# Chapter 8

# Finite speed, local uniqueness

## 8.1 Invariant definitions. The incoming spaces

Recall the notations.The symbol $L(i\tilde{\xi}) = \sum i\xi_j A_j + B$ acts from $\mathbb{E}$ to $\mathbb{F}$ with $\dim \mathbb{E} = \dim \mathbb{F} = N$. We denote by $p(\tilde{\xi}) = \det L(i\tilde{\xi})$. The principal symbol is $L_0(i\tilde{\xi}) = \sum i\tilde{\xi}_j A_j$. $L$ is assumed to be hyperbolic in some direction $\nu$ and $\Gamma \subset \mathbb{R}^{1+d}$ denotes the open convex cone of hyperbolic directions. We consider the domain $\Omega = \{x_n > 0\}$ where $x_n = n \cdot x$, and $n \in \mathbb{R}^{1+d}$ is the inner conormal to the boundary. The boundary matrix is $A_n = L_0(n)$, supposed to be invertible, and we denotes by $G(\tilde{i\xi}) = A_n^{-1} L(i\tilde{\xi})$.

By Theorem 2.4.2, there is $\gamma_0 > 0$ such that

$$\tilde{\xi} \in \mathbb{R}^{1+d}, \ \vartheta \in \Gamma \qquad \Rightarrow \qquad p(\tilde{\xi} - i\gamma_0\nu - i\vartheta) \neq 0 \tag{8.1.1}$$

We can normalize $\nu$ so that $\gamma_0 = 1$ so that, denoting by $\Gamma_\nu = \nu + \Gamma \subset \Gamma$,

$$\operatorname{Im} \tilde{\xi} \in \Gamma_\nu \qquad \Rightarrow \qquad p(\tilde{\xi}) \neq 0. \tag{8.1.2}$$

This implies that for $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma_\nu$, $G(\tilde{\xi})$ has no purely imaginary eigenvalue and hence the definition of incoming spaces has the following extension:

**Definition 8.1.1.** *For $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma_\nu$, the incoming space $\mathbb{E}^{in}(\tilde{\xi})$ is the invariant space of $G(\tilde{\xi})$ associated to the eigenvalues in $\{\operatorname{Re} \lambda > 0\}$.*

The dimension of $\mathbb{E}^{in}$ is constant, and was computed above.

**Lemma 8.1.2.** *$\mathbb{E}^{in}(\tilde{\xi})$ is an holomorphic vector bundle over $\mathbb{R}^{1+d} - i\Gamma_\nu$ of dimension $N_+$, the number of positive eigenvalues of $A_n^{-1} L(\nu)$.*

*In particular, if $n \in \Gamma$ [resp. $n \in -\Gamma$], then $\mathbb{E}^{in} = \mathbb{C}^N$ [resp. $\mathbb{E}^{in} = \{0\}$ ]*

From now on, we assume that $\pm n \notin \Gamma$ otherwise $\mathbb{E}^{in} = \mathbb{C}^N$ or $\mathbb{E}^{in} = \{0\}$ and all what follows is trivial.

Because

$$G(i\tilde{\xi} + sn) = G(i\tilde{\xi}) + s\mathrm{Id}$$

the incoming spaces have the property that

$$\mathbb{E}^{in}(\tilde{\xi} + sn) = \mathbb{E}^{in}(\tilde{\xi}) \tag{8.1.3}$$

if the segment $[\xi, \xi + sn]$ is contained in $\mathbb{R}^{1+d} - i\Gamma_\nu$. (This is trivial if $s \in \mathbb{R}$; in general, the assumption is that for $t \in [0,1]$ the eigenvalues of $G(i\tilde{\xi} + itsn)$ do not cross the imaginary axis, implying that the invariant space associated to the eigenvalues in $\{\mathrm{Re}\,\lambda > 0\}$ is constant).

