

ON THE ARITHMETIC AND GEOMETRY OF SPACES $L_{m+1,n}$

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ABSTRACT. In this paper, we prove several new results on certain \mathbb{F}_p -vector spaces of logarithmic differential forms in characteristic p called *spaces* $L_{m+1,n}$. Expanding the previous work by the first two authors, we prove positive and negative results for the existence of spaces $L_{m+1,n}$ in many situations, as well as a classification of all spaces $L_{12,2}$ and $L_{15,2}$ for $p = 3$. The novel tools used are Moore determinants and computational algebra.

1. INTRODUCTION

Algebraic geometry in positive characteristic is a rich subject that has been intensively studied ever since the foundations of algebraic geometry were rigorously established.¹ As a result, a plethora of positive characteristic methods have been developed. These mathematical tools not only unlock novel insights into the geometry of algebraic varieties but also have surprising applications to other disciplines. For example, they form the foundations for the development of advanced coding techniques and cryptographic protocols used nowadays. Moreover, the richness of algebraic geometry in positive characteristic has been also exploited to prove results in characteristic zero. Roughly speaking, this is made possible by the use of two complementary procedures: *lifting to characteristic 0*, that assigns an object in characteristic zero with a given object in positive characteristic, and *reduction modulo p* , that assigns an object in positive characteristic with a given object in characteristic zero.

In this paper, we focus on a particular phenomenon in positive characteristic, namely the existence of the so-called *spaces* $L_{m+1,n}$.

Definition. Given k an algebraically closed field of characteristic $p > 0$ and strictly positive integers $n, m \in \mathbb{N}$, a set Ω of differential forms on $\mathbb{P}_k^1 = k \cup \infty$ is called a *space* $L_{m+1,n}$ if it satisfies the following conditions

- The set Ω is a n -dimensional vector space over \mathbb{F}_p ;
- Every $\omega \in \Omega - \{0\}$ is logarithmic;
- Every $\omega \in \Omega - \{0\}$ has a unique zero at ∞ of order $m - 1$.

Classically, the motivation for studying spaces $L_{m+1,n}$ arose from the following lifting problem: let $G \subset \text{Aut}_k k[[z]]$ be a finite order subgroup of k -automorphisms of the ring $k[[z]]$ of formal power series over k . It is said that G *lifts to characteristic zero* if there exists a finite extension R of the ring of Witt vectors $W(k)$ with uniformizer π and a commutative diagram

$$(1.1) \quad \begin{array}{ccc} & \text{Aut}_R R[[Z]] & \\ & \nearrow j & \downarrow r \\ G & \xrightarrow{i} & \text{Aut}_k k[[z]] \end{array}$$

¹Most notably, studying varieties over finite fields was the main motivation that prompted Weil to write his seminal book *Foundations of Algebraic Geometry* [1].

where i is the inclusion and r is the reduction modulo π . The *local lifting problem for curves* is the question of determining which embeddings $G \subset \text{Aut}_k k[[z]]$ admit a lifting. This problem has now a long tradition and can be studied from many angles: the interested reader is referred to the surveys [2] and [3] for a thorough discussion. In this context, the spaces $L_{m+1,n}$ arise in the elementary abelian case, that is, when $G \cong (\mathbb{Z}/p\mathbb{Z})^n$. This connection is now well established, thanks to the results obtained in [4], [5] (for $n = 1$), [6] and [7] (for $n > 1$). Thanks to these investigations, we know that the existence of a space $L_{m+1,n}$ for a given triple (p, n, m) is equivalent to the existence of a lifting of $(\mathbb{Z}/p\mathbb{Z})^n$ with equidistant branch locus and $m + 1$ fixed points.² We also know that $m + 1$ is necessarily of the form λp^{n-1} for some $\lambda > 0$, a fact that leads naturally to the following question

Question. For which triples (p, n, λ) does there exist a space $L_{\lambda p^{n-1}, n}$?

