Optimal High-Order Elements in H(curl) and H(div) for Pyramids, Prisms and Hexahedra

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Bibliography and motivation

- Nedelec's first family not optimal on non-affine hexahedra and prisms
  R.S. Falk, P. Gatto and P. Monk
  Hexahedral H(div) and H(curl) finite elements

- Difficult case of finite elements on pyramids
  N. Nigam, J. Phillips
  Higher-order finite elements on pyramids
  J.-L. Coulomb, F.-X. Zgainski and Y. Maréchal
  A pyramidal element to link hexahedral, prismatic and tetrahedral edge finite elements

- Is it possible to construct finite elements providing an optimal H(div) / H(curl) estimate in $O(h^r)$?

Polynomial spaces

$$\mathbb{Q}_{m,n,p}(x,y,z) = Span\left\{x^i y^j z^k,\ 0 \le i \le m,\ 0 \le j \le n,\ 0 \le k \le p\right\}$$

$$\mathbb{P}_r(x,y,z) = Span\left\{x^i y^j z^k,\ i,j,k \ge 0,\ i+j+k \le r\right\}$$

$$\mathbb{B}_r = \mathbb{P}_r(x,y,z) \oplus \sum_{k=0}^{r-1} \mathbb{P}_k(x,y)\left(\frac{xy}{1-z}\right)^{r-k}$$

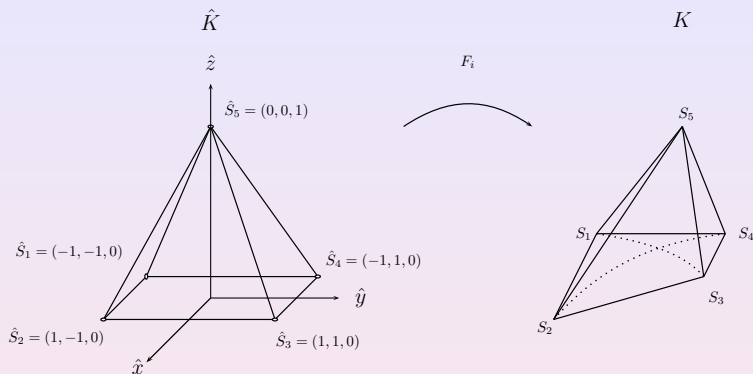$$\widetilde{\mathbb{P}}_r(x,y,z) = Span\left\{x^i y^j z^k,\ i,j,k \ge 0,\ i+j+k = r\right\}$$

$$\mathbb{D}_r(x,y,z) = \mathbb{P}_{r-1}^3 \oplus \begin{bmatrix} x \\ y \\ z \end{bmatrix} \widetilde{\mathbb{P}}_{r-1}(x,y,z)$$

$$\mathcal{S}_r(x,y,z) = \left\{u \in \widetilde{\mathbb{P}}_r^{\,3} \text{ so that } u_1\,x + u_2\,y + u_3\,z = 0\right\}$$

$$\mathcal{R}_r(x,y,z) = \mathbb{P}_{r-1}^3 \oplus \mathcal{S}_r$$

$\mathcal{R}_r(x,y,z), \mathbb{D}_r(x,y,z)$: Nedelec's first family on tetrahedra

Condition of optimality for edge elements

Expression of *F* for the pyramid :

$$F(\hat{x}, \hat{y}, \hat{z}) \,=\, A + B\hat{x} + C\hat{y} + D\hat{z} + \frac{\hat{x}\hat{y}}{4(1-\hat{z})}\left(S_1 + S_3 - S_2 - S_4\right)$$

F affine if the basis of the pyramid is a parallelogramm.

Condition of optimality for edge elements

Finite element space :

$$V_h = \{u \in H(\text{curl}, \Omega) \text{ so that } u|_K \in P_r^F(K)\}$$

Condition of optimality for edge elements

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$$V_h = \{u \in H(\text{curl}, \Omega) \text{ so that } u|_K \in P_r^F(K)\}$$

Use of Piola transform to write the space in the reference element $\hat{K}$:

$$P_r^F(K) = \{u \text{ such that } DF^* u \circ F \in \hat{P}_r(\hat{K})\}$$

Condition of optimality for edge elements

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$$P_r^F(K) = \{u \text{ such that } DF^* u \circ F \in \hat{P}_r(\hat{K})\}$$

Condition of optimality for a given choice of $\hat{P}_r(\hat{K})$:

$$\forall K, \quad \mathcal{R}_r(x, y, z) \subset P_r^F(K)$$

This condition is sufficient to obtain optimal estimates in $O(h^r)$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Hexahedra

$$\mathbb{Q}_{r-1,r+1,r+1} \times \mathbb{Q}_{r+1,r-1,r+1} \times \mathbb{Q}_{r+1,r+1,r-1}$$

