, \quad \Phi_\tau : \begin{cases} \mathcal{U}_\tau \rightarrow X \\ u \mapsto \int_0^\tau T_{-1}(\tau-s)Bu(s) ds \end{cases} \quad \text{and} \quad \mathcal{F}_\tau : \begin{cases} \mathcal{U}_\tau \rightarrow \mathcal{Y}_\tau, \\ u \mapsto CT_{-1}(\cdot)B * u \end{cases}$$

where $*$ denotes convolution. Requiring continuity of these maps leads to the notions of *admissibility* for the observation operator C , for the control operator B , respectively, and the notion of *wellposedness* for the input–output map \mathcal{F}_τ of system (1). In case of uniformly exponentially stable semigroups and $\mathcal{Y}_\tau = L^p([0, \tau], Y)$ and $\mathcal{U}_\tau = L^p([0, \tau], U)$, one may also consider wellposedness on $\mathbb{R}_+ = (0, \infty)$, i.e. the spaces $\mathcal{Y} = L^p(\mathbb{R}_+, Y)$ and $\mathcal{U} = L^p(\mathbb{R}_+, U)$ and continuity of the maps

$$\Psi : \begin{cases} X \rightarrow \mathcal{Y} \\ x_0 \mapsto CT(\cdot)x_0 \end{cases}, \quad \Phi : \begin{cases} \mathcal{U} \rightarrow X \\ u \mapsto \int_0^\infty T_{-1}(t)Bu(t) dt \end{cases} \quad \text{and} \quad \mathcal{F} : \begin{cases} \mathcal{U} \rightarrow \mathcal{Y}, \\ u \mapsto CT_{-1}(\cdot)B * u \end{cases}.$$

Notice that in the second mapping the convolution is replaced by an integration of the semigroup against the right hand side (cf. [31] and Remark 1.4 below).

If X, Y and U are Hilbert spaces it is natural to take $\mathcal{Y} = L^2(\mathbb{R}_+, Y)$ and $\mathcal{U} = L^2(\mathbb{R}_+, U)$ (cf. [15, 31, 32]). In the Banach space case one can consider $\mathcal{Y} = L^p(\mathbb{R}_+, Y)$ and $\mathcal{U} = L^p(\mathbb{R}_+, U)$ where $p \in [1, \infty]$ (cf. [8, 15, 28, 31]). For the case $p = 2$, the notion of *α -admissibility* for observation and control operators was introduced in [12] meaning that the L^2 -space on \mathbb{R}_+ with values in Y or U , respectively, is taken with respect to a polynomial weight on \mathbb{R}_+ . In this paper, we will extend that notion for observation operators from L^2 -norms to L^p -norms with $p \in [1, \infty]$. For control operators our definition differs from that given in [12] (cf. Remarks 1.2 and 1.4 below). Furthermore, we will introduce and study the new notion of *L^p -wellposedness of type α* for the system (1). For a short discussion on our motivation we refer to Theorem 1.16 and Remark 1.18.

The main basic question in modelling a given system in order to obtain a wellposed system of the form (1) is, of course, how to check for admissibility of the operators C and B and the wellposedness of the input–output map \mathcal{F} . For Hilbert spaces X, Y, U , the well-known *Weiss conjecture* (cf. [15, 33]) relates L^2 -admissibility on \mathbb{R}_+ of C and B to boundedness of

$$W_C := \{\lambda^{1/2}C(\lambda + A)^{-1} : \lambda > 0\} \subseteq B(X, Y) \quad \text{and} \\ W_B := \{\lambda^{1/2}(\lambda + A_{-1})^{-1}B : \lambda > 0\} \subseteq B(U, X),$$

respectively. Assuming that $T(\cdot)$ is bounded analytic and that $A^{1/2}$ is L^2 -admissible for A , boundedness of W_C actually characterises L^2 -admissibility of C (cf. [20] for this result and a detailed discussion of the Weiss conjecture). In [12], this characterisation was extended to cover the case of L^2 -admissibility of type α for a certain range of α .

For $\omega \in (0, \pi)$, let $S(\omega) := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\}$ and let $S(0) := (0, \infty)$. We recall that a *sectorial operator A of type $\omega \in [0, \pi)$* in a Banach space X is a closed linear operator A satisfying $\sigma(A) \subseteq \overline{S(\omega)}$ and, for any $\nu \in (\omega, \pi)$,

$$\sup\{\|\lambda R(\lambda, A)\| : |\arg \lambda| \geq \nu\} < \infty.$$

Observe that $-A$ generates a bounded analytic semigroup in X if and only if A is a densely defined sectorial operator in X of type $< \pi/2$. Moreover, we recall that a sectorial operator

with dense range is actually injective (cf. [22, Thm. 3.8]). Occasionally we shall use fractional powers of sectorial operators or their functional calculus. We refer to, e.g., [14, 19, 21].

L^p -admissibility of type α . In this paper, we assume that A is a densely defined sectorial operator of type $< \pi/2$ with dense range, and we characterise L^p -admissibility of type α for observation and control operators, the range of values of α depending on $p \in [1, \infty]$. For a fixed $p \in [1, \infty]$ we denote by p' the dual exponent given by $1/p + 1/p' = 1$. In order to formulate our first definition we introduce, for any $k \in \mathbb{N}$ and a given generator $-A$ of a bounded strongly continuous semigroup $T(\cdot)$ in X , the spaces $X_k := X_k(A)$, which is $\mathcal{D}(A^k)$ equipped with the norm $\|(\text{Id} + A)^k \cdot\|_X$, and $X_{-k} := X_{-k}(A)$, which is the completion of X with respect to the norm $\|(\text{Id} + A)^{-k} \cdot\|_X$. Observe that $1 \in \rho(-A)$ and that replacing $\text{Id} + A$ by $\lambda_0 + A$ in these expressions leads to equivalent norms whenever $\lambda_0 \in \rho(-A)$ (again, we refer to [9, Sect. II.5]). The semigroup $T(\cdot)$ has an extension to a bounded strongly continuous semigroup $T_{-k}(\cdot)$ on X_{-k} whose generator A_{-k} is an extension of the operator A .

We denote, for $\alpha \in \mathbb{R}$ and intervals $I \subseteq \mathbb{R}_+$,

$$L_\alpha^p(I, X) := \{f : I \rightarrow X : t \mapsto t^\alpha f(t) \in L^p(I, X)\}.$$

The space $L_\alpha^p(\mathbb{R}_+, X)$ is abbreviated $L_\alpha^p(X)$.

Definition 1.1. Let X, U, Y be Banach spaces and for some $k \in \mathbb{N}$ let $C \in B(X_k, Y)$ and $B \in B(U, X_{-k})$. Given $p \in [1, \infty]$ and a bounded analytic semigroup $T(\cdot)$ on X ,

- (a) C is called *finite-time L^p -admissible of type $\alpha > -1/p$* , if for any $\tau > 0$ there is a constant $M_\tau > 0$ such that for all $x \in X_k$ we have

$$\|t \mapsto CT(t)x\|_{L_\alpha^p((0, \tau), Y)} \leq M_\tau \|x\|_X. \quad (2)$$

- (b) B is called *finite-time L^p -admissible of type $\alpha < 1/p'$* (or $\alpha \leq 0$ for $p = 1$), if for any $\tau > 0$ there is a constant $K_\tau > 0$ such that for all $u \in L_\alpha^p((0, \tau), U)$ we have

$$\|T_{-k}B * u\|_{L^\infty((0, \tau), X)} \leq K_\tau \|u\|_{L_\alpha^p((0, \tau), U)}. \quad (3)$$

If the above estimates hold with constants M_τ and K_τ that can be chosen independently of $\tau > 0$, the operators B and C are called (infinite-time) *L^p -admissible of type α* .

Remark 1.2. In part (b) of the definition, inequality (3) means $\text{ess sup}_{t \in (0, \tau)} \|\Phi_t(u)\|_X \leq K_\tau \|u\|_{L_\alpha^p((0, \tau), U)}$. This seems to be better suited when dealing with weighted spaces than requiring just $\|\Phi_\tau(u)\|_X \leq K_\tau \|u\|_{L_\alpha^p((0, \tau), U)}$.

Notice that our definition differs from that in [12]: for observation operators, L^2 -admissibility of type α is called 2α -admissibility there. For control operators, our definition is different from the notion studied in [12]. This is due to the application to nonlinear problems in Theorem 1.16. See also Remark 1.4 for a study of both notions.

Lemma 1.3. *The notion of finite-time L^p -admissibility of type α for A is independent of the underlying interval $[0, \tau]$ in the definition.*

If the semigroup is uniformly exponentially stable and $\alpha \geq 0$ or $p = \infty$, finite-time L^p -admissibility of type α for control operators is equivalent to (infinite-time) L^p -admissibility of type α . For observation operators this is true for all α . Moreover, an observation operator C is (infinite-time) L^p -admissible of type α of A if and only if

$$\left(\int_0^\infty \|t^\alpha CT(t)x\|^p dt \right)^{1/p} \leq M_{\tau, \alpha} \|x\|. \quad (4)$$

holds.

Notice that the requirement $\alpha \geq 0$ or $p = \infty$ in the above equivalence assertion for control operators is necessary. We provide a short counterexample in case $\alpha < 0$ in Example 3.1.

Remark 1.4 (Dualisation). Let X, Y be reflexive Banach spaces and $p \in (1, \infty)$ and assume that $C \in B(X_1, Y)$ is L^p -admissible of type α for A on \mathbb{R}_+ , Then, for $u \in L^p_{-\alpha}(\mathbb{R}_+, Y')$,

$$\begin{aligned} \langle t \mapsto CT(t)x, u \rangle_{L^p_{-\alpha} \times L^p_{-\alpha}} &= \int_0^\infty \langle t^\alpha CT(t)x, t^{-\alpha} u(t) \rangle_{Y \times Y'} dt \\ &= \langle x, \int_0^\infty T(t)' C' u(t) dt \rangle_{X \times X'}. \end{aligned}$$

One can therefore consider the following dual condition to (4), that was introduced in [12].

$$\left\| \int_0^\infty T(t)Bu(t) dt \right\|_X \leq K \|u\|_{L^p_{-\alpha}(\mathbb{R}_+, U)}, \quad (5)$$

where the integral is considered as a Pettis integral in X_{-k} taking values in X . Notice, that in case $\alpha \neq 0$ the reflection $R_\tau u = u(\tau - \cdot)$ is not bounded on $L^p_\alpha([0, \tau])$. Therefore, one cannot hope that (5) might be equivalent to infinite-time L^p -admissibility of type α unless $\alpha = 0$. We shall see in Theorem 1.8 below that condition (5) is indeed a stronger notion than L^p -admissibility of type α . However, for $\alpha > 0$, L^p -admissibility of an observation operator C implies L^p -admissibility of C' on X' by Theorem 1.8 below.

L^p -estimates and the real interpolation method. As mentioned above, the crucial condition in [20] (besides $T(\cdot)$ being bounded and analytic) for the Weiss conjecture to hold was that $A^{1/2}$ is admissible for A , i.e., the existence of a constant $L > 0$ such that

$$\int_0^\infty \|A^{1/2}T(t)x\|^2 dt = \int_0^\infty \|(tA)^{1/2}T(t)x\|^2 \frac{dt}{t} = \|\psi(tA)x\|_{L^2(\mathbb{R}_+, dt/t, X)}^2 \leq L \|x\|_X^2,$$

where $\psi(z) = z^{1/2}e^{-z}$. In our situation this corresponds to

$$\int_0^\infty \|t^\alpha A^{\alpha+1/p}T(t)x\|^p dt = \int_0^\infty \|(tA)^{\alpha+1/p}T(t)x\|^p \frac{dt}{t} = \|\psi(tA)x\|_{L^p(\mathbb{R}_+, dt/t, X)}^p \leq \tilde{L} \|x\|_X^p \quad (6)$$

for $\psi(z) = z^{\alpha+1/p} \exp(-z)$. We denote $L^p_*(\mathbb{R}_+, X) := L^p(\mathbb{R}_+, dt/t, X)$. It is known that the property of a sectorial operator A of type ω on a Banach space X , to satisfy an estimate $\|\psi(\cdot A)x\|_{L^p_*(\mathbb{R}_+, Y)} \leq L \|x\|_X$ does not depend upon the particular choice of the function $\psi \in H_0^\infty(S(\nu)) \setminus \{0\}$, $\nu > \omega$, where

$$H_0^\infty(S(\nu)) = \{f \in H^\infty(S(\nu)) : \exists c, s > 0 : |f(z)| \leq c \frac{|z|^s}{1+|z|^{2s}}\},$$

see [22, Thm. 5], [2, Thm. 4.1], [13, Thm. 4.3] or [14, Thm 6.4.3]. Therefore, we say that A satisfies L^p_* -estimates on X if (6) holds.

