LOCALLY ANALYTIC VECTORS, ANTICYLOTOMIC EXTENSIONS AND A CONJECTURE OF KEDLAYA

by

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Abstract. — Let K be a finite extension of $\mathbf{Q}_p$ and let $\mathcal{G}_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. Fontaine has constructed a useful classification of p-adic representations of $\mathcal{G}_K$ in terms of cyclotomic (φ, Γ)-modules. Lately, interest has risen around a generalization of the theory of (φ, Γ)-modules, replacing the cyclotomic extension with an arbitrary infinitely ramified p-adic Lie extension. Computations from Berger suggest that locally analytic vectors should provide such a generalization for any arbitrary infinitely ramified p-adic Lie extension, and this has been conjectured by Kedlaya.

In this paper, we focus on the case of $\mathbf{Z}_p$-extensions, using recent work of Berger-Rozensztajn and Porat on an integral version of locally analytic vectors, and prove that Kedlaya's conjecture does not hold for anticyclotomic extensions. This also provide an example of an extension for which there is no overconvergent lift of its field of norms and for which there exist nontrivial higher locally analytic vectors (questions for which almost nothing was known until now).

Contents

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Introduction

Let p be a prime, and let K be a finite extension of $\mathbf{Q}_p$. One of the main ideas to study p-adic representations and $\mathbf{Z}_p$-representations of $\mathcal{G}_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ is to use an intermediate extension $K \subset K_\infty \subset \overline{\mathbf{Q}}_p$ such that K_∞/K is nice enough but still deeply ramified (in the sense of [**CG96**]), so that $\overline{\mathbf{Q}}_p/K_\infty$ is almost étale and "contains almost all the ramification of the extension $\overline{\mathbf{Q}}_p/K$". If K_∞/K is an infinitely ramified p-adic Lie extension then those assumptions are satisfied. Classically, one lets K_∞ be the cyclotomic extension $K(\mu_{p^\infty})$ of K.

One striking result following this idea has been the construction of cyclotomic (φ,Γ)-modules. Fontaine has constructed in [**Fon90**] an equivalence of categories $V \mapsto \mathbf{D}(V)$ between the category of all p-adic representations of $\mathcal{G}_K$ and the category of étale (φ,Γ)-modules. Different versions of cyclotomic (φ,Γ)-modules theory can be defined: one can define them over a 2-dimensional local ring $\mathbf{B}_K$, over a subring $\mathbf{B}_K^\dagger$ of $\mathbf{B}_K$ consisting of so-called overconvergent elements, or over the Robba ring. In every case, a (φ,Γ)-module is a finite free module over the corresponding ring, equipped with semilinear actions of φ and $\Gamma = \mathrm{Gal}(K_{\mathrm{cycl}}/K)$ commuting one to another (the ring itself being equipped with such actions).

Thanks to a theorem of Cherbonnier and Colmez [**CC98**] and a theorem of Kedlaya [**Ked05**], these different theories are equivalent. Moreover, the theories over both $\mathbf{B}_K^\dagger$ and $\mathbf{B}_K$ come with their integral counterparts, so that free $\mathbf{Z}_p$-representations of $\mathcal{G}_K$ are equivalent to étale (φ,Γ)-modules over some integral subring of either $\mathbf{B}_K^\dagger$ or $\mathbf{B}_K$.

Lately, there has been an increasing interest in generalizing both (φ,Γ)-modules theory [**Ber14, Car13, KR09**] and more generally in understanding how to replace the cyclotomic extension by an arbitrary infinitely ramified p-adic Lie extension in p-adic Hodge theory [**BC16, Poy22b**].

One could try to define (φ,Γ)-modules attached to an almost totally ramified p-adic Lie extension by copying the constructions in the cyclotomic case. This strategy relies on finding a "lift of the field of norms" and happens to work in the Lubin-Tate setting [**KR09**]. Under some strong assumptions (which are not always met even in the cyclotomic case), namely that the lift is of "finite height", Berger showed in [**Ber14**] that there were some restrictions on the kind of extensions one could consider in this case (and proving for example that there was no finite height lift of the field of norms in the anticyclotomic setting). The author proved that, under the same strong assumptions, the only extensions for which one could lift the field of norms were actually only the Lubin-Tate ones [**Poy22a**]. A more natural and less constraining assumption would be to ask for which extensions one could have an overconvergent lift, but in this case almost nothing is known.

An other idea to generalize (φ,Γ)-modules theory, and which has been used with success by Berger and Colmez [**BC16**] to generalize Sen theory, has been to use the theory of locally analytic vectors, initially introduced by Schneider and Teitelbaum [**ST03**]. Berger and Colmez have shown that Sen theory could be completely generalized to any arbitrary infinitely ramified p-adic Lie extension by using locally analytic vectors. Computations from Berger [**Ber16**] showed that locally analytic vectors in the cyclotomic setting recovered the cyclotomic (φ,Γ)-modules over the Robba ring, and suggested that the theory of locally analytic vectors should be able to define a theory of (φ, Γ)-modules for any arbitrary infinitely ramified p-adic Lie extension. In [**Ked13**], Kedlaya conjectured that indeed, locally analytic vectors should provide a nice (φ, Γ)-module theory for any such p-adic Lie extension, and that the theory should even be defined at an integral level.

Up until recently, locally analytic vectors were only defined in a setting in which p is inverted, so it was difficult to use them in an integral setting (and even more in characteristic p). One could say that an element x in a free $\mathbf{Z}_p$-algebra was locally analytic if it became locally analytic after inverting p, which is what Kedlaya does in the statement of his conjecture, but this definition is not very practical and does not extend for characteristic p algebras.

Recently, Berger-Rozensztajn [**BR22a, BR22b**], Gulotta [**Gul19**], Johansson and Newton [**JN19**] and Porat [**Por24**] have generalized the classical notion of locally analytic vectors (denoted by "Super-Hölder vectors" in the works of Berger and Rozensztajn) to a characteristic p and integral setting, by using classical tools of p-adic analysis like Mahler expansions. In [**Por24**], Porat has proven that these new integral locally analytic vectors can be used to recover cyclotomic (φ, Γ)-modules, thus generalizing the computations of Berger [**Ber16**] to an integral setting. This makes it possible to reinterpret Kedlaya's conjecture in terms of those new integral locally analytic vectors.

In this paper, we focus on the particular case of $\mathbf{Z}_p$-extensions, of which both the cyclotomic and the anticyclotomic extensions are a particular case, and try to give a description on what the locally analytic vectors in the rings used to define (φ, Γ)-modules are. We thus let K_∞/K be a totally ramified $\mathbf{Z}_p$-extension, and we look at the structure of the ring $(\widetilde{\mathbf{A}}^\dagger)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty),\mathrm{Gal}(K_\infty/K)-\mathrm{la}}$, which we write $(\widetilde{\mathbf{A}}_K^\dagger)^{\mathrm{la}}$ in what follows.

Our first result is that only two very different situations may occur:

1. either there is no nontrivial locally analytic vectors in $\widetilde{\mathbf{A}}_K^\dagger$, that is $(\widetilde{\mathbf{A}}_K^\dagger)^{\mathrm{la}} = \mathcal{O}_K$;
2. or $(\widetilde{\mathbf{A}}_K^\dagger)^{\mathrm{la}} = \varphi^{-\infty}(\mathbf{A}_K^\dagger)$, where $\mathbf{A}_K^\dagger$ is a ring of overconvergent functions in one variable.

In the second case, we prove in the meantime that everything behaves as if K_∞/K was the cyclotomic extension of K. In particular, we obtain an overconvergent lift of the field of norms. Moreover, we also obtain a description of the rings $(\widetilde{\mathbf{B}}_K^I)^{\mathrm{la}}$ which matches the specialization to the cyclotomic setting of theorem 4.4 of [**Ber16**].

Of course, if one believes in Kedlaya's conjecture, then the first situation above should never arise. In the cyclotomic setting, we exhibit a locally analytic vector of a ring of the form $(\widetilde{\mathbf{B}}_K^I)^{\mathrm{la}}$ (with I well chosen) which can't be written as a power series in the variable defining the ring $\mathbf{A}_K^\dagger$, so that the first situation does arise for anticyclotomic extensions. As a consequence, we obtain the following theorem:

Theorem 0.1. — *Kedlaya's conjecture does not hold for anticyclotomic extensions.*

Moreover, still for $\mathbf{Z}_p$-extensions, we are also able to relate the existence of an overconvergent lift of the field of norms with the existence of nontrivial locally analytic vectors, and we thus obtain the following theorem:

Theorem 0.2. — *If K_∞/K is a $\mathbf{Z}_p$-extension, then there exists an overconvergent lift of the field of norms of K_∞/K if and only if there exists a nontrivial locally analytic* *vector in* $\widetilde{\mathbf{A}}_K^{\dagger}$. *In particular, there is no overconvergent lift of the field of norms in the anticyclotomic case.*

Finally, still using these results coming from the anticyclotomic setting, we are able to "exhibit" nontrivial higher locally analytic vectors in $R^1_{\mathrm{la}}(\widetilde{\mathbf{A}}_K^{\dagger})$, which is (as far as the author knows) the first case where this is done.

Notations

For the rest of the paper, we fix a prime p and we let K be a finite extension of $\mathbf{Q}_p$, with residue field k_K of cardinal $q = p^f$, and ramification index e. We let $\mathcal{O}_K$ denote the ring of integers of K, and we let π be a uniformizer of $\mathcal{O}_K$.

1. Lubin-Tate and anticylotomic extensions

Let LT be a Lubin-Tate formal $\mathcal{O}_K$-module attached to the uniformizer π of $\mathcal{O}_K$. For $a \in \mathcal{O}_K$, we let $[a](T)$ denote the power series giving the multiplication by a map on LT. Let T be a local coordinate on LT such that $[\pi](T) = T^q + \pi T$, except in the particular case where $K = \mathbf{Q}_p$ and $\pi = p$, where we choose instead a local coordinate T such that $[p](T) = (1+T)^p - 1$. We let $K_n = K(\mathrm{LT}[\pi^n])$ be the extension of K generated by the π^n-torsion points of LT, and we let $K_{\mathrm{LT}} = \cup_{n \geq 1} K_n$. We let $\Gamma_{\mathrm{LT}} = \mathrm{Gal}(K_{\mathrm{LT}}/K)$ and $H_{\mathrm{LT}} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_{\mathrm{LT}})$. By Lubin-Tate theory (see [**LT65**]), if $g \in \Gamma_{\mathrm{LT}}$ then there exists a unique $a_g \in \mathcal{O}_K^{\times}$ such that g acts on the torsion points of LT through the power series $[a_g](T)$, and the map $\chi_\pi : g \in \Gamma_{\mathrm{LT}} \mapsto a_g \in \mathcal{O}_K^{\times}$ is a group isomorphism called the Lubin-Tate character attached to π.

For $n \geq 1$, we let $\Gamma_n = \mathrm{Gal}(\mathrm{LT}/K_n)$ so that $\Gamma_n = \{g \in \Gamma_{\mathrm{LT}}, \chi_\pi(g) \in 1 + \pi^n \mathcal{O}_K\}$. We let $u_0 = 0$ and for $n \geq 1$ we let $u_n \in \overline{\mathbf{Q}}_p$ be such that $[\pi](u_n) = u_{n-1}$, with $u_1 \neq 0$. We have $K_n = K(u_n)$, and u_n is a uniformizer of K_n. We also let $Q_n(T)$ be the minimal polynomial of u_n over K, so that $Q_0(T) = T$, $Q_1(T) = [\pi](T)/T$ and $Q_{n+1} = Q_n([\pi](T))$ if $n \geq 1$.

We let $\log_{\mathrm{LT}} = T + O(\deg \geq 2) \in K[\![T]\!]$ denote the Lubin-Tate logarithm map, which converges on the open unit disk and is such that $\log_{\mathrm{LT}}([a](T)) = a \cdot \log_{\mathrm{LT}}(T)$ for $a \in \mathcal{O}_K$. We recall that $\log_{\mathrm{LT}}(T) = T \cdot \prod_{k \geq 1} Q_k(T)/\pi$, and we let $\exp_{\mathrm{LT}}$ denote the inverse of $\log_{\mathrm{LT}}$.

When $K = \mathbf{Q}_{p^2}$, the unramified extension of $\mathbf{Q}_p$ of degree 2, and $\pi = p$, then K_{LT} contains two special and particularly interesting sub-$\mathbf{Z}_p$-extensions: the cyclotomic extension $K_{\mathrm{cycl}} = K(\mu_{p^\infty})$ of K, which is defined, Galois and abelian over $\mathbf{Q}_p$, and the anticyclotomic extension K_{ac} which is the unique $\mathbf{Z}_p$-extension of K, defined, Galois and pro-dihedral over $\mathbf{Q}_p$: the Frobenius σ of $\mathrm{Gal}(K/\mathbf{Q}_p)$ acts on $\mathrm{Gal}(K_{\mathrm{ac}}/K)$ by inversion. It is linearly disjoint from K_{cycl} over K, and the compositum $K_{\mathrm{cycl}} \cdot K_{\mathrm{ac}}$ is equal to K_{LT}. If we let χ_p denote the Lubin-Tate character corresponding to K_{LT}, then $\chi_{\mathrm{cycl}} = N_{K/\mathbf{Q}_p}(\chi_p) = \sigma(\chi_p) \cdot \chi_p$. One defines an anticyclomic character $\chi_{\mathrm{ac}} : \mathrm{Gal}(K_{\mathrm{ac}}/K) \to \mathcal{O}_K^{\times}$ by $g \mapsto \frac{\chi_p(g)}{\sigma(\chi_p(g))}$ which is an isomorphism on to its image, and the anticyclotomic extension is the subfield of K_{LT} fixed by the elements $g \in \Gamma_{\mathrm{LT}}$ such that $\chi_{\mathrm{ac}}(g) = 1$.

If $K/\mathbf{Q}_p$ is of degree 2, and if π is a uniformizer of $\mathcal{O}_K$, then K_{LT} contains an unramified twist of the cyclotomic extension of K, denoted by K_∞^η, and we have an identification between $\mathrm{Gal}(K_\infty^\eta/K)$ and an open subgroup of $\mathbf{Z}_p^\times$, given by $g \mapsto \eta\chi_{\mathrm{cycl}}(g)$. Moreover, if $\sigma \in Gal(K/\mathbf{Q}_p)$ denote the nontrivial automorphism of K, then $\eta\chi_{\mathrm{cycl}} = N_{K/\mathbf{Q}_p}(\chi_\pi(g)) = \chi_\pi(g)\sigma(\chi_\pi(g))$. By analogy with the classical anticyclotomic case, we define an "anticyclotomic" character $\chi_{\mathrm{LT,ac}} : \Gamma_{\mathrm{LT}} \to \mathcal{O}_K^\times$ by $g \mapsto \frac{\chi_\pi(g)}{\sigma(\chi_\pi(g))}$ and we define $K_{\mathrm{LT,ac}}$, the "anticyclotomic extension of K in K_{LT} by $K_{\mathrm{LT,ac}} = K_{\mathrm{LT}}^{\chi_{\mathrm{LT,ac}}=1}$. This is a $\mathbf{Z}_p$-extension, linearly disjoint from K_∞^η over K.