Prisms

$$(\mathcal{R}_r(\hat{x},\hat{y}) \otimes \mathbb{P}_{r+1}(\hat{z})) \times (\mathbb{P}_{r+1}(\hat{x},\hat{y}) \otimes \mathbb{P}_{r-1}(\hat{z}))$$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Pyramids

$$\mathbb{B}^3_{r-1} \oplus \left\{ \frac{\hat{x}^p \hat{y}^p}{(1-\hat{z})^{p+2}} \begin{bmatrix} \hat{y}(1-\hat{z}) \\ \hat{x}(1-\hat{z}) \\ \hat{x}\hat{y} \end{bmatrix}, \ 0 \le p \le r-1 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^m \hat{y}^{n+2}}{(1-\hat{z})^{m+2}} \begin{bmatrix} (1-\hat{z}) \\ 0 \\ \hat{x} \end{bmatrix}, \ \frac{\hat{x}^{n+2} \hat{y}^m}{(1-\hat{z})^{m+2}} \begin{bmatrix} 0 \\ (1-\hat{z}) \\ \hat{y} \end{bmatrix}, \ 0 \le m \le n \le r-2 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^p \hat{y}^q}{(1-\hat{z})^{p+q+1-r}} \begin{bmatrix} (1-\hat{z}) \\ 0 \\ \hat{x} \end{bmatrix}, \ \frac{\hat{x}^q \hat{y}^p}{(1-\hat{z})^{p+q+1-r}} \begin{bmatrix} 0 \\ (1-\hat{z}) \\ \hat{y} \end{bmatrix}, \ \begin{matrix} 0 \le p \le r-1 \\ 0 \le q \le r+1 \end{matrix} \right\}$$

Optimal Finite Element spaces

When expressed on the cube $[-1, 1]^3$, it is more friendly :

Pyramids (expressed on the cube)

$$\left(\mathbb{B}_{r-1} \circ T\left(\tilde{x}, \tilde{y}, \tilde{z}\right)\right)^3 \oplus \left\{ \tilde{x}^p \tilde{y}^p (1 - \tilde{z})^p \left[\begin{array}{c} \tilde{y} \\ \tilde{x} \\ \tilde{x}\tilde{y} \end{array} \right], \quad 0 \le p \le r - 1 \right\}$$

$$\oplus \left\{ \tilde{x}^m \tilde{y}^{n+2} (1 - \tilde{z})^{n+1} \left[\begin{array}{c} 1 \\ 0 \\ \tilde{x} \end{array} \right], \ \tilde{x}^{n+2} \tilde{y}^m (1 - \tilde{z})^{n+1} \left[\begin{array}{c} 0 \\ 1 \\ \tilde{y} \end{array} \right], \quad 0 \le m \le n \le r - 2 \right\}$$

$$\oplus \left\{ \tilde{x}^p \tilde{y}^q (1 - \tilde{z})^r \left[\begin{array}{c} 1 \\ 0 \\ \tilde{x} \end{array} \right], \ \tilde{x}^q \tilde{y}^p (1 - \tilde{z})^r \left[\begin{array}{c} 0 \\ 1 \\ \tilde{y} \end{array} \right], \quad \begin{array}{l} 0 \le p \le r - 1 \\ 0 \le q \le r + 1 \end{array} \right\}$$

F and DF are polynomial in the cube $[-1, 1]^3$

$$(x, y, z) = F(\tilde{x}, \tilde{y}, \tilde{z}) = A_0 + A_1\tilde{x}(1-\tilde{z}) + A_2\tilde{y}(1-\tilde{z}) + A_3\tilde{z} + C\tilde{x}\tilde{y}(1-\tilde{z})$$

where A_0, A_1, A_2, A_3, C can be adjusted at any value by changing the vertices of the pyramid.

$$\frac{\partial F}{\partial \hat{x}} = A_1 + C\tilde{y}, \quad \frac{\partial F}{\partial \hat{y}} = A_2 + C\tilde{x}, \quad \frac{\partial F}{\partial \hat{z}} = A_3 + C\tilde{x}\tilde{y}$$

We choose a polynomial $p \in \mathcal{R}_r(x, y, z)$ and express DF^*p in $(\tilde{x}, \tilde{y}, \tilde{z})$ variables and determine the sufficient and necessary basis functions of $(\tilde{x}, \tilde{y}, \tilde{z})$ needed to generate polynomials p for any set of coefficients A_0, A_1, A_2, A_3, C.

Nedelec's first family

Nedelec's first family $\hat{P}_r^1(\hat{K})$:

Hexahedra

$$\mathbb{Q}_{r-1,r,r} \times \mathbb{Q}_{r,r-1,r} \times \mathbb{Q}_{r,r,r-1}$$

Prisms

$$(\mathcal{R}_r(\hat{x}, \hat{y}) \otimes \mathbb{P}_r(\hat{z})) \ \times \ (\mathbb{P}_r(\hat{x}, \hat{y}) \otimes \mathbb{P}_{r-1}(\hat{z}))$$

Pyramids

Same expression with $0 \le p \le r - 1$, $0 \le q \le r$
This finite element space is new.