In this paper we make progress on this largely open question, as well as introduce tools that can be used to see it in a new light. Let us recall what was known prior to this paper: if $n = 2$ and $p = 2$ there are spaces $L_{2\lambda, 2}$ for every $\lambda > 0$ ([7, Théorème 2.2.4]). If $n = 2$ and $p \geq 3$ then there are no spaces $L_{p, 2}$ and no spaces $L_{3p, 2}$, while spaces $L_{2p, 2}$ exist if, and only if $p = 3$ ([7, Théorème 2.2.5]). If $n = 2$ and $\lambda \geq 4$ the only examples of spaces $L_{\lambda p, 2}$ that were known before this paper have the special property that λ is a multiple of $p - 1$ and obey a strict geometric constraint. Similarly, in the case $n \geq 3$ the known examples (see Section 5.1) satisfy $p - 1 | \lambda$ and are of a very special nature.

Our contributions to the question above solve the problem of existence of spaces $L_{\lambda p^{n-1}, n}$ in three distinct cases:

- The case $n = 2$, $p = 3$ and $\lambda = 4, 5$;
- The case $n = 2$ and $p > 3\lambda$;
- The case $n \geq 2$ and $p = 2$.

More precisely, we provide a complete classification of spaces $L_{\lambda p, 2}$ for $p = 3$ and $\lambda = 4, 5$, we show that for $p > 3\lambda$ there are no spaces $L_{\lambda p, 2}$. Moreover, we prove new results on the existence of spaces $L_{\lambda p^{n-1}, n}$ for $n \geq 3$ that, when applied to the case $p = 2$, lead to construction of large classes of spaces $L_{\lambda 2^{n-1}, n}$ when λ is either even or congruent to 1 modulo $2^n - 2$. In proving the results above, we make use of techniques that have not been exploited in this context before, and that we believe to be of independent interest, such as computational commutative algebra, étale pullbacks, and Moore determinants. The first main result presented in the paper is the case $n = 2$ and $p > 3\lambda$.

Theorem (cf. Theorem 3.7). *There are no spaces $L_{\lambda p, 2}$ when $p > 3\lambda$.*

The proof of this theorem relies on a result of Pagot³, that shows how the existence of Ω a space $L_{\lambda p, 2}$ for fixed λ and p is equivalent to the existence of two polynomials $Q_1, Q_2 \in k[X]$ of degree λ satisfying three conditions. The first condition is that Q_1 and Q_2 are \mathbb{F}_p -linearly independent. The second condition is that the set of poles of Ω coincides with the set of zeroes of the polynomial $Q_1 Q_2^p - Q_1^p Q_2$. The third condition can be expressed as a multivariate polynomial system, whose indeterminates are the coefficients of Q_1 and Q_2 . This system is in general very complicated, but a subsystem of necessary conditions can be extracted thanks to the fact that the nonzero elements of Ω are logarithmic and hence their poles have residues in \mathbb{F}_p^\times . In the context of our theorem, the necessary conditions can be expressed in terms of the coefficients of Q_1 and Q_2 and become tractable enough to get a contradiction under the assumptions of the theorem.

This result is a big progress towards the classification of spaces $L_{\lambda p, 2}$: thanks to it, the existence of a space $L_{\lambda p, 2}$ for a fixed λ needs to be checked only at a finite number of primes, which in principle can be done by solving the polynomial system in the coefficients of Q_1 and Q_2 given by Proposition 3.1. The use of computational tools, such as one of the many algorithms to compute Gröbner bases,

²See [8, Théorème 11] for the construction of a lifting with equidistant branch locus given a space $L_{m+1, n}$.

³Proposition 3.1, first appearing as [8, Proposition 7]

is essential to perform this task, but not enough to conclude. In fact, the resulting computations are of a high complexity and a straightforward implementation does not yield a solution in a reasonable time when $\lambda > 3$. Hence, some simplifications are needed to reduce the number of variables in the polynomial system under consideration. A first simplification is made possible by the fact that a space $L_{\lambda p^{n-1}, n}$ can be transformed by applying to \mathbb{P}_k^1 a homography that fixes ∞ . The resulting space is again a space $L_{\lambda p^{n-1}, n}$ that is said to be *equivalent* to the first. Another useful construction is obtained by considering the action of the relative Frobenius morphism on a space $L_{\lambda p^{n-1}, n}$ (see Section 2.3). The resulting space is again a space $L_{\lambda p^{n-1}, n}$ that is said to be *Frobenius equivalent* to the first.