Let A be a densely defined sectorial operator of type ω with dense range on X . Let $\mathbb{X} := (X, \|A(I+A)^{-2} \cdot\|)^\sim$. For some holomorphic function ψ on $S(\nu)$, $\nu > \omega$ and $\theta \in (-1, 1)$ such that $z^{-\theta}\psi(z) \in H_0^\infty(S(\nu)) \setminus \{0\}$ consider the space

$$X_{\theta, \psi, p} := \left\{ x \in \mathbb{X} : t^{-\theta}\psi(tA)x \in L^p_*(\mathbb{R}_+, X) \right\}.$$

Since $X_{\theta, \psi, p}$ does not depend on $\psi \in H_0^\infty(S(\nu)) \setminus \{0\}$ (see above), we write for short $X_{\theta, p} := X_{\theta, \psi, p}$. Resorting to the space \mathbb{X} in this setting allows explicitly that $X_{\theta, p}$ may be a larger space than X . For the background of this construction we refer e.g. to [18], [11, Section 2] or [14, Chapter 6.3]. Notice that in this terminology L^p_* -estimates for A read as $X \hookrightarrow X_{0, p}$. These spaces are strongly connected to real interpolation spaces between the homogeneous spaces \dot{X}_{-1} and \dot{X}_1 . Here, for $\sigma \in \mathbb{Z}$, we denote by $\dot{X}_\sigma := \dot{X}_\sigma(A)$ the completion of the space $\mathcal{D}(A^\sigma)$

with respect to the homogeneous norm $\|A^\sigma \cdot\|_X$. We refer to [17], [19, Sect. 15], [11] or [13] for more details on these spaces. Observe that – due to our assumptions – the space $\mathcal{D}(A^\sigma)$ is dense in X , and that $\dot{X}_\sigma = X_\sigma$ in case $0 \in \rho(A)$. With these notations the following result holds true (cf. [2] for the case $p = 2$ and [14, Thm. 6.4.6], [13, Thm. 5.2] for the general case).

Theorem 1.5. *Let X be a Banach space and A be a densely defined sectorial operator of type ω with dense range on X . Then*

$$(\dot{X}_{-1}(A), \dot{X}_1(A))_{\theta,p} = X_{2\theta-1,p}$$

for all $\theta \in (0, 1)$ and $p \in [1, \infty]$.

Remark 1.6. In the situation of the above theorem, A has L_*^p -estimates if and only if $X \hookrightarrow (\dot{X}_{-1}(A), \dot{X}_1(A))_{1/2,p}$. This can be used to establish L_*^p -estimates for given operators in concrete cases, as we will show in Section 2.

We start our main results with the following characterisations of L^p -admissibility of type α for observation and control operators that extend the results in [12].

Theorem 1.7. *Let $p \in [1, \infty]$ and A be a densely defined sectorial operator of type $\omega < \pi/2$ with dense range on X . Let $C \in B(X_k, Y)$ be an observation operator for some $k \geq 1$. Let $\alpha \in (-1/p, k - 1/p)$ and consider the set*

$$W_C := \{\lambda^{k-\alpha-1/p} C(\lambda + A)^{-k} : \lambda > 0\} \subseteq B(X, Y). \quad (7)$$

Then the following assertions hold:

- (a) *If C is L^p -admissible of type α for A , then W_C is bounded in $B(X, Y)$.*
- (b) *If A satisfies L_*^p -estimates and W_C is bounded in $B(X, Y)$, then C is L^p -admissible of type α for A .*

Notice that, for the operator $C := A^{\alpha+1/p}$, the set W_C in (7) is bounded, whence the assumption of L_*^p -estimates in the above characterisation cannot be improved (cf. (6)).

Theorem 1.8. *Let $p \in [1, \infty]$ and A be a densely defined sectorial operator of type $\omega < \pi/2$ with dense range on X . Let $B \in B(U, X_{-k})$ be a control operator for some $k \geq 1$ and let $\alpha \in (1/p' - k, 1/p')$ and consider the set*

$$W_B := \{\lambda^{k+\alpha-1/p'} (\lambda + A_{-k})^{-k} B : \lambda > 0\} \subseteq B(U, X). \quad (8)$$

Then the following assertions hold:

- (a) *If B is L^p -admissible of type α for A , then W_B is bounded.*
- (b) *If W_B is bounded and $\alpha > 0$ (for $p > 1$) or $\alpha = 0$ (for $p = 1$) then B is L^p -admissible of type α for A .*
- (a)' *If the dual condition (5) holds, then W_B is bounded.*
- (b)' *If $p < \infty$ and W_B is bounded and the adjoint operator A' satisfies $L_*^{p'}$ -estimates, then (5) holds.*

As mentioned above, in case $\alpha = 0$, L^p -admissibility of type α and (5) are equivalent. In particular, a modified Weiss conjecture on control operators holds for $p = 1$.

Also the boundedness conditions on the sets W_C and W_B in (7) and (8) may be characterised by real interpolation methods. In fact, they are conditions on the *domain* of (a continuous extension of) the observation operator C and the *range* of the control operator B , respectively. This way of viewing the boundedness conditions on W_C and W_B , respectively, is not new for control operators (cf. [33]) under the supplementary assumption of $0 \in \varrho(A)$. The general equivalence for observation and control operators was first studied in [11]. The proof of the following is very similar to the arguments used there and thus omitted.

Theorem 1.9. *Let A be an injective sectorial operator on the Banach space X , and let $k \in \mathbb{N}$. Let $C \in B(X_k, Y)$ and $B \in B(U, X_{-k})$ be bounded operators, where U, Y are Banach spaces, and let $\theta \in (0, 1)$. Then the following equivalences hold true:*

- (a) $\sup_{0 < \lambda < \infty} \|\lambda^{k\theta}(\lambda + A_{-k})^{-k}B\|_{U \rightarrow X} < \infty$ if and only if $B : U \rightarrow X_{-k}$ is bounded in the norm $U \rightarrow (\dot{X}_{-k}, X)_{\theta, \infty}$.
- (b) $\sup_{0 < \lambda < \infty} \|\lambda^{k(1-\theta)}C(\lambda + A)^{-k}\|_{X \rightarrow Y} < \infty$ if and only if $C : X_k \rightarrow Y$ is bounded in the norm $(X, \dot{X}_k)_{\theta, 1} \rightarrow Y$.

Remark 1.10. Of course, if for some $k \in \mathbb{N}$, $C \in B(X_k, Y)$ and if the operators $\lambda^{k(1-\theta)}C(\lambda + A)^{-k}$, $\lambda > 0$, are uniformly bounded, then for any $n \in \mathbb{N}_0$, $C \in B(X_{k+n}, Y)$ and by the sectoriality of A the operators $\lambda^{n+k(1-\theta)}C(\lambda + A)^{-k-n}$, $\lambda > 0$, are uniformly bounded. However, by reiteration (cf. [29, 1.10.2]),

$$(X, \dot{X}_k)_{\theta, 1} = (X, \dot{X}_{k+n})_{\sigma, 1}$$

for $\sigma = \frac{n+k(1-\theta)}{k+n}$, whence the assertions of Theorem 1.9 do not depend on the question, for which $k \in \mathbb{N}$ the given boundedness conditions are satisfied. The same arguments apply to the conditions on control operators.

Wellposedness of the full system. We now study the full system (1) for $\mathcal{Y} = L_\alpha^p(\mathbb{R}_+, Y)$ and $\mathcal{U} = L_\alpha^p(\mathbb{R}_+, U)$.

Definition 1.11. Let X, U, Y be Banach spaces and let $k \in \mathbb{N}$, $p \in [1, \infty]$ and $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$. Let $T(\cdot)$ be a bounded analytic semigroup on X generated by $-A$. Then the system (1) is called L^p -wellposed of type α , if $C \in B(X_k, Y)$ and $B \in B(U, X_{-k})$ are L^p -admissible of type α , and $\mathcal{F}^\alpha : L_\alpha^p(U) \rightarrow L_\alpha^p(Y)$, $\mathcal{F}^\alpha u = CT_{-k}(\cdot)B * u$ is bounded.

If we have finite-time L^p -admissibility of type α of B and C on $I := [0, \tau]$ and if $\mathcal{F}_\tau^\alpha : L_\alpha^p(I, U) \rightarrow L_\alpha^p(I, Y)$ is bounded, we call the system *finite-time L^p -wellposed of type α* . By resorting to $(e^{-\omega t}T(t))$ we may then assume $0 \in \varrho(A)$. The next lemma shows that the only possible singularity of the convolution kernel of \mathcal{F}_τ^α is at $t = 0$, whence the notion of finite-time L^p -wellposedness of type α does not depend on $\tau > 0$.

Notice that by analyticity of the semigroup $T(\cdot)$, we have $T_{-k}(t)X_{-k} \subseteq X_k$ for $t > 0$. Therefore the convolution kernel $CT_{-k}(\cdot)B$ of \mathcal{F}^α is a well defined bounded operator from U to Y (see next lemma for norm estimates). From Theorems 1.7, 1.8 and 1.9 we know that L^p -admissibility of type α of B and C yields $(X, \dot{X}_k)_{\theta, 1} \hookrightarrow \mathcal{D}(C)$ and $\mathcal{R}(B) \subseteq (X, \dot{X}_{-k})_{\sigma, \infty}$ for $\theta = (\alpha + \frac{1}{p})/k$ and $\sigma = (\frac{1}{p'} - \alpha)/k$. Notice that $k(\sigma + \theta) = 1$. The next lemma is well known to specialists. We use it for $q = 1$ and $r = \infty$ and provide a proof in Section 3.

Lemma 1.12. *Let X be a Banach space and $T(\cdot)$ be a bounded analytic semigroup on X . Let $k \in \mathbb{N}$ and $\sigma, \theta \in (0, 1)$ such that $k(\sigma + \theta) = 1$. Let $q, r \in [1, \infty]$ and $Z := (X, \dot{X}_k)_{\theta, q}$ and $W := (\dot{X}_{-k}, X)_{1-\sigma, r}$. Then, there exists a constant $M > 0$ such that $\|T(t)\|_{W \rightarrow Z} \leq M/t$ for all $t > 0$.*

To sum up the above considerations: whenever C and B are L^p -admissible of type α , we have $\|CT(t)B\|_{U \rightarrow Y} \leq M/t$ for $t > 0$ for some constant $M > 0$. A corresponding condition in the following theorem seems thus very natural.

Theorem 1.13. *Let $p \in (1, \infty)$, $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$ and $k \in \mathbb{N}$. Let X, U, Y be Banach spaces and let $T(\cdot)$ be a bounded analytic semigroup on X , $C \in B(X_k, Y)$ and $B \in B(U, X_{-k})$ such that $\|CT(t)B\|_{U \rightarrow Y} \leq M/t$. Then $\mathcal{F}^\alpha := CT(\cdot)B*$ is bounded $L_\alpha^p(U) \rightarrow L_\alpha^p(Y)$ if and only if $\mathcal{F} = \mathcal{F}^0$ is bounded $L^p(U) \rightarrow L^p(Y)$.*

The above theorem is basically a reformulation of [24, Thm. 2.4]. However, our proof allows also negative values of α . It relies on the following result:

Proposition 1.14. *Let $p \in (1, \infty)$ and $\alpha < 1 - \frac{1}{p}$. Let U, Y be Banach spaces and suppose that $K \in C(\mathbb{R}_+, B(U, Y))$ satisfies $\|K(t)\| \leq M/t$ for some $M > 0$. Let*

$$(Tf)(t) := \int_0^t K(t-s) \left[\left(\frac{t}{s} \right)^\alpha - 1 \right] f(s) ds, \quad f \in L^p(\mathbb{R}_+, X).$$

Then $T \in B(L^p(\mathbb{R}_+, X), L^p(\mathbb{R}_+, Y))$ with norm bound cM where $c = c(p, \alpha)$.