Tangential restrictions on triangular faces :

$$\mathcal{R}_r(x, y)$$

Tangential restrictions on quadrilateral faces :

$$\mathbb{Q}_{r-1,r+1}(x, y) \times \mathbb{Q}_{r+1,r-1}(x, y)$$

Condition of optimality for facet elements

Finite element space :

$$V_h = \{u \in H(\text{div}, \Omega) \text{ so that } u|_K \in P_r^F(K)\}$$

Use of Piola transform to write the space in the reference element $\hat{K}$:

$$P_r^F(K) = \{u \text{ such that } |DF|DF^{-1} u \circ F \in \hat{P}_r(\hat{K})\}$$

Condition of optimality for a given choice of $\hat{P}_r(\hat{K})$:

$$\forall K, \quad \mathbb{D}_r(x, y, z) \subset P_r^F(K)$$

This condition is sufficient to obtain optimal estimates in $O(h^r)$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Hexahedra

$$\mathbb{Q}_{r+2,r,r}(\hat{x}, \hat{y}, \hat{z}) \times \mathbb{Q}_{r,r+2,r}(\hat{x}, \hat{y}, \hat{z}) \times \mathbb{Q}_{r,r,r+2}(\hat{x}, \hat{y}, \hat{z})$$

Prisms

$$\mathbb{D}_{r+1}(\hat{x}, \hat{y}) \otimes \mathbb{P}_r(\hat{z}) \times \mathbb{W}_{r-1,r+2}(\hat{x}, \hat{y}, \hat{z})$$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Pyramids

$$\mathbb{B}_{r-1}(\hat{x}, \hat{y}, \hat{z})^3$$

$$\oplus \begin{bmatrix} \dfrac{\hat{x}^{n+1}\,\hat{y}^m}{(1-\hat{z})^{m+1}} \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \dfrac{\hat{x}^m\,\hat{y}^{n+1}}{(1-\hat{z})^{m+1}} \\ 0 \end{bmatrix} \quad 0 \le m \le n \le r-1$$

$$\oplus \begin{bmatrix} \dfrac{\hat{x}^{m+1}\,\hat{y}^{n+1}}{(1-\hat{z})^{m+2}} \\ 0 \\ -\dfrac{\hat{x}^m\hat{y}^{n+1}}{(1-\hat{z})^{m+1}} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \dfrac{\hat{x}^{n+1}\,\hat{y}^{m+1}}{(1-\hat{z})^{m+2}} \\ -\dfrac{\hat{x}^{n+1}\hat{y}^m}{(1-\hat{z})^{m+1}} \end{bmatrix} \quad 0 \le m \le n \le r-1$$

$$\oplus \frac{\hat{x}^i\hat{y}^j}{(1-\hat{z})^{i+j+1-r}} \begin{bmatrix} \hat{x} \\ \hat{y} \\ -(1-\hat{z}) \end{bmatrix}, \quad 0 \le i, j \le r$$

Optimal Finite Element spaces

When expressed on the cube $[-1, 1]^3$, it is more friendly :

Pyramids (expressed on the cube)

$$
\left(\mathbb{B}_{r-1} \circ T\left(\tilde{x}, \tilde{y}, \tilde{z}\right) \right)^3
$$
$$
\oplus \begin{bmatrix} \tilde{x}^{k+1}\tilde{y}^m(1-\tilde{z})^k \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \tilde{x}^m\tilde{y}^{k+1}(1-\tilde{z})^k \\ 0 \end{bmatrix}, \quad 0 \le m \le k \le r-1
$$
$$
\oplus \begin{bmatrix} \tilde{x}^{m+1}\tilde{y}^{n+1}(1-\tilde{z})^n \\ 0 \\ -\tilde{x}^m\tilde{y}^{n+1}(1-\tilde{z})^n \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \tilde{x}^{n+1}\tilde{y}^{m+1}(1-\tilde{z})^n \\ -\tilde{x}^{n+1}\tilde{y}^m(1-\tilde{z})^n \end{bmatrix}, \quad 0 \le m \le n \le r-1
$$
$$
\oplus \begin{bmatrix} \tilde{x}^{i+1}\tilde{y}^j(1-\tilde{z})^r \\ \tilde{x}^i\tilde{y}^{j+1}(1-\tilde{z})^r \\ -\tilde{x}^i\tilde{y}^j(1-\tilde{z})^r \end{bmatrix}, \quad 0 \le i, j \le r
$$

Normal components on triangular faces :

$$\mathbb{P}_{r-1}(x, y)$$

Normal components on quadrilateral faces :

$$\mathbb{Q}_{r,r}(x, y)$$

Super-optimality for facet elements

Necessary condition to obtain optimal estimates in $O(h^r)$:

$$\forall K, \quad \mathbb{P}^3_{r-1}(x, y, z) \subset P_r^F(K)$$

$$\forall K, \quad \mathbb{P}_{r-1}(x, y, z) \subset \text{div} P_r^F(K)$$

Super-optimal spaces are the minimal spaces $\hat{P}_r$ satisfying these conditions.

Because of the condition with the divergence, non-unicity of these spaces.