Consider the projection $\varpi : \mathbb{R}^{1+d} \mapsto \mathbb{R}^{1+d}/\mathbb{R}n \approx T^*\partial\Omega$ and its complex extension $\mathbb{C}^{1+d} \mapsto \mathbb{C}^{1+d}/\mathbb{C}n \approx \mathbb{C} \otimes T^*\partial\Omega$. Let $\Gamma^\flat$ denote the projection of $\Gamma$:

$$\Gamma^\flat = \{\zeta \ : \ \exists\tilde{\xi} \in \Gamma \ , \ \zeta = \varpi\tilde{\xi}\} \subset T^*\partial\Omega\backslash\{0\}. \tag{8.1.4}$$

It is an open convex cone in $T^*\partial\Omega$. Let $\Gamma^\flat_\nu = \nu^\flat + \Gamma^\flat = \varpi\Gamma_\nu$. It is convex and for $\zeta \in \Gamma^\flat_\nu$, $\varpi^{-1}(\zeta)$ is a segment in $\Gamma$. Thus the invariance (8.1.3) implies that $\mathbb{E}^{in}$ depends only on $\varpi\tilde{\xi}$ and legitimates the following definition:

**Definition 8.1.3.** *For $\zeta \in T^*\partial\Omega - i\Gamma^\flat_\nu$, we set*

$$\mathbb{E}^{in}(\zeta) = \mathbb{E}^{in}(\tilde{\xi}), \qquad \xi \in \mathbb{R}^{1+d} - i\Gamma_\nu, \quad \varpi\tilde{\xi} = \zeta. \tag{8.1.5}$$

In coordinates $(t, x', x_n)$ with dual variables $(\tau, \xi', \xi_n) \in \mathbb{R}^d \times \mathbb{R}$, one can identify $T^*\partial\Omega$ with the first factor $\mathbb{R}^d$. This is what we did in the previous chapter, and this is why we used the notation $\zeta$ for element of $T^*\partial\Omega$. More importantly, we have extended the definition of $\mathbb{E}^{in}$ to the complex domain $\{\mathrm{Im}\,\zeta \in \Gamma^\flat_\nu\}$.

When $L = L_0$ is homogeneous, then $\mathbb{E}^{in}$ is clearly homogeneous of degree 0 and defined in $\mathbb{R}^{1+d} - i\Gamma$. In general, because $L_0$ is hyperbolic with the same cone of hyperbolic directions $\Gamma$, we can introduce the incoming spaces associated to $L_0$, which we denote by. $\mathbb{E}^{in}_0(\tilde{\xi})$. For $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma$ and $\varepsilon > 0$ small, we have

$$\begin{aligned} \Pi^{in}(\tilde{\xi}/\varepsilon) =& \frac{1}{2i\pi} \int_{\mathcal{C}^+} (G_0(i\tilde{\xi}) + \varepsilon A_n^{-1} B - z\mathrm{Id})^{-1} dz \\ &\to \Pi^{in}_0(\tilde{\xi}) \qquad \textit{as } \varepsilon \to 0. \end{aligned} \tag{8.1.6}$$

This property is still true in the quotient $\tilde{\xi} \mapsto \zeta$. Note that these convergences hold for $\operatorname{Im} \zeta \in \Gamma^\flat$, which means in particular that $\operatorname{Im} \tilde{\zeta} \neq 0$. No uniformity in $\operatorname{Im} \zeta$ is claimed as $\operatorname{Im} \zeta \to 0$.

In the homogeneous case the domain of definition of $\mathbb{E}^{in}$ can be extended, using the following remark:

**Lemma 8.1.4.** *For all complex number $a$,*

$$\text{(8.1.7)} \qquad \operatorname{Im} \zeta \in -\Gamma^\flat, \ \operatorname{Im}(a\zeta) \in -\Gamma^\flat \qquad \Rightarrow \qquad \mathbb{E}_0^{in}(a\zeta) = \mathbb{E}_0^{in}(\zeta).$$

*Proof.* Because $\Gamma^\flat$ is an open convex cone, one has $a \neq 0$ and $a \neq -1$. With $a_t = ta + (1-t) \neq 0$, we prove that $\mathbb{E}_0^{in}(a_t \zeta)$ is constant.