2. Locally analytic and super-Hölder vectors

In this section, we recall the classical notion of locally analytic vectors, following [**Eme17**] and [**Ber16**, §2], along with the notion of locally analytic vectors for $\mathbf{Z}_p$-Tate algebras as introduced by Porat [**Por24**].

Let G be a p-adic Lie group, and let G_0 be an open subgroup of G which is a uniform pro-p-group (see §4 of [**DDSMS03**] for the definition of a uniform prop-p-group and Interlude A of ibid for the statement). The main interest of such a subgroup G_0 is that it provides a nice specific fundamental system of open neighborhoods of G, along with coordinates $\mathbf{c} : G_0 \to \mathbf{Z}_p^d$, where d is the dimension of G as a p-adic Lie group. Namely, if we let $G_i = \{g^{p^i}, g \in G_0\}$ then we have the following properties (see §4 of [**DDSMS03**] for the proof):

1. for $i \geq 0$, G_i is an open normal uniform subgroup of G_0;
2. $[G_i : G_{i+1}] = p^d$;
3. there is a coordinate $\mathbf{c} : G_0 \to \mathbf{Z}_p^d$ such that for $i \geq 0$, $\mathbf{c}(G_i) = (p^i\mathbf{Z}_p)^d$;
4. For $g, h \in G_0$, we have $gh^{-1} \in G_i$ if and only if $\mathbf{c}(g) - \mathbf{c}(h) \in (p^i\mathbf{Z}_p)^d$.

In the rest of this article, if G is a p-adic Lie group then we assume that we also have chosen such a subgroup G_0, along with coordinates $\mathbf{c}$ and the $(G_i)_{i\geq 0}$ as a fundamental system of open neighborhoods of G.

Let H be an open subgroup of G which is uniform pro-p, with coordinate $\mathbf{c} : H \to \mathbf{Z}_p^d$. Let W be a $\mathbf{Q}_p$-Banach representation of G. We say that $w \in W$ is an H-analytic vector if there exists a sequence $\{w_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^d}$ such that $w_{\mathbf{k}} \to 0$ in W and such that $g(w) = \sum_{\mathbf{k}\in\mathbf{N}^d} \mathbf{c}(g)^{\mathbf{k}} w_{\mathbf{k}}$ for all $g \in H$. We let $W^{H-\mathrm{an}}$ be the space of H-analytic vectors. This space injects into $\mathcal{C}^{\mathrm{an}}(H, W)$, the space of all analytic functions $f : H \to W$. Note that $\mathcal{C}^{\mathrm{an}}(H, W)$ is a Banach space equipped with its usual Banach norm, so that we can endow $W^{H-\mathrm{an}}$ with the induced norm, that we will denote by $||\cdot||_H$. With this definition, we have $||w||_H = \sup_{\mathbf{k}\in\mathbf{N}^d} ||w_{\mathbf{k}}||$ and $(W^{H-\mathrm{an}}, ||\cdot||_H)$ is a Banach space.

The space $\mathcal{C}^{\mathrm{an}}(H, W)$ is endowed with an action of $H \times H \times H$, given by

$$((g_1, g_2, g_3) \cdot f)(g) = g_1 \cdot f(g_2^{-1} g g_3)$$

and one can recover $W^{H-\mathrm{an}}$ as the closed subspace of $\mathcal{C}^{\mathrm{an}}(H, W)$ of its $\Delta_{1,2}(H)$-invariants, where $\Delta_{1,2} : H \to H \times H \times H$ denotes the map $g \mapsto (g, g, 1)$ (see [**Eme17**, §3.3] for more details).

We say that a vector w of W is locally analytic if there exists an open subgroup H as above such that $w \in W^{H-\mathrm{an}}$. Let W^{la} be the space of such vectors, so that $W^{\mathrm{la}} = \varinjlim_H W^{H-\mathrm{an}}$, where H runs through a sequence of open subgroups of G. The space W^{la} is naturally endowed with the inductive limit topology, so that it is an LB space.

Let W be a Fréchet space whose topology is defined by a sequence $\{p_i\}_{i\geq 1}$ of seminorms. Let W_i be the Hausdorff completion of W at p_i, so that $W = \varprojlim_{i\geq 1} W_i$. The space W^{la} can be defined but as stated in [**Ber16**] and as showed in §7 of [**Poy22b**], this space is too small in general for what we are interested in, and so we make the following definition, following [**Ber16**, Def. 2.3]:

Definition 2.1. — If $W = \varprojlim_{i\geq 1} W_i$ is a Fréchet representation of G, then we say that a vector $w \in W$ is pro-analytic if its image $\pi_i(w)$ in W_i is locally analytic for all i. We let W^{pa} denote the set of all pro-analytic vectors of W.

We extend the definition of W^{la} and W^{pa} for LB and LF spaces respectively.

Because the classical definition of locally analytic vectors involves denominators in p, it may seem difficult to generalize this notion for $\mathbf{Z}_p$-algebras where p is not invertible (and may even be 0). The main idea to generalize the classical notion of locally analytic vectors to this setting is (as often in p-adic analysis) to replace Taylor expansions with Mahler expansions, using binomial coefficients. This is explained and used in [**BR22a**] and [**Por24**]. Following those two papers, we place ourselves in the following setting: R is a $\mathbf{Z}_p$-algebra, which is a Tate ring endowed with a valuation $\mathrm{val}_R : R \to (-\infty, \infty]$ satisfying the following properties:

1. $\mathrm{val}_R(x) = \infty$ if and only if $x = 0$ (meaning that R is separated for the topology induced by val_R);
2. $\mathrm{val}_R(xy) \geq \mathrm{val}_R(x) + \mathrm{val}_R(y)$ for all $x, y \in R$;
3. $\mathrm{val}_R(x+y) \geq \inf(\mathrm{val}_R(x), \mathrm{val}_R(y))$ for all $x, y \in R$;
4. $\mathrm{val}_R(p) > 0$.

We extend this definition to R-modules.

In what follows, G is a uniform pro-p-group. For an R-module M, endowed with a compatible valuation val_M, we write $\mathcal{C}^0(G, M)$ for the set of continuous functions from G to M.

Following [**Por24**], we make the following definition:

Definition 2.2. —
1. Let $\lambda, \mu \in \mathbf{R}$. We let $\mathcal{C}^{\mathrm{an}-\lambda,\mu}(G, M)$ denote the set of functions $G \to M$ such that $\mathrm{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^i + \mu$ whenever $gh^{-1} \in G_i$ for all $g, h \in G$ and $i \geq 0$. Note that it is contained in $\mathcal{C}^0(G, M)$ (see §2 of [**Por24**]).
2. We let $\mathcal{C}^{\mathrm{an}-\lambda}(G, M)$ denote the set of functions $f : G \to M$ such that there exists $\mu \in \mathbf{R}$ such that $f \in \mathcal{C}^{\mathrm{an}-\lambda,\mu}(G, M)$.
3. We let $\mathcal{C}^{\mathrm{la}}(G, M)$ be the colimit of the cofinal system $\{\mathcal{C}^{\mathrm{an}-\lambda,\mu}(G, M)\}_{\lambda,\mu}$, or equivalently, of the cofinal system $\{\mathcal{C}^{\mathrm{an}-\lambda}(G, M)\}_{\lambda}$.

We refer the reader to §2 of [**Por24**] to see different characterization of those sets of functions.

We now assume that G is a uniform pro-p-group, acting on M by isometries. As in the Banach-space setting, the space $\mathcal{C}^0(G, M)$ is endowed with an action of $G \times G \times G$, given by

$$((g_1, g_2, g_3) \cdot f)(g) = g_1 \cdot f(g_2^{-1} g g_3)$$

and we define $M^{G,\mathrm{la}}$ (resp. $M^{G,\lambda-\mathrm{an}}$ resp. $M^{G,\lambda-\mathrm{an},\mu}$) as the subspace of $\mathcal{C}^{\mathrm{la}}(G, M)$ (resp. $\mathcal{C}(G, M)^{G,\lambda-\mathrm{an}}$ resp. $\mathcal{C}(G, M)^{G,\lambda-\mathrm{an},\mu}$) of its $\Delta_{1,2}(G)$-invariants, where $\Delta_{1,2} : G \to G \times G \times G$ denotes the map $g \mapsto (g, g, 1)$.

We define the locally analytic vectors of M as the elements of

$$M^{\mathrm{la}} := \varinjlim_i M^{G_i-\mathrm{la}}.$$

As explained in Example 2.1.3 of [**Por24**], when $R = \mathbf{Q}_p$, M is a $\mathbf{Q}_p$-Banach space and we recover the classical locally analytic vectors. We can actually give a more precise statement. Let $\mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$ be the space of functions $f : \mathbf{Z}_p \to \mathbf{Q}_p$ whose restriction to any ball of the form $a + p^h\mathbf{Z}_p$ is the restriction of an analytic function $f_{a,h}$. This is a Banach space with the obvious norm. If W is a $\mathbf{Q}_p$-Banach space we define $\mathrm{LA}_h(\mathbf{Z}_p, W) := W \widehat{\otimes}_{\mathbf{Q}_p} \mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$. Theorem 3 of [**Ami64**] and theorem I.4.7 of [**Col10**] have the following corollary:

Corollary 2.3. — *If $f \in \mathcal{C}^0(\mathbf{Z}_p, \mathbf{Q}_p)$, the following are equivalent:*
- *$f \in \mathrm{LA}_h(\mathbf{Z}_p, \mathbf{Q}_p)$;*
- *$f \in \mathcal{C}^{\mathrm{an}-\lambda}(\mathbf{Z}_p, \mathbf{Q}_p)$ for all $\lambda > -h - \frac{\log(p-1)}{\log(p)}$.*

Proof. — See the proof of [**Col10**, Coro. I.4.8] and proposition 1.14 of [**BR22a**] □

In particular, for all $n \geq 0$, there exist $\lambda \in \mathbf{R}$, depending only on n, such that if M be a $\mathbf{Q}_p$-Banach space on which G acts by isometry, then $x \in M^{G_0-\mathrm{an},\lambda}$ if and only if $x \in M^{G_n-\mathrm{an}}$ (in the sense of the classical definition).

Finally, one may define higher locally analytic vectors, coming from the derived functor induced by $M \mapsto M^{\mathrm{la}}$. Once again, we follow Porat [**Por24**, §2.3] by setting

$$R^i_{\mathrm{la}}(M) := \varinjlim_j H^i(G_j, \mathcal{C}^{\mathrm{la}}(G_j, M)),$$

where the cocycles considered are continuous, and we take the inductive topology on $\mathcal{C}^{\mathrm{la}}(G_j, M)$ induced from that of its submodules $\mathcal{C}^{\lambda-\mathrm{an}}(G_j, M)$. These groups form what we call the higher locally analytic vectors of M, and if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence (in the appropriate category) then we have a long exact sequence

$$0 \to M_1^{\mathrm{la}} \to M_2^{\mathrm{la}} \to M_3^{\mathrm{la}} \to R^1_{\mathrm{la}}(M_1) \to \cdots$$

We finish this section with some properties needed in the rest of the paper.

Lemma 2.4. — *If $x, y \in R^{G,\lambda-\mathrm{an}}$ then $xy \in R^{G,\lambda-\mathrm{an}}$.*

Proof. — See item 1 of [**BR22a**, Prop. 1.11]. □

Lemma 2.5. — *If $x \in R^{G,\lambda-\mathrm{an}} \cap R^{\times}$, then $1/x \in R^{G,\lambda-\mathrm{an}}$.*

Proof. — See item 2 of [**BR22a**, Prop. 1.11]. □

3. Locally analytic vectors for classical rings of periods

In this section we quickly recall the definition of some classical rings of periods, and then recall several results regarding the locally analytic vectors attached to p-adic Lie extensions (and especially in the cyclotomic and Lubin-Tate cases) in those rings. We also explain how the normalization of the valuation may affect the "radius of analyticity" of the elements considered.

3.1. Some rings of p-adic periods. — In this section, we recall the definition of some rings of p-adic periods, defined in [**Fon90, Fon94**], [**Ber02**] and [**Col02**]. We also recall the definitions of some rings of periods attached to Lubin-Tate extensions, which can be specialized to recover the rings appearing in the cyclotomic setting.

We let $\widetilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^q} \mathcal{O}_{\mathbf{C}_p}/\pi$ be the tilt of $\mathcal{O}_{\mathbf{C}_p}$. It is a perfect ring of characteristic p which is equipped with a valuation $v_{\mathbf{E}}$ coming from the one of $\mathbf{C}_p$, and is complete for this valuation. We let $\widetilde{\mathbf{E}}$ denote the fraction field of $\widetilde{\mathbf{E}}^+$. If F is a subfield of $\mathbf{C}_p$, let $\mathfrak{a}_F^c$ be the set of elements x of F such that $v_K(x) \geq c$, and for any $c > 0$ we identify $\widetilde{\mathbf{E}}^+$ with $\varprojlim_{x \mapsto x^q} \mathcal{O}_{\mathbf{C}_p}/\mathfrak{a}^c_{\mathbf{C}_p}$.

If $\{u_n\}_{n \geq 0}$ are as in §1, then the sequence $\overline{u} := (\overline{u_0}, \overline{u_1}, \cdots) \in (\mathcal{O}_{\mathbf{C}_p}/\pi)^{\mathbf{N}}$ belongs to $\widetilde{\mathbf{E}}^+$, and we have $v_{\mathbf{E}}(\overline{u}) = q/(q-1)e$.

We let $\widetilde{\mathbf{A}}^+ = \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} W(\widetilde{\mathbf{E}}^+)$, and $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/\pi]$. We also let $\widetilde{\mathbf{A}} = \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} W(\widetilde{\mathbf{E}})$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/\pi]$. We write $[\cdot]$ for the Teichmüller map. We endow these rings with the Frobenius map $\varphi_q = \mathrm{id} \otimes \varphi^h$.

By §9.2 of [**Col02**], there exists $u \in \widetilde{\mathbf{A}}^+$, whose image is $\overline{u}$, and such that $\varphi_q(u) = [\pi](u)$ and $g(u) = [\chi_\pi(g)](u)$ if $g \in \Gamma_K$. If $K = \mathbf{Q}_p$ and $\pi = p$, then $u = [\varepsilon] - 1$, where $\varepsilon \in \widetilde{\mathbf{E}}^+$ is a compatible sequence of q^n-th roots of 1. We let $Q_k = Q_k(u) \in \widetilde{\mathbf{A}}^+$.

Recall that we have a map $\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p}$ which is a ring homomorphism, whose kernel is a principal ideal generated by $\varphi_q^{-1}(Q_1)$ or by $[\widetilde{\pi}] - [\pi]$ (see proposition 8.3 of [**Col02**]), where $\widetilde{\pi} \in \widetilde{\mathbf{E}}^+$ is a compatible sequence of q^n-th roots of π. In particular, $\varphi_q^{-1}(Q_1)/([\widetilde{\pi}]-\pi)$ is a unit of $\widetilde{\mathbf{A}}^+$ and so are the elements $Q_k/([\widetilde{\pi}]^{q^k} - \pi)$ for all $k \geq 1$.