Super-optimal spaces for H(div)

Minimal spaces satisfying the super optimality condition :

Hexahedra

$$\begin{bmatrix} \hat{x}^i \hat{y}^j \hat{z}^k \\ 0 \\ 0 \end{bmatrix}, \ i \leq r+1; \ j,k \leq r; j+k \neq 2r, \text{ and symmetric counterparts}$$

$$\begin{bmatrix} \hat{x}^{r+2} \hat{y}^j \hat{z}^k \\ 0 \\ 0 \end{bmatrix}, \ 0 \leq j,k \leq r, \text{ and symmetric counterparts}$$

$$\begin{bmatrix} \hat{x}^{r+1} \hat{y}^r \hat{z}^r \\ \hat{x}^r \hat{y}^{r+1} \hat{z}^r \\ \hat{x}^r \hat{y}^r \hat{z}^{r+1} \end{bmatrix}$$

Dimension $= \dim \hat{P}_r^{opt} \ - \ (3r+5)$

Super-optimal spaces for H(div)

Minimal spaces satisfying the super optimality condition :

Prisms

$$\hat{P}_r^{opt} \setminus \left[\left(\widetilde{P}_r(\hat{x}, \hat{y})\hat{z}^r \right)^2 \times \{0\} \right] \ + \ \widetilde{P}_{r-1}(\hat{x}, \hat{y})\hat{z}^r \left[\begin{array}{c} \hat{x} \\ \hat{y} \\ 0 \end{array} \right]$$

$$\text{Dimension} = \dim \hat{P}_r^{opt} \ - \ (r+2)$$

Pyramids

$$\hat{P}_r^{opt} \text{ is minimal}$$

Basis functions for super-optimal hexahedra

$P_i^{\alpha,\beta}$ being Jacobi polynomials :

Functions associated with a face

$$\left[\begin{array}{c} (1-x)P_j^{0,0}(2y-1)P_k^{0,0}(2z-1) \\ 0 \\ 0 \end{array} \right], \ j,k \leq r; \ j+k \neq 2r$$

Interior separated functions

$$\left[\begin{array}{c} x(1-x)P_i^{1,1}(2x-1)P_j^{0,0}(2y-1)P_k^{0,0}(2z-1) \\ 0 \\ 0 \end{array} \right]$$

$$i,j,k \leq r; \ i=r \text{ or } j \neq r \text{ or } k \neq r$$

Basis functions for super-optimal hexahedra

$P_i^{\alpha,\beta}$ being Jacobi polynomials :

Functions associated with a face

$$\left[\begin{array}{c} (1-x)P_j^{0,0}(2y-1)P_k^{0,0}(2z-1) \\ 0 \\ 0 \end{array} \right], \; j,k \leq r; \; j+k \neq 2r$$

Interior linked function

$$\left[\begin{array}{c} x(1-x)P_{r-1}^{1,1}(2x-1)P_r^{0,0}(2y-1)P_r^{0,0}(2z-1) \\ y(1-y)P_{r-1}^{1,1}(2y-1)P_r^{0,0}(2x-1)P_r^{0,0}(2z-1) \\ z(1-z)P_{r-1}^{1,1}(2z-1)P_r^{0,0}(2x-1)P_r^{0,0}(2y-1) \end{array} \right]$$

Dispersion error for wave equation

For regular hexahedra :
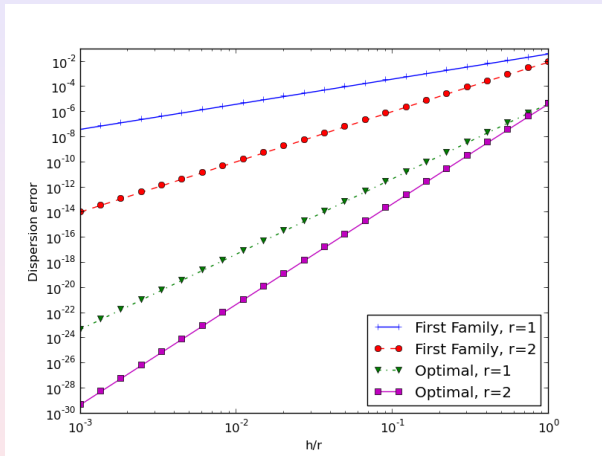
Dispersion error equal to $O(h^{2r})$ for first family, to $O(h^{2r+4})$ for optimal and super-optimal elements

Dispersion error for wave equation

For non-affine hexahedra :
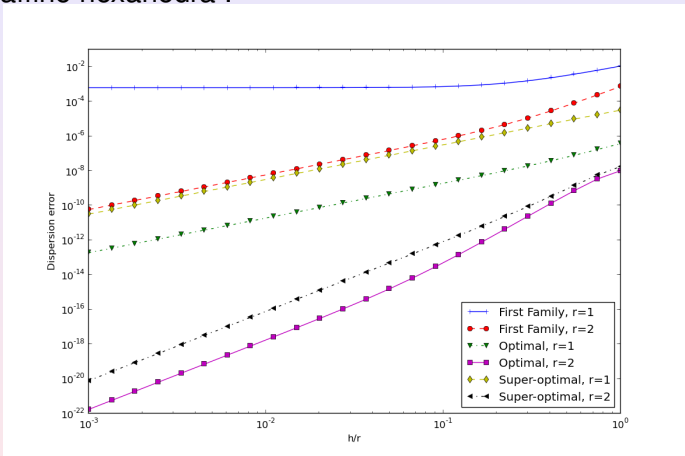
Dispersion error equal to $O(h^{2r-2})$ for first family, to $O(h^{2r})$ for optimal and super-optimal elements