The assumptions are that $\zeta = \varpi \tilde{\xi}$ and $a\zeta = \varpi \tilde{\eta}$ with $\operatorname{Im} \tilde{\xi} \in -\Gamma$ and $\operatorname{Im} \tilde{\eta} \in -\Gamma$. Thus $\tilde{\eta} = a\tilde{\xi} + sn$, for some complex number $s$. For $t \in [0,1]$, $a_t \zeta = \varpi(\tilde{\xi}_t)$ with $\tilde{\xi}_t = t\tilde{\eta} + (1-t)\tilde{\xi} = a_t \tilde{\xi} + tsn$. Because

$$G_0(i\xi_t) = a_t G_0(i\xi) + its\mathrm{Id},$$

the invariant spaces of $G_0(\tilde{\xi}_t)$ are those of $G(\tilde{\xi})$. Moreover, since $\Gamma$ is convex, $\operatorname{Im} \tilde{\xi}_t \in -\Gamma$ and the eigenvalues of $G(i\xi_t)$ do not cross the imaginary axis. Hence $\mathbb{E}_0^{in}(\tilde{\xi}_t) = \mathbb{E}_0^{in}(\tilde{\xi})$. $\square$

Introduce the open set

$$\text{(8.1.8)} \qquad \mathcal{G} = \{a\zeta, \ \operatorname{Im} \zeta \in -\Gamma^\flat, a \in \mathbb{C}\backslash\{0\}\} \subset \mathbb{C} \otimes T^*\partial\Omega \approx \mathbb{C}^{1+d}/\mathbb{C}n.$$

This set is conic and stable by multiplication by complex numbers $a \neq 0$, but is *not* convex. If $a\zeta = b\zeta'$, with $\operatorname{Im} \zeta$ and $\operatorname{Im} \zeta'$ in $-\Gamma\flat$, then $\zeta' = \alpha\zeta$ with $\alpha = a/b$ and (8.2.7) implies that $\mathbb{E}_0^{in}(\zeta) = \mathbb{E}_0^{in}(\zeta')$. Therefore, it makes sense to extend the definition of $\mathbb{E}_0^{in}$ to the domain $\mathcal{G}$ in such a way that

$$\text{(8.1.9)} \qquad \forall \zeta \in \mathcal{G}, \ \forall a \in \mathbb{C}\backslash\{0\}: \qquad \mathbb{E}_0^{in}(a\zeta) = \mathbb{E}_0^{in}(\zeta).$$

In particular, the incoming space $\mathbb{E}^{in}(\zeta)$ is defined when $\tilde{\zeta} \in \Gamma^\flat$. We show that we can also extend the definition of $\mathbb{E}^{in}$ to this region.

**Lemma 8.1.5.** *When $\tilde{\zeta} = \varpi\tilde{\xi}$ and $\tilde{\xi} \in \Gamma$, the eigenvalues of $G_0(\tilde{\xi})$ are real and exactly $N_+$ are positive. The associated invariant space is $\mathbb{E}_0^{in}(\zeta)$ and has a holomorphic extension to a neighborhood of $\zeta$.*

*Moreover, there are $\varepsilon_0 > 0$ and a complex neighborhood $\mathscr{V}$ of $\zeta$ such that $\mathbb{E}^{in}$ extends holomorphical to the cone $\{\varepsilon^{-1}\zeta', \varepsilon < \varepsilon_0, \zeta' \in \mathscr{V}\}$ and*

$$\text{(8.1.10)} \qquad \forall \zeta' \in \mathscr{V}: \qquad \Pi^{in}(\varepsilon^{-1}\zeta') \to \Pi_0^{in}(\zeta')$$

*One has similar results when $\theta \in -\Gamma^\flat$, with $\mathbb{E}_0^{in}(-\theta)$ associated to the negative eigenvalues of $G_0(-\theta)$, so that $\mathbb{E}_0^{in}(-\theta) = \mathbb{E}^{in}(\theta)$ in accordance with (8.1.9).*

*Proof.* The eigenvalues of $G_0(\tilde{\xi}) = A_n^{-1} L_0(\tilde{\xi})$ are the inverse of those of $L_0(i\tilde{\xi})^{-1} A_n$ which are real since we assumed that $\tilde{\xi}$ is in the cone $\Gamma$. And they do not vanish since the matrices are invertible.