Every element of $\widetilde{\mathbf{B}}^+[1/[\overline{u}]]$ can be written as $\sum_{k \gg -\infty} \pi^k [x_k]$, where $(x_k)_{k \in \mathbf{Z}}$ is a bounded sequence of $\widetilde{\mathbf{E}}$. For $r > 0$, we define a valuation $V(\cdot, r)$ on $\widetilde{\mathbf{B}}^+[1/[\overline{u}]]$ by the formula

$$V(x,r) = \inf_{k \in \mathbf{Z}} \left(\frac{k}{e} + \frac{p-1}{pr} v_{\mathbf{E}}(x_k) \right) \text{ if } x = \sum_{k \gg -\infty} \pi^k [x_k].$$

If I is a closed subinterval of $[0, +\infty[$, $I \neq [0,0]$, we let $V(x, I) = \inf_{r \in I, r \neq 0} V(x, r)$ (one can take a look at remark 2.1.9 of [**GP19**] to understand why we avoid defining $V(\cdot, 0)$). We define $\widetilde{\mathbf{B}}^I$ as the completion of $\widetilde{\mathbf{B}}^+[1/[\overline{u}]]$ for $V(\cdot, I)$ if $0 \notin I$. If $0 \in I$, we let $\widetilde{\mathbf{B}}^I$ be the completion of $\widetilde{\mathbf{B}}^+$ for $V(\cdot, I)$. We let $\widetilde{\mathbf{A}}^I$ be the ring of integers of $\widetilde{\mathbf{B}}^I$ for $V(\cdot, I)$. By §2 of [**Ber02**], we have that $\widetilde{\mathbf{A}}^{[r,s]}$ is also the p-adic completion of $\widetilde{\mathbf{A}}^+[\frac{p}{[\overline{u}]^r}, \frac{[\overline{u}]^s}{p}]$.

For $k \geq 1$, we let $r_k = p^{kh-1}(p-1)$. The map $\theta \circ \varphi_q^{-k} : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p}$ extends by continuity to $\widetilde{\mathbf{A}}^I$, provided that $r_k \in I$, in which case we have that $\theta \circ \varphi_q^{-k}(\widetilde{\mathbf{A}}^I) \subset \mathcal{O}_{\mathbf{C}_p}$.

For $r > 0$, we define $\widetilde{\mathbf{B}}^{\dagger,r}$ the subset of overconvergent elements of "radius" r of $\widetilde{\mathbf{B}}$, by

$$\widetilde{\mathbf{B}}^{\dagger,r} = \left\{ x = \sum_{k \gg -\infty} \pi^k [x_k] \text{ such that } \lim_{k \to +\infty} v_{\mathbf{E}}(x_k) + \frac{pr}{(p-1)e} k = +\infty \right\}.$$

Note that $\widetilde{\mathbf{B}}^{\dagger,r}$ can naturally be identified with a subring of $\widetilde{\mathbf{B}}^{[r,r]}$ and we endow it with the valuation $V(\cdot, r)$. We let

$$\widetilde{\mathbf{A}}^{\dagger,r} = \left\{ x = \sum_{k \geq 0} \pi^k [x_k] \in \widetilde{\mathbf{A}} \cap \widetilde{\mathbf{B}}^{\dagger,r} \text{ such that } \forall k \geq 0, v_{\mathbf{E}}(x_k) + \frac{pr}{(p-1)e} k \geq 0 \right\}$$

and we also endow it with the valuation $V(\cdot, r)$. Note that $\widetilde{\mathbf{A}}^{\dagger,r}$ is also the p-adic completion of $\widetilde{\mathbf{A}}^+[\frac{p}{[\overline{u}]^r}]$. If $\rho = \frac{r_0}{r}$, we let $\widetilde{\mathbf{A}}^{(0,\rho]} := \widetilde{\mathbf{A}}^{\dagger,r}[1/[\overline{u}]]$. We endow $\widetilde{\mathbf{A}}^{(0,\rho]}$ with the valuation v_ρ given by the $[\overline{u}]$-adic valuation, so that $\rho v_\rho = V(\cdot, r)$ and $v_\rho = \frac{r}{r_0} V(\cdot, r)$. Note that $\widetilde{\mathbf{A}}^{\dagger,r}$ is the ring of integers of $\widetilde{\mathbf{A}}^{(0,\rho]}$ for v_ρ and also for $V(\cdot, r)$. Moreover, for any $\rho > 0$, we have $\widetilde{\mathbf{A}}^{(0,\rho]}/(\pi) = \widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{A}}_K^{(0,\rho]}/(\pi) = \widetilde{\mathbf{E}}_K$.

We let $\widetilde{\mathbf{B}}^\dagger := \cup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}$ and $\widetilde{\mathbf{A}}^\dagger = \cup_{\rho>0} \widetilde{\mathbf{A}}^{(0,\rho]}$.

For $\rho > 0$, let $\rho' = \rho \cdot e \cdot p/(p-1) \cdot (q-1)/q$. Note that we have $V(u^i, r) = i/r'$. Let I be a subinterval of $[0, +\infty[$ which is either a subinterval of $]1, +\infty[$ or such that $0 \in I$. Let $f(Y) = \sum_{k \in \mathbf{Z}} a_k Y^k$ be a power series with $a_k \in K$ and such that $v_p(a_k) + k/\rho' \to +\infty$ when $|k| \to +\infty$ for all $\rho \in I$. The series $f(v)$ converges in $\widetilde{\mathbf{B}}^I$ and we let $\mathbf{B}_K^I$ denote the set of all $f(v)$ with f as above. It is a subring of $\widetilde{\mathbf{B}}_K^I = (\widetilde{\mathbf{B}}^I)^{H_K}$ which is stable under the action of Γ_K. The Frobenius map gives rise to a map $\varphi_q : \mathbf{B}_K^I \to \mathbf{B}_K^{qI}$.

We shall write $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ for $\widetilde{\mathbf{B}}^{[r,+\infty[}$ and $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ for $\mathbf{B}_K^{[r;+\infty[}$. We let $\mathbf{B}_K^{\dagger,r}$ denote the set of $f(u) \in \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ such that the sequence $\{a_k\}_{k \in \mathbf{Z}}$ is bounded. This is a subring of $\widetilde{\mathbf{B}}_K^{\dagger,r} = (\widetilde{\mathbf{B}}^{\dagger,r})^{H_K}$. We also define $\mathbf{A}_K^{\dagger,r} = \mathbf{B}_K^{\dagger,r} \cap \widetilde{\mathbf{A}}^{\dagger,r}$.

Lemma 3.1. — *An element $x = \sum_{k \geq 0} \pi^k [x_k] \in \widetilde{\mathbf{A}}^{\dagger,r}$ is a unit of $\widetilde{\mathbf{A}}^{\dagger,r}$ if and only if $v_{\mathbf{E}}(x_0) = 0$. Moreover, if $x \in \widetilde{\mathbf{A}}^\dagger$ is such that $v_{\mathbf{E}}(x_0) \geq 0$ then :*

1. *there exists $r > 0$ such that $x \in \widetilde{\mathbf{A}}^{\dagger,r}$;*
2. *there exists $s \geq r$ such that $\frac{x}{[x_0]}$ belongs to $\widetilde{\mathbf{A}}^{\dagger,s}$ and is a unit of $\widetilde{\mathbf{A}}^{\dagger,s}$.*

Proof. — The first statement is [**CC98**, Lemm. II.1.4].

For item 1, let us write $x = \sum_{k=0}^{\infty} p^k [x_k]$. Since $x \in \widetilde{\mathbf{A}}^\dagger$, there exists $t > 0$ such that $\frac{k}{e} + \frac{p-1}{pt} v_{\mathbf{E}}(x_k)$ goes to $+\infty$ when $k \to +\infty$, so that the sequence $(\frac{k}{e} + \frac{p-1}{pt} v_{\mathbf{E}}(x_k))$ is bounded below by some constant C. If $C \geq 0$ then $x \in \widetilde{\mathbf{A}}^{\dagger,r}$ so the first item is satisfied. Otherwise, it is bounded by $-D$ for some $D > 0$. Then if $s \geq t \cdot (eD+1)$, we have $\frac{k}{e} + \frac{p-1}{ps} v_{\mathbf{E}}(x_k) \geq 0$ for $k \geq 1$, and since $v_{\mathbf{E}}(x_0) \geq 0$, this means that $V(x, s) \geq 0$.

For item 2, one uses item 1 to find $s \geq r$ such that $\frac{x}{[x_0]}$ belongs to $\widetilde{\mathbf{A}}^{\dagger,s}$, and then this element is a unit of $\widetilde{\mathbf{A}}^{\dagger,s}$ by the first statement of the lemma. □

Lemma 3.2. — *Let $k \geq 0$. The kernel of $\theta : \widetilde{\mathbf{A}}^{\dagger,r_k} \to \mathcal{O}_{\mathbf{C}_p}$ is a principal ideal generated by Q_k if $k \geq 1$ and by $\varphi_q^{-1}(Q_1)$ if $k = 0$.*

Proof. — Up to composing by a suitable power of φ_q, it suffices to prove the case for $k = 0$. In this case, we know that $\varphi_q^{-1}(Q_1)/\pi$ generates the kernel of the map $\theta : \widetilde{\mathbf{A}}^{[r_0,r_0]} \to \mathcal{O}_{\mathbf{C}_p}$. This implies that $\varphi^{-1}(Q_1)$ generates the kernel of $\theta : \widetilde{\mathbf{B}}^{[r_0,r_0]} \to \mathbf{C}_p$.

Since we have the inclusions $\widetilde{\mathbf{A}}^{\dagger,r_0} \subset \widetilde{\mathbf{B}}^{\dagger,r_0} \subset \widetilde{\mathbf{B}}^{[r_0,r_0]}$, this means that if $x \in \ker(\theta : \widetilde{\mathbf{A}}^{\dagger,r_0} \to \mathcal{O}_{\mathbf{C}_p})$ then $\varphi_q^{-1}(Q_1)$ divides x in $\widetilde{\mathbf{B}}^{[r_0,r_0]}$. Since $\widetilde{\mathbf{B}}^{\dagger}$ is a field, we know that $\varphi_q^{-1}(Q_1)$ divides x in $\widetilde{\mathbf{B}}^{\dagger}$, and thus $\varphi_q^{-1}(Q_1)$ divides x in $\widetilde{\mathbf{B}}^{\dagger,r_0} = \widetilde{\mathbf{B}}^{\dagger} \cap \widetilde{\mathbf{B}}^{[r_0,r_0]}$. This proves that the kernel of $\theta : \widetilde{\mathbf{B}}^{\dagger,r_0} \to \mathbf{C}_p$ is a principal ideal generated by $\varphi_q^{-1}(Q_1)$.

Since $\widetilde{\mathbf{B}}^{\dagger,r_0} = \widetilde{\mathbf{A}}^{\dagger,r_0}[1/\pi]$, this means that $\varphi_q^{-1}(Q_1)$ generates a prime ideal of $\widetilde{\mathbf{A}}^{\dagger,r_0}$: since localizations and quotients commute, $\widetilde{\mathbf{A}}^{\dagger,r_0}/(\varphi_q^{-1}(Q_1))[1/\pi] \simeq \widetilde{\mathbf{B}}^{\dagger,r_0}/(\varphi_q^{-1}(Q_1)) = \mathbf{C}_p$ and thus $\widetilde{\mathbf{A}}^{\dagger,r_0}/(\varphi_q^{-1}(Q_1))$ is an integral domain (since it has no π-torsion).

Now if $x \in \ker(\theta : \widetilde{\mathbf{A}}^{\dagger,r_0} \to \mathcal{O}_{\mathbf{C}_p})$, then there exists $n \geq 0$ and $z \in \widetilde{\mathbf{A}}^{\dagger,r_0}$ such that $\pi^n \cdot x = \varphi_q^{-1}(Q_1) \cdot z$. Since the ideal generated by $\varphi_q^{-1}(Q_1)$ is prime in $\widetilde{\mathbf{A}}^{\dagger,r_0}$ and since $\pi \notin (\varphi_q^{-1}(Q_1)) \cdot \widetilde{\mathbf{A}}^{\dagger,r_0}$, this means that $\varphi_q^{-1}(Q_1)$ divides x in $\widetilde{\mathbf{A}}^{\dagger,r_0}$ and so we are done. □

Lemma 3.3. — *Let $x \in \widetilde{\mathbf{A}}$, whose image modulo π is $\overline{x} = (x_n)_{n \geq 0}$ in $\widetilde{\mathbf{E}}$, and assume that there exists $n \geq 0$ such that $x \in \widetilde{\mathbf{A}}^{\dagger,r_n}$, so that $\overline{x} \in \widetilde{\mathbf{E}}^+$. Then for $m > n$, $\theta \circ \varphi_q^{-m}(x) = x_n$ in $\mathcal{O}_{\mathbf{C}_p}/\mathfrak{a}^c_{\mathbf{C}_p}$ where $c = \frac{q-1}{qe}$.*

Proof. — If $x \in \widetilde{\mathbf{A}}^{\dagger,r_n}$, $x = \sum_{k \geq 0} \pi^k [x_k]$ in $\widetilde{\mathbf{A}}$, then $\theta \circ \varphi_q^{-m}(x)$ is well defined for $m \geq n$ and given by $\theta \circ \varphi_q^{-m}(x) = \sum_{k \geq 0} \pi^k x_k^{(m)}$ (this is a direct consequence of lemma 5.18 of [**Col08**]). But then the fact that $x \in \widetilde{\mathbf{A}}^{\dagger,r_n}$ implies that for $m > n$, the $\pi^k x_k^{(m)}$, $k \neq 0$ have p-adic valuation $\geq \frac{q-1}{qe}$. Thus $\theta \circ \varphi_q^{-m}(x) = x_0^{(m)} \mod \mathfrak{a}^c_{\mathbf{C}_p}$. □

Proposition 3.4. — *Let $k \geq 0$ and let $r = r_k$. If $y \in \widetilde{\mathbf{A}}^{[0,r]} + \pi \cdot \widetilde{\mathbf{A}}^{[r,r]}$ and if $\{y_i\}_{i \geq 0}$ is a sequence of elements of $\widetilde{\mathbf{A}}^{\dagger,r}$ such that $y - \sum_{i=0}^{j-1} y_i \cdot (Q_k/\pi)^i$ belongs to $\ker(\theta \circ \varphi_q^{-k})^j$ for all $j \geq 1$, then there exists $j \geq 1$ such that $y - \sum_{i=0}^{j-1} y_i \cdot (Q_k/\pi)^i \in \pi \cdot \widetilde{\mathbf{A}}^{[r,r]}$.*

Proof. — This is almost the same proposition as proposition 3.3 of [**Ber16**] except that we allow the y_i to belong to $\widetilde{\mathbf{A}}^{\dagger,r}$. Our proof follows the one of ibid almost verbatim.