Nodal basis functions

$(\hat{M}_i)$: position of degrees of freeedom

$\hat{t}_i$ the associated direction
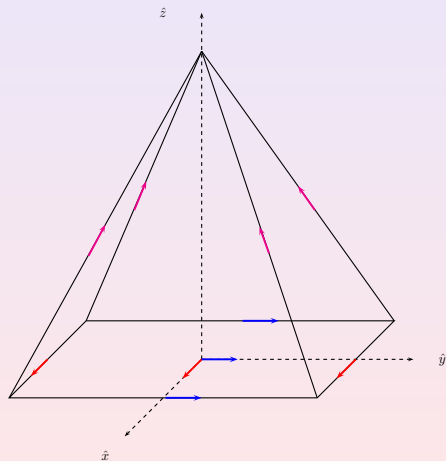
Nodal basis functions

$(\hat{M}_i)$: position of degrees of freeedom

$\hat{t}_i$ the associated direction

$(\hat{\psi}_i)$ a basis of the finite element space $\hat{P}_r$

Vandermonde matrix :

$$VDM_{i,j} = \hat{\psi}_i(\hat{M}_j) \cdot \hat{t}_j$$

The basis function $\hat{\varphi}_i$ associated with the point $\hat{M}_i$:

$$\hat{\varphi}_i = \sum_j (VDM^{-1})_{i,j} \, \hat{\psi}_j.$$

Nodal basis functions

Vandermonde matrix :

$$VDM_{i,j} = \hat{\psi}_i(\hat{M}_j) \cdot \hat{t}_j$$
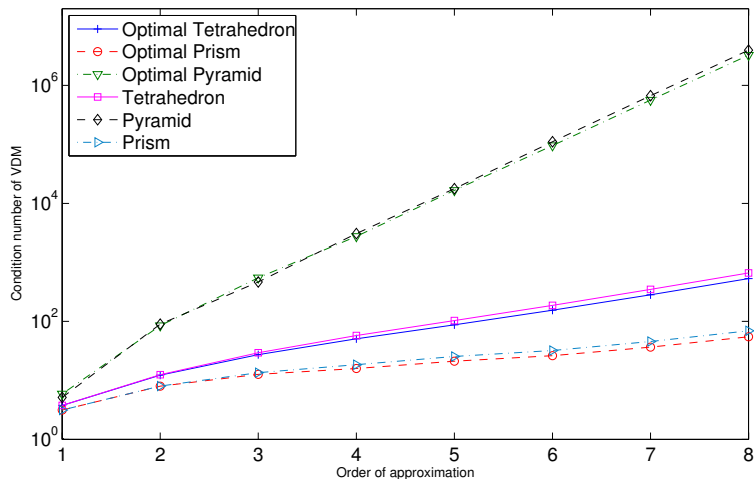
Hierarchical basis functions for the pyramid

Parameters β_i are associated with triangular faces
Parameters λ_i are associated with vertices of the pyramid

$$\begin{cases} \beta_1 = \dfrac{1 - \hat{x} - \hat{z}}{2} \\[2mm] \beta_2 = \dfrac{1 - \hat{y} - \hat{z}}{2} \\[2mm] \beta_3 = \dfrac{1 + \hat{x} - \hat{z}}{2} \\[2mm] \beta_4 = \dfrac{1 + \hat{y} - \hat{z}}{2} \end{cases} \qquad \begin{cases} \lambda_1 = \dfrac{\beta_1 \, \beta_2}{1 - \hat{z}} \\[2mm] \lambda_2 = \dfrac{\beta_2 \, \beta_3}{1 - \hat{z}} \\[2mm] \lambda_3 = \dfrac{\beta_3 \, \beta_4}{1 - \hat{z}} \\[2mm] \lambda_4 = \dfrac{\beta_4 \, \beta_1}{1 - \hat{z}} \\[2mm] \lambda_5 = \hat{z} \end{cases}$$

Hierarchical basis functions for the pyramid

γ_i are parametrizations of vertical edges
γ_i are parametrizations of horizontal edges

$$\begin{cases} \gamma_1 = \dfrac{2\hat{z} + \hat{x} + \hat{y}}{2} \\[2mm] \gamma_2 = \dfrac{2\hat{z} - \hat{x} + \hat{y}}{2} \\[2mm] \gamma_3 = \dfrac{2\hat{z} - \hat{x} - \hat{y}}{2} \\[2mm] \gamma_4 = \dfrac{2\hat{z} + \hat{x} - \hat{y}}{2} \end{cases} \qquad \begin{cases} \delta_1 = \delta_3 = \hat{x} \\[2mm] \delta_2 = \delta_4 = \hat{y} \end{cases}$$

Hierarchical basis functions for the pyramid

Use of Jacobi polynomials $P_i^{\alpha,\beta}(x)$ orthogonal with respect to weight $(1-x)^\alpha(1+x)^\beta$