Moreover, $\mathrm{Im}\,(-ia\xi) \in -\Gamma$ if $\mathrm{Re}\, a > 0$ and thus, by (8.1.7), $\mathbb{E}_0^{in}(-ia\tilde{\xi}) = \mathbb{E}_0^{in}(-i\tilde{\xi})$, which means that the invariant space of $G_0(-ia\tilde{\xi})$ associated to eigenvalues in $\mathrm{Re}\,\lambda > 0$ is constant an equal to $\mathbb{E}^{in}(-i\tilde{\xi})$. When $a \to i$ in $\mathrm{Re}\, a > 0$, $G_0(-ia\tilde{\xi}) \to G_0(\xi)$, which has real and non vanishing eigenvalues. Thus $\mathbb{E}^{in}(-i\tilde{\xi})$ is the invariant space of $G_0(\xi)$ associated to eigenvalues in $\mathrm{Re}\,\lambda > 0$. Since 0 is not an eigenvalue of $G_0(i\tilde{\xi})$ the invariant space can be continued analytical for all small perturbations of $G_0(\tilde{\xi})$ and the remaining part of the lemma follows. $\square$

## 8.2 The Lopatinski determinant(s)

We consider boundary conditions $M : \mathbb{E} \mapsto \mathbb{G}$, with with $\dim \mathbb{G} = N_+$ as above. The question under discussion is to know wether $\mathbb{E}^{in}(\zeta) \cap \ker M$ is trivial or not. There are several ways to express this condition. First, given an arbitrary scalar product in $\mathbb{E}$, one can measure the angle between $\ker M$ and $\mathbb{E}^{in}(\xi')$ through the quantity

$$D(\zeta) = \big| \det(\mathbb{H}, \mathbb{E}^{in}(\zeta)) \big| \tag{8.2.1}$$

where the determinant is computed by taking orthonormal bases in each space. This quantity does not depend on the choice of the bases, but it depends only on the choice of a scalar product on $\mathbb{E}$. One has

$$\mathbb{E}^{in}(\zeta) \cap \ker M = \{0\} \qquad \Leftrightarrow \qquad D(\zeta) \neq 0. \tag{8.2.2}$$

However, this choice ignores an important feature of the problem, which is the analytic dependence of $\mathbb{E}^{in}$. Locally in $T^*\partial\Omega - i\Gamma_\nu^\flat$, one can choose a holomorphic basis $e_k^{in}(\zeta)$ of $\mathbb{E}^{in}(\zeta)$, and form the (local) Lopatinski determinant

$$\ell(\zeta) = \det \Big[ g_1, \dots, g_{N-N_+}, e_1^{in}(\zeta), \dots, e_{N_+}^{in}(\zeta) \Big] \tag{8.2.3}$$

where the $g_j$ form a basis of $\ker M$. This function has the advantage of being holomorphic in $\zeta$, and locally there are constants $0 < c \le C$ such that

$$c|\ell(\zeta)| \le D(\zeta) \le C|\ell(\zeta)|. \tag{8.2.4}$$

The function $\ell$ can be globalized using analytic continuation and the property that $T^*\partial\Omega - i\Gamma'$ is simply connected, but the global properties of the extended function do not seem obvious.