When $p = 3$ and $n = 2$, by performing the above simplifications and applying considerations of symmetry under change of variables, we can find enough relations between the coefficients of Q_1 and Q_2 to make the polynomial system treatable in the cases $\lambda = 4$ and $\lambda = 5$. In this way, we not only find instances of, but we can also classify all possible spaces $L_{12,2}$ and $L_{15,2}$.

Theorem (cf. Theorem 6.2). *Let $p = 3$, $n = 2$, and $\lambda = 4$. Up to equivalence, a pair (Q_1, Q_2) of polynomials satisfies the conditions of Proposition 3.1 (and hence determines a space $L_{12,2}$) if, and only if, it is of the form*

$$\begin{aligned} Q_1 &= a(X^4 + (a^4 - a^2 - 1)X^2 + a^8) \\ Q_2 &= X^4 - (a^4 + a^2 - 1)X^2 + 1, \end{aligned}$$

for some $a \in k$ such that $a^2 \notin \mathbb{F}_3$.

Theorem (cf. Theorem 6.7). *Let $p = 3$, $n = 2$, and $\lambda = 5$. Up to equivalence and Frobenius equivalence, a pair (Q_1, Q_2) of polynomials satisfies the conditions of Proposition 3.1 (and hence determines a space $L_{15,2}$) if, and only if, it is either of the form*

$$\begin{aligned} Q_1 &= (\mu^2 - \mu - 1)X^5 + X^3 - (\mu^2 - \mu - 1)X^2 - \mu X \\ Q_2 &= -\mu X^5 + (\mu^2 + \mu - 1)X^3 + (\mu^2 + \mu)X^2 + (\mu^2 - 1)X - (\mu^2 + \mu + 1), \end{aligned}$$

where $\mu \in \mathbb{F}_{27}$ is such that $\mu^3 - \mu + 1 = 0$, or of the form

$$\begin{aligned} Q_1 &= a(X^5 - X^3 - X^2 + aX - (a + 1)) \\ Q_2 &= (-a - 1)(X^5 - (a + 1)X^3 + (a + 1)X^2 + X + a), \end{aligned}$$

where $a \in \mathbb{F}_9$ is such that $a^2 + 1 = 0$.

In particular, we have infinite equivalence classes of spaces $L_{12,2}$ and only finitely many equivalence classes of spaces $L_{15,2}$. Finding a geometric explanation of this phenomenon would be very interesting.

In the case $p > 3$ the above simplifications are not enough to conclude. However, a finer strategy can be employed to obtain a classification of all spaces $L_{4p,2}$ for all prime p 's, even though it requires substantial more work. In fact, in this case Theorem 3.7 allows us to consider only the cases $p = 3$ (discussed above) and $p = 5, 7, 11$. In these cases, we can use a combination of elementary arguments to find previously undiscovered relations between the zeroes of $Q_1 Q_2^p - Q_1^p Q_2$. In almost all cases (i.e. except when Q_1 and Q_2 are of a very specific form) these relations take the form of vanishing of certain Schur polynomials, that can be turned, using Jacobi-Trudi relations, into the vanishing of certain Toeplitz determinants involving symmetric polynomials in some of the zeroes. These new relations are easier to work out and show that, for $p = 5$ all the spaces $L_{20,2}$ are of a previously known form, and for $p = 7, 11$ there are no spaces $L_{4p,2}$. A discussion of the remaining outstanding case necessitates additional arguments and the computation of Grobner bases, but can be achieved and confirms that there are no spaces $L_{4p,2}$ other than the previously known cases in general. The write up of this result is in preparation.

