De la théorie des groupes au Rubik's Cube

Lycée Max Linder - 23 Février 2023

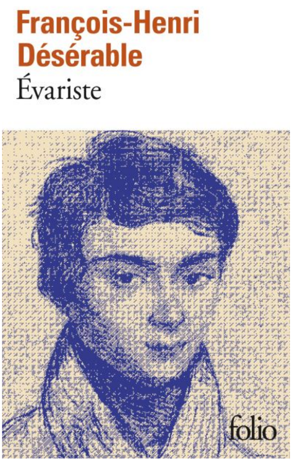
Théorie des groupes

Le Rubik's Cube

Un peu de théorie

Un peu d'algorithmique

Un peu d'originalité



Théorie des groupes

Début XIX-ème siècle

Permutations des racines d'un polynôme

Un groupe $(G, *)$:

- G est un ensemble.
- $*$ est une application $G^2 \to G$. (loi interne)

Tels que:

- $\forall (g, g', g'') \in G^3,\ (g * g') * g'' = g * (g' * g'')$. (associat.)
- $\exists e \in G,\ \forall g \in G,\ e * g = g * e = g$. (elt. neutre)
- $\forall g \in G,\ \exists g' \in G,\ g * g' = g' * g = e$. (elt symétrique)

Exemples: $(\mathbb{Z}, +)$, $(\mathbb{R}^*, \times)$, $(\mathfrak{S}_n, \circ)$, $(\mathbb{Z}/n\mathbb{Z}, +)$.

Un exemple de groupe de Galois !!

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid (a, b) \in \mathbb{Q}^2\}.$$

$$P(X) = X^2 - 2 = (X - \sqrt{2})(X + \sqrt{2}).$$

$$\sigma \left| \begin{array}{ccc} \mathbb{Q}(\sqrt{2}) & \rightarrow & \mathbb{Q}(\sqrt{2}) \\ a + b\sqrt{2} & \mapsto & a - b\sqrt{2} \end{array} \right.$$

$$\sigma(x + y) = \sigma(x) + \sigma(y),\ \sigma(x.y) = \sigma(x).\sigma(y).$$

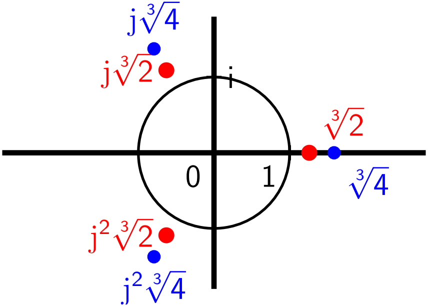
$$\mathrm{Gal}(\mathbb{Q}(\sqrt{2})) = \mathrm{Aut}(\mathbb{Q}(\sqrt{2})) = (\{\mathrm{Id}_{\mathbb{Q}(\sqrt{2})}, \sigma\}, \circ)$$

Remarque: C'est la même chose que la conjugaison complexe pour $X^2 + 1 = (X - \mathrm{i})(X + \mathrm{i})$.

Plus compliqué !!

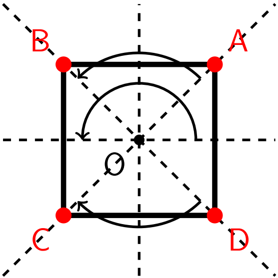
$$P(X) = X^3 - 2 = (X - \sqrt[3]{2})(X - \mathrm{j}\sqrt[3]{2})(X - \mathrm{j}^2\sqrt[3]{2}).$$

$$\mathbb{Q}(\sqrt[3]{2}, \mathrm{j})$$



Le groupe $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \mathrm{j}))$ contient 6 éléments.

Le groupe du carré



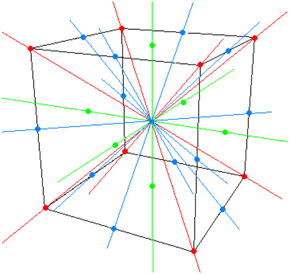
Le groupe des isométries conservant le carré est:

$$\{\mathrm{Id}, -\mathrm{Id}, \mathrm{rot}(O, \pi/2), \mathrm{rot}(O, -\pi/2), \mathrm{Sym}_{d_1}, \mathrm{Sym}_{d_2}, \mathrm{Sym}_{d_3}, \mathrm{Sym}_{d_4}\}.$$