There is an alternate way to preserve analyticity. Fix a basis $e_k$ of $\mathbb{E}$ and for all subset $J = \{j_1, \ldots, j_{N_+}\} \subset \{1, \ldots, N\}$ of $N_+$ elements consider

$$\ell_J(\zeta) = \det\left[g_1, \ldots, g_{N-N_+}, \Pi^{in}(\zeta)e_{j_1}, \ldots, \Pi^{in}(\zeta)e_{N_+}.\right] \tag{8.2.5}$$

These functions are clearly defined and holomorphic in $T^*\partial\Omega - i\Gamma_\nu^\flat$ and

$$\mathbb{E}^{in}(\zeta) \cap \ker M \neq \{0\} \qquad \Leftrightarrow \qquad \forall J,\ \ell_J(\zeta) = 0. \tag{8.2.6}$$

Considering the principal part $L_0$ which is hyperbolic with the same cone of hyperbolic directions $\Gamma$, one can form the quantities $D_0$ and $\ell_{J,0}$ associated to $L_0$ and $M$. The following properties are immediate consequences of (8.1.9), (8.1.6) and Lemma 8.1.5.

**Proposition 8.2.1.** *i) $D_0$ and $\ell_{J,0}$ are defined on the set $\mathcal{G}$ and*

$$\forall \zeta \in \mathcal{G},\ \forall a \in \mathbb{C}\backslash\{0\}: \qquad D_0(a\zeta) = D_0(\zeta), \quad \ell_{J,0}(a\zeta) = \ell_{J,0}(\zeta). \tag{8.2.7}$$

*ii) For all $\zeta \in T^*\partial\Omega - \Gamma^\flat$,*

$$D(\zeta/\varepsilon) \to D_0(\zeta), \quad \ell_J(\zeta/\varepsilon) \to \ell_{J,0}(\zeta) \quad \textit{as } \varepsilon \to 0. \tag{8.2.8}$$

*iii) if $\theta \in \Gamma^\flat$, there are $\varepsilon_0$ and a complex neighborhood $\mathscr{V}$ of $\theta$ such that $D$ and the $\ell_J$ are defined for $\zeta/\varepsilon$ if $\zeta \in \mathscr{V}$ and $\varepsilon < \varepsilon_0$ and the convergence above is true on $\mathscr{V}$.*

## 8.3 The Lopatinski condition

First remark that if $\vartheta \in \Gamma^\flat$, then there is $\gamma_0$ such that $\gamma\vartheta \in \Gamma_\nu^\flat$ when $\gamma \geq \gamma_0$. This legitimates the following definition:

**Definition 8.3.1.** *The (weak) Lopatinski condition is satisfied in the direction $\vartheta \in \Gamma^\flat$ if and only if there is $\gamma_0$ such that $D(\zeta - i\gamma\vartheta) \neq 0$ for all $\zeta \in T^*\partial\Omega$ and $\gamma > \gamma_0$.*

**Lemma 8.3.2.** *If $L$ satisfies the Lopatinski condition in the direction $\vartheta \in \Gamma^\flat$, then $L_0$ also satisfies the Lopatinski condition.*

*Proof.* Suppose that $D_0(\zeta) = 0$ at some $\zeta \in T^*\partial\Omega - i\gamma\vartheta$. For $\varepsilon$ small enough, the function $g_\varepsilon(z) = D(\zeta + z\vartheta/\varepsilon)$ is defined for $z$ in a disc centered at the origin and $g_\varepsilon \to D_0(\zeta + z\vartheta)$. Moreover, $D_0$ is not identically 0. Hence, by Lemma 8.3.5 (Hurwitz lemma if we replace $D$ by an holomorphic local version), $g_\varepsilon$ vanishes in a neighborhood of the origin. $\square$

**Theorem 8.3.3.** *Suppose that the Lopatinski condition is satisfied in the direction $\vartheta \in \Gamma^\flat$. Let $\Sigma$ denote the component of $\vartheta$ in $\{\zeta \in \Gamma^\flat, D_0(-i\zeta) \neq 0\}$. Then $\Sigma$ is an open convex subcone of $\Gamma^\flat$ in $T^*\partial\Omega$ and the Lopatinski condition is satisfied in all direction $\theta \in \Sigma$.*