We shortly discuss boundedness of \mathcal{F} for $\alpha = 0$. Suppose that $C : Z \rightarrow Y$ and $B : U \rightarrow W$ are bounded where Z and W are Banach spaces satisfying $X_1 \subseteq Z \subseteq X$ and $X \subseteq W \subseteq X_{-1}$. Suppose further that the restriction A_W of A_{-1} to W is sectorial with $\mathcal{D}(A_W) = Z$ (equivalent norms). Then $CT(\cdot)B^* : L^p(U) \rightarrow L^p(Y)$ is bounded if $T(\cdot)^* : L^p(W) \rightarrow L^p(Z)$ is bounded. But it is well-known that the latter is equivalent to A_W having the property of *maximal L^p -regularity* (we refer to [1, 6, 7, 19, 30] for this relation, the problem of maximal regularity, characterisation results and further references on the subject). Thus we have obtained

Theorem 1.15. *Let $p \in (1, \infty)$ and W be a Banach space $X \subseteq W \subseteq X_{-1}$ such that A_W has maximal L^p -regularity. Let $C : Z \rightarrow Y$ and $B : U \rightarrow W$ be bounded where Z denotes $\mathcal{D}(A_W)$ equipped with the graph norm. Then \mathcal{F}^α is bounded $L^\alpha_\alpha(U) \rightarrow L^\alpha_\alpha(Y)$ for any $\alpha \in (-\frac{1}{p}, \frac{1}{p'})$.*

Our results can be used to establish existence and uniqueness of solutions to some nonlinear systems in feedback form. Let $p \in (1, \infty)$, let X, U, Y be Banach spaces and let $x_0 \in X$. Let $F : X \rightarrow B(Y, U)$ be a Lipschitz-continuous operator-valued function, let A, B, C be linear operators such that $-A_0 := -(A - BF(x_0)C)$ generates an analytic semigroup $T(\cdot)$ in X and such that B and C are finite-time L^p -admissible of type α for A_0 . We consider the closed loop system

$$\begin{cases} x'(t) + Ax(t) &= Bu(t), \\ x(0) &= x_0, \\ y(t) &= Cx(t), \\ u(t) &= F(x(t))y(t), \end{cases} \quad (9)$$

on $[0, \tau]$ which we rewrite as

$$\begin{cases} x'(t) + A_0x(t) &= B(F(x(t)) - F(x_0))Cx(t) & t \in [0, \tau], \\ x(0) &= x_0. \end{cases} \quad (10)$$

We are interested in mild solutions of (10), i.e. solutions of

$$x = T(\cdot)x_0 + T * B(F(x) - F(x_0))Cx \quad \text{on } [0, \tau]. \quad (11)$$

The following result shall also be proved in Section 3.

Theorem 1.16. *Assume in addition to the preceding assumptions that $Z \hookrightarrow X$ is a Banach space such that the system corresponding to (A_0, B, Id_Z) is finite-time L^p -wellposed of type α and that $C \in B(Z, Y)$. Then there exists $\tau = \tau(x_0) > 0$ such that (9) has a unique mild solution $x \in C([0, \tau], X) \cap L^\alpha_\alpha([0, \tau], Z)$.*

Remark 1.17. The reason for introducing the space Z here is that $x \mapsto Cx$ may not induce a closed operator from $C([0, \tau], X)$ into $L^\alpha_\alpha([0, \tau], Y)$. If C is closed as an operator from X to Y , then we can replace $C([0, \tau], X) \cap L^\alpha_\alpha([0, \tau], Z)$ by $\{x \in C([0, \tau], X) : x(t) \in D(C) \text{ a.e.}, Cx \in L^\alpha_\alpha([0, \tau], Y)\}$ in the assertion, and the assumption on the space Z is not needed.

Remark 1.18. Our results on L^p -admissibility of type α and L^p -wellposedness of type α give more flexibility in the choice of Banach spaces X, U , and Y for the modelling of a given problem. This is important for the study of nonlinear systems via fixed point arguments, e.g. via Theorem 1.16, where an appropriate choice of α allows to choose state spaces X as function

spaces with little regularity (cf. also [24, Rem. 3.3(b)]). For the example of the controlled heat equation that we study in Section 2 we refer to Remarks 2.11 and 2.12 where we describe how to obtain state spaces with negative smoothness index by suitable choices of α . In Example 2.14 we give an application of Theorem 1.16 in a nonlinear feedback situation. We mention in this context that Besov spaces of negative order have become relevant as spaces for initial values in the study of other nonlinear partial differential problems, e.g. Navier-Stokes equations (cf. [5]).

In the next section we provide examples and applications of our results. In Section 3 we shall give proofs of the results presented so far.

2. EXAMPLE: A CONTROLLED HEAT EQUATION

In this section we illustrate our results with a controlled heat equation. In [4], the problem has been studied in the state space $X = L^2(\Omega)$ and for $\alpha = 0$. Below we discuss Lebesgue and Besov spaces.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^\infty$. Denote the outer normal unit vector on $\partial\Omega$ by $\nu : \partial\Omega \rightarrow \mathbb{R}^n$. We consider the following problem

$$\begin{cases} x'(t) - \Delta x(t) &= 0, & (t > 0) \\ \frac{\partial x(t)}{\partial \nu} \Big|_{\partial\Omega} &= u(t), & (t > 0) \\ x(0) &= x_0 \\ y(t) &= x(t) \Big|_{\partial\Omega}, & (t > 0), \end{cases} \quad (12)$$

where $x(\cdot)$ takes values in some function space X . The functions $u(\cdot)$ and $y(\cdot)$ take values in function spaces on the boundary. For the modelling we follow closely [4] and set $A := -\Delta$ with Neumann boundary condition. In the state spaces we shall consider below, the operator A is sectorial of type 0, but not injective. We aim for finite-time L^c -admissibility of type γ for observation operators and finite-time L^b -admissibility of type β for control operators, where $b, c \in [1, \infty]$. Sufficient for this is $L^{b/c}$ -admissibility of type β/γ on \mathbb{R}_+ for $\text{Id} + A$. In order to use our characterisations of admissibility, we assume

$$\beta \in (-\frac{1}{b}, \frac{1}{b'}) \quad \text{and} \quad \gamma \in (-\frac{1}{c}, \frac{1}{c'}).$$

First, we start with

The L^q -case. Consider $X := L^q(\Omega)$, $1 < q < \infty$. Due to the smoothness of $\partial\Omega$ we then have $\mathcal{D}(A) = \{x \in W_q^2(\Omega) : \frac{\partial x}{\partial \nu} \Big|_{\partial\Omega} = 0\}$ and $-A$ generates a bounded analytic semigroup. For to ensure L_*^c -estimates, we have to impose

$$X \hookrightarrow (\dot{X}_{-1}(\text{Id} + A), \dot{X}_1(\text{Id} + A))_{\frac{1}{2}, c} = (X_{-1}(A), X_1(A))_{\frac{1}{2}, c} = (X_{-\delta}(A), X_\delta(A))_{\frac{1}{2}, c} \quad (13)$$

where the last equality holds for any $\delta > 0$ by reiteration. In the case $X = L^q(\Omega)$, we have $X_\delta = H_q^{2\delta}(\Omega)$ for small $\delta > 0$ (cf. [26]) and $X_{-\delta} = H_q^{-2\delta}(\Omega)$ by dualisation. Therefore, L_*^c -estimates for $\text{Id} + A$ on $L^q(\Omega)$ are equivalent to the continuous embedding $L^q(\Omega) \hookrightarrow B_{q,c}^0(\Omega)$ (cf. [29, Thm 2.4.1, 4.3.1]). We use the following lemma which shall be proved in Section 3.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^\infty$. If $p, q \in [1, \infty]$, we have $L^q(\Omega) \hookrightarrow B_{q,p}^0(\Omega)$ if and only if $p \geq \max(2, q)$.*

For an application of Theorem 1.7 for L^c -admissibility of C in $X = L^q(\Omega)$ we thus need $c \geq \max(2, q)$. For an application of Theorem 1.8 for L^b -admissibility of B in $X = L^q(\Omega)$ in case $\beta = 0$, we also need $L_*^{b'}$ -estimates for $(I+A)'$ on X' . By the arguments above these are equivalent to $L^{q'}(\Omega) \hookrightarrow B_{q',b'}^0(\Omega)$ which means by Lemma 2.1 that, for $\beta = 0$, we have to

suppose $b \leq \min(q, 2)$. Obviously, if $\beta = 0$, then application of both Theorems 1.7 and 1.8 for $b = c$ would require $b = c = q = 2$ and we are back in the Hilbert space situation. We shall come back to this below.

Admissibility. Denoting the Dirichlet trace operator $\gamma_0 : x \mapsto x|_{\partial\Omega}$ by C we are looking for a space Y on $\partial\Omega$ such that $C : \mathcal{D}(A) \rightarrow Y$ is L^c -admissible of type γ for $\text{Id} + A$. If $c \geq \max(2, q)$, then $\text{Id} + A$ has L^c_* -estimates and we know by Theorem 1.7 that L^c -admissibility of type γ of $C : \mathcal{D}(A) \rightarrow Y$ is equivalent to uniform boundedness of the operators

$$\lambda^{1-\gamma-\frac{1}{c}} C(\lambda + \text{Id} + A)^{-1}, \quad \lambda > 0.$$

By Theorem 1.9 this is equivalent to C having a continuous extension to the Banach space Z where

$$\begin{aligned} Z &:= (X, \dot{X}_k(\text{Id} + A))_{\gamma+\frac{1}{c}, 1} \\ &= (X, X_k(A))_{\gamma+\frac{1}{c}, 1} \\ &\stackrel{(*)}{=} \begin{cases} B_{q,1}^s(\Omega) & \text{if } s < 1 + \frac{1}{q} \\ \{x \in B_{q,1}^s(\Omega) : \frac{\partial x}{\partial \nu}|_{\partial\Omega} = 0\} & \text{if } s > 1 + \frac{1}{q}, \end{cases} \end{aligned}$$

where $s = 2\gamma + \frac{2}{c}$. For equality $(*)$ we refer to [29, Thm 4.3.3] or [10, Thm 3.5]. It is known that γ_0 is bounded from $B_{q,1}^s(\Omega)$ to $B_{q,1}^{s-\frac{1}{q}}(\partial\Omega)$ if $s > \frac{1}{q}$ (cf. [29, Thm. 4.7.1]). We thus have almost proved the following

Proposition 2.2. *Let $\gamma \in \mathbb{R}$, $c \in [1, \infty]$ and $q \in (1, \infty)$ satisfy $c \geq \max(q, 2)$ and $2\gamma + \frac{2}{c} \in (\frac{1}{q}, 1 + \frac{1}{q})$, and let Y be a Banach space and $X = L^q(\Omega)$. Then $\gamma_0 : \mathcal{D}(A) \rightarrow Y$ is L^c -admissible of type γ for $\text{Id} + A$ if and only if $B_{q,1}^{2\gamma+\frac{2}{c}-\frac{1}{q}}(\partial\Omega) \hookrightarrow Y$.*

Proof. Let $s := 2\gamma + \frac{2}{c}$. The arguments above show that $B_{q,1}^{s-\frac{1}{q}}(\partial\Omega) \hookrightarrow Y$ is sufficient. To prove necessity we compose $\gamma_0 : B_{q,1}^s(\Omega) \rightarrow Y$ with a continuous extension operator $E_0 : B_{q,1}^{s-\frac{1}{q}}(\partial\Omega) \rightarrow B_{q,1}^s(\Omega)$ such that $\gamma_0 E_0 = \text{Id}$ (cf. [29, Thm. 4.7.1]). \square

Remark 2.3. The upper bound $2\gamma + \frac{2}{c} < 1 + \frac{1}{q}$ appears only for simplicity of formulation. The calculation of the space Z above indicates how to proceed in other cases.

To obtain the representation of the control operator B we follow again ideas in [4] and multiply the state equation in (12) with a fixed function $v \in C^\infty(\bar{\Omega})$. Then, integrating by parts gives

$$\langle x'(t), v \rangle_\Omega + \langle \nabla x(t), \nabla v \rangle_\Omega = \int_{\partial\Omega} u(t)v \, d\sigma,$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the usual duality pairing on $L^q(\Omega) \times L^q(\Omega)$ and σ denotes the surface measure on $\Gamma := \partial\Omega$. Denoting extensions of the usual $L^q(\Gamma) \times L^q(\Gamma)$ -duality by $\langle \cdot, \cdot \rangle_\Gamma$ we thus have

$$\int_{\partial\Omega} u(t)v \, d\sigma = \langle u(t), \gamma_0 v \rangle_\Gamma,$$

which means that $B = \gamma'_0$ if we identify $X_{-1}(A)$ with the dual space of $(X')_1(A')$. Notice that $A' = -\Delta$ with Neumann boundary conditions in $X' = L^q(\Omega)$.