For two horizontal edges :

$$(\lambda_1\nabla(\lambda_2+\lambda_3)-\lambda_2\nabla(\lambda_1+\lambda_4))\,P_i^{0,0}(\delta_1),\, 0\le i\le r-1$$

$$(\lambda_1\nabla(\lambda_3+\lambda_4)-\lambda_4\nabla(\lambda_1+\lambda_2))\,P_i^{0,0}(\delta_2),\, 0\le i\le r-1$$

For a vertical edge :

$$(\lambda_1\nabla\lambda_5-\lambda_5\nabla\lambda_1)\,P_i^{0,0}(\gamma_1),\quad 0\le i\le r-1$$

Hierarchical basis functions for the pyramid

For the base:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4)) \; \beta_4 \, P_i^{0,0}(\frac{\beta_3 - \beta_1}{1 - \hat{z}}) P_j^{1,1}(\frac{\beta_4 - \beta_2}{1 - \hat{z}})(1 - \hat{z})^{\max(i,j)-1}$$

$$(\lambda_1 \nabla(\lambda_3 + \lambda_4) - \lambda_4 \nabla(\lambda_2 + \lambda_1)) \; \beta_3 \, P_j^{1,1}(\frac{\beta_3 - \beta_1}{1 - \hat{z}}) P_i^{0,0}(\frac{\beta_4 - \beta_2}{1 - \hat{z}})(1 - \hat{z})^{\max(i,j)-1}$$

$$0 \leq i, j \leq r - 1$$

For a triangular face:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4)) \; \lambda_5 \, P_i^{0,0}(\delta_1) P_j^{0,0}(\gamma_1)$$

$$(\lambda_1 \nabla \lambda_5 - \lambda_5 \nabla \lambda_1) \; \beta_3 \, P_i^{0,0}(\delta_1) P_j^{0,0}(\gamma_1)$$

$$0 \leq i + j \leq r - 2$$

Hierarchical basis functions for the pyramid

For interior functions:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4))\ \beta_4\ \lambda_5\ P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$(\lambda_1 \nabla(\lambda_3 + \lambda_4) - \lambda_4 \nabla(\lambda_2 + \lambda_1))\ \beta_3\ \lambda_5\ P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$(\lambda_1 \nabla\lambda_5 - \lambda_5 \nabla\lambda_1)\ \beta_3\ \beta_4\ P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$0 \leq i, j \leq r - 2,$$
$$0 \leq k \leq r - 2 - max(i, j)$$

$$P_{ijk}(\hat{x}, \hat{y}, \hat{z}) = \ P_i^{0,0}(\frac{\beta_3 - \beta_1}{1 - \hat{z}}) P_j^{0,0}(\frac{\beta_4 - \beta_2}{1 - \hat{z}})$$

$$P_k^{2\,max(i,j)+2,0}(2\hat{z} - 1)(1 - \hat{z})^{max(i,j)-1}$$

Comparison with other pyramidal edge elements

- $\hat{P}_r^1$ is the same space as proposed by Coulomb et al, Graglia and Gheorma, Gradinaru and Hiptmair, Doucet et al, Nigam and Phillips for $r = 1$.
- First space proposed by Nigam and Phillips contains more degrees of freedom than $\hat{P}_r^1$ while providing the same order of convergence
- Second space proposed by Nigam and Phillips contains $r(r-1)$ less degrees of freedom but is not consistent for non-affine pyramids
- Basis functions of Coulomb et al, Graglia and Gheorma for $r = 2$ induce spurious modes and are providing only first-order convergence even for affine pyramids

Dispersion analysis

Maxwell's equations

$$-\omega^2 E + \text{curl}(\text{curl} E) = f$$

$$-\omega^2 \int_\Omega E \cdot \varphi_i + \text{curl} E \cdot \text{curl} \varphi_i \, dx = \int_\Omega f \cdot \varphi_i \, dx$$

$$-\omega^2 M_h E + K_h E = F_h$$

Dispersion analysis

Maxwell's equations

$$-\omega^2 E + \text{curl}(\text{curl}E) = f$$

$$-\omega^2 \int_\Omega E \cdot \varphi_i + \text{curl}E \cdot \text{curl}\varphi_i \, dx = \int_\Omega f \cdot \varphi_i \, dx$$

$$-\omega^2 M_h E + K_h E = F_h$$

Research of eigenvalues (ω, E) with quasi-periodic conditions

$$E(\vec{x} + \vec{h}) = exp^{i\vec{k}\cdot\vec{h}} E(\vec{x})$$

Dispersion error $= \dfrac{\omega - ||\vec{k}||}{\omega}$

Dispersion analysis

Research of eigenvalues (ω, E) with quasi-periodic conditions

$$E(\vec{x} + \vec{h}) = exp^{i\vec{k}\cdot\vec{h}} E(\vec{x})$$

Dispersion analysis on purely pyramidal mesh

Eqn: −0.98x −0.17y −0.077z = 0
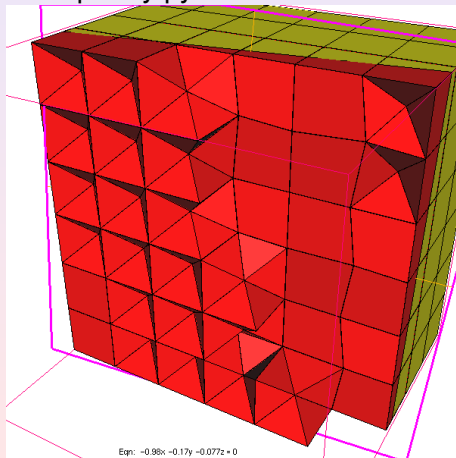
Dispersion analysis