*Proof.* **a)** For $\zeta \in T^*\partial\Omega$, we look at the function of the complex variable $z$, $F_\zeta(z) = D_0(\zeta + z\vartheta)$. It is defined when $\zeta + z\vartheta \in \mathcal{G}$, in particular when $\operatorname{Im} z < 0$ since then $\zeta + z\vartheta \in T^*\partial\Omega - i\Gamma^\flat$ and, by assumption, $F_\zeta$ does not vanish there. Moreover, $-\zeta - z\vartheta \in T^*\partial\Omega - i\Gamma^\flat$ when $\operatorname{Im} z > 0$, and thus $\zeta + z\vartheta \in \mathcal{G}$. By (8.2.7), $F_\zeta(z) = D_0(-\zeta - z\vartheta)$ wich is $\neq 0$ by assumption. This shows that for $\zeta \in T^*\partial\Omega$, $F_\zeta$ is defined and does not vanish when $\operatorname{Im} z \neq 0$.

**b)** When $\theta \in \Sigma$, $\operatorname{Im}(-i(\theta + z\vartheta)) = -\theta - \operatorname{Re} z\vartheta \in -\Gamma^\flat$ when $\operatorname{Re} z \geq 0$ thus $-i(\theta + z\vartheta) \in T^*\partial\Omega - i\Gamma^\flat$ and $\theta + z\nu \in \mathcal{G}$. Thus, $F_\theta$ is defined for $\operatorname{Re} z \geq 0$. It does not vanish when $\operatorname{Im} z \neq 0$ by step a), and it does not vanish when $z = 0$ since $F_\theta(0) = D_0(\theta) = D_0(-i\theta)$ which is $\neq 0$ by assumption. Therefore, $F_\theta(z) \neq 0$ when $\operatorname{Re} z = 0$.

Moreover, for $|z|$ large in $\operatorname{Re} z \geq 0$, one has $\vartheta + z^{-1}\zeta \in \Gamma^\flat \subset \mathcal{G}$ and $F_\zeta(z) = D_0(\vartheta + z^{-1}\zeta) = D_0(-i(\vartheta + z^{-1}\zeta)) \neq 0$ since $D_0(\vartheta \neq 0)$.

This shows that $F_\theta$ does not vanish when $\operatorname{Re} z = 0$ or when $\operatorname{Re} z \geq 0$ and $lz|$ is large. Since $F_\vartheta(z) = D_0((1+z)\vartheta = D_0(\vartheta) \neq 0$ for all $z$ such that $\operatorname{Re} z \geq 0$, Lemma 8.3.6 by deformation that $F_\theta$ does not vanish either on the domain $\{\operatorname{Re} z \geq 0\}$:

$$\forall \theta \in \Sigma,\ \forall z,\ \operatorname{Re} z \geq 0 \quad \Rightarrow \quad D_0(\theta + z\vartheta) \neq 0. \tag{8.3.1}$$

Because $\operatorname{Re} 1/z \geq 0$ when $\operatorname{Re} z \geq 0$, the homogeneity of $D_0$, implies that $D_0(\vartheta + z\theta) \neq 0$ when $\operatorname{Re} z \geq 0$ and $z \neq 0$. This property is also true at $z = 0$, and hence

$$\forall \theta \in \Sigma,\ \forall z,\ \operatorname{Re} z \geq 0 \quad \Rightarrow \quad D_0(\vartheta + z\theta) \neq 0. \tag{8.3.2}$$

In particular, this applies to $z$ real nonnegative, and by homogeneity, one has $D_0(t\theta' + s\nu') \neq 0$ when $t > 0$ and $s \geq 0$. This extends to $t = 0$. Thus the segment $[\nu, \theta']$ is contained in $\Sigma$ and $\Sigma$ is star shaped with respect to $\nu$.

**c)** Let $\zeta \in T^*\partial\Omega$ and $\theta \in \Sigma$. For $\gamma > \gamma_0$, we look at the function of $z$, $G_\gamma(z) = D(\zeta - i\gamma\vartheta - iz\theta)$, which is defined for $\operatorname{Re} z \geq 0$ since then $\operatorname{Im}(\zeta - i\gamma\theta^\flat - iz\theta) = -\gamma\theta^\flat - \operatorname{Re} z\theta \in -\Gamma^\flat_\nu - \Gamma^\flat \subset -\Gamma^\flat_\nu$. It does not vanish when $\operatorname{Re} z = 0$, since the Lopatinski condition is satisfied in the direction $\vartheta$.