By lemma 3.1 of ibid. there exist $j \geq 1$ and $a_0, \ldots, a_{j-1}$ elements of $\widetilde{\mathbf{A}}^+$ such that we have

$$y - \left(a_0 + a_1 \cdot (Q_k/\pi) + \cdots + a_{j-1} \cdot (Q_k/\pi)^{j-1}\right) \in \pi \widetilde{\mathbf{A}}^{[r,s]}. \tag{3.1}$$

We have $a_0 \in \widetilde{\mathbf{A}}^+$, $y_0 \in \widetilde{\mathbf{A}}^{\dagger,r}$ so that both belong to $\widetilde{\mathbf{A}}^{\dagger,r}$. By assumption, we have $\theta \circ \varphi_q^{-k}(y_0 - a_0) \in \pi \mathcal{O}_{\mathbf{C}_p}$ so that by lemma 3.2 there exists $c_0, d_0 \in \widetilde{\mathbf{A}}^{\dagger,r}$ such that $a_0 = y_0 + Q_k c_0 + \pi d_0$. This implies that the identity (3.1) holds if we replace a_0 by y_0.

We now assume that $f \leq j-1$ is such that the identity (3.1) holds if we replace each a_i by y_i, for $i \leq f-1$. The element

$$\sum_{i=0}^{j-1} a_i \cdot (Q_k/\pi)^i - \sum_{i=0}^{j-1} y_i \cdot (Q_k/\pi)^i$$

belongs to $\pi\widetilde{\mathbf{A}}^{[r,r]} + (Q_k/\pi)^j\widetilde{\mathbf{A}}^{[r,r]}$. By assumption, the element

$$\sum_{i=f}^{j-1-f} a_i \cdot (Q_k/\pi)^i - \sum_{i=f}^{j-1-f} y_i \cdot (Q_k/\pi)^i$$

belongs to $\pi\widetilde{\mathbf{A}}^{[r,r]} + (Q_k/\pi)^{j-f}\widetilde{\mathbf{A}}^{[r,r]}$ since $\pi\widetilde{\mathbf{A}}^{[r,r]} \cap (Q_k/\pi)^f\widetilde{\mathbf{A}}^{[r,r]} = \pi(Q_k/\pi)^f\widetilde{\mathbf{A}}^{[r,r]}$ by applying enough times item (2) of lemma 3.2 of ibid. Now $a_f \in \widetilde{\mathbf{A}}^+$, $y_f \in \widetilde{\mathbf{A}}^{\dagger,r}$ so that both belong to $\widetilde{\mathbf{A}}^{\dagger,r}$, and we have that $\theta \circ \varphi_q^{-k}(y_f - a_f) \in \pi\mathcal{O}_{\mathbf{C}_p}$ so that by lemma 3.2 there exists $c_f, d_f \in \widetilde{\mathbf{A}}^{\dagger,r}$ such that $a_f = y_f + Q_k c_f + \pi d_f$, so that the identity (3.1) also holds by replacing a_f with y_f.

By induction, this shows that $y - \sum_{i=0}^{j-1} y_i \cdot (Q_k/\pi)^i$ belongs to $\pi\widetilde{\mathbf{A}}^{[r,r]}$. □

3.2. Locally analytic vectors in those rings and a conjecture of Kedlaya. — We now explain the relations between the classical point of view of locally analytic vectors in Banach representations of p-adic Lie groups and the new point of view of locally analytic vectors in mixed characteristic, in the context of the ring $\widetilde{\mathbf{A}}^\dagger$.

In the rest of this subsection, we let K_∞/K be an infinitely ramified p-adic Lie extension, with Galois group Γ_K, a p-adic Lie group of rank d. We also choose coordinates $\mathbf{c}$ along with a nice fundamental system $(\Gamma_n)_{n \geq 1}$ of open neighborhoods of the identity of Γ_K as in §2. If R is a ring endowed with an action of $\mathcal{G}_K$ we write R_K for R^{H_K}.

Note that if $\rho' \leq \rho$, then $v_{\rho'} \geq v_\rho$ by definition. Therefore, if $x \in \widetilde{\mathbf{A}}_K^{(0,\rho]}$ is such that it is λ, μ-analytic for Γ_m, then it is also λ, μ-analytic for Γ_m as an element of $\widetilde{\mathbf{A}}_K^{(0,\rho']}$ for all $\rho' \leq \rho$. It therefore makes sense to define $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_m-\mathrm{an},\lambda,\mu} = \varinjlim_{\rho>0}(\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_m-\mathrm{an},\lambda,\mu}$, and we also define $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_m-\mathrm{an},\lambda}$ and $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}}$ in the same way.

Lemma 3.5. — *We have $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_K-\mathrm{an},\lambda}$ if and only if $\varphi_q(x) \in (\widetilde{\mathbf{A}}_K^{(0,p^{-1}\rho]})^{\Gamma_K-\mathrm{an},f+\lambda}$.*

Proof. — This just follows from the fact that $v_{p^{-1}\rho}(\varphi_q(x)) = qv_\rho(x)$ (which is item (v) of [**Col08**, Prop. 5.4]). □

Lemma 3.6. — *Let $x \in \widetilde{\mathbf{A}}_K^{(0,\rho]}$. Then $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_K-\mathrm{la}}$ if and only if $x \in (\widetilde{\mathbf{B}}^{[r,r]})^{\Gamma_K-\mathrm{la}}$, where $r = r_0/\rho$.*

Proof. — Let $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_K-\mathrm{la}}$. Therefore, there exists $\lambda \in \mathbf{R}$ such that $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_K,\lambda-\mathrm{an}}$. This means that for all $m \geq 0$, $v_\rho(g \cdot x - x) \geq p^\lambda \cdot p^n + \mu$ for all $g \in \Gamma_n$ and for some $\mu \in \mathbf{R}$. Since $v_\rho = \frac{r}{r_0}V(\cdot, r)$, this means that $x \in (\widetilde{\mathbf{B}}^{[r,r]})^{\Gamma_m-\mathrm{an},\lambda'}$, where $\lambda' = \lambda - \alpha$ with α such that $p^\alpha = \frac{r}{r_0}$, and is thus locally analytic as an element of $(\widetilde{\mathbf{B}}^{[r,r]})$ by corollary 2.3.

For the converse, the reasoning is the same: by corollary 2.3, if $x \in \widetilde{\mathbf{A}}_K^{(0,\rho]}$ belongs to $(\widetilde{\mathbf{B}}^{[r,r]})^{\Gamma_K-\mathrm{la}}$ then there exist $\lambda \in \mathbf{R}$ such that $x \in (\widetilde{\mathbf{B}}_K^{[r,r]})^{\Gamma_K,\lambda-\mathrm{an}}$. The relation $v_\rho = \frac{r}{r_0}V(\cdot,r)$ implies that $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_m-an,\lambda'}$ where $\lambda' = \lambda+\alpha$ and so we are done. □

Proposition 3.7. — *Let $x \in \widetilde{\mathbf{A}}_K^\dagger$. Then $x \in (\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}}$ if and only if $x \in (\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger)^{\Gamma_K-\mathrm{pa}}$.*

Proof. — Let $\rho > 0$, $m \geq 0$, and $\lambda, \mu \in \mathbf{R}$ be such that $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_m-\mathrm{an},\lambda,\mu}$. Let $r = r_0/\rho$. By the remark above, if $s \geq r$ and $\rho' = r_0/s$, then $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho']})^{\Gamma_m-\mathrm{an},\lambda,\mu}$, so that by lemma 3.6, there exist $m' \geq m$, λ', $\mu' \in \mathbf{R}$ such that $x \in (\widetilde{\mathbf{B}}_K^{[s,s]})^{\Gamma_m-\mathrm{an},\lambda',\mu'}$. Using the maximum principle (see corollary 2.20 of [**Ber02**]), this implies that $x \in (\widetilde{\mathbf{B}}_K^{[r,s]})^{\Gamma'_m-\mathrm{an},\lambda'',\mu''}$ for $\lambda'' = \max(\lambda,\lambda')$ and $\mu'' = \max(\mu',\mu'')$. Therefore, x belongs to $(\widetilde{\mathbf{B}}_K^{[r,s]})^{\Gamma_K-\mathrm{la}}$ by corollary 2.3. Since this is true for every $s \geq r$, we deduce that $x \in (\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r})^{\Gamma_K-\mathrm{pa}}$.

For the converse, assume that $x \in (\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger)^{\Gamma_K-\mathrm{pa}}$. Then $x \in (\widetilde{\mathbf{B}}_K^{[r,r]})^{\Gamma_K-\mathrm{la}}$ for any $r > 0$ such that $x \in \widetilde{\mathbf{B}}^{\dagger,r}$. Therefore, $x \in (\widetilde{\mathbf{A}}_K^{(0,\rho]})^{\Gamma_K-\mathrm{la}}$ for $\rho = r_0/r$ by lemma 3.6 and thus $x \in (\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}}$. □

Corollary 3.8. — *We have $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}} = \widetilde{\mathbf{A}}^\dagger \cap (\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger)^{\Gamma_K-\mathrm{pa}}$.*

Remark 3.9. — Note that since the valuations on $\widetilde{\mathbf{A}}_K^{(0,\rho]}$ and $\widetilde{\mathbf{B}}_K^{[r,r]}$ are not normalized in the same way, we do not have that $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}} = \widetilde{\mathbf{A}}^\dagger \cap (\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger)^{\Gamma_K-\mathrm{la}}$. Actually, one can show that in the cyclotomic case, the ring $(\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger)^{\Gamma_K-\mathrm{la}}$ is quite small (see §7 of [**Poy22b**]) and does not contain $u = [\varepsilon]-1$, which clearly belongs to $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}}$.

We now recall the conjecture of Kedlaya [**Ked13**, Conjecture 12.13].

***Conjecture 3.10* (Kedlaya)**. — *Let T be a finite free $\mathbf{Z}_p$-module equipped with a continuous action of $\mathcal{G}_K$. For $r > 0$, let $\mathbf{D}_K^{\dagger,r}(T) = (\widetilde{\mathbf{A}}^{\dagger,r} \otimes_{\mathbf{Z}_p} T)^{H_K}$ and let $\mathbf{D}_K^{\dagger,r}(T)^{\Gamma_K-\mathrm{la}}$ denote the set of locally analytic elements of $\mathbf{D}_K^{\dagger,r}(T)$ for the action of Γ_K as defined in §2, which is a module over $(\widetilde{\mathbf{A}}_K^{\dagger,r})^{\Gamma_K-\mathrm{la}}$. Then for any $r > 0$, the natural map*

$$\widetilde{\mathbf{A}}_K^{\dagger,r} \otimes_{(\widetilde{\mathbf{A}}_K^{\dagger,r})^{\Gamma_K-\mathrm{la}}} \mathbf{D}_K^{\dagger,r}(T)^{\Gamma_K-\mathrm{la}} \to \mathbf{D}_K^{\dagger,r}(T)$$

is an isomorphism.

Note that the natural map $\widetilde{\mathbf{A}}^{\dagger,r} \otimes_{\widetilde{\mathbf{A}}_K^{\dagger,r}} \mathbf{D}_K^{\dagger,r}(T) \to \widetilde{\mathbf{A}}^{\dagger,r} \otimes_{\mathbf{Z}_p} T$ is an isomorphism thanks to for example §8 of [**KL15**]. We quickly remark that thanks to lemma 3.6 it is easy to check that the definitions of locally analytic elements in $\widetilde{\mathbf{A}}^{\dagger,r}$ used in [**Ked13**] coincide with ours.

4. Overconvergent lifts of the field of norms

Let K_∞ be an infinite totally ramified Galois extension of K whose Galois group is a p-adic Lie group. The main theorem of [**Sen72**] shows that K_∞/K is "strictly arithmetically profinite" (or strictly APF) in the terminology of [**Win83**] and we can thus apply the field of norms construction of ibid. to K_∞/K. We let $X_K(K_\infty)$ be the field of norms attached to the extension K_∞/K, which is a local field of characteristic p with residue field k_K by theorem 2.1.3 of ibid. In particular there exists a uniformizer u of $X_K(K_\infty)$ such that $X_K(K_\infty) = k_K(\!(u)\!)$. Moreover, this field comes equipped with an action of Γ_K and of the absolute Frobenius $\varphi : x \mapsto x^p$.

If we let $\mathcal{E}_{K_\infty}$ denote the set of finite subextensions $K \subset E \subset K_\infty$, then by definition, elements of $X_K(K_\infty)$ are norm-compatible sequences $(x_E)_{E\in\mathcal{E}(K_\infty)}$ such that $x_E \in E$ for all $E \in \mathcal{E}(K_\infty)$, and $N_{F/E}(x_F) = x_E$ whenever $E, F \in \mathcal{E}(K_\infty)$, $E \subset F$.

Since K_∞/K is strictly APF, there exists by [**Win83**, 4.2.2.1] a constant $c = c(K_\infty/K) > 0$ such that for all $F \subset F'$ finite subextensions of K_∞/K, and for all $x \in \mathcal{O}_{F'}$, we have

$$v_K\left(\frac{N_{F'/F}(x)}{x^{[F':F]}} - 1\right) \geq c.$$

We can always assume that $c \leq v_K(p)/(p-1)$ and we do so in what follows. By §2.1 and §4.2 of [**Win83**], there is a canonical $\mathcal{G}_K$-equivariant embedding $\iota_K : A_K(K_\infty) \hookrightarrow \widetilde{\mathbf{E}}^+$, where $A_K(K_\infty)$ is the ring of integers of $X_K(K_\infty)$. We can extend this embedding into a $\mathcal{G}_K$-equivariant embedding $X_K(K_\infty) \hookrightarrow \widetilde{\mathbf{E}}$ where $\widetilde{\mathbf{E}}$ is the fraction field of $\widetilde{\mathbf{E}}^+$, and we note $\mathbf{E}_K$ its image. We also let $\mathbf{E}_K^+$ denote the ring of valuation of $\widetilde{\mathbf{E}}_K$. We can actually give an explicit description of this embedding.

Proposition 4.1. — *Let* $0 < c \leq c(K_\infty/K)$.

1. *the map* $\iota_K : A_K(K_\infty) \to \varprojlim_{x\mapsto x^q} \mathcal{O}_{K_\infty}/\mathfrak{a}^c_{K_\infty} = \widetilde{\mathbf{E}}^+_K$ *is injective and isometric;*
2. *the image of* ι_K *is* $\varprojlim_{x\mapsto x^q} \mathcal{O}_{K_n}/\mathfrak{a}^c_{K_n}$.

Proof. — This is proven in §4.2 of [**Win83**]. □

Let E be a finite extension of $\mathbf{Q}_p$, with residue field $k_E = k_K$. Let ϖ_E be a uniformizer of E, and let $\mathbf{A}_K$ denote the ϖ_E-adic completion of $\mathcal{O}_E[\![T]\!][1/T]$ (the notation $\mathbf{A}_K$ is used here for compatibility witht the action of $\mathcal{G}_K$ but be mindful that this is actually dependent on E even if it does not appear in the notation). The ring $\mathbf{A}_K$ is a ϖ_E-Cohen ring of $X_K(K_\infty) = k_K(\!(\pi_K)\!)$, and following the definition of [**Ber14**], we say that the action of Γ_K is liftable if there exists such a field E and power series $\{F_g(T)\}_{g\in\Gamma_K}$ and $P(T)$ in $\mathbf{A}_K$ such that:

1. $\overline{F}_g(\pi_K) = g(\pi_K)$ and $\overline{P}(\pi_K) = \pi_K^q$;
2. $F_g \circ P = P \circ F_g$ and $F_g \circ F_h = F_{hg}$ for all $g, h \in \Gamma_K$;

where the notations $\overline{F}_g$ and $\overline{P}$ stand for the reduction of the power series mod ϖ_E.