We are interested in L^b -admissibility of type β for $\gamma'_0 : U \rightarrow X_{-1}$ and $\text{Id} + A$ in $X = L^q(\Omega)$. Assuming $b \leq \min(q, 2)$ in case $\beta = 0$ we may use Theorem 1.8 and only have to check boundedness of $\gamma'_0 : U \rightarrow W$ where the extrapolation space W is given by

$$W = (\dot{X}_{-1}(\text{Id}+A), X)_{\beta+\frac{1}{b}, \infty} = (X_{-1}, X)_{\beta+\frac{1}{b}, \infty} = (((X')_1, X')_{\beta+\frac{1}{b}, 1})' = ((X', (X')_1)_{1-(\beta+\frac{1}{b}), 1})'$$

(cf. [3, Sect.3.7]). As we have seen above, we have

$$(X', (X')_1)_{1-(\beta+1/b),1} = B_{q',1}^{2-2\beta-2/b}(\Omega)$$

for $2-2\beta-2/b < 1+1/q' = 2-1/q$. Now $\gamma_0 : B_{q',1}^{2-2\beta-2/b}(\Omega) \rightarrow B_{q',1}^{2-2\beta-2/b-1/q'}(\partial\Omega)$ is continuous for $2-2\beta-2/b > 1/q' = 1-1/q$ and $\partial\Omega$ is without boundary, hence $(B_{q',1}^s(\partial\Omega))' = B_{q,\infty}^{-s}(\partial\Omega)$. Thus we have proved one implication in the following

Proposition 2.4. *Let $\beta \in \mathbb{R}$, $b \in [1, \infty]$ and $q \in (1, \infty)$ satisfy $2/b + 2\beta \in (1/q, 1 + 1/q)$. If $\beta = 0$ assume in addition $b \leq \min(q, 2)$. Let U be a Banach space and $X = L^q(\Omega)$. Then $\gamma'_0 : U \rightarrow X_{-1}$ is L^b -admissible of type β for $\text{Id} + A$ if and only if $U \hookrightarrow B_{q,\infty}^{2/b+2\beta-1-1/q}(\partial\Omega)$.*

For the remaining implication we make use of E'_0 where E_0 is the extension map from the proof of Proposition 2.2.

Wellposedness of the full system. We study L^p -wellposedness of type α in finite time for the system (12) in the state space $X = L^q(\Omega)$ where $\alpha \in \mathbb{R}$, $p \in [1, \infty]$ and $q \in (1, \infty)$. Hence we have $c = b = p$ and $\gamma = \beta = \alpha$ in the admissibility situations above. Again, it is sufficient to study infinite-time L^p -admissibility of type α for $\text{Id} + A$. The assumptions of Propositions 2.2 and 2.4 lead, for $\alpha \neq 0$, to the restrictions $p \geq \max(q, 2)$ and $2\alpha + 2/p \in (1/q, 1 + 1/q)$, and for $\alpha = 0$ to the restriction $p = q = 2$.

For $p = q = 2$ we thus can state the following result extending the case $\alpha = 0$, $r = 2$, considered in [4].

Proposition 2.5. *Let $|\alpha| < 1/4$, $r \in [1, \infty]$ and $X = L^2(\Omega)$. Let Y and U be Banach spaces on $\partial\Omega$ such that $B_{2,r}^{2\alpha+1/2}(\partial\Omega) \hookrightarrow Y$ and $U \hookrightarrow B_{2,r}^{2\alpha-1/2}(\partial\Omega)$. Then the system (12) is L^2 -wellposed of type α .*

Proof. By Propositions 2.2 and 2.4, C and B are L^p -admissible of type α for $\text{Id} + A$. Notice that the semigroup associated to $\text{Id} + A$ is $S(t) := e^{-t}T(t)$.

Since $B = \gamma_0 : B_{2,r}^{2\alpha+1/2}(\partial\Omega) \rightarrow \widetilde{W} := (X_{-1}, X)_{\beta+1/b,r} = (X_{-1}, X)_{\alpha+1/2,r}$ (recall $\beta = \alpha$ and $b = p = 2$) and $C = \gamma_0 : \widetilde{Z} \rightarrow B_{2,r}^{2\alpha+1/2}(\partial\Omega)$ where $\widetilde{Z} := B_{2,r}^{2\alpha+1}(\Omega) = (X, X_1)_{\alpha+1/2,r}$ it suffices to prove that $S(\cdot)* : L^p_\alpha(\mathbb{R}_+, \widetilde{W}) \rightarrow L^p_\alpha(\mathbb{R}_+, \widetilde{Z})$. Theorem 1.13 applies by $|\alpha| < 1/4 < 1/2$. Now notice that by $\widetilde{Z} = \mathcal{D}(A_{\widetilde{W}})$. Hence we are left to check that $A_{\widetilde{W}}$ has maximal L^p -regularity in \widetilde{W} which holds by [6, Thm. 4.7]. \square

With essentially the same arguments we can study problem (12) in state spaces $X = H_q^\delta(\Omega)$ where we restrict to $\delta \in (-1/q', 1/q)$ and are thus not bothered by additional boundary conditions. Here we have $X_1 = \mathcal{D}(A) = \{x \in H_q^{2+\delta}(\Omega) : \frac{\partial x}{\partial \nu}|_{\partial\Omega} = 0\}$, and a repetition of the arguments above yields for the corresponding interpolation spaces $Z = B_{q,1}^{\delta+2(\alpha+1/p)}(\Omega)$ if $\delta + 2(\alpha + 1/p) < 1 + 1/q$ and $W = (B_{q',1}^{2-\delta-2(\alpha+1/p)}(\Omega))'$ if $2 - \delta - 2(\alpha + 1/p) < 1 + 1/q'$, i.e. if $1/q < \delta + 2(\alpha + 1/p)$. Thus we obtain

Proposition 2.6. *Let $q \in (1, \infty)$ and $p \in [1, \infty]$ satisfy $p \geq \max(2, q)$. Let $\delta \in (-1/q', 1/q)$, and $\delta + 2(\alpha + 1/p) \in (1/q, 1 + 1/q)$ where $\alpha \neq 0$. Let $X = H_q^\delta(\Omega)$ and let Y and U be Banach spaces on $\partial\Omega$.*

- (a) *The operator $C = \gamma_0 : \mathcal{D}(A) \rightarrow Y$ is L^p -admissible of type α for $\text{Id} + A$ if and only if $B_{q,1}^{\delta+2(\alpha+1/p)-1/q}(\partial\Omega) \hookrightarrow Y$.*
- (b) *The operator $B = \gamma'_0 : U \rightarrow X_{-1}$ is L^p -admissible of type α for $\text{Id} + A$ if and only if $U \hookrightarrow B_{q,\infty}^{\delta+2(\alpha+1/p)-1/q-1}(\partial\Omega)$.*

(c) Let $r \in [1, \infty]$ and $B_{q,r}^{\delta+2(\alpha+1/p)-1/q}(\partial\Omega) \hookrightarrow Y$ and $U \hookrightarrow B_{q,r}^{\delta+2(\alpha+1/p)-1/q-1}(\partial\Omega)$.

Then the system (12) is finite-time L^p -wellposed of type α .

Observe that the assumptions imply $\alpha + 1/p \in (0, 1)$.

We see that δ may be chosen arbitrarily close to $-1/q'$ by taking p large and adjusting α . Moreover, one still has free choice of $r \in [1, \infty]$.

The Besov space case. We continue the study of (12), but now we take as state space the Besov space $X := B_{q,v}^0(\Omega)$ where $1 < q, v < \infty$ are fixed. Again we put $A := -\Delta$ with homogeneous Neumann boundary conditions, i.e., A has domain $\mathcal{D}(A) = \{x \in B_{q,v}^2(\Omega) : \frac{\partial x}{\partial \nu}|_{\partial\Omega} = 0\}$. As in the L^q -case, $-A$ generates a bounded analytic semigroup in X .

Admissibility. Still denoting by γ_0 the Dirichlet trace on $\partial\Omega$, we study L^c -admissibility of type γ for $C = \gamma_0$ and L^b -admissibility of type β for $B = \gamma_0'$ where $\beta, \gamma \in \mathbb{R}$ and $b, c \in [1, \infty]$. First we note that $(X_{-1}, X_1)_{1/2, c} = B_{q,c}^0(\Omega)$, whence $\text{Id} + A$ has L_*^c -estimates on X if and only if $B_{q,v}^0(\Omega) \hookrightarrow B_{q,c}^0(\Omega)$ which is equivalent to $c \geq v$. Notice that this condition does not depend on q . If $c \geq v$, then the characterising Theorem 1.7 applies, and L^c -admissibility of type γ of the observation operator $C = \gamma_0$ may be checked by verifying the boundedness condition on the set W_C . Again, we use Theorem 1.9 and find that, for $2\gamma + 2/c < 1 + 1/q$, W_C is bounded if and only if γ_0 has a continuous extension to the very same Banach space

$$Z = B_{q,1}^{2\gamma+2/c}(\Omega)$$

we calculated above. Hence the proof of the following can be done as in the case $X = L^q(\Omega)$.

Proposition 2.7. *Let $\gamma \in \mathbb{R}$, $c \in [1, \infty]$ and $q, v \in (1, \infty)$ satisfy $c \geq v$ and $2\gamma + 2/c \in (1/q, 1 + 1/q)$. Let Y be a Banach space and $X = B_{q,v}^0(\Omega)$. Then $\gamma_0 : \mathcal{D}(A) \rightarrow Y$ is L^c -admissible of type γ for $\text{Id} + A$ if and only if $B_{q,1}^{2\gamma+2/c-1/q}(\partial\Omega) \hookrightarrow Y$.*

We turn to L^b -admissibility of type β for $B = \gamma_0'$. For an application of Theorem 1.8 in $X = B_{q,v}^0(\Omega)$ in case $\beta = 0$ we need $L_*^{b'}$ -estimates for $(\text{Id} + A)'$, which are, by the argument above, equivalent to $b \leq v$. This yields

Proposition 2.8. *Let $\beta \in \mathbb{R}$, $b \in [1, \infty]$ and $q, v \in (1, \infty)$ satisfy $2/b + 2\beta \in (1/q, 1 + 1/q)$. Assume additionally $b \leq v$ if $\beta = 0$. Let U be a Banach space and $X = B_{q,v}^0(\Omega)$. Then $\gamma_0' : U \rightarrow X_{-1}$ is L^b -admissible of type β for $\text{Id} + A$ if and only if $U \hookrightarrow B_{q,\infty}^{2/b+2\beta-1-1/q}(\partial\Omega)$.*

Wellposedness of the full system. We study L^p -wellposedness of type α for the system (12) where $\alpha \in \mathbb{R}$, $p \in [1, \infty]$ and the state space $X = B_{q,v}^0(\Omega)$ with $q, v \in (1, \infty)$. Again we have $c = b = p$ and $\gamma = \beta = \alpha$ in the admissibility situations above. The assumptions of Propositions 2.7 and 2.8 now lead, for $\alpha \neq 0$, to the restrictions $p \geq v$ and $2\alpha + 2/p \in (1/q, 1 + 1/q)$. In case $\alpha = 0$ we are led to $p = v$. For example, if $p = q$, then we are led to $2\alpha \in (-1/q, 1/q')$ which, for $q = 2$, corresponds to the condition $|\alpha| < 1/4$ in Proposition 2.5 (recall $L^2(\Omega) = B_{2,2}^0(\Omega)$). By the arguments used above we hence obtain

Proposition 2.9. *Let $\alpha \in \mathbb{R}$, $p \in (1, \infty)$ and $q \in (1, \infty)$ satisfy $2\alpha + 2/p \in (1/q, 1 + 1/q)$. Let $X = B_{q,p}^0(\Omega)$ and let Y and U be Banach spaces on $\partial\Omega$ such that $B_{q,p}^{2\alpha+2/p-1/q}(\partial\Omega) \hookrightarrow Y$ and $U \hookrightarrow B_{q,p}^{2/p+2\alpha-1-1/q}(\partial\Omega)$. Then the system (12) is finite-time L^p -wellposed of type α .*

Proof. The proof is similar to the case $X = L^q(\Omega)$. We check here that Theorem 1.13 applies, i.e., that $\alpha \in (-1/p, 1/p')$. The condition on α in the assumption may be rephrased as $\alpha \in (1/2q - 1/p, 1/2 + 1/2q - 1/p)$. In particular $\alpha > 1/2q - 1/p \geq -1/p$ and $\alpha < 1/2 + 1/2q - 1/p \leq 1 - 1/p$. \square

We also give an analogue of Proposition 2.6 for Besov spaces, e.g. for state spaces $X = B_{q,p}^\delta(\Omega)$ where $q, p \in (1, \infty)$ and we restrict to $\delta \in (-\frac{1}{q'}, \frac{1}{q})$ for the same reasons as before.