Moreover, when $z$ is large, setting $\hat{z} = z/|z|$, one has

$$G_\gamma(z) = D\big(-i\hat{z}\theta + |z|^{-1}(\zeta - i\gamma\vartheta)\big)$$

By iii) of Proposition 8.2.1, since $\theta \in \Gamma^\flat$, this converges to $D_0(-i\hat{z}\theta) = D_0(-i\theta) \neq 0$ if $\operatorname{Re} \hat{z} \geq 0$ . This implies that $G_\gamma$ does not vanish in the half space $\operatorname{Re} z \geq 0$, either when $\operatorname{Re} z = 0$ or when $|z| \geq R_0(1+\gamma)$, for some $R_0$ large enough.

Therefore, applying Lemma 8.3.6, to prove that

$$\text{(8.3.3)} \quad \forall \zeta \in T^*\partial\Omega,\ \forall \gamma > \gamma_0, \forall z,\ \operatorname{Re} z \geq 0 \quad \Rightarrow \quad D(\zeta - i\gamma\vartheta - iz\theta) \neq 0.$$

it is sufficient to show that for $\gamma_1$ large

$$\text{(8.3.4)} \qquad \gamma \geq \gamma_1,\ |z| \leq R_0(1+\gamma): \qquad D(\zeta - i\gamma\vartheta - iz\theta) \neq 0.$$

Here we factor out $\gamma$ and use again the Proposition 8.2.1 which implies that

$$G_\gamma(z) = D(\gamma(-i\vartheta - i\hat{z}\theta + \gamma^{-1}\zeta) \to D_0(-i(\vartheta + \hat{z}\theta)),$$

where $\hat{z} = z/\gamma$ is bounded. By step (8.3.2) the limit does not vanish and is bounded from below since $|\hat{z}|$ is bounded. Therefore, (8.3.4) and (8.3.3) follow.

**d)** Because $\Sigma$ is open, one can replace $\theta$ by $\theta - \delta\vartheta$ for some $\delta > 0$ small, and (8.3.3) implies that

$$\text{(8.3.5)} \qquad \forall \zeta \in T^*\partial\Omega,\ ,\forall z,\ \operatorname{Re} z > \beta \quad \Rightarrow \quad D(\zeta - iz\theta) \neq 0.$$

This shows that the Lopatinski condition is satisfied in the direction $\theta'$.

Applying step a), this implies that $\Sigma$ is star shaped with respect to $\theta'$ and the proof of the theorem is complete. □

**Theorem 8.3.4.** *If $M$ satisfies the uniform Lopatinski condition in a direction $\vartheta \in \Gamma^\flat$, then $\Sigma = \Gamma^\flat$ and the uniform Lopatinski condition is satisfied in all directions $\theta \in \Gamma^\flat$.*

*Proof.* We have seen that $\Gamma^\flat - i\Gamma^\flat \in \mathcal{C}$ and that $\Delta$ is continuous there. The uniform Lopatinski condition implies that $|\Delta(a\theta)| \geq c$ when $\theta \in \Gamma^\flat \operatorname{Im} a < 0$. Hence by continuity $|\Delta(\theta)| \geq c$, implying that $\theta \in \Sigma$.

By Proposition **??**, there is $\varepsilon > 0$ such that $M'$ satisfies the Lopatinski condition in the direction $\vartheta$ if $|M - M'| \leq \varepsilon$, and thus in all direction $\theta \in \Sigma = \Gamma^\flat$ by Theorem 8.3.3 and the remark above. By Proposition **??**, this implies that the uniform Lopatinski condition is satisfied in all directions $\theta \in \Gamma^\flat$. □