When the action of Γ_K is liftable we get a (φ,Γ)-module theory as in Fontaine's classical cyclotomic theory [**Fon90**] in order to study $\mathcal{O}_E$-representations of $\mathcal{G}_K$, replacing the cyclotomic extension in the theory of Fontaine by the extension K_∞/K. In particular, if the action of Γ_K is liftable, then there is an equivalence of categories between étale (φ_q,Γ_K)-modules on $\mathbf{A}_K$ and $\mathcal{O}_E$-linear representations of $\mathcal{G}_K$ (see [**Ber14**, Thm. 2.1]).

Proposition 4.2. — *There is a* $\mathcal{G}_K$*-equivariant embedding* $\mathbf{A}_K \hookrightarrow \widetilde{\mathbf{A}}_K$ *and compatible with* φ_q *that lifts the embedding* $\iota_K : X_K(K_\infty) \hookrightarrow \widetilde{\mathbf{E}}_K = \widetilde{\mathbf{E}}^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)}$.

Proof. — See [**Fon90**, A.1.3] or [**Ber14**, §3]. □

Let $\mathcal{A}_K^{\dagger,r}$ denote the set of Laurent series $\sum_{k\in\mathbf{Z}} a_kT^k$ with coefficients in $\mathcal{O}_K$ such that $v_p(a_k)+kr/e \geq 0$ for all $k \in \mathbf{Z}$ and such that $v_p(a_k)+kr/e \to +\infty$ when $k \to -\infty$. We say that a lift of the field of norms is overconvergent if the power series $P(T)$ giving the lift of the Frobenius belongs to $\mathcal{A}_K^{\dagger,s}$ for some $r>0$.

We now assume that there is an overconvergent lift of the field of norms. Let $u \in \widetilde{\mathbf{A}}_K$ be the image of T by the embedding given by proposition 4.2, so that $\varphi_q(u)=P(u)$ and $g(u)=F_g(u)$ for $g \in \Gamma_K$.

Lemma 4.3. — *If $P(T) \in \mathcal{A}_K^{\dagger,r}$ then $u \in \widetilde{\mathbf{A}}^{\dagger,r}$.*

Proof. — We assume without any loss of generality that $P(T) \in \mathcal{A}_K^{\dagger,r}$ with $r \geq \frac{p-1}{p}$.

We have $u \in \widetilde{\mathbf{A}}$, and we write $u=(u-[\overline{u}])+[\overline{u}]$. Since $\overline{u} \in \widetilde{\mathbf{E}}^+$, we have $u \in (\widetilde{\mathbf{A}}^+ + \varpi_E\widetilde{\mathbf{A}})$. Let us write $P(T)=P^+(T)+P^-(1/T)$, with $P^+(T) \in T^q+\mathfrak{m}_E[\![T]\!]$ and $P^-(T)=\sum_{n>0}a_nT^n \in \mathfrak{m}_E[\![T]\!]$ with $v_p(a_n) \geq ne/r$. We thus have

$$P^+(u) \in (P^+([\overline{u}])+\varpi_E^2\widetilde{\mathbf{A}}) \subset (\widetilde{\mathbf{A}}^+ + \varpi_E^2\widetilde{\mathbf{A}})$$

and

$$P^-\left(\frac{1}{u}\right) = P^-\left(\frac{1}{[\overline{u}]}\frac{1}{1+\frac{u-[\overline{u}]}{[\overline{u}]}}\right) \in (P^-(\frac{1}{[\overline{u}]})+\varpi_E^2\widetilde{\mathbf{A}})$$

since $\frac{1}{1+\frac{u-[\overline{u}]}{[\overline{u}]}} \in 1+\varpi_E\widetilde{\mathbf{A}}$. Thus $P(u) \in (P([\overline{u}])+\varpi_E^2\widetilde{\mathbf{A}}) \subset (\widetilde{\mathbf{A}}^{\dagger,r}+\varpi_E^2\widetilde{\mathbf{A}})$.

Therefore, $u=\varphi_q^{-1}(P(u)) \in (\widetilde{\mathbf{A}}^{\dagger,r/q}+\varpi_E^2\widetilde{\mathbf{A}}) \subset (\widetilde{\mathbf{A}}^{\dagger,r}+\varpi_E^2\widetilde{\mathbf{A}})$. Now let us assume that $u \in (\widetilde{\mathbf{A}}^{\dagger,r}+\varpi_E^k\widetilde{\mathbf{A}})$ for some $k \geq 2$. Let us write $u=a+b$, with $a \in \widetilde{\mathbf{A}}^{\dagger,r}$ and $b \in \varpi_E^k\widetilde{\mathbf{A}}$. Since $\overline{u}=\overline{a}$, we have that $\frac{a}{[\overline{a}]}$ belongs to and is a unit of $\widetilde{\mathbf{A}}^{\dagger,r'}$ with $r'=r+\frac{p-1}{p}$ by lemma 3.1. Writing $a=\frac{a}{[\overline{a}]}[\overline{a}]$ shows that $b/a \in \varpi_E^k\widetilde{\mathbf{A}}$ and thus

$$\frac{1}{u}=\frac{1}{a}\left(\frac{1}{1+\frac{b}{a}}\right) \in (\frac{[\overline{a}]}{a}\frac{1}{[\overline{a}]}+\varpi_E^k\widetilde{\mathbf{A}}).$$

Therefore, we have

$$P(u) \in P(\frac{[\overline{a}]}{a}\frac{1}{[\overline{a}]})+\varpi_E^{k+1}\widetilde{\mathbf{A}}) \subset \widetilde{\mathbf{A}}^{\dagger,r'}+\varpi_E^{k+1}\widetilde{\mathbf{A}})$$

and thus $u=\varphi_q^{-1}(P(u)) \in (\widetilde{\mathbf{A}}^{\dagger,r'/q}+\varpi_E^{k+1}\widetilde{\mathbf{A}}) \subset (\widetilde{\mathbf{A}}^{\dagger,r}+\varpi_E^{k+1}\widetilde{\mathbf{A}})$. We can now conclude by using the fact that for any $r>0$, we have $\bigcap_{k\geq 0}(\widetilde{\mathbf{A}}^{\dagger,r}+\varpi_E^k\widetilde{\mathbf{A}})=\widetilde{\mathbf{A}}^{\dagger,r}$ (which follows from the definition of $\widetilde{\mathbf{A}}^{\dagger,r}$). □

Remark 4.4. — If one looks closely at the proof of lemma 4.3, one could improve the radius of surconvergence of u, but we don't need this level of precision here.

By [**Ber14**, Rem. 4.3], we have the following:

Proposition 4.5. — *For all $g \in \Gamma_K$, $F_g(T) \in T\cdot(\mathcal{A}_K^{\dagger,r})^\times$.*

Proposition 4.6. — *We have $u \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K-\mathrm{la}}$.*

Proof. — Recall that $\mathcal{A}_K^{\dagger,r}$ is endowed with a valuation v_r given by $V_r(f(T)) = \inf_{k\in\mathbf{Z}}(v_p(a_k)+kr/e)$ if $f(T) = \sum_{k\in\mathbf{Z}} a_kT^k$. Since the Galois action on $\widetilde{\mathbf{A}}_K^{\dagger,r}$ is continuous, there exists $n \gg 0$ such that for all $g \in \Gamma_n$, $V_r(g(u)-u,r) > \frac{1}{p-1}$. Up to increasing r if needed, we can assume that $V_r(F_g(T)-T) > \frac{1}{p-1}$ for all $g \in \Gamma_n$.

For $g \in \Gamma_n$, let $\Delta_g : \mathcal{A}_K^{\dagger,r} \to \mathcal{A}_K^{\dagger,r}$ be the map defined by $h(T) \mapsto h(F_g(T)) - h(T)$. We claim that the target of this map is indeed contained in $\mathcal{A}_K^{\dagger,r}$ (i.e. that $\Delta_g(\mathcal{A}_K^{\dagger,r}) \subset \mathcal{A}_K^{\dagger,r}$) and that it satisfies $\|\Delta_g(h)\|_r \leq |p|^c \cdot \|h\|_r$ for any $c < \frac{1}{p-1}$. In order to prove this claim, we write $h = h^+ + h^-$, where $h(T) = \sum_{n\in\mathbf{Z}} a_nT^n$, $h^+(T) = \sum_{n\geq 0} a_nT^n$ and $h^-(T) = \sum_{n<0} a_nT^n$, which we rewrite as $h^-(T) = \sum_{n>0} b_nT^{-n}$. Then

$$h^+(F_g(T)) - h^+(T) = \sum_{n\geq 0} a_n\left((F_g(T)^n - T^n\right) = \sum_{n>0} a_n(F_g(T)-T)(\sum_{k=0}^{n} F_g(T)^kT^{n-k})$$

and since $\|F_g(T)\|_r = \|T\|_r$ by proposition 4.5, this means that

$$|\Delta_g(h^+(T))|_r \leq \sum_{n>0} |a_n|_p \|\Delta_g(T)\|_r \|h^+\|_r.$$

We do the same for h^-: we have $h^-(F_g(T)) - h^-(T) = \sum_{n\geq 1} b_n(\frac{1}{F_g(T)^n} - \frac{1}{T^n})$. We write $B(T) = \frac{T}{F_g(T)} \in (\mathcal{A}_K^{\dagger,r})^\times$. Thus

$$\frac{1}{F_g(T)^n} - \frac{1}{T^n} = \frac{T^n - F_g(T)^n}{(TF_g(T))^n} = \frac{T^n - F_g(T)^n}{T^{2n}} B(T)^n.$$

and hence

$$\frac{1}{F_g(T)^n} - \frac{1}{T^n} = (T - F_g(T)) \sum_{k=0}^{n} \left(\frac{F_g(T)}{T}\right)^k \frac{B(T)^n}{T^n}.$$

Since $\frac{F_g(T)}{T}$ is a unit of $\mathcal{A}_K^{\dagger,r}$ (and so is $B(T)$), we obtain that

$$\|(T - F_g(T)) \sum_{k=0}^{n} \left(\frac{F_g(T)}{T}\right)^k \frac{B(T)^n}{T^n}\|_r = \|T - F_g(T)\|_r \|T^{-n}\|_r$$

so that

$$\|\Delta_g(h^-(T))\|_r \leq \|T - F_g(T)\|_r \sum_{n>0} |b_n|_p \|T^{-n}\|_r = \|T - F_g(T)\|_r \|h^-(T)\|_r.$$

Therefore, $\|\Delta_g(h(T))\|_r \leq \|T - F_g(T)\|_r \|h(T)\|_r$, which gives us exactly the result claimed.

To conclude, let $B := \{f(u), f(T) \in \mathcal{A}_K^{\dagger,r}[1/p]\} \subset \widetilde{\mathbf{B}}_K^{\dagger,r}$. Then the completion of B for $V(\cdot,r)$ is a $\mathbf{Q}_p$-Banach space (contained in $\widetilde{\mathbf{B}}_K^{[r,r]}$) which is Γ_n-stable, and over which we showed that the Γ_n-action satisfies $\|\gamma - 1\| < p^{-\frac{1}{p-1}}$ for any $\gamma \in \Gamma_n$, so that the Γ_K-action on this Banach space is locally analytic by [**BSX15**, Lemm. 2.14]. Thus $u \in (\widetilde{\mathbf{B}}_K^{[r,r]})^{\Gamma_K-\mathrm{la}}$, which finishes the proof thanks to lemma 3.6. □

5. Structure of locally analytic vectors in $\widetilde{\mathbf{A}}^{\dagger}$ for $\mathbf{Z}_p$-extensions

In this section, we assume that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension, with Galois group $\Gamma_K \simeq \mathbf{Z}_p$. The goal of this section is to prove that if there are nontrivial locally analytic vectors in $\widetilde{\mathbf{A}}_K^{\dagger}$, that is if $(\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K-\mathrm{la}} \neq \mathcal{O}_K$, then everything behaves just as if K_∞/K was the cyclotomic extension.

Let K_∞/K be a totally ramified $\mathbf{Z}_p$-extension, with Galois group $\Gamma_K \simeq \mathbf{Z}_p$. We assume furthermore that $(\widetilde{\mathbf{A}}_K^{\dagger})^{\mathrm{la}} \neq \mathcal{O}_K$, which means that it contains a nontrivial locally analytic vector. For $n \geq 1$ we let K_n/K be the subextension of K_∞/K such that $\mathrm{Gal}(K_n/K) = \mathbf{Z}/q^n\mathbf{Z}$ and we let $\Gamma_n = \mathrm{Gal}(K_\infty/K_n) \subset \Gamma_K$. We also let $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$. Note that, up to extending the field K, we can always assume without loss of generality that $K/\mathbf{Q}_p$ is Galois, and we do so in what follows.

We let $s : \widetilde{\mathbf{A}} \to \widetilde{\mathbf{A}}/\pi\widetilde{\mathbf{A}} \simeq \widetilde{\mathbf{E}}$ denote the projection map given by the reduction modulo π. Note that it induces by restriction projections that we will still denote by s: $\widetilde{\mathbf{A}}_K \to \widetilde{\mathbf{E}}_K$, $\widetilde{\mathbf{A}}^{\dagger} \to \widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{A}}_K^{\dagger} \to \widetilde{\mathbf{E}}_K$ and whose kernel is still generated by π.

Proposition 5.1. — *We have $(\widetilde{\mathbf{E}}_K)^{\Gamma_K,0-\mathrm{an}} \subset \mathbf{E}_K$, and we have $(\widetilde{\mathbf{E}}_K)^{\Gamma_K-\mathrm{la}} = \bigcup_{n\geq 0} \varphi_q^{-n}(\mathbf{E}_K)$.*

Proof. — This is theorem 2.2.3 of [**BR22b**]. □

In what follows, we choose the smallest integer λ such that $(\widetilde{\mathbf{E}}_K)^{G_0,\lambda-\mathrm{an}} \subset \mathbf{E}_K$. In particular, $\lambda \leq 0$.