Dispersion on affine pyramids

Dispersion on non-affine pyramids
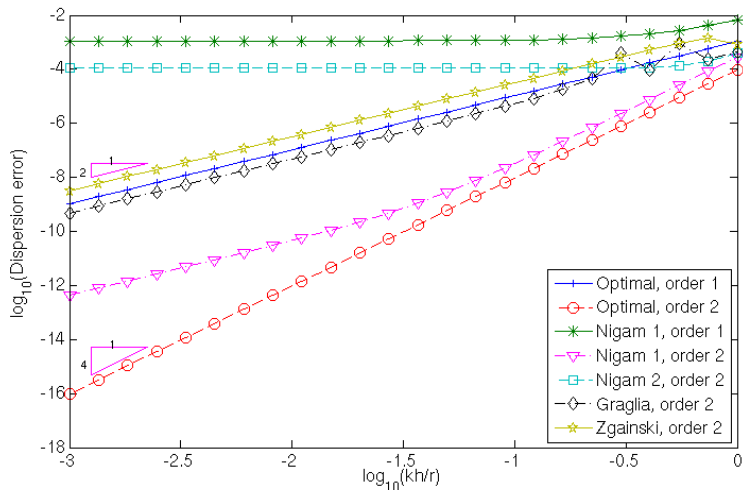
Convergence for the cube

Gaussian source inside a cube and non-affine pyramids :

Convergence for the sphere
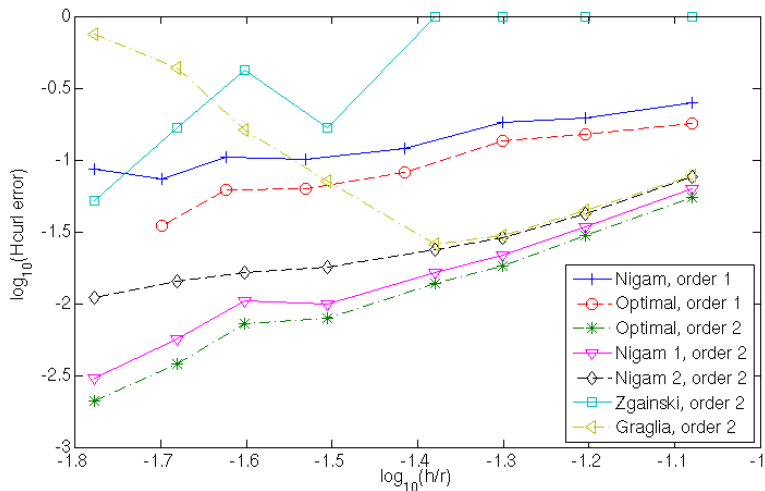
Scattering of a perfectly conducting object

$$
\begin{cases}
-\omega^2 \varepsilon\, E + \text{curl}(\dfrac{1}{\mu}\,\text{curl}\, E) \,=\, f \text{ in } \Omega \\[2mm]
E \times n = -E^{\text{inc}} \times n \text{ on } \Gamma \\[2mm]
\text{curl} E \times n \,=\, i\, k (n \times E) \times n \text{ on } \Sigma
\end{cases}
$$