### 8.3.1 The analogue of Rouché's theorem

**Lemma 8.3.5.** *Suppose that $D_n$ is a sequence of functions on $\dot{H} = \{\mathrm{Re}\, z > \}$, which converge uniformly to $D$ on compact subsets of $\dot{H}$. Suppose that for all $\underline{z} \in \dot{H}$ there is a neighborhood $\omega$ of $\underline{z}$, a sequence of holomorphic functions $\ell_n$ on $\omega$ for $n \geq n_0$, which converge to $\ell$, and a constant $C > 1$ such that*

$$\forall z \in \omega, \forall n \geq n_0, \quad \frac{1}{C}|\ell_n(z)| \leq D_n(z) \leq C|\ell_n(z)| \tag{8.3.6}$$

*and $\ell_n \to \ell$ Suppose that $D$ is not identically zero. Then, if $D$ vanishes at $z_0 \in \dot{H}$, there is a sequence $z_n \to z$ such that $D_n(z_n) = 0$.*

*Proof.* **a)** From the lemma above, we know that $D(\cdot)$ cannot vanish identically on any open set since it does not vanish at infinity $\dot{H}$.

**b )** If $D(z) = 0$, then by assumption there are holomorphic functions $\ell_n \to \ell$ on a neighborhood $\omega$ such that the zeros of $D_n$ [resp. $D$] in $\omega$ are the zeros of the $\ell_n$. Since $\ell$ is not identically zero, $z$ is a zero of finite order $m$ and on a possibly smaller neighborhood of $z$, for $n$ large enough, $\ell_n$ has the $m$ zeros, counted with their multiplicities. □

**Lemma 8.3.6.** *Suppose that $D$ is a continuous function on $\mathcal{H} := [0, 1] \times H$ where $H = \{z \in \mathbb{C}, \mathrm{Re}\, z \geq 0\}$. Suppose that for all $(t_0, z_0) \in \mathcal{H}$, there is a neighborhood of $(t_0, z_0)$, a function $\ell$ on this neighborhood, continuous in $t$ and holomorphic in $z$, and a constant $C > 1$ such that*

$$\frac{1}{C}|\ell(t,z)| \leq D(t,z) \leq C|\ell(t,z)|. \tag{8.3.7}$$

*Suppose that there is $R > 0$ such that for all $t \in [0, 1]$, $D(t, z) \neq 0$ when* $\mathrm{Re}\, z = 0$ *and when $|z| \geq R$. Suppose that $D(0, z) \neq 0$ for all $z \in H$. Then if $D(1, \cdot)$ does not vanish on $H$.*

*Proof.* a) We show that $D(t, \cdot)$ cannot vanish identically on any open set. If it would, let $Z$ denote the non empty set of points $z \in \dot{H}$ such that $D(t, \cdot)$ vanishes identically on a neighborhood of $z$. It is open by definition. If $z_n$ is a sequence of points in $Z$ which converge to $z \in \dot{H}$, the assumption implies that on a neighborhood $\omega$ of $z$, the zeros of $D$ are zeros of an holomorphic function $\ell$. In particular, for $n$ large $z_n \in \omega$ and $\ell(z_n) = 0$. Therefore, the zeros of $\ell$ have an accumulation at point, implying that $\ell$ and therefore $D$ must vanish identically on $\omega$. Therefore $Z$ is open and closed and $Z = \dot{H}$, which contradicts the assumption that $D(t, \cdot)$ does not vanish at infinity. .

b) The set $N$ of $(t,z)$ such that $D(t,z) = 0$ is compact in $]0,1] \times \dot{H}$ where $\dot{H} = \{\operatorname{Re} z > 0\}$ is the interior of $H$. If it is not empty, let $t_0 = \min\{t, (t,z) \in N\}$ and let $z_0 \in \dot{H}$ such that $D(t_0, z_0) = 0$. Then $t_0 > 0$.

Let $\ell$ be a function satisfying (8.3.7) on a neighborhood of $(t_0, z_0)$. By a), $\ell(t_0, \cdot)$ it is not identically 0, and therefore it is has a zero of finite order at $z_0$ and therefore does not vanish on the boundary of a small disc containing $z_0$. Hence, by Rouché's theorem, $\ell(t, \cdot)$ has a root in this disc for $t - t_0$ small, which contradicts the definition of $t_0$. $\square$