Proposition 2.10. *Let $q, p \in (1, \infty)$, $\delta \in (-\frac{1}{q'}, \frac{1}{q})$, $2\alpha + \frac{2}{p} + \delta \in (\frac{1}{q}, 1 + \frac{1}{q})$, and $X = B_{q,p}^\delta(\Omega)$. Let Y and U be Banach spaces on $\partial\Omega$.*

- (a) *The operator $C = \gamma_0 : \mathcal{D}(A) \rightarrow Y$ is L^p -admissible of type α for $\text{Id}+A$ if and only if $B_{q,1}^{2\alpha + \frac{2}{p} + \delta - \frac{1}{q}}(\partial\Omega) \hookrightarrow Y$.*
- (b) *The operator $B = \gamma'_0 : U \rightarrow X_{-1}$ is L^p -admissible of type $-\alpha$ for $\text{Id}+A$ if and only if $U \hookrightarrow B_{q,\infty}^{2\alpha + \frac{2}{p} + \delta - 1 - \frac{1}{q}}(\partial\Omega)$.*
- (c) *Let $r \in [1, \infty]$ and $B_{2,r}^{2\alpha + \frac{2}{p} + \delta - \frac{1}{q}}(\partial\Omega) \hookrightarrow Y$ and $U \hookrightarrow B_{2,r}^{2\alpha + \frac{2}{p} + \delta - 1 - \frac{1}{q}}(\partial\Omega)$. Then the system (12) is finite-time L^p -wellposed of type α .*

Discussion. We discuss our results for the system (12) by starting from $Y := B_{q,r}^s(\partial\Omega)$ and $U := B_{q,r}^{s-1}(\partial\Omega)$ where $s \in (0, 1)$, $q \in (1, \infty)$ and $r \in [1, \infty]$ are fixed. We now look for α , p and a state space X such that (12) is L^p -wellposed of type α . This should be compared with the situation studied in [4] where $p = q = r = 2$, $\alpha = 0$, $s = \frac{1}{2}$, and $X = L^2(\Omega)$.

Remark 2.11. In case $q = 2$, we may take $p = 2$ and $X = H_2^\delta(\Omega)$ where the restrictions may be read off of Proposition 2.6: $2\alpha + \delta + \frac{1}{2} = s$ and $|\delta| < \frac{1}{2}$. Thus we see that a suitable choice of α always allows to have δ arbitrarily close to $-\frac{1}{2}$. In particular this applies to the ‘‘classical’’ case $s = \frac{1}{2}$ where the restriction $\alpha = 0$ forces $\delta = 0$ and $X = L^2(\Omega)$.

Remark 2.12. For general $q \in (1, \infty)$, we may apply Proposition 2.10 and take $X = B_{q,p}^\delta(\Omega)$ under the restrictions $s = 2\alpha + \frac{2}{p} + \delta - \frac{1}{q}$ and $\delta \in (-\frac{1}{q'}, \frac{1}{q})$. We see that, taking p arbitrarily large and δ arbitrarily close to $-\frac{1}{q'}$, we still obtain finite-time L^p -wellposedness of type α for the system with state space $X = B_{q,p}^\delta(\Omega)$ by adjusting α . Observe that the state space $X = H_q^\delta(\Omega)$ would have required the additional restriction $p \geq \max(2, q)$ for $\alpha \neq 0$.

Remark 2.13. We let $s = \frac{1}{2}$ and $U := H_2^{-\frac{1}{2}}(\partial\Omega)$, $Y := H_2^{\frac{1}{2}}(\partial\Omega)$, i.e. we take $q = r = 2$ in the situation discussed above. Then, for $\delta \in (-\frac{1}{2}, \frac{1}{2})$ and $X = B_{2,p}^\delta(\Omega)$, we have that (12) is finite-time L^p -wellposed of type α if $\delta = 1 - 2(\alpha + \frac{1}{p})$. The restrictions are $p \in (1, \infty)$ and $\alpha + \frac{1}{p} \in (\frac{1}{4}, \frac{3}{4})$. This means in other words that, for $\epsilon \in (0, 1)$, $p \in [2, \infty)$, and $X = B_{2,p}^{-\frac{1}{2} + \epsilon}(\Omega)$, the system (12) is finite-time L^p -wellposed of type $\alpha = \frac{3}{4} - \frac{\epsilon}{2} - \frac{1}{p}$.

Now we consider a nonlinear feedback in the setting of the above remark.

Example 2.14. Since $Y = H_2^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^{\frac{2n-2}{n-2}}(\partial\Omega)$ and $L^{2-\frac{2}{n}}(\partial\Omega) \hookrightarrow H_2^{-\frac{1}{2}}(\partial\Omega)$, Hölder’s inequality yields that any $g \in L^{n-1}(\partial\Omega)$ induces a bounded multiplication operator $Y \rightarrow U, y \mapsto g \cdot y$. We take a smooth open subset Ω_0 with $\overline{\Omega_0} \subset \Omega$, e.g. a small ball, and let $\psi(x) := \int_{\Omega_0} x(\omega) d\omega = \langle x, 1_{\Omega_0} \rangle$ for $x \in X$. Observe that $\psi \in X'$ for all spaces X we mentioned above (cf. [29]). Taking a Lipschitz-continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, we let $F(x)y := f(\psi(x))g \cdot y$. Then $F : X \rightarrow B(Y, U)$ is Lipschitz-continuous. We interpret $x \mapsto \psi(x)$ as a distributed measurement in Ω affecting via $f(\psi(x))$ the intensity of the linear feedback $y \mapsto g \cdot y$.

For $\epsilon \in (0, 1)$, $p \in [2, \infty)$, $X = B_{2,p}^{-\frac{1}{2} + \epsilon}(\Omega)$, and $\alpha = \frac{3}{4} - \frac{\epsilon}{2} - \frac{1}{p}$ we take $x_0 \in X$ satisfying (for simplicity) $f(\psi(x_0)) = 0$ and show that Theorem 1.16 applies. We have $F(x_0) = 0$, hence $A_0 = A$. We take $Z = H_2^1(\Omega)$ whence by an application of Theorem 1.7 and Theorem 1.9, Id_Z is finite-time L^p -admissible of type α (cf. [29, Remark 2.8.1]). Since $B = \gamma'_0 : H_2^{-\frac{1}{2}}(\partial\Omega) \rightarrow (H_2^1(\Omega))' =: W$ is bounded and $\text{Id}+A$ has maximal L^p -regularity in W we obtain by Remark 1.4 and Theorem 1.13 that the system (A_0, B, Id_Z) is L^p -wellposed of type α in finite time. Hence

we may apply Theorem 1.16 and obtain that the nonlinear system

$$\begin{cases} x'(t) - \Delta x(t) &= 0, & t \in [0, \tau], \\ x(0) &= x_0, \\ \frac{\partial x(t)}{\partial \nu} &= f\left(\int_{\Omega_0} x(t) d\omega\right) g \cdot x(t)|_{\partial\Omega}, \end{cases} \quad (14)$$

has, for some $\tau(x_0) > 0$, a unique mild solution $x(\cdot)$ in $C([0, \tau], B_{2,p}^{-1/2+\epsilon}(\Omega)) \cap L^p_\alpha([0, \tau], H^1_2(\Omega))$. This means that we obtain solutions also for rather rough initial data x_0 .

In the example above we made the assumption $f(\psi(x_0)) = 0$ for simplicity. For initial data x_0 with $f(\psi(x_0)) \neq 0$ one may resort to perturbation results in [11].

3. PROOFS

Finite-time admissibility of type α .

Proof of Lemma 1.3. Assume C to be finite-time L^p -admissible of type α for A . For $b \geq 0$ we then have

$$\begin{aligned} \left(\int_{b+\tau/2}^{b+\tau} \|t^\alpha CT(t)x\|^p dt \right)^{1/p} &= \left(\int_{\tau/2}^{\tau} \|s^\alpha (1 + \frac{b}{s})^\alpha CT(s)T(b)x\|^p dt \right)^{1/p} \\ &\leq \left(\int_0^{\tau} \|s^\alpha CT(s)T(b)x\|^p dt \right)^{1/p} \max\left(\left(1 + \frac{2b}{\tau}\right)^\alpha, 1\right) \\ &\leq M_\tau \|T(b)x\| \max\left(\left(1 + \frac{2b}{\tau}\right)^\alpha, 1\right). \end{aligned}$$

If $T(\cdot)$ is uniformly exponentially stable, i.e. if there are $c, \epsilon > 0$ such that $\|T(t)\| \leq ce^{-\epsilon t}$, $t \geq 0$, then we obtain, for $b = (k-1)\tau/2$ and $k \in \mathbb{N}_0$,

$$\left(\int_{k\tau/2}^{(k+1)\tau/2} \|t^\alpha CT(t)x\|^p dt \right)^{1/p} \leq c M_\tau \max\left(\left(1 + \frac{2b}{\tau}\right)^\alpha, 1\right) e^{-\epsilon(k-1)\tau/2} \|x\|.$$

Thus (4) holds.

A similar reasoning shows that the notion of finite-time L^p -admissibility of type α for control operators is independent of τ : let $a = \tau/2$ and $b = \frac{3}{2}\tau$. Since

$$\int_0^b T(b-s)Bu(s) ds \leq T(a) \int_0^{\tau} T(\tau-s)Bu(s) ds + \int_a^{\tau} T(\tau-s)Bu(s+a) ds,$$

L^p -admissibility of type α on $[0, \tau]$ gives for $p < \infty$

$$\begin{aligned} \left\| \int_0^b T(b-s)Bu(s) ds \right\| &\leq K_\tau \|T(a)\| \|u\|_{L^p_\alpha([0, \tau])} + K_\tau \|\mathbb{1}_{[a, \tau]}(\cdot)u(a+\cdot)\|_{L^p_\alpha([0, \tau])} \\ &\leq K_\tau \|T(a)\| \|u\|_{L^p_\alpha([0, \tau])} + K_\tau \tilde{c}_\alpha \|u\|_{L^p_\alpha([\tau, b])} \\ &\leq K_\tau (\tilde{c}_\alpha + \|T(\tau/2)\|) \|u\|_{L^p_\alpha([0, b])}, \end{aligned}$$

where $\tilde{c}_\alpha = \max\left(\left(\frac{1}{2}\right)^\alpha, \left(\frac{2}{3}\right)^\alpha\right)$ does not depend on τ . In case $p = \infty$, we obtain the same estimate with $\tilde{c}_\alpha = 1$ by directly regarding the first line of the above inequalities. Thus, B is L^p -admissible of type α on $[0, \frac{3}{2}\tau]$ as required. An iteration of the argument shows that

$$K_{(\frac{3}{2}\tau)^n} \leq K_\tau (\tilde{c}_\alpha)^n \prod_{j=1}^n (1 + \tilde{c}_\alpha^{-1} \|T(3\tau(\frac{3}{2})^j)\|).$$

If $T(\cdot)$ is uniformly exponentially stable and if $\alpha \geq 0$ or if $p = \infty$, the left hand side of the above inequality is bounded since $\sum_{j \geq 1} \exp(-3\epsilon\tau(\frac{3}{2})^j) < \infty$ and $|\tilde{c}_\alpha| \leq 1$. \square

If $\alpha < 0$ and $p < \infty$, equivalence of finite-time and infinite-time L^p -admissibility of type α fails in general:

Example 3.1. Take $U = X = \mathbb{C}$, $-\alpha = \beta > 0$ and $T(t) = e^{-ct}$. Then, for any $\tau > 0$,

$$|T * u(\tau)| \leq \|u\|_{L_\alpha^p(0,\tau)} \|e^{-c(\cdot)}\|_{L_{\beta'}^{p'}(0,\tau)} \leq c_{p'} \tau^{\beta + \frac{1}{p'}} \|u\|_{L_\alpha^p(0,\tau)},$$

and the identity is finite-time L^p -admissible of type α .

On the other hand, letting $u_k := \mathbb{1}_{[k,k+1]} e^{-c(\cdot-k)}$ for $k \in \mathbb{N}$, we have $\|u_k\|_{L_\alpha^p} \leq k^{-\beta} \|u_k\|_{L^p} = ck^{-\beta}$ and for $\delta \in (0, 1]$,

$$(T * u_k)(k + \delta) = \int_k^{k+\delta} e^{-c(k+\delta-s)} e^{-c(s-k)} ds = \delta e^{-c\delta}.$$

Hence the identity is not infinite-time L^p -admissible of type α .

Characterisation of L^p -admissibility of type α . In the proof of Theorems 1.7 and 1.8 we make use of the following lemma.

Lemma 3.2 ([12, Lem. 4.1]). *Let $\sigma \in (0, \pi)$, let $\varphi \in H_0^\infty(S(\sigma))$, and let $m \geq 1$ be an integer. There exist a function $f \in H_0^\infty(S(\sigma))$ and a constant $a \in \mathbb{C}$ such that*

$$\varphi(z) = z^m f^{(m)}(z) + a \frac{z^m}{(1+z)^{m+1}}, \quad z \in S(\sigma). \quad (15)$$

Furthermore, if $\delta, \epsilon \in (0, 1)$ are positive numbers such that

$$|\varphi(z)| = O(|z|^{-\delta}) \quad \text{at } \infty \quad \text{and} \quad |\varphi(z)| = O(|z|^\epsilon) \quad \text{at } 0,$$

then f can be chosen so that we also have $|f(z)| = O(|z|^{-\delta})$ at ∞ , and $|f(z)| = O(|z|^\epsilon)$ at 0.