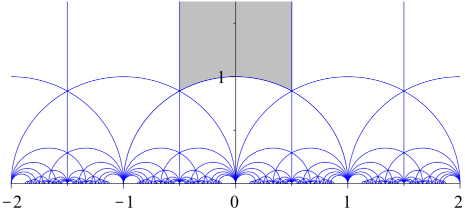
Action de groupe

Groupes d'isométries

Du cube



Du domaine fondamental



Le groupe Symétrique

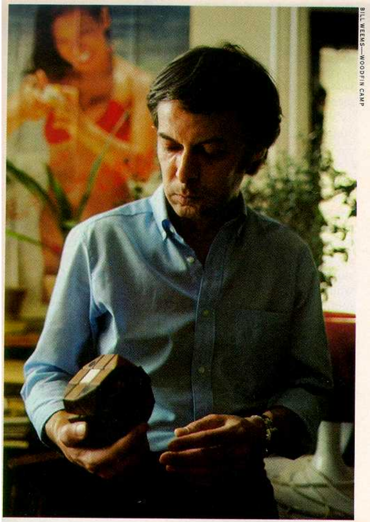
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 7 & 8 & 6 & 4 & 5 \end{pmatrix} \in \mathfrak{S}_8$$

$$\text{Cardinal :} |\mathfrak{S}_n| = n!$$

$$\text{Signature :} s(\sigma) = (-1)^{\mathrm{Inv}} \in \{\pm 1\}$$

$$\text{Commutateurs :} aba^{-1}b^{-1}$$

Action de groupe, sous-groupe distingué, morphisme, centre, groupe dérivé, produit de groupes . . .



Rubik longs for the Golden Age, when the Cube's delights were his alone.

Ernő Rubik

En 1974

Un mois sans solution


1
2
3
4
5
6
7
8
9
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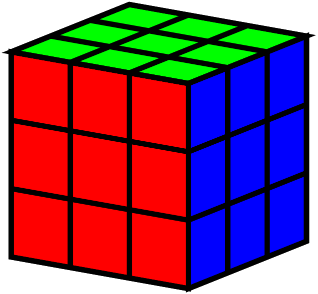
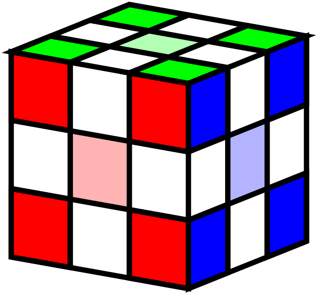
11
12
13
14
15
16

Démontage



- 3 axes rigides
- 8 coins
- 12 arêtes

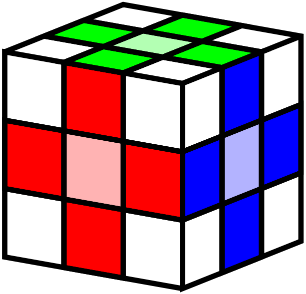
Démontage



$$\mathfrak{S}_8 \times (\mathbb{Z}/3\mathbb{Z})^8$$

$$8!.3^8$$

Démontage



$$\mathfrak{S}_{12} \times (\mathbb{Z}/2\mathbb{Z})^{12}$$

$$12!.2^{12}$$

Démontage

$$D = \mathfrak{S}_8 \times (\mathbb{Z}/3\mathbb{Z})^8 \times \mathfrak{S}_{12} \times (\mathbb{Z}/2\mathbb{Z})^{12}$$

$$(\sigma, \underline{o_c}, \tau, \underline{o_a}) \in D$$

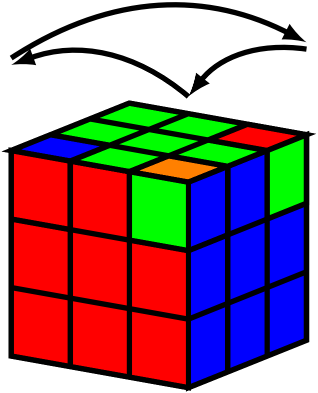
$$|D| = 8!.3^8.12!.2^{12}$$

Des invariants

- $(\sigma, \underline{o_c}, \tau, \underline{o_a}) \mapsto s(\sigma).s(\tau) \in \{\pm 1\}$.
- $(\sigma, \underline{o_c}, \tau, \underline{o_a}) \mapsto \Sigma(\underline{o_c}) \in \mathbb{Z}/3\mathbb{Z}$.
- $(\sigma, \underline{o_c}, \tau, \underline{o_a}) \mapsto \Sigma(\underline{o_a}) \in \mathbb{Z}/2\mathbb{Z}$.