Corollary 5.2. — *We have $s((\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K,\lambda-\mathrm{an}}) \subset \mathbf{E}_K$ and $s((\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K-\mathrm{la}}) \subset \varphi_q^{-\infty}(\mathbf{E}_K)$.*

Proof. — Let $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K,\lambda-\mathrm{an}}$. Then $s(x) \in \widetilde{\mathbf{A}}_K^{\dagger}/\pi\widetilde{\mathbf{A}}_K^{\dagger} \simeq \widetilde{\mathbf{E}}_K$ is λ-analytic (for Γ_K) for the valuation induced on $\widetilde{\mathbf{E}}_K$ by the one on $\widetilde{\mathbf{A}}_K^{\dagger}$. Proposition 5.1 shows that $s(x) \in \mathbf{E}_K$, so this proves the first part of the corollary. The second part comes from the fact that $x \in \widetilde{\mathbf{A}}^{(0,\rho]}$ is κ-analytic (for Γ_K) if and only if $\varphi_q^{\ell}(x)$ is $(\kappa - f\ell)$-analytic (for Γ_K) by lemma 3.5. □

Lemma 5.3. — *There exists $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K,\lambda-\mathrm{an}}$ whose image by s belongs to $\mathbf{E}_K^+ \setminus k_K$.*

Proof. — Since we assumed that $(\widetilde{\mathbf{A}}_K^{\dagger})^{\mathrm{la}}$ is non trivial, there exists $\kappa \geq 0$ and $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\kappa-\mathrm{an}}$ whose image by s is an element of $\varphi_q^{-k}(\mathbf{E}_K) \setminus k_K$ for some $k \geq 0$, and by applying φ_q^{ℓ} to this element for $\ell \gg 0$, that there exists $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\lambda-\mathrm{an}}$ whose image by s is an element of $\mathbf{E}_K \setminus k_K$. By lemma 2.5, we can assume up to replacing x by its inverse that $s(x)$ belongs to $\mathbf{E}_K^+$: since $s(x) \neq 0$, the inverse of x belongs to $\widetilde{\mathbf{A}}^{\dagger}$ and not just $\widetilde{\mathbf{B}}^{\dagger}$. □

Definition 5.4. — We let $\alpha := \min\{v_{\mathbf{E}}(s(x)), v_{\mathbf{E}}(s(x)) > 0, x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K,\lambda-\mathrm{an}}\}$.

Note that the set of elements x in $(\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K,\lambda-\mathrm{an}}$ such that the valuation of $s(x)$ is nonzero is nonempty by lemma 5.3, and that the set of $s(x)$ such that $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K,\lambda-\mathrm{an}}$ is included in $\mathbf{E}_K^+$ by corollary 5.2. Since the valuation on $\mathbf{E}_K^+$ is discrete, this means that α is well defined, and that the minimum is reached for some element in $(\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K,\lambda-\mathrm{an}}$ which will be denoted by v.

Since $\alpha = v_{\mathbf{E}}(s(v)) > 0$, the sequence $(v^n)_{n \geq 0}$ goes to 0 in $\widetilde{\mathbf{A}}_K$ for the $(\pi, [s(v)])$-adic topology (for which $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}_K$ are complete), and thus $\mathcal{O}_{K_0}(\!(v)\!)$ is naturally a subring of $\widetilde{\mathbf{A}}_K$. We let $\mathbf{A}_K$ denote the π-adic completion of $\mathcal{O}_{K_0}(\!(v)\!)$ in $\widetilde{\mathbf{A}}_K$ (we recall that $\widetilde{\mathbf{A}}_K$ is π-adically complete).

In the definition 5.4 above, we can thanks to lemma 3.1 assume thanks to lemma 3.1 that our choice of v satisfies the additional assumption that there exists $n \geq 0$ such that $v \in \widetilde{\mathbf{A}}^{\dagger, r_n}$ and such that $\frac{v}{[s(v)]}$ belongs to $\widetilde{\mathbf{A}}^{\dagger, r_n}$ and is a unit of this ring. In order to avoid additional notations we write r for r_n in the rest of this section.

Lemma 5.5. — *We have* $s((\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}}) \subset k_K(\!(s(v))\!)$.

Proof. — Let $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{G_0, \lambda-\mathrm{an}}$. By corollary 5.2, we know that $s(x) \in \mathbf{E}_K$. Let $k = v_{\mathbf{E}}(s(x))$, and let $k = q_0 \alpha + r$ be the euclidean division of k by α. We have $v^{-q_0} x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{G_0, \lambda-\mathrm{an}}$ by lemma 2.4, and $0 \leq v_{\mathbf{E}}(s(v^{-q_0} x)) = r < \alpha$, so that $r = 0$ by definition of α. There exists therefore $c_0 \in k_K$ such that $z := [c_0] v^{q_0}$ satisfies $v_{\mathbf{E}}(s(x) - s(z)) > v_{\mathbf{E}}(s(x))$. Now $x - z \in (\widetilde{\mathbf{A}}_K^{\dagger})^{G_0, \lambda-\mathrm{an}}$ and thus we can apply the same reasoning to $x - z$ instead of x. This yields $c_1 \in k_K$ and $q_1 > q_0$ such that $z_1 := x - [c_0] v^{q_0} + [c_1] v^{q_1}$ is such that $v_{\mathbf{E}}(s(z_1)) > v_{\mathbf{E}}(s(x - z))$. Applying the same process inductively gives us $(q_i)_{i \geq 0} \in \mathbf{Z}^{\mathbf{N}}$ an increasing sequence and $(c_i)_{i \geq 0} \in k_K^{\mathbf{N}}$ such that $s(x) = \sum_{i=0}^{+\infty} c_i s(v)^{q_i}$ and thus $s(x) \in k_K(\!(s(v))\!)$. □

Lemma 5.6. — *We have* $(\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}} \subset \mathbf{A}_K$.

Proof. — Let $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}}$. By lemma 5.5, we know that $s(x) \in k_K(\!(s(v))\!)$. Therefore, there exists $P_0(T) \in O_K(\!(T)\!)$ such that $x - P_0(v) \in \pi \widetilde{\mathbf{A}}_K^{\dagger}$. Moreover, since $x, v \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}}$, this implies that $x - P_0(v) \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}} \cap \pi \widetilde{\mathbf{A}}_K^{\dagger} = \pi (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}}$. Let $z = \frac{x - P_0(v)}{\pi} \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}}$. Then applying the same process for z instead of x yields $P_1(T) \in O_K(\!(T)\!)$ such that $x - P_0(v) - \pi \cdot P_1(v) \in \pi^2 \widetilde{\mathbf{A}}_K^{\dagger}$. Inductively, we find a sequence $(P_i(T))_{i \geq 0}$ of elements of $O_K(\!(T)\!)$ such that $x = \sum_{i=0}^{\infty} \pi^i \cdot P_i(v)$, and this series converges in $\widetilde{\mathbf{A}}_K$ since it is π-adically complete, to an element of $\mathbf{A}_K$ by definition of $\mathbf{A}_K$. □

Lemma 5.7. — *We have* $(\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K-\mathrm{la}} \subset \varphi_q^{-\infty}(\mathbf{A}_K)$.

Proof. — Let $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K-\mathrm{la}}$. Therefore there exists $m \geq 0$ and $\mu \in \mathbf{R}$ such that $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_m, \mu-\mathrm{an}}$. Note that by lemma 1.10 of [**BR22a**], this is equivalent to the existence of $\mu' \in \mathbf{R}$ such that $x \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \mu'-\mathrm{an}}$. If k is an integer such that $kf + \mu' \geq \lambda$, then $\varphi_q^k(x) \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, (\mu'+kf)-\mathrm{an}} \subset (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K, \lambda-\mathrm{an}}$ so that $\varphi_q^k(x) \in \mathbf{A}_K$ by lemma 5.6, and thus $x \in \varphi_q^{-k}(\mathbf{A}_K)$. □

Recall that $\mathcal{A}_K^{\dagger, s}$ denote the set of Laurent series $\sum_{k \in \mathbf{Z}} a_k T^k$ with coefficients in $\mathcal{O}_K$ such that $v_p(a_k) + ks/e \geq 0$ for all $k \in \mathbf{Z}$ and such that $v_p(a_k) + ks/e \to +\infty$ when $k \to -\infty$.

Recall that there exists some $r > 0$ such that $v \in \widetilde{\mathbf{A}}^{\dagger, r}$ and such that $\frac{v}{[s(v)]}$ is a unit in $\widetilde{\mathbf{A}}^{\dagger, r}$. For $s \geq r$, we let $\mathbf{A}_K^{\dagger, s}$ denote the set of $P(v) \in \widetilde{\mathbf{A}}$ such that $P \in \mathcal{A}_K^{\dagger, s}$. We also let $\mathbf{A}_K^{\dagger} = \cup_{s \geq r}(\mathbf{A}_K^{\dagger, s}[1/v])$.

Proposition 5.8. — *For $s \geq r$, we have $(\widetilde{\mathbf{A}}_K^{\dagger,s})^{\Gamma_K,\lambda-\mathrm{an}} = \mathbf{A}_K^{\dagger,s}$.*

Proof. — By lemma 3.1 and the choice of r we made, v belongs to $\widetilde{\mathbf{A}}^{\dagger,r}$ and is such that $\frac{v}{[s(v)]}$ is a unit in $\widetilde{\mathbf{A}}^{\dagger,r}$.

Now the proof of item (i) of [**Col08**, Prop. 7.5] carries over and shows that $\mathbf{A}_K \cap \widetilde{\mathbf{A}}_K^{\dagger,r} = \mathbf{A}_K^{\dagger,r}$ for $r \geq s$. To finish the proof, it suffices to notice that if $x \in (\widetilde{\mathbf{A}}_K^{\dagger,r})^{Gamma_K,\lambda-\mathrm{an}}$, then $x \in \mathbf{A}_K$ by lemma 5.6, and x belongs to $\widetilde{\mathbf{A}}_K^{\dagger,r}$. □

Corollary 5.9. — *There exists $P(T)$ in $\mathcal{A}_K^{\dagger,qr}$ such that $\varphi_q(v) = P(v)$, and for each $g \in \Gamma_K$, there exists a series $F_g(T)$ in $\mathcal{A}_K^{\dagger,r}$ such that $g(v) = F_g(v)$.*

Corollary 5.10. — *We have $(\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K-\mathrm{la}} = \varphi_q^{-\infty}(\mathbf{A}_K^{\dagger})$.*

Proposition 5.11. — *There exist $k \geq 0$ and $w \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\Gamma_K-\mathrm{la}}$ such that $\varphi^{-k}(s(w))$ is a uniformizer of $\mathbf{E}_K$.*

Proof. — Let u be a uniformizer of $\mathbf{E}_K$, and let us write $\overline{v}$ for $s(v)$. Note that $k_K((u))/k_K((\overline{v}))$ is a finite extension of local fields of characteristic p. It can thus be decomposed as a purely inseparable extension of a separable extension of $k_K((\overline{v}))$, so that there exists $k \geq 0$ and a separable monic polynomial P with coefficients in $k_K((\overline{v}))$ such that $\varphi^k(u)$ is a root of P. Now let $y := \varphi^k(u)$ and let $\widetilde{P}(T) \in \mathcal{O}_K((v))[T] \subset \widetilde{\mathbf{B}}_K^{\dagger}$ be a lift of P which is monic. Since $\mathbf{B}_K^{\dagger} := \mathbf{A}_K^{\dagger}[1/p]$ is a Henselian field (cf §2 of [**Mat95**]), and since $\widetilde{\mathbf{B}}^{\dagger}$ is absolutely unramified and has $\widetilde{\mathbf{E}}$ as a residue field which contains $\mathbf{E}_K$, there exists $\widetilde{y} \in \widetilde{\mathbf{B}}_K^{\dagger}$ lifting y such that $\widetilde{P}(\widetilde{y}) = 0$ and by construction $\widetilde{y} \in \widetilde{\mathbf{A}}_K^{\dagger}$ and $\widetilde{P}'(\widetilde{y}) \neq 0$.

Since $\widetilde{P}'(\widetilde{y}) \neq 0$ and since $\widetilde{\mathbf{B}}_K^{\dagger}$ is a field, there exists $r > 0$ such that $\widetilde{P}'(\widetilde{y})$ is invertible in $\widetilde{\mathbf{B}}_K^{\dagger,r}$ and such that all the coefficients of $\widetilde{P}$ belong to $\mathbf{B}_K^{\dagger,r} \subset \widetilde{\mathbf{B}}_K^{\dagger,r}$ (up to increasing r if needed for the last inclusion to make sense). Since the coefficients of $\widetilde{P}$ belong to $\mathbf{B}_K^{\dagger,r}$, they are locally analytic for the action of Γ_K as elements of $\widetilde{\mathbf{B}}^{[r,r]}$ by lemma 3.6. Thus there exists $k \gg 0$ such that for $g \in G_k$, we have that the coefficients of gP are analytic functions of G_k. Moreover, we have the equality $(g\widetilde{P})(g(\widetilde{y}) = 0$ and $\widetilde{P}'(\widetilde{y})$ is invertible in $\widetilde{\mathbf{B}}_K^{[r,r]}$ so that $\widetilde{y} \in (\widetilde{\mathbf{B}}_K^{[r,r]})^{\Gamma_K-\mathrm{la}}$ by the implicit function theorem for analytic functions (which follows from the inverse function theorem given on page 73 of [**Ser92**]). Using once again lemma 3.6, this shows that $\widetilde{y} \in (\widetilde{\mathbf{A}}_K^{\dagger})^{G_0-\mathrm{la}}$ and thus $w = \widetilde{y}$ satisfies the claim. □

Corollary 5.12. — *In definition 5.4, $s(v)$ is actually a uniformizer of $\mathbf{E}_K$.*

Proof. — Let w be as in proposition 5.11. Since we assumed at the beginning of the §that $K/\mathbf{Q}_p$ is Galois, we can find $\tau \in \mathrm{Gal}(K/\mathbf{Q}_p)$ whose image in $\mathrm{Gal}(k_K/\mathbf{F}_p)$ is the absolute Frobenius φ. We let $\iota_\tau : \widetilde{\mathbf{A}} = \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} W(\widetilde{\mathbf{E}}) \to \widetilde{\mathbf{A}}$ be the map defined by $(\tau \otimes \varphi)$. Note that this map preserves locally analytic vectors, but that there is a shift in terms of "level of analyticity" coming from lemma 3.5.

We have $\iota_\tau^{-k}(w) \in (\widetilde{\mathbf{A}}_K^{\dagger})^{\mathrm{la}}$ and thus by corollary 5.10 $\iota_\tau^{-k}(w) \in \varphi_q^{-\ell}(\mathbf{A}_K^{\dagger})$ for some $\ell \geq 0$. Therefore there exists $r > 0$ and $R(T) \in \mathcal{A}^{\dagger,r}[1/T]$ such that $\iota_\tau^{-k}(w) = \varphi_q^{-\ell}(R(v))$.

We also know by proposition 5.11 that $\iota_\tau^{-k}(w)$ lifts a uniformizer u of $\mathbf{E}_K$, so that if $\overline{R} \in k_K(\!(T)\!)$ denotes the Laurent series obtained by reducing the coefficients of R modulo p, we have $u = \varphi_q^{-\ell}(\overline{R}(s(v)))$. Since $s(v) \in \mathbf{E}_K^+$ and since u is a uniformizer of $\mathbf{E}_K$, there exists $f(T) \in k_K[T]$ such that $s(v) = f(u)$. This means that we have the inclusions

$$k_K(\!(\varphi_q^\ell(u))\!) \subset k_K(\!(s(v))\!) \subset k_K(\!(u)\!)$$

and thus since the extension $k_K(\!(u)\!)/k_K(\!(\varphi^\ell(u))\!)$ is purely inseparable, so is $k_K(\!(u)\!)/k_K(\!(s(v))\!)$. This means that there exists $h \geq 0$ such that $\varphi^{-h}(s(v))$ is a uniformizer of $k_K(\!(u)\!)$.