Convergence for the sphere

Eqn: −0.91x −0.35y −0.21z −0.25 = 0
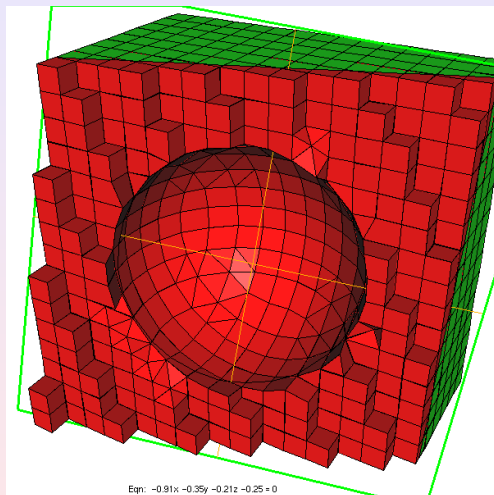
Convergence for the sphere
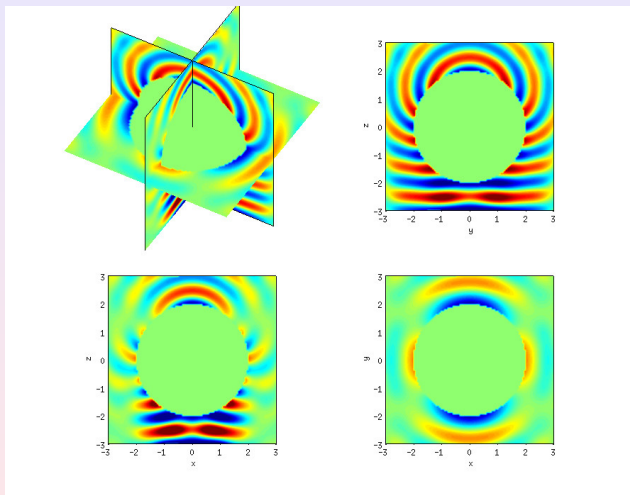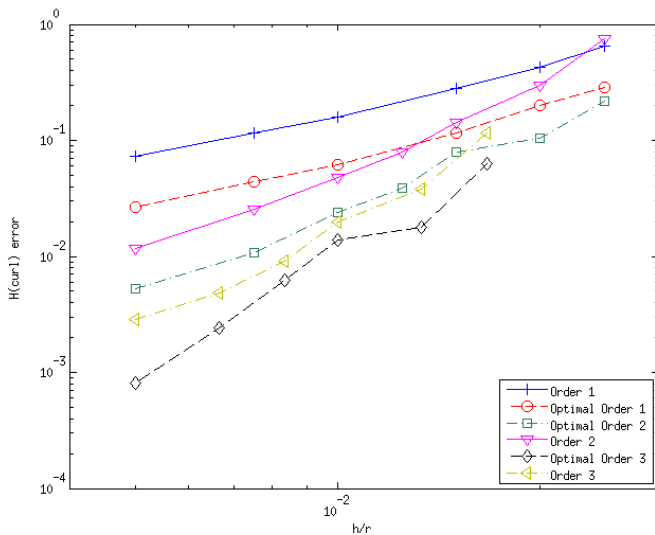
Convergence for the sphere
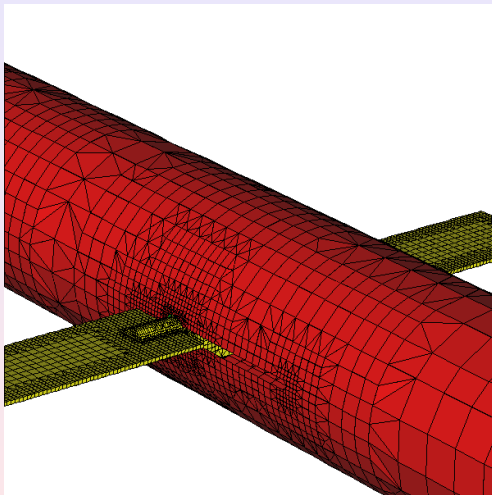
Scattering by a satellite

Each color is associated with an order of approximation
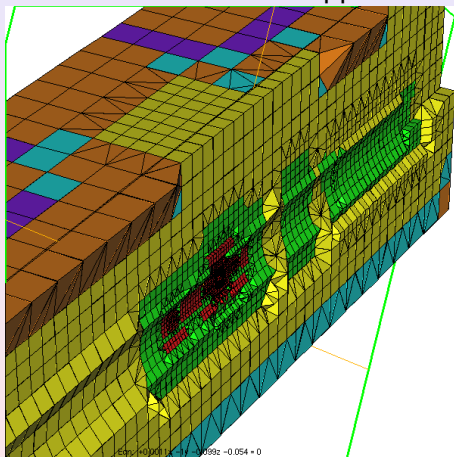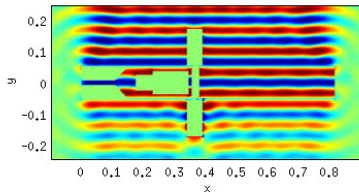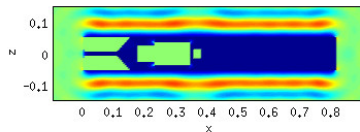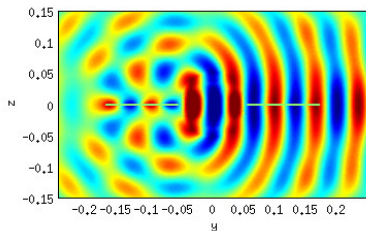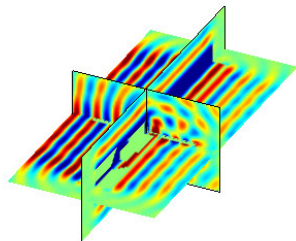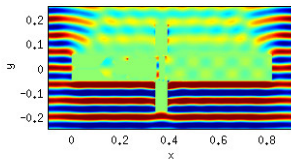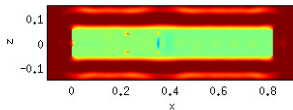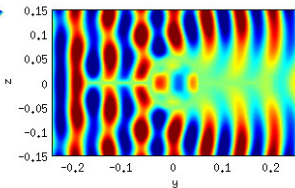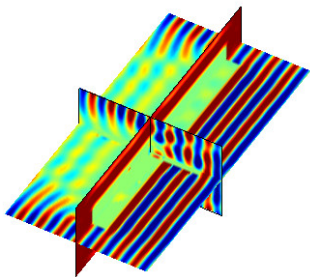
Scattering by a satellite

The mesh contains 35006 tetrahedra, 50390 hexahedra (40 659 affine hexahedra), 48865 pyramids (40 508 affine pyramids), 4582 wedges. We use $\hat{P}_r^1$ and there are 2 570 034 dofs.

END