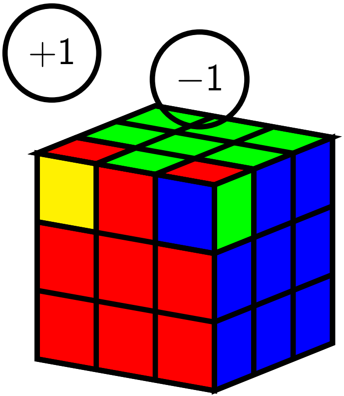
$$\Phi : D \to \{\pm 1\} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

Placer les coins



$$R(U'L'U)R'(U'LU)$$

Orientations des coins



$$(L'DLD')^2U'(L'DLD')^{-2}U$$

Le groupe du Rubik's Cube

$$R = \mathrm{Ker}(\Phi) \lhd D$$

$$|R| = \frac{8!.3^8.12!.2^{12}}{2.3.2} = 43\,252\,003\,274\,489\,856\,000$$

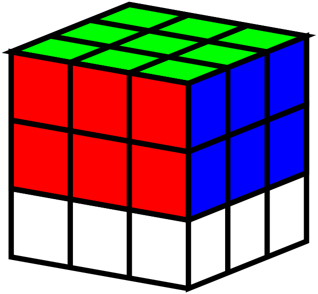
Naïvement

On sait faire !!

les mouvements élémentaires

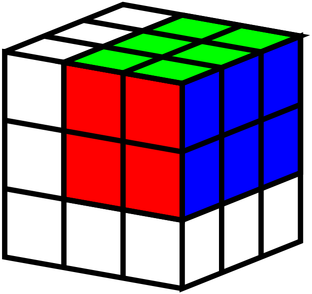
C'est très long

"Layer by Layer"



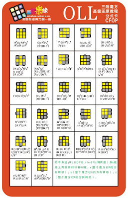
Puis les mouvements élémentaires

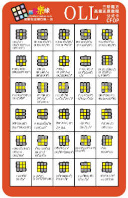
Les méthodes Fridrich et Petrus



Listes de séquences

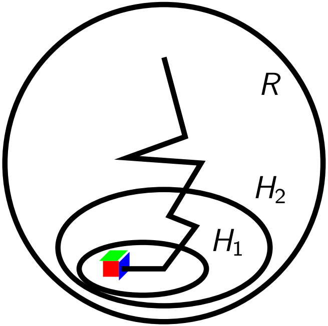
- OLL
- PLL



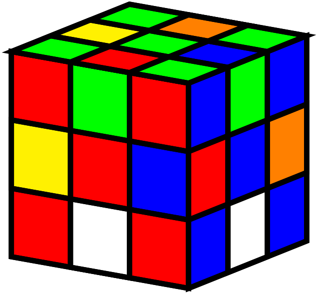


Et Dieu dans tout ça

En au plus 20 mouvements



Le Superflip



Les variations $3 \times 3 \times 3$

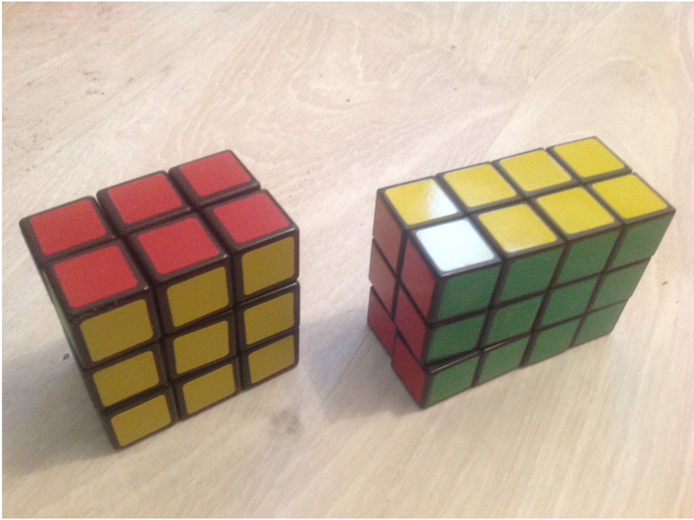
Avec des centres

$$D' = D \times (\mathbb{Z}/4\mathbb{Z})^6$$

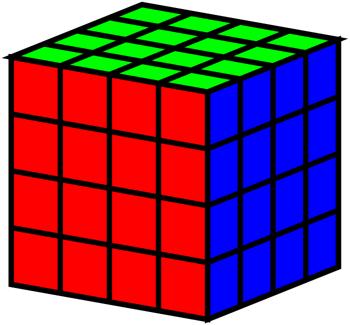
- $(\sigma, \underline{o_c}, \tau, \underline{o_a}, \underline{o'_c}) \mapsto \Sigma(\underline{o'_c}) \in \mathbb{Z}/4\mathbb{Z}$.