Proof of Theorem 1.7. (a) Notice that for any $\lambda \in \mathbb{C}$ with positive real part and for any $x \in X$, we have

$$(\lambda + A)^{-k} x = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} T(t) x dt.$$

For $x \in X_k = \mathcal{D}(A^k)$, the integrand $t \mapsto t^{k-1} e^{-\lambda t} T(t) x$ belongs to $L^1(\mathbb{R}_+, X_k)$, and so continuity of C on X_k shows

$$C(\lambda + A)^{-k} x = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} C T(t) x dt.$$

By Hölder's inequality we thus have for $x \in X_k$

$$\begin{aligned} \|C(\lambda + A)^{-k} x\| &\leq \frac{1}{(k-1)!} \int_0^\infty \|t^\alpha C T(t) x\| t^{k-1-\alpha} e^{-\operatorname{Re}(\lambda)t} dt \\ &\leq \frac{1}{(k-1)!} \left\| t \mapsto t^\alpha C T(t) x \right\|_{L^p(\mathbb{R}_+, Y)} \left(\int_0^\infty t^{(k-1-\alpha)p'} e^{-\operatorname{Re}(\lambda)p't} dt \right)^{\frac{1}{p'}} \\ &\stackrel{s=\operatorname{Re}(\lambda)p't}{\leq} \frac{M}{(k-1)!} \|x\|_X \left((p' \operatorname{Re}(\lambda))^{-1-(k-1-\alpha)p'} \Gamma(1+(k-1-\alpha)p') \right)^{\frac{1}{p'}} \\ &= K \operatorname{Re}(\lambda)^{-k+\alpha+\frac{1}{p}} \|x\|_X, \end{aligned}$$

where Γ is the usual Gamma function and the number K depends only on k, p and the admissibility constant M . By density of X_k in X this shows the first assertion.

(b) Without loss of generality we may assume $k \geq 2$ since by sectoriality of A , whenever W_C is bounded for some $k \in \mathbb{N}$, it is also bounded when k is replaced by $k+1$ (see Remark 1.10).

We make use of the (unbounded) operator A^{-1} that is densely defined on the range of A . We set $F_k(z) := z^{k-1}e^{-z}$. Then for any $x \in X_k$ and any $t > 0$, we have

$$t^\alpha CT(t)x = t^{\alpha-k+1}CA^{-k+1}F_k(tA)x. \quad (16)$$

For some $\epsilon \in (0, 1)$ that we will precise later on consider the decomposition $F_k(z) = \varphi(z)\psi(z)$ where

$$\varphi(z) = z^\epsilon(1+z)^{-1}, \quad \text{and} \quad \psi(z) = z^{k-1-\epsilon}(1+z)e^{-z}. \quad (17)$$

Note that $\psi \in H_0^\infty(S(\theta))$ for any $\theta < \frac{\pi}{2}$, whereas $\varphi \in H_0^\infty(S(\sigma))$ for any $\sigma < \pi$. By (16), we have

$$\int_0^\infty \|t^\alpha CT(t)x\|_Y^p dt \leq \int_0^\infty \underbrace{\| [t^{\alpha+1/p-k+1}CA^{-k+1}\psi(tA)] t^{-1/p}\varphi(tA)x \|_X^p}_{=:K(t)} dt.$$

Now we will show that the operator family $K(t)$, $t > 0$, is uniformly bounded. Once this is done, the assertion of the theorem follows immediately from the assumed L_*^p -estimate for A (cf. (6)).

We fix $\sigma \in (\omega, \pi)$ and apply Lemma 3.2 to ψ with $m = k - 1$ and $\delta = 1 - \epsilon$. Let $f \in H_0^\infty(S(\sigma))$ denote the corresponding function satisfying equation (15). Note that according to that equation, $z \mapsto z^{k-1}f^{(k-1)}(z)$ belongs to $H_0^\infty(S(\sigma))$. Let $\theta = \theta_\sigma$ for some $\theta \in (\omega, \sigma)$ and let Γ denote the positively orientated boundary of $S(\theta)$. Then, as in the proof of [12, Thm. 4.2] the following representation formula for x in the dense subspace $Z := \text{ran}(A^{k-1}(I+A)^{-k})$ holds:

$$CA^{-k+1}[z^{k-1}f^{(k-1)}(z)](tA)x = \frac{(k-1)!}{2\pi i} \int_\Gamma f(\lambda)t^{k-1}CR(\lambda, tA)^k x d\lambda, \quad t > 0. \quad (18)$$

For $\lambda \in \Gamma$, by the resolvent equation we have

$$\lambda^{k-\alpha-1/p}CR(\lambda, A)^k = |\lambda|^{k-\alpha-1/p}C(|\lambda| + A)^{-k} [2 \cosh(\pm\theta/2)\lambda R(\lambda, A) - I]^k, \quad (19)$$

and thus $\lambda^{k-\alpha-1/p}CR(z, A)^k$ is uniformly bounded by sectoriality of A . Now, by the representation (18),

$$K(t) = \frac{(k-1)!}{2\pi i} \int_\Gamma t^{\alpha+1/p} f(\lambda) CR(\lambda, tA)^k x d\lambda + at^{\alpha+1/p}C(I+tA)^{-k}$$

on Z . Next we show that for an appropriately chosen $\epsilon \in (0, 1)$, $K(t) \in B(X, Y)$ and moreover the operators $K(t)$ are uniformly bounded for $t > 0$. To this end, write

$$\begin{aligned} K(t) &= \frac{(k-1)!}{2\pi i} \int_\Gamma f(\lambda)\lambda^{1-k+\alpha+1/p} \left[\left(\frac{\lambda}{t}\right)^{k-\alpha-1/p} CR\left(\frac{\lambda}{t}, A\right)^k x \right] \frac{d\lambda}{\lambda} \\ &\quad + a \left[\left(\frac{1}{t}\right)^{k-\alpha-1/p} C\left(\frac{1}{t} + A\right)^{-k} \right]. \end{aligned}$$

By our assumption (7) and scaling invariance of Γ and the measure $d\lambda/\lambda$ we obtain that $K(t)$ is uniformly bounded, provided that the integral

$$\int_\Gamma |f(\lambda)| |\lambda|^{-k+\alpha+1/p} d|\lambda|$$

is finite. By the estimates in Lemma 3.2 we know that $f \in O(|z|^{k-1-\epsilon})$ at zero and $f \in O(|z|^{-n})$ for any $n \in \mathbb{N}$ at infinity. Therefore the above integral is finite if

$$k - 1 - \epsilon - k + \alpha + 1/p > -1, \quad \text{i.e. if} \quad \epsilon < \alpha + 1/p.$$

This however, due to our assumption on α , may always be satisfied by an appropriate choice of $\epsilon \in (0, 1)$, and the proof is done. \square

To analyse L^p -admissibility for control operators, assume that $\|T_{-k}(t)B\|_{U \rightarrow X} \leq Mt^{-\gamma}$ for some $\gamma \in \mathbb{R}$. For $t > 0$ fixed we thus have

$$\begin{aligned} \left\| \int_0^t T(t-s)Bu(s) ds \right\|_X &\leq \int_0^t \|T(t-s)B\|_{U \rightarrow X} s^{-\alpha} \|s^\alpha u(s)\|_U ds \\ &\leq c \int_0^t (t-s)^{-\gamma} s^{-\alpha} \|s^\alpha u(s)\|_U ds. \end{aligned}$$

Let $k_{\alpha,\gamma}(t,s) = \mathbb{1}_{(0,t)}(s)(t-s)^{-\gamma} s^{-\alpha}$ for $s, t \in (0, \tau)$. Thus the study of the kernel $k_{\alpha,\gamma}$ which may or may not induce a bounded integral operator $K_{\alpha,\gamma} : L^p(0, \tau) \rightarrow L^\infty(0, \tau)$ gives a sufficient criterion for (in)finite-time L^p -admissibility of type α of control operators.

For $p = 1$, $K_{\alpha,\gamma}$ is bounded if and only if $k_{\alpha,\gamma}$ is uniformly bounded, which, for finite τ , is equivalent to $\gamma \leq 0$ and $\alpha \leq 0$. In this case we have

$$\|K_{\alpha,\gamma}\|_{L^\infty(0,\tau) \rightarrow L^\infty(0,\tau)} = \|k_{\alpha,\gamma}\|_\infty = \frac{|\alpha|^{|\alpha|} |\gamma|^{|\gamma|}}{|\alpha + \gamma|^{|\alpha + \gamma|}} \tau^{|\alpha + \gamma|}.$$

For $\tau = \infty$, $k_{\alpha,\gamma}$ is uniformly bounded if and only if $\alpha = \gamma = 0$.

For $p > 1$ we use Hölder's inequality and obtain

$$\begin{aligned} \int_0^t k_{\alpha,\gamma}(t,s)|f(s)| ds &\leq \left(\int_0^t (k_{\alpha,\gamma}(t,s))^{p'} ds \right)^{1/p'} \|f\|_p \\ (\text{subst. } s = t\sigma) &= \left(t^{1-(\gamma+\alpha)p'} \int_0^1 (1-\sigma)^{-\gamma p'} \sigma^{-\alpha p'} d\sigma \right)^{1/p'} \|f\|_p. \end{aligned}$$

Convergence of the integral in the last line is equivalent to $\gamma < 1/p'$ and $\alpha < 1/p'$. Taking the sup over $t \in (0, \tau)$ we see that, for finite τ ,

$$\|K_{\alpha,\gamma}\|_{L^p(0,\tau) \rightarrow L^\infty(0,\tau)} \leq c_{\alpha,\gamma,p} \tau^{1/p' - (\gamma + \alpha)}$$

provided that $\gamma < 1/p'$, $\alpha < 1/p'$ and $\gamma + \alpha \leq 1/p'$, whereas, for $\tau = \infty$,

$$\|K_{\alpha,\gamma}\|_{L^p(0,\infty) \rightarrow L^\infty(0,\infty)} \leq c_{\alpha,\gamma,p}$$

provided that $\gamma + 1/p < 1$, $\alpha + 1/p < 1$ and $\gamma + \alpha + 1/p = 1$. Since, in fact (cf. [16]),

$$\|K_{\alpha,\gamma}\|_{L^p(0,\tau) \rightarrow L^\infty(0,\tau)} = \sup_{t \in (0,\tau)} \|k_{\alpha,\gamma}(t, \cdot)\|_{L^{p'}(0,\tau)},$$

we have just proven the following result.

Proposition 3.3. *Assume that for some γ , $\|T(t)B\|_{U \rightarrow X} \leq t^{-\gamma}$, $t > 0$. Then $K_{\alpha,\gamma}$ is bounded $L^p(0, \tau) \rightarrow L^\infty(0, \tau)$ if and only one of the following conditions holds:*

$$\begin{aligned} (i) \quad & p = 1 \quad \tau < \infty \quad \alpha \leq 0 \quad \gamma \leq 0 \\ (ii) \quad & p = 1 \quad \tau = \infty \quad \alpha = 0 \quad \gamma = 0 \\ (iii) \quad & p > 1 \quad \tau < \infty \quad \alpha + 1/p < 1 \quad \gamma + 1/p < 1 \quad \alpha + \gamma + 1/p \leq 1 \\ (iv) \quad & p > 1 \quad \tau = \infty \quad \alpha + 1/p < 1 \quad \gamma + 1/p < 1 \quad \alpha + \gamma + 1/p = 1. \end{aligned} \tag{20}$$

Observe that condition (iv) implies $\alpha > 0$ and $\gamma > 0$.

Proof of Theorem 1.8. (a). Consider for $t > 0$ the function $u(s) := \mathbb{1}_{(t/2,t)}(s)u_0$. Then

$$\left\| \int_0^t T(t-s)Bu(s) ds \right\|_X = \left\| \int_{t/2}^t AT(t-s)Bu(s) ds \right\|_{\dot{X}_{-1}} = \|[T(t) - T(t/2)]Bu_0\|_{\dot{X}_{-1}}.$$

We have $\|u\|_{L_\alpha^p([0,t],U)} = \frac{1}{1+\alpha p} (t/2)^{\alpha+1/p}$. Let $f(z) := \exp(z) - \exp(z/2)$. Notice that the function $z \mapsto z^{-\alpha-1/p} f(z) \in H_0^\infty(S(\sigma))$ for all $\sigma < \pi/2$, so Theorem 1.5 applies. We obtain

$$\mathcal{R}(B) \subseteq ((\dot{X}(A_{-1}))_{-1}, (\dot{X}(A_{-1}))_1)_{1/2(\alpha+1/p+1), \infty} = (\dot{X}_{-2}, X)_{1/2(\alpha+1/p+1), \infty} = (\dot{X}_{-k}, X)_{1-\theta, \infty}$$

with $\theta = 1/k (1/p' - \alpha)$, $k \in \mathbb{N}$. The claim now follows from Theorem 1.9.