But now $s\left(\iota_\tau^{-h}(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K,\lambda-\mathrm{an}}\right) \subset k_K(\!(\varphi^{-h}(s(v)))\!) = \mathbf{E}_K$ by lemma 5.5, and thus $\lambda' = \lambda - h$ satisfies proposition 5.1. However, this contradicts our choice of λ, so that $h = 0$ and $s(v)$ is a uniformizer of $\mathbf{E}_K$. □

Remark 5.13. — In particular, corollaries 5.9 and 5.12 show that the existence of a nontrivial locally analytic vector implies the existence of an overconvergent lift of the field of norms as defined in §4. Note that this only holds *a priori* for $\mathbf{Z}_p$-extensions, because super-Hölder vectors in this case recover exactly the perfectization of the corresponding field of norms. As pointed out in remark 2.2.4 of [**BR22b**], as soon as K_∞/K is a p-adic Lie extension whose Galois group is of dimension (as a p-adic Lie group) at least 2, then the set of super-Hölder vectors of $\widetilde{\mathbf{E}}_K$ contains the field of norms $X_K(L_\infty)$ of any p-adic Lie extension L_∞/K contained in K_∞ and is thus no longer generated by a single element over k_K.

Recall that $n \geq 0$ is such that $v \in \widetilde{\mathbf{A}}^{\dagger,r_n}$ and such that $\frac{v}{[s(v)]}$ is a unit of $\widetilde{\mathbf{A}}^{\dagger,r_n}$. Since $v \in \widetilde{\mathbf{A}}^{\dagger,r_n}$ and since $\theta : \widetilde{\mathbf{A}}^{\dagger,r_0} \to \mathcal{O}_{\mathbf{C}_p}$ is well defined, we can consider $\theta \circ \varphi_q^{-m}(v)$ for all $m \geq n$. By lemma 3.3 and proposition 4.1, we have $v_p(\varphi_q^{-n}(v)) \to 0$ when $m \to +\infty$, so up to increasing n we can always assume that for all $m \geq n$, $v_{\mathbf{E}}(\theta \circ \varphi_q^{-m}(v)) < c$ where $c = c(K_\infty/K)$ is as in §4.

Lemma 5.14. — *We have* $(\widehat{K_\infty})^{\Gamma_m-\mathrm{an}} = K_m$.

Proof. — One can follow the first part of the proof of [**BC16**, Thm. 3.2]. □

Proposition 5.15. — *Let* $m \geq n$. *Then:*

1. $\varphi^{-m}(v) \in (\widetilde{\mathbf{B}}_K^{[r_0,r_0]})^{\Gamma_m-\mathrm{an}}$;
2. $v \in (\widetilde{\mathbf{B}}_K^{[r_m,r_m]})^{\Gamma_m-\mathrm{an}}$;
3. $\theta \circ \varphi^{-m}(v)$ *is a uniformizer of* $\mathcal{O}_{K_m}$.

Proof. — We start with the first item. Let $m \geq n$ and let $v_m = \varphi_q^{-m}(v) \in \widetilde{\mathbf{A}}$. The fact that $v \in \widetilde{\mathbf{A}}^{\dagger,r_n}$ and that $m \geq n$ implies that $v_m \in \widetilde{\mathbf{A}}^{\dagger,r_0}$. Recall that we have $v \in (\widetilde{\mathbf{A}}^{\dagger,r_m})^{\Gamma_K,\lambda-\mathrm{an}}$ with $\lambda \leq 0$. Therefore $v_m \in (\widetilde{\mathbf{A}}^{\dagger,r_0})^{\Gamma_K,(\lambda-m)-\mathrm{an}}$ by lemma 3.5. This means that there exists $\mu \in \mathbf{R}$ such that for all $g \in \Gamma_n$, $v_1((g-1)(v_m)) \geq q^{-m}q^n + \mu$. Therefore, for all $g \in \Gamma_n$, $V((g-1)(v_m), r_0) \geq q^{-m}q^n + \mu$. By corollary 2.3 and since $p > 2$, this means that $v_m \in (\widetilde{\mathbf{B}}_K^{[r_0,r_0]})^{\Gamma_m-\mathrm{an}}$. This proves item (i).

Item 2 follows directly from item 1 by applying φ_q^m to v_m and using the fact that φ_q is a continuous $\mathbf{Q}_p$-linear map from $\widetilde{\mathbf{B}}_K^{[r_0,r_0]}$ to $\widetilde{\mathbf{B}}_K^{[r_m,r_m]}$.

For item 3, we proceed as follows: first, we note that since $v_m \in (\widetilde{\mathbf{B}}_K^{[r_0,r_0]})^{\Gamma_m-\mathrm{an}}$ and since $\theta : \widetilde{\mathbf{B}}_K^{[r_0,r_0]} \to \mathcal{O}_{\widehat{K_\infty}}$ is a continuous $\mathbf{Q}_p$-linear map, this implies that $\theta(v_m) \in (\widehat{K_\infty})^{\Gamma_m-\mathrm{an}} = K_m$ by lemma 5.14. But since we chose n such that $v_{\mathbf{E}}(\theta \circ v_n) < c$, $\theta(v_m)$ and since $m \geq n$, we know by lemma 3.3 and proposition 4.1 that $\theta(v_m)$ has the same valuation as a uniformizer of $\mathcal{O}_{K_m}$, and thus is itself a uniformizer of $\mathcal{O}_{K_m}$. □

Proposition 5.16. — *For all $m \geq n$, there exists $\beta_m \in \mathbf{A}_K^{\dagger,r_n}$ such that β_m generates the kernel of the map $\theta \circ \varphi_q^{-m} : \widetilde{\mathbf{A}}^{\dagger,r_m} \to \mathcal{O}_{\mathbf{C}_p}$.*

Proof. — By proposition 5.15, we know that $\theta \circ \varphi_q^{-m}(v)$ is a uniformizer of $\mathcal{O}_{K_m}$. Since K_m/K is totally ramified of degree q^m, there exists $P(T) \in \mathcal{O}_K[T]$ a monic Eisenstein polynomial of degree q^m such that $P(v_m) = 0$. In particular, we have $P(v) \in \ker(\theta \circ \varphi_q^{-m} : \widetilde{\mathbf{A}}^{\dagger,r_m} \to \mathcal{O}_{\mathbf{C}_p})$ and $P(0)$ is a uniformizer of $\mathcal{O}_K$. We can write $P(v) = \sum_{k \geq 0}[a_k]p^k$ in $\widetilde{\mathbf{A}}$, and the fact that P is monic Eisenstein of degree q^m implies that $a_0 = [s(v)]^{q^m}$.

Moreover, since $P(v) \in \ker(\theta \circ \varphi_q^{-m} : \widetilde{\mathbf{A}}^{\dagger,r_m} \to \mathcal{O}_{\mathbf{C}_p})$ we know that $([\widetilde{\pi}]^{q^m} - \pi)$ divides $P(v)$ in $\widetilde{\mathbf{A}}^{\dagger,r_m}$ by lemma 3.2. Moreover, $v_{\mathbf{E}}(\widetilde{\pi}^{q^m}) = v_{\mathbf{E}}(s(v)^{q^m})$ since v lifts a uniformizer of the field of norms of K_∞/K. This means that $P(v)/([\widetilde{\pi}]^{q^m} - \pi)$ is an element of $\widetilde{\mathbf{A}}^{\dagger,r_m}$ whose image by s has valuation 0. By lemma 3.1, it is therefore a unit of $\widetilde{\mathbf{A}}^{\dagger,r_m}$ so that $P(v)$ is also a generator of the ideal $\ker(\theta \circ \varphi_q^{-m} : \widetilde{\mathbf{A}}^{\dagger,r_m} \to \mathcal{O}_{\mathbf{C}_p})$. □

6. Locally analytic vectors in the rings $\widetilde{\mathbf{B}}^I$ for good $\mathbf{Z}_p$-extensions

Our goal is now to derive results regarding the structure of the rings $(\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$, assuming that K_∞/K is a "good" $\mathbf{Z}_p$-extension, i.e. for which there are nontrivial locally analytic vectors in $\widetilde{\mathbf{A}}_K^\dagger$. Since K_∞/K is a $\mathbf{Z}_p$-extension, it is abelian and by local class field theory, there exists a Lubin-Tate extension K_{LT}/K such that $K_\infty \subset K_{\mathrm{LT}}$. We let $\Gamma_{\mathrm{LT}} = \mathrm{Gal}(K_{\mathrm{LT}}/K)$ and $H_{\mathrm{LT}} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_{\mathrm{LT}})$ and we keep the notations from §1 and from the previous section.

In particular, there exists $n \geq 0$ and $v \in \widetilde{\mathbf{A}}_K^{\dagger,r_n}$ such that v lifts a uniformizer of the field of norms of K_∞/K. Moreover, for all $m \geq n$, there exists $\beta_m \in \mathbf{A}_K^{\dagger,r_n}$ such that β_m generates the kernel of the map $\theta \circ \varphi_q^{-m} : \widetilde{\mathbf{A}}^{\dagger,r_m} \to \mathcal{O}_{\mathbf{C}_p}$.

For I a subinterval of $]1,+\infty[$ such that $\min(I) \geq r_n$, let $f(Y) = \sum_{k \in \mathbf{Z}} a_k Y^k$ be a power series with $a_k \in \mathcal{O}_K$ and such that $v_p(a_k) + kr_0/re \to +\infty$ when $|k| \to +\infty$ for all $r \in I$. The series $f(v)$ converges in $\widetilde{\mathbf{B}}^I$ and we let $\mathbf{B}_K^I$ denote the set of $f(u)$ where f is as above. Note that this is a subring of $\widetilde{\mathbf{B}}_K^I$ which is stable by the action of Γ_K. The Frobenius map gives rise to a map $\varphi_q : \mathbf{B}_K^I \to \mathbf{B}_K^{qI}$. If $m \geq 0$, then $\varphi_q^{-m}(\mathbf{B}_K^{q^m I}) \subset \widetilde{\mathbf{B}}_K^I$ and we let $\mathbf{B}_{K,m}^I = \varphi_q^{-m}(\mathbf{B}_K^{q^m I})$, and $\mathbf{B}_{K,\infty}^I = \bigcup_{m \geq 0} \mathbf{B}_{K,m}^I$. We let $\mathbf{A}_{K,m}^I$ denote the ring of integers of $\mathbf{B}_{K,m}^I$ for $V(\cdot, I)$.

Lemma 6.1. — *Let $I = [r_h, r_k]$ with $h \geq n$, and let $m_0 \geq 0$ be such that $t_\pi, t_\pi/Q_k$ and Q_k/β_k belong to $(\widetilde{\mathbf{B}}_{\mathrm{LT}}^I)^{\Gamma_{\mathrm{LT},m_0}-\mathrm{an}}$. If $m \geq m_0$ and if $a \in \widetilde{\mathbf{B}}_K^I$ is such that $\beta_k \cdot a \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$ then $a \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$.*

Proof. — Let us write $\beta_k \cdot a = Q_k \cdot \frac{\beta_k}{Q_k} \cdot a$. By [**Ber16**, Lemm. 4.3], we know that $\frac{\beta_k}{Q_k} \cdot a \in (\widetilde{\mathbf{B}}^I_{\mathrm{LT}})^{\Gamma_{\mathrm{LT},m_0}-\mathrm{an}}$. Since $Q_k/\beta_k \in (\widetilde{\mathbf{B}}^I_{\mathrm{LT}})^{\Gamma_{\mathrm{LT},m_0}-\mathrm{an}}$, this implies that a itself belongs to $(\widetilde{\mathbf{B}}^I_{\mathrm{LT}})^{\Gamma_{\mathrm{LT},m_0}-\mathrm{an}}$, and since $a \in (\widetilde{\mathbf{B}}^I_K)$ this finishes the proof. □

The following theorem, relying on the exact same ideas as theorem 4.4 of [**Ber16**], gives a description of the locally analytic vectors of $\widetilde{\mathbf{B}}^I_K$:

Theorem 6.2. — *Let $I = [r_h, r_k]$ and let $m \geq 0$ be such that $t_\pi, t_\pi/Q_k$ and Q_k/β_k belong to $(\widetilde{\mathbf{B}}^I_{\mathrm{LT}})^{\Gamma_{m+k}-\mathrm{an}}$. Then:*

1. $(\widetilde{\mathbf{B}}^I_K)^{\Gamma_{m+k}-\mathrm{an}} \subset \mathbf{B}^I_K$;
2. $(\widetilde{\mathbf{B}}^I_K)^{\mathrm{la}} = \mathbf{B}^I_{K,\infty}$.

Proof. — This is basically the same proof as the one of [**Ber16**, Theo 4.4] (note that there's a slight gap in ibid. which is fixed in [**Ber18**]) once we have the same "ingredients". Note that the second item follows directly from the first one, so we only need to prove the first item.

We start by proving the result when $h = k$ so that $I = [r_k, r_k]$. Let $x \in \widetilde{\mathbf{A}}^I_K \cap (\widetilde{\mathbf{B}}^I_K)^{\Gamma_{m+k}-\mathrm{an}}$.

If $d = q^{\ell-1}(q-1)$ then by a straightforward generalization of corollary 2.2 of [**Ber02**], we have $\widetilde{\mathbf{A}}^{[r_k;r_k]} = \widetilde{\mathbf{A}}^{[0;r_k]}\{\pi/[s(v)]\}$. Thus for all $n \geq 1$, there exists $k_n \geq 0$ such that $([s(v)]^d/\pi)^{k_n} \cdot x \in \widetilde{\mathbf{A}}^{[0;r_k]} + \pi^n \widetilde{\mathbf{A}}^{[r_k,r_k]}$. Since $v/[s(v)]$ is a unit of $\widetilde{\mathbf{A}}^{\dagger,r_k}$ we have $(v^d/\pi)^{k_n} \cdot x \in (\widetilde{\mathbf{A}}^{\dagger,r_k})^\times \cdot (\widetilde{\mathbf{A}}^{[0;r_k]} + \pi^n \widetilde{\mathbf{A}}^{[r_k,r_k]})$. Note that if $x \in \widetilde{\mathbf{A}}^I_K \cap (\widetilde{\mathbf{B}}^I_K)^{\Gamma_{m+k}-\mathrm{an}}$ and if $x_n = (v^d/\pi)^{k_n} \cdot x$ then by proposition 5.15 we have that $x_n \in \widetilde{\mathbf{A}}^{[r_k,r_k]}_K \cap (\widetilde{\mathbf{B}}^{[r_k,r_k]}_K)^{\Gamma_{m+k}-\mathrm{an}}$ so that $\theta \circ \varphi^{-k}(x_n) \in \mathcal{O}^{\Gamma_{m+k}-\mathrm{la}}_{\widehat{K_\infty}} = \mathcal{O}_{K_{m+k}}$ by lemma 5.14.