$$|R'| = \frac{8!.3^8.12!.2^{12}.4^6}{2.3.2.2}$$
$$= 88\,580\,102\,706\,155\,225\,088\,000$$

Les variations de tailles



4x4x4



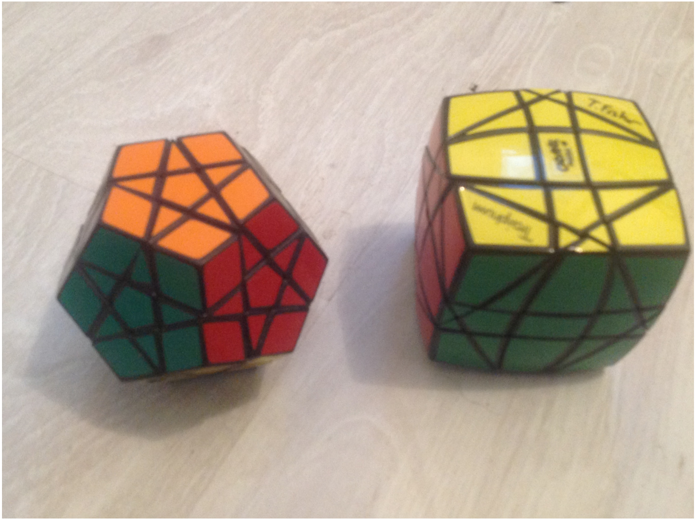
Solution non unique !!
Sous-groupe (non distingué)

$$\frac{\frac{8!.3^8.24!.24!}{2.3}}{\frac{(4!)^6}{2}}$$

Les variations de tailles

Les variations de formes

Les variations de formes



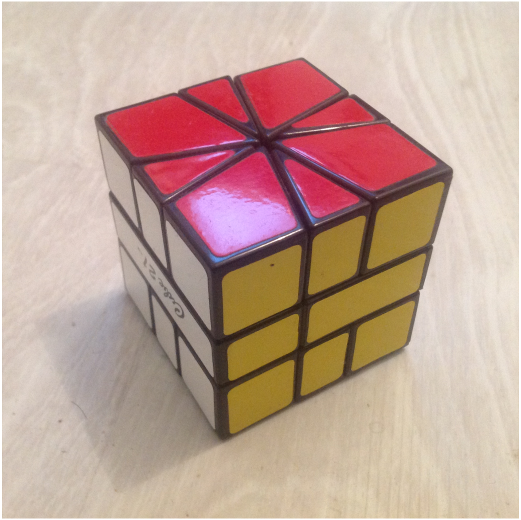
Les fusions

Les fusions

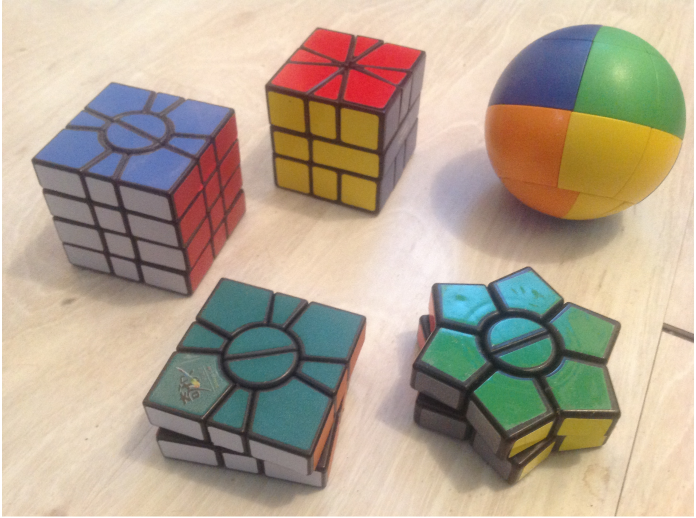
Les "Skewb"

Les "Skewb"

Les Square-1/Cube-21



Les Square-1/Cube-21



Les "Gear"

Les “Gear”

Et encore d'autres

Toujours plus loin…