(b) . By Theorem 1.9, boundedness of W_B is equivalent to $B : U \rightarrow (\dot{X}_{-k}, X)_{\theta, \infty}$ with $\theta = 1 + 1/k (\alpha - 1/p')$. By analyticity of the semigroup this implies $\|T(t)B\| \leq ct^{-\gamma}$ with $\gamma = 1/p' - \alpha$ (cf. the arguments in the proof of Lemma 1.12). Hence, if additionally to the assumptions of the theorem, $\alpha > 0$ in case $p > 1$ (or $p = 1$ in case $\alpha = 0$, respectively), Proposition 3.3 gives the claim.

(a)'. Let (5) hold. For $Re(\lambda) > 0$ and $u \in U$ we have

$$(\lambda + A_{-k})^{-k} Bu = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-\lambda t} T_{-k}(t) Bu dt.$$

Then, by assumption

$$\begin{aligned} & \|Re(\lambda)^{k+\alpha-1/p'} (\lambda + A)^{-k-1} Bu\| \\ & \leq \frac{1}{(k-1)!} \left\| \int_0^\infty t^{k-1} Re(\lambda)^{k+\alpha-1/p'} e^{-\lambda t} T_{-k}(t) Bu dt \right\| \\ & = \frac{1}{(k-1)!} \left\| \int_0^\infty T_{-k}(t) B [t^{k-1} Re(\lambda)^{k+\alpha-1/p'} e^{-\lambda t} \otimes u] dt \right\| \\ & \leq \frac{K}{(k-1)!} \|h_\lambda(t) \otimes u\|_{L_\alpha^p(\mathbb{R}_+, U)} \leq \tilde{K} \|u\|_U. \end{aligned}$$

Here, the uniform boundedness of the functions $h_\lambda(t) := Re(\lambda)^{k+\alpha-1/p'} t^{k-1} e^{-\lambda t}$, $\lambda > 0$ in $L_\alpha^p(\mathbb{R}_+)$ is shown similar to the proof of Theorem 1.7.

(b)'. Without loss of generality we assume $k \geq 2$. We proceed in two steps.

Step 1: For to show the existence of the integral in (5), we choose some $x' \in (X_{-k})'$, $x' \neq 0$. Notice that $(X_{-k})'$ may be identified with $(X')_k$, that is the domain $\mathcal{D}((A')^k)$ with graph norm. We consider

$$\int_0^\infty |\langle T_{-k}(t) Bu(t), x' \rangle| dt = \int_0^\infty |\langle t^{-k+1} A^{-k+1} (tA)^{k-1} T_{-k}(t) Bu(t), x' \rangle| dt.$$

As we did in the proof of Theorem 1.7, we decompose $F_k(z) = z^{k-1} e^{-z}$ as $F_k(z) = \varphi(z) \psi(z)$ with φ, ψ as in (17). We obtain

$$\begin{aligned} & \int_0^\infty |\langle T_{-k}(t) Bu(t), x' \rangle| dt \\ & = \int_0^\infty |\langle t^{-k+1} A_{-k}^{-k+1} \varphi(tA_{-k}) \psi(tA_{-k}) Bu(t), x' \rangle| dt. \end{aligned}$$

Notice that by sectoriality of A , the operators $A_{-k}(\mu + A_{-k})^{-1}$, $\mu > 0$ are uniformly bounded. Moreover, $\lim_{\mu \rightarrow 0^+} A_{-k}(\mu + A_{-k})^{-1} Bu = Bu$ in X_{-k} since A_{-k} has dense range in X . Therefore, applying Fatou's lemma and writing $B_\mu := A_{-k}(\mu + A_{-k})^{-1} B$, we have

$$\leq \liminf_{\mu \rightarrow 0^+} \int_0^\infty |\langle t^{-k+1} A_{-k}^{-k+1} \varphi(tA_{-k}) \psi(tA_{-k}) B_\mu u(t), x' \rangle| dt$$

Notice that $\psi(tA_{-k}) Bu(t) \in \mathcal{D}(A_{-k}^{-k+1+\epsilon})$ whereas $\psi(tA_{-k}) B_\mu u(t) \in \mathcal{D}(A_{-k}^{-k+\epsilon})$. This observation allows to interchange the operators $\varphi(tA_{-k})$ and A_{-k}^{-k+1} as follows

$$\begin{aligned}
&= \liminf_{\mu \rightarrow 0^+} \int_0^\infty |\langle \varphi(tA_{-k})t^{-k+1}A_{-k}^{-k+1}\psi(tA_{-k})B_\mu u(t), x' \rangle| dt \\
&= \liminf_{\mu \rightarrow 0^+} \int_0^\infty |\langle t^{1+1/p'}-kA_{-k}^{-k}\psi(tA_{-k})B_\mu u(t), t^{-1/p'}\varphi(tA_{-k})'x' \rangle| dt \\
&\leq \liminf_{\mu \rightarrow 0^+} \left\| t \mapsto \underbrace{[t^{1/p'-\alpha-k+1}A_{-k}^{-k-1}\psi(tA_{-k})B_\mu]}_{=:L_\mu(t)} t^\alpha u(t) \right\|_{L^p(\mathbb{R}_+, X)} \|\varphi(tA')x'\|_{L^{p'}(\mathbb{R}_+, dt/t, X')}.
\end{aligned}$$

Notice that by assumption on A' , the $L^{p'}$ -norm has an estimate against the norm of x' , whence the existence of the integral is proven if we show the uniform boundedness of the operators $L_\mu(t)$ for $t > 0$ and $\mu > 0$. This step is very similar to the proof of uniform boundedness of the family $K(t)$, $t > 0$ in the proof of Theorem 1.7: we apply Lemma 3.2 with $m = k - 1$ to the function $\psi(z)$ and obtain for fixed $t > 0$

$$A_{-k}^{-k+1}\psi(tA_{-k})B_\mu u(t) = \frac{(k-1)!}{2\pi i} \int_\Gamma f(\lambda)t^{k-1}R(\lambda, tA_{-k})^k B_\mu u(t) d\lambda + at^{k-1}(I + tA_{-k})^{-k} B_\mu u(t).$$

Now for $u \in U$ write

$$\begin{aligned}
L_\mu(t)u &= \frac{(k-1)!}{2\pi i} \int_\Gamma f(\lambda)t^{1/p'-\alpha}R(\lambda, tA_{-k})^k B_\mu u d\lambda + at^{1/p'-\alpha}(I + tA_{-k})^{-k} B_\mu u(t) \\
&= \frac{(k-1)!}{2\pi i} \int_\Gamma f(\lambda)\lambda^{-k-\alpha+1/p'} [A_{-k}(\mu+A_{-k})^{-1}] \left[\left(\frac{\lambda}{t}\right)^{k+\alpha-1/p'+1} R\left(\frac{\lambda}{t}, A_{-k}\right)^k B \right] u \frac{d\lambda}{\lambda} \\
&\quad + a [A_{-k}(\mu+A_{-k})^{-1}] \left[\left(\frac{1}{t}\right)^{k+\alpha-1/p'} \left(\frac{1}{t} + A_{-k}\right)^{-k} B \right] u.
\end{aligned}$$

Therefore, by the assumption (8) and a similar calculation to (19) the set $\{L_\mu(t) : t > 0, \mu > 0\}$ is bounded in $B(U, X)$ provided that the integral

$$\int_\Gamma |f(\lambda)| |\lambda|^{-k-\alpha+1/p'} d|\lambda|$$

is finite. Since $f \in O(|z|^{k-1-\epsilon})$ in zero and $f \in O(|z|^{-n})$ for any $n \in \mathbb{N}$, this boils down to

$$k - 1 - \epsilon - k - \alpha + 1/p' > -1, \quad \text{i.e. to } \alpha < 1/p' - \epsilon,$$

which, due to our assumption on α , always may be satisfied by some $\epsilon \in (0, 1)$.

Step 2: Now let $x' \in X'$. We show that $t \mapsto \langle T(t)Bu(t), x' \rangle \in L^1(\mathbb{R}_+)$ with a norm estimate against $K'\|u\|_{L_\alpha^p(\mathbb{R}_+, U)}\|x'\|_{X'}$. To this end we first notice that by analyticity of the semigroup, $T_{-k}(t)Bu(t) \in X$ for $t > 0$. Moreover, for t positive, $T_{-k}(t)Bu(t) = \lim_{\epsilon \rightarrow 0} T_{-k}(t + \epsilon)Bu(t)$ in X . Therefore, Fatou's lemma yields

$$\int_0^\infty |\langle T_{-k}(t)Bu(t), x' \rangle| dt \leq \liminf_{\epsilon \rightarrow 0} \int_0^\infty |\langle T_{-k}(t)Bu(t), T_{-k}(\epsilon)'x' \rangle| dt.$$

Notice that $y'_\epsilon := T_{-k}(\epsilon)'x' \in \mathcal{D}((A')^k)$ and by step one,

$$\int_0^\infty |\langle T(t)Bu(t), y'_\epsilon \rangle| dt \leq K\|u\|_{L_\alpha^p} \|y'_\epsilon\|_{X'}.$$

Since $\|T_{-k}(\epsilon)'x'\| \leq K_0\|x'\|$ the integral $\int_0^\infty T(t)Bu(t) dt$ exists as a Pettis integral in X . The above argumentation shows that we have a bounded linear mapping $\Phi : L^p(\mathbb{R}_+, U) \rightarrow X''$, $\Phi(u) = \int_0^\infty T_{-k}(t)Bu(t) dt$. If u is a step function with compact support that does not contain zero, the integral in question even exists as a Bochner integral. In this case, it takes values in X by analyticity of the semigroup $T(\cdot)$. Since such step functions are dense in $L^p(\mathbb{R}_+, U)$ (recall $p < \infty$) we obtain $\mathcal{R}\Phi \subseteq X$ and thus Φ is necessarily bounded from $L^p(\mathbb{R}_+, U)$ to X . This finishes our proof. \square

Regularity and Wellposedness.

Proof of Lemma 1.12. It is not hard to see, that for bounded analytic semigroups and $k \in \mathbb{N}$, $\|t^k A^k T(t)\| \leq c_k < \infty$. Indeed, by the elementary functional calculus for sectorial operators (cf. [21]) and substituting $tz = \lambda$ one has

$$\begin{aligned} \|(tA)^k T(t)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma} (tz)^k e^{-tz} R(z, A) dz \right\| \\ &\leq M \int_{\Gamma} |tz|^k e^{-t \operatorname{Re}(z)} \frac{|dz|}{|z|} = M \int_{\Gamma} |\lambda|^k e^{-\operatorname{Re} \lambda} \frac{|d\lambda|}{|\lambda|} =: c_k < \infty. \end{aligned}$$

This shows $\|T(t)\|_{X \rightarrow \dot{X}_k} \leq c_k t^{-k}$, $t > 0$. On the other hand, clearly $\|T(t)\|_{X \rightarrow X} \leq c_0$, $t \geq 0$. By real interpolation we obtain immediately $\|T(t)\|_{X \rightarrow Z} \leq k_1 t^{-\theta k}$. Similarly, considering \dot{X}_{-k} and X in place of X and \dot{X}_k , one obtains $\|T(t)\|_{W \rightarrow X} \leq k_2 t^{-\sigma k}$. Both estimates together give the claim by the semigroup property. \square

Proof of Proposition 1.14. Let $\phi(s) := |(1+s)^\alpha - 1|$. Then

$$\|(Tf)(t)\|_Y \leq M \int_0^t \frac{1}{t-s} \phi\left(\frac{t-s}{s}\right) \|f(s)\|_U ds,$$

whence T is pointwise bounded by a multiple of the scalar integral operator

$$(\tilde{T}u)(t) := \int_0^t \frac{1}{t-s} \phi\left(\frac{t-s}{s}\right) u(s) ds$$

which has the kernel $k(t, s) = \mathbb{1}_{[0,t]}(s) \frac{1}{t-s} \phi\left(\frac{t-s}{s}\right)$. Notice that, substituting $s = t\sigma$,

$$\|k(t, \cdot)\|_{p'}^{p'} = \int_0^t \left| \frac{(1 + \frac{t-s}{s})^\alpha - 1}{t-s} \right|^{p'} ds = t^{1-p'} \int_0^1 \left| \frac{\sigma^{-\alpha} - 1}{\sigma - 1} \right|^{p'} d\sigma \leq t^{1-p'} \tilde{c}$$

where $\tilde{c} = \tilde{c}(p, \alpha) < \infty$ since letting $g(\sigma) := \sigma^{-\alpha}$, the limit for $\sigma \rightarrow 1$ equals $g'(1) = -\alpha$. It follows by Hölders inequality that $|\tilde{T}u(t)| \leq cM \|u\|_p t^{-1/p}$ (notice $(1-p')/p' = -1/p$). Therefore,

$$\lambda \mu(\{t > 0 : |\tilde{T}u(t)| \geq \lambda\})^{1/p} \leq \lambda \mu(\{t > 0 : M \|u\|_p t^{-1/p} \geq \lambda\})^{1/p} = M \|u\|_p,$$

showing that \tilde{T} is of weak type (p, p) for every $p \in (1, \infty)$. By Marcinkiewicz interpolation, \tilde{T} is bounded on $L^p(\mathbb{R}_+)$ for $p \in (1, \infty)$, which implies the assertion. \square