By proposition 5.15 there exists $y_{n,0} \in \mathcal{O}_K[\varphi_q^{-m}(v)]$ such that $\theta \circ \varphi_q^{-k}(x_n) = \theta \circ \varphi_q^{-k}(y_{n,0})$. By 3.2, there exists $x_{n,1} \in \widetilde{\mathbf{A}}^{[r_k,r_k]}_K$ such that $x_n - y_{n,0} = (\beta_m/\pi) \cdot x_{n,1}$. By lemma 6.1, $x_{n,1} \in (\widetilde{\mathbf{B}}^{[r_k,r_k]}_K)^{\Gamma_{m+k}-\mathrm{an}}$. Applying this procedure inductively gives us a sequence $\{y_{n,i}\}_{i \geq 0}$ of elements of $\mathcal{O}_K[\varphi_q^{-m}(v)]$ such that for all $j \geq 1$, we have

$$x_n - \left(y_{n,0} + y_{n,1} \cdot (\beta_k/\pi) + \cdots + y_{n,j-1} \cdot (\beta_k/\pi)^{j-1}\right) \in \ker(\theta \circ \varphi_q^{-k})^j.$$

Since the $y_{n,i}$ belong to $\widetilde{\mathbf{A}}^{\dagger,r_n}$ and since β_m/Q_m is a unit by proposition 5.16, we can apply proposition 3.4 so that there exists $j \gg 0$ such that

$$x_n - \left(y_{n,0} + y_{n,1} \cdot (\beta_k/\pi) + \cdots + y_{n,j-1} \cdot (\beta_k/\pi)^{j-1}\right) \in \pi \widetilde{\mathbf{A}}^{[r_k,r_k]},$$

and thus belongs to $\pi(\widetilde{\mathbf{A}}^{[0,r_k]} + \pi^{n-1}\widetilde{\mathbf{A}}^{[r_k,r_k]})$ by item (3) of [**Ber16**, Lemm. 3.2]. We can thus write $x_n - (y_{n,0} + y_{n,1} \cdot (\beta_k/\pi) + \cdots + y_{n,j-1} \cdot (\beta_k/\pi)^{j-1}) = \pi x_n'$ with $x_n' \in (\widetilde{\mathbf{A}}^{[0,r_k]} + \pi^{n-1}\widetilde{\mathbf{A}}^{[r_k,r_k]})$. By proposition 5.15, x_n' belongs to $(\widetilde{\mathbf{B}}^{[r_k,r_k]}_K)^{\Gamma_{m+k}-\mathrm{an}}$. We can now do the same with x_n' instead of x_n and we thus find some $j \gg 0$ and elements $\{y_{n,i}\}_{i \leq j}$ of $\mathcal{O}_K[\varphi_q^{-m}(v)]$ such that if $y_n = y_{n,0} + y_{n,1} \cdot (\beta_k/\pi) + \cdots + y_{n,j-1} \cdot (\beta_k/\pi)^{j-1}$ then $y_n - x_n \in \pi^n \widetilde{\mathbf{A}}^{[r_k,r_k]}$. If $z_n = (\pi/v^d)^{k_n} y_n$ then $z_n - x = (\pi/v^d)^{k_n}(y_n - x_n) \in \pi^n \widetilde{\mathbf{A}}^{[r_k,r_k]}$ and thus $(z_n)_{n \geq 1}$ converges p-adically to x, and for all $n \geq 0$ z_n belongs to $\mathbf{A}^{[r_k,r_k]}_{K,m}$ so that $x \in \mathbf{A}^{[r_k,r_k]}_{K,m}$.

This proves the result when $h = k$. Assume now that $h \neq k$. The same proof shows that if $x \in \widetilde{\mathbf{A}}^I_K \cap (\widetilde{\mathbf{B}}^I_K)^{\Gamma_{m+k}-\mathrm{an}}$ then $x = \varphi_q^{-m}(v)$ where f converges on the annulus corresponding to the interval $[q^m r_k, q^m r_k]$. Let us write $f(Y) = f^+(Y) + f^-(Y)$, where $f^+(Y)$ is the positive part and converges on $[0, q^m r_k]$ (note that we only know that $f^+(\varphi_q^{-m}(v))$ belongs to $\widetilde{\mathbf{B}}^{[r_n,r_k]}$) and $f^-(Y)$ is the negative part and converges and is bounded on $[q^m r_k; +\infty[$. If we let $x^- = \varphi_q^{-m}(f^-(v))$ then it belongs to both $\widetilde{\mathbf{B}}^{[r_h;r_k]}$ (by the fact that $x^- = x - x^+$ where $x^+ = \varphi_q^{-m}(f^+(v)) \in \widetilde{\mathbf{B}}^{[r_n,r_k]} \subset \widetilde{\mathbf{B}}^{[r_h,r_k]}$) and to $\widetilde{\mathbf{B}}^{[r_k,+\infty[}$ so that it belongs to $\widetilde{\mathbf{B}}^{[r_h,+\infty[}$.

The final result needed to conclude is that if the power series $f^-(Y)$ converges on $[q^m r_k, +\infty[$ and if $f^-(v)$ belongs to $\widetilde{\mathbf{B}}^{[q^m r_h,+\infty[}$ then $f^-(v)$ converges on $[q^m r_h, +\infty[$. The proof is the same as the one of [**Col08**, Prop. 7.5] (see also [**CC98**, Lemm. II.2.2]). Therefore, we have $x \in \mathbf{A}_{K,m}^{[r_h,r_k]}$, as claimed. □

7. The case of anticyclotomic extensions

In this section, we explain how to use the results from the previous section to show that anticylotomic extensions provide a counterexample to Kedlaya's conjecture. This also provides an example where there are some nontrivial higher locally analytic vectors attached to a totally ramified p-adic Lie extension in the rings of periods considered.

Proposition 7.1. — *There is no nontrivial locally analytic vectors in $\widetilde{\mathbf{A}}_K^\dagger$ in the anticyclotomic case.*

Proof. — Let us assume that there exist nontrivial locally analytic vectors in $\widetilde{\mathbf{A}}_K^\dagger$ in the anticylotomic case. Therefore, there exists $v \in \widetilde{\mathbf{A}}_K^\dagger$, locally analytic and lifting a uniformizer of $\mathbf{E}_K^+$, as in §5.

Let $\sigma : K \to K$ denote the nontrivial automorphism of K, and let $y = \frac{t_{\mathrm{id}}}{t_\sigma}$, where t_σ is as in [**Ber16**, §5]. Since t_σ has no zeroes on $I = [r_m, r_m]$, y defines an element of $\widetilde{\mathbf{B}}_{\mathrm{LT}}^{[r_m,r_m]}$. Moreover, we have $g(y) = \chi_{\mathrm{LT}}(g)\sigma(\chi_{\mathrm{LT}}(g))^{-1}y$ so that y is invariant by all the elements $g \in \mathrm{Gal}(K_{\mathrm{LT}}/K)$ such that $\chi_{\mathrm{LT}}(g) = \sigma(\chi_{\mathrm{LT}}(g))$ and is clearly analytic for the action of Γ_{LT}. Therefore, $y \in (\widetilde{\mathbf{B}}_{K_{ac}}^{[r_m,r_m]})^{\Gamma_K-\mathrm{an}}$. By theorem 6.2, there exists $m \gg 0$ and $k \geq 0$ such that $y \in \mathbf{B}_{K_{ac},k}^{[r_m,r_m]}$. Therefore, there exists $R(T) \in \mathcal{A}_{K_{ac}}^{[q^k r_m, q^k r_m]}$ such that $y = R(\varphi_q^{-k}(u))$.

We now apply φ_q to y. Since $\varphi_q(t_{\mathrm{id}}) = \pi t_{\mathrm{id}}$ and $\varphi_q(t_\sigma) = \sigma(\pi)t_\sigma$, we have $\varphi_q(y) = \frac{\pi}{\sigma(\pi)}y$ (note that in the classical anticyclotomic case, we have $\varphi_q(y) = y$). This means that $R(\varphi_q(u)) = a \cdot R(u)$ with $a = \frac{\pi}{\sigma(\pi)}$. Letting $P(T)$ be the power series such that $P(u) = \varphi_q(u)$, we get that $(R \circ P)(\varphi_q^{-m}(v)) = a \cdot R(\varphi_q^{-m}(v))$ and in particular $R(T)$ also belongs to $\mathcal{A}_{K_{ac}}^{[q^{k+1} r_m, q^{k+1} r_m]}$. Since a is a unit this is not possible because the valuation $V(\cdot, q r_m)$ is not the same on both sides. □

In particular, using propositions 4.6 and 7.1, we obtain the following result:

Theorem 7.2. — *There is no overconvergent lift of the field of norms in the anticyclotomic setting.*

Proposition 7.3. — *Let K_∞/K be a $\mathbf{Z}_p$ extension with Galois group Γ_K, and assume that $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}} = \mathcal{O}_K$. Then Kedlaya's conjecture is false for K_∞/K.*

Proof. — Let us assume that $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}} = \mathcal{O}_K$ and that Kedlaya's conjecture is true. This means that if T is a free $\mathcal{O}_K$-representation of $\mathcal{G}_K$ then $\mathbf{D}_K^{\dagger,\mathrm{an}}(T) := (\widetilde{\mathbf{A}}^\dagger \otimes_{\mathbf{Z}_p} T)^{H_K,\Gamma_K-\mathrm{la}}$ is an $\mathcal{O}_K$-module such that $\widetilde{\mathbf{A}}^\dagger \otimes_{\mathcal{O}_K} \mathbf{D}_K^{\dagger,\mathrm{an}}(T) \simeq \widetilde{\mathbf{A}}^\dagger \otimes_{\mathbf{Z}_p} T$ and thus $(\widetilde{\mathbf{A}}^\dagger \otimes_{\mathcal{O}_K} \mathbf{D}_K^{\dagger,\mathrm{an}}(T))^{\varphi_q=1} \simeq T$.

Moreover, since $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}} = \mathcal{O}_K$, we can assume that K_∞/K is not (an unramified twist of) the cyclotomic extension of K. Now let T be a rank 1 $\mathcal{O}_K$-representation of $\mathcal{G}_K$, with basis e. By Kedlaya's conjecture, there exists $y \in \widetilde{\mathbf{A}}^\dagger$ such that $(T \otimes_{\mathcal{O}_K} \widetilde{\mathbf{A}}^\dagger)^{H_K,\Gamma_K-\mathrm{la}}$ is a rank 1 $\mathcal{O}_K$-module generated by $e \otimes y$, and comes equipped with an $\mathcal{O}_K$-linear action of Γ_K and φ_q. In particular, there exists $a \in \mathcal{O}_K^\times$ (since φ_q is an isomorphism) such that $\varphi_q(e \otimes y) = a \cdot (e \otimes y)$, and Γ_K acts on $e \otimes y$ by multiplication by some character $\eta : \Gamma_K \to \mathcal{O}_K^\times$.

By local class field theory, there exists z in $\mathcal{O}_{\widehat{K^{\mathrm{unr}}}}$, the ring of integers of the p-adic completion of the maximal unramified extension of K, such that $\frac{z}{\varphi_q(z)} = a$. Since $\mathcal{O}_{\widehat{K^{\mathrm{unr}}}} \subset \widetilde{\mathbf{A}}^+ \subset \widetilde{\mathbf{A}}^\dagger$, we have that $z \in \widetilde{\mathbf{A}}^\dagger$ and if $x = e \otimes y \otimes z \in \mathbf{D}_K^{\dagger,\mathrm{an}}(T) \otimes_{\mathcal{O}_K} \widetilde{\mathbf{A}}^\dagger \simeq T \otimes_{\mathcal{O}_K} \widetilde{\mathbf{A}}^\dagger$, we get that $\varphi_q(x) = x$ so that $yz \in \widetilde{\mathbf{A}}^\dagger$ is invariant by φ_q and thus belongs to $\mathcal{O}_K$.

This means that $y \in \mathcal{O}_{\widehat{K^{\mathrm{unr}}}}$, and since Γ_K acts on $e \otimes y$ by multiplication by some character $\eta : \Gamma_K \to \mathcal{O}_K^\times$, this means that $\mathcal{G}_K$ acts on e by multiplication by a character which factors through $\mathrm{Gal}(K_\infty \cdot K^{\mathrm{unr}}/K)$. Since this is true for any rank 1 representation T of $\mathcal{G}_K$, this means by local class field theory that $K^{\mathrm{ab}} = K_\infty \cdot K^{\mathrm{unr}}$, which is possible if and only if K_∞ is a Lubin-Tate extension of K. Since K_∞/K is a $\mathbf{Z}_p$-extension, this means that $K = \mathbf{Q}_p$ and that K_∞/K is an unramified twist of the cyclotomic extension of K, which as stated above is ruled out by the assumption that $(\widetilde{\mathbf{A}}_K^\dagger)^{\Gamma_K-\mathrm{la}} = \mathcal{O}_K$. □

As a corollary of propositions 7.1 and 7.3, we obtain the following theorem:

Theorem 7.4. — *Anticyclotomic extensions provide a counterexample to Kedlaya's conjecture.*

We finish this section and paper by exhibiting nontrivial locally analytic vectors in the anticyclotomic setting:

Proposition 7.5. — *Let K_∞/K be an anticyclotomic extension. Then for any $n \geq 0$, we have an embedding $\mathcal{O}_{K_\infty} \subset R^1_{\mathrm{la}}(\widetilde{\mathbf{A}}_K^{\dagger,r_n})$.*

Proof. — Let $n \geq 0$, and let x be a generator of $\ker(\theta : \widetilde{\mathbf{A}}_K^{\dagger,r_n} \to \mathcal{O}_{\widehat{K_\infty}})$ given by lemma 3.2. Consider the following exact sequence:

$$0 \to \widetilde{\mathbf{A}}_K^{\dagger,r_n} \to (\frac{1}{x}\widetilde{\mathbf{A}}_K^{\dagger,r_n}) \to \widetilde{\mathbf{A}}_K^{\dagger,r_n}/x\widetilde{\mathbf{A}}_K^{\dagger,r_n} \to 0$$

and note that $\widetilde{\mathbf{A}}_K^{\dagger,r_n}/x\widetilde{\mathbf{A}}_K^{\dagger,r_n} \simeq \mathcal{O}_{\widehat{K_\infty}}$. Taking Γ_K-analytic vectors, we obtain:

$$0 \to (\widetilde{\mathbf{A}}_K^{\dagger,r_n})^{\Gamma_K-\mathrm{la}} \to ((\frac{1}{x}\widetilde{\mathbf{A}}_K^{\dagger,r_n}))^{\Gamma_K-\mathrm{la}} \to \mathcal{O}_{\widehat{K_\infty}}^{\Gamma_K-\mathrm{la}} \to R^1_{\mathrm{la}}(\widetilde{\mathbf{A}}_K^{\dagger,r_n})$$

By proposition 7.1, we have $(\widetilde{\mathbf{A}}_K^{\dagger,r_n})^{\Gamma_K-\mathrm{la}} = \mathcal{O}_K$. Moreover, we have $(\frac{1}{x}\widetilde{\mathbf{A}}_K^{\dagger,r_n}) \subset \widetilde{\mathbf{A}}^\dagger$, so that still by proposition 7.1, we have $((\frac{1}{x}\widetilde{\mathbf{A}}_K^{\dagger,r_n}))^{\Gamma_K-\mathrm{la}} = \mathcal{O}_K$. Finally, $\mathcal{O}_{\widehat{K_\infty}}^{\Gamma_K-\mathrm{la}} \simeq \mathcal{O}_{K_\infty}$ by [**BC16**, Thm. 1.6]. This gives us the result we wanted. □

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