Proof of Theorem 1.13. Let \mathcal{F} denote the convolution operator acting $L^p(\mathbb{R}_+, U) \rightarrow L^p(\mathbb{R}_+, Y)$ and \mathcal{F}^α denote the same operator acting $L_\alpha^p(\mathbb{R}_+, U) \rightarrow L_\alpha^p(\mathbb{R}_+, Y)$. Further let $\Phi_\alpha : L_\alpha^p(\mathbb{R}_+, \cdot) \rightarrow L^p(\mathbb{R}_+, \cdot)$ be the canonical isometric isomorphism given by $(\Phi_\alpha f)(t) := t^\alpha f(t)$. Then $T := \Phi_\alpha \mathcal{F}^\alpha \Phi_\alpha^{-1} - \mathcal{F}$ satisfies

$$(Tu)(t) = \int_0^t \left[\left(\frac{t}{s}\right)^\alpha - 1 \right] K(t-s) u(s) ds$$

with $K(t-s) = CT(t-s)B$. By analyticity of the semigroup, $K \in C(\mathbb{R}_+, B(U, Y))$ and by hypothesis $\|K(t)\| \leq M/t$. Therefore, Proposition 1.14 applies and yields boundedness of $T : L^p(\mathbb{R}_+, U) \rightarrow L^p(\mathbb{R}_+, Y)$. Hence \mathcal{F} is bounded if and only if \mathcal{F}^α is. \square

Proof of Theorem 1.16. We let $v := T(\cdot)x_0$ and denote, for $\rho, \tau > 0$ to be fixed later,

$$\Sigma_{\rho, \tau} = \{x \in C([0, \tau], X) \cap L_\alpha^p([0, \tau], Z) : x(0) = x_0, \|x - w\|_\Sigma \leq \rho\}$$

where $\|x\|_\Sigma := \max\{\|x\|_{C([0, \tau], X)}, \|x\|_{L_\alpha^p([0, \tau], Z)}\}$. Observe that we dropped τ in notation of the norm, and that $\Sigma_{\rho, \tau}$ is complete for the metric induced by $\|x\|_\Sigma$. We let $\Gamma x := v + T(\cdot) * B(F(x) - F(x_0))Cx$ for $x \in \Sigma_{\rho, \tau}$ and shall choose ρ and τ such that Γ is a contraction on $\Sigma_{\rho, \tau}$. Then Banach's fixed point theorem ends the proofs.

If f is a simple function with values in U with compact support in $(0, \tau)$, then by analyticity of the semigroup $T(\cdot)B * f$ is a continuous, X -valued function. Since such functions are dense in $L_\alpha^p(U)$ (recall $p < \infty$), $T(\cdot)B *$ maps $L_\alpha^p(U)$ boundedly into $C([0, \tau], X)$. We let $c_v(\tau) := \|v - x_0\|_{C([0, \tau], X)}$ and $l_v(\tau) := \|v\|_{L_\alpha^p([0, \tau], Z)}$. In the following we shall drop the time interval in the norms. The assumptions imply that $\|T(\cdot)B * u\|_\Sigma \leq K\|u\|_{L_\alpha^p(U)}$ for some $K > 0$. We write L for the Lipschitz-constant of the function F . Then we have, for $x \in \Sigma_{\rho, \tau}$,

$$\begin{aligned} \|\Gamma x - v\|_\Sigma &\leq K\|(F(x) - F(x_0))Cx\|_{L_\alpha^p(U)} \\ &\leq KL\|C\|\|x - x_0\|_{L^\infty(X)}\|x\|_{L_\alpha^p(Z)} \\ &\leq KL\|C\|(\|x - v\|_{C(X)} + \|v - x_0\|_{C(X)})(\|x - v\|_{L_\alpha^p(Z)} + \|v\|_{L_\alpha^p(Z)}) \\ &\leq KL\|C\|(\rho + c_v(\tau))(\rho + l_v(\tau)). \end{aligned}$$

Similarly we obtain, for $x, \tilde{x} \in \Sigma_{\rho, \tau}$,

$$\begin{aligned} \|\Gamma x - \Gamma \tilde{x}\|_\Sigma &\leq K\|(F(x) - F(x_0))Cx - (F(\tilde{x}) - F(x_0))C\tilde{x}\|_{L_\alpha^p(U)} \\ &\leq K(\|(F(x) - F(x_0))C(x - \tilde{x})\|_{L_\alpha^p(U)} + \|(F(x) - F(\tilde{x}))C\tilde{x}\|_{L_\alpha^p(U)}) \\ &\leq KL\|C\|(\|x - x_0\|_{L^\infty(X)}\|x - \tilde{x}\|_{L_\alpha^p(Z)} + \|x - \tilde{x}\|_{C(X)}\|\tilde{x}\|_{L_\alpha^p(Z)}) \\ &\leq KL\|C\|(\rho + c_v(\tau) + \rho + l_v(\tau))\|x - \tilde{x}\|_\Sigma. \end{aligned}$$

Now we choose $\rho > 0$ such that $\eta := 4KL\|C\|\rho < 1$ and then $\tau > 0$ such that $\max\{c_v(\tau), l_v(\tau)\} \leq \rho$. Thus we obtain

$$\|\Gamma x - v\|_\Sigma \leq \eta\rho < \rho \quad \text{and} \quad \|\Gamma x - \Gamma \tilde{x}\|_\Sigma \leq \eta\|x - \tilde{x}\|_\Sigma$$

for $x, \tilde{x} \in \Sigma_{\rho, \tau}$, as desired. \square

L^q -spaces and Besov spaces.

Proof of Lemma 2.1. The definition of Besov and Triebel-Lizorkin spaces together with Minkowski's inequality yield, for any $s \in \mathbb{R}$,

$$\begin{aligned} B_{q,p}^s(\Omega) &\hookrightarrow F_{q,p}^s(\Omega) \quad \text{provided that } q \geq p \text{ and} \\ F_{q,p}^s(\Omega) &\hookrightarrow B_{q,p}^s(\Omega) \quad \text{provided that } q \leq p. \end{aligned}$$

This will be used in the sequel. First we show the 'if'-part, that is we show $L^q(\Omega) \hookrightarrow B_{q,p}^0(\Omega)$ in the case $p \geq \max(2, q)$, that is $(\frac{1}{q}, \frac{1}{p}) \in \text{I}$ where area I is as depicted below. By the above embeddings of Besov and Triebel-Lizorkin spaces, $L^q(\Omega) = F_{q,2}^0(\Omega) \hookrightarrow F_{q,p}^0 \hookrightarrow B_{q,p}^0$.

Next, we consider the case $(\frac{1}{q}, \frac{1}{p}) \in \text{III}$, that is $p \leq \min(q, 2)$ and $(p, q) \neq (2, 2)$. If $p \leq 2 \leq q$ and if $p < q$ we have

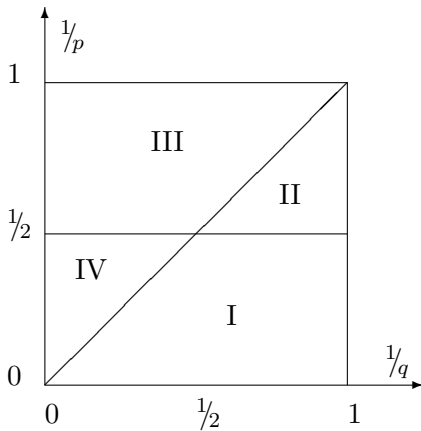
$$B_{q,p}^0(\Omega) \hookrightarrow B_{q,2}^0 \xrightarrow{\neq} F_{q,2}^0(\Omega) = L^q(\Omega),$$

and if $p, q < 2$ and $p \leq q$ we have

$$B_{q,p}^0(\Omega) \hookrightarrow F_{q,p}^0(\Omega) \xrightarrow{\neq} F_{q,2}^0(\Omega) = L^q(\Omega).$$

Therefore, obviously $L^q(\Omega) \not\hookrightarrow B_{q,p}^0(\Omega)$.

For counterexamples in area II and IV we construct specific functions $f \in L^q$ by wavelet decompositions (cf. [23]), that show why the Besov norm cannot be estimated by the L^q -norm.



Let Λ be the set of all points $\lambda = 2^{-j}k + 2^{-j-1}\epsilon$ where $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $0 \neq \epsilon \in \{0, 1\}^n$. Then every $\lambda \in \Lambda$ corresponds to unique j , k and ϵ . Let Q_λ be the dyadic cube defined by $Q_\lambda := \{x \in \mathbb{R}^n : 2^j x - k \in [0, 1)^n\}$. Finally, by [23, Thm. III.8.1] choose some 1-regular

wavelet basis (ψ_λ) with compact support. Then $\text{supp } \psi_\lambda \subset cQ_\lambda$ for some $c > 0$. We let $\Lambda' := \{\lambda \in \Lambda : \text{supp } \psi_\lambda \subset \Omega\}$.

First we treat $(\frac{1}{q}, \frac{1}{p})$ in area II, that is $q < p < 2$. By [23, Thm VI.2.1], for $f = \sum_\lambda \alpha(\lambda)\psi_\lambda(x)$ in $L^q(\mathbb{R}^n)$, we have equivalence

$$\|f\|_{L^q} \sim \left\| \left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |Q_\lambda|^{-1} \mathbb{1}_{Q_\lambda} \right)^{1/2} \right\|_{L^q}. \quad (21)$$

In the following, it will be sufficient to consider only functions f that decompose in a finite sum. If $Q \subset \Omega$ for some $Q = Q_{\lambda_0}$, $\lambda_0 \in \Lambda'$, set $\alpha(\cdot)$ such that only dyadic sub-cubes of Q are considered in the above summation: if Q_λ belongs to the j^{th} dyadic subdivision of Q then let $\alpha(\lambda) := \alpha_j$, otherwise let $\alpha(\alpha) := 0$. Then the expression on the right hand side of (21) L^q -norm of f simplifies to

$$\left\| \left(\sum_{j=0}^N \sum_{\Lambda'_j} |\alpha(\lambda)|^2 \mathbb{1}_{Q_\lambda} \right)^{1/2} \right\|_{L^q} = |Q|^{1/q} \left(\sum_{j=0}^N |\alpha_j|^2 \right)^{1/2}.$$

On the other hand side, an equivalent $B_{q,p}^0$ -norm of $f = \sum_\lambda \alpha(\lambda)\psi_\lambda(x)$ is given by

$$\|f\|_{B_{q,p}^0} \sim \left(\sum_{j=0}^N \left(\left(\sum_{\lambda \in \Lambda'_j} |\alpha(\lambda)|^q \right)^{1/q} 2^{-nj(1/q-1/2)} \right)^p \right)^{1/p}, \quad (22)$$

see [23, VI.10.5]. But, for $\lambda \in \Lambda'_j$ such that $Q_\lambda \subset Q$, $|\alpha(\lambda)| = |\alpha_j|2^{jn/q}$, whence

$$\|f\|_{B_{q,p}^0} \sim \left(\sum_{j=0}^N |\alpha_j| 2^{njp/2} \right)^{1/p}.$$

Therefore, setting $\alpha_j := 2^{njp/2}$ for $j = 0, \dots, N$ and letting $N \rightarrow \infty$ reveals that $L^q(\Omega) \hookrightarrow B_{q,p}^0$ implies $p \geq 2$.

Finally, consider the case IV, that is $q > 2$ and $2 < p < q$. If we set the wavelet coefficients $\alpha(\lambda)$ of f in (21) such that the cubes in $\{Q(\lambda) : \alpha(\lambda) \neq 0\}$ are piecewise disjoint, then

$$\begin{aligned} \|f\|_q &\sim \left\| \left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |Q_\lambda|^{-1} \mathbb{1}_{Q_\lambda} \right)^{1/2} \right\|_{L^q} = \left\| \sum_{\lambda \in \Lambda} |\alpha(\lambda)| |Q_\lambda|^{-1/2} \mathbb{1}_{Q_\lambda} \right\|_{L^q} \\ &= \left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^q |Q_\lambda|^{-q/2} |Q_\lambda| \right)^{1/q} \\ &= \left(\sum_j \left(\left(\sum_{\lambda \in \Lambda_j} |\alpha(\lambda)|^q \right)^{1/q} |Q_\lambda|^{1/q-1/2} \right)^q \right)^{1/q}. \end{aligned}$$

On the other hand side, notice that $|Q_\lambda| = 2^{-nj}$. Thus, comparing the L^q -norm of f with the equivalent $B_{q,p}^0$ -norm given by (22), we find $L^q(\Omega) \hookrightarrow B_{q,p}^0(\Omega)$ requires $q \geq p$ contradicting the assumption $p < q$. \square

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