We then turn our attention to the case of spaces $L_{\lambda p^{n-1}, n}$. The crucial new tool that we use in this setting is the *Moore determinant* $\Delta_n(\underline{a})$ of a n -uple of elements $\underline{a} := (a_1, \dots, a_n)$ in a field of characteristic p . This is defined as the determinant of the associated Moore matrix, namely we have

$$\Delta_n(\underline{a}) := \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^p & a_2^p & \dots & a_n^p \\ \vdots & \vdots & \dots & \vdots \\ a_1^{p^{n-1}} & a_2^{p^{n-1}} & \dots & a_n^{p^{n-1}} \end{vmatrix}.$$

The Moore determinant is a basic object in arithmetic in characteristic p , due to its relationship with the theory of additive polynomials (cf. [9, Chapter 1]). In this paper, we apply a mix of classical and recent results (obtained in [10]) on Moore determinants to describe effectively the differential forms belonging to a space $L_{\lambda p^{n-1}, n}$. Our exposition of Moore determinants is self-interested and self-contained: we put all and only the results we need in the dedicated Appendix A, and include there further, more complete, references for the interested reader.

Moore determinants allow us to prove two main new results about spaces $L_{\lambda p^{n-1}, n}$. These take the form of a necessary and sufficient criterion for the existence of spaces $L_{\lambda p^{n-1}, n}$, which for $n = 2$ boils down to Proposition 3.1.

Theorem (cf. Theorems 4.7 and 4.8). *Let $n \geq 2$ and let Ω be a n -dimensional \mathbb{F}_p -vector space of differential forms on \mathbb{P}_k^1 , generated by elements $\omega_1, \dots, \omega_n$. Then Ω is a space $L_{\lambda p^{n-1}, n}$ if, and only if, the following conditions are met:*

- (i) *There exists a non-zero logarithmic form $\omega \in \Omega$.*
- (ii) *There exist a n -uple of polynomials $Q := (Q_1, \dots, Q_n) \in k[X]^n$ such that*
 - *The polynomials Q_i are of degree λ for every $i = 1, \dots, n$.*
 - *The leading coefficients q_1, \dots, q_n of Q_1, \dots, Q_n satisfy $\Delta_n(q_1, \dots, q_n) \neq 0$.*
 - *We have*

$$\omega_i = \frac{(-1)^{i-1} \Delta_{n-1}(\widehat{Q}_i)}{\Delta_n(Q)} dX$$

for every $i = 1, \dots, n$, where \widehat{Q}_i denotes the $n - 1$ -uple obtained by removing Q_i from Q .

This condition simplifies our task of classifying spaces $L_{\lambda p^{n-1}, n}$ insofar as it gives us a practical recipe to use the polynomials Q_i 's to build a basis of Ω . In Section 4.2 we develop a constructive strategy to build suitable Q_i 's, that is fruitfully applied in the case $p = 2$ to create a large class of new examples. Our main result in this sense is the following:

Theorem (cf. Theorem 4.26 and Corollary 4.28). *Let λ be either even or $\lambda \equiv 1 \pmod{2^n - 2}$. Then there exist infinitely many equivalence classes of spaces $L_{\lambda 2^{n-1}, n}$.*

Then, we turn our attention to the examples of spaces $L_{\lambda p^{n-1}, n}$ for $n \geq 3$ that were known in the literature prior to the present paper. We remark that all these spaces share a very special structure: each of them is an étale pullback of a space $L_{(p-1)p^{n-1}, n}$ whose set of poles is the set of nonzero elements in a n -dimensional \mathbb{F}_p -subvector space of k . We call *standard* any space $L_{(p-1)p^{n-1}, n}$ whose poles satisfy the property above, and we apply Theorems 4.7 and 4.8 in the special case of standard spaces, shedding new light on their arithmetic properties. As a result, we are able to characterize the subspaces of standard spaces as étale pullbacks of standard spaces of lower dimension.

Proposition (cf. Proposition 5.6). *Let Ω be a standard $L_{(p-1)p^{n-1}, n}$ space and let $\widetilde{\Omega}$ be a proper r -dimensional subspace of Ω . Then, there exists a standard $L_{(p-1)p^{r-1}, r}$ space Ω' and a degree p^{n-r} morphism $\sigma : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ ramified only at ∞ such that $\widetilde{\Omega}$ is the pullback of Ω' under σ .*

When $p = 2$, we can show that all standard $L_{2^{n-1},n}$ spaces and their étale pullbacks arise from the construction of Section 4.2.

Finally, we remark that the spaces $L_{12,2}$ and $L_{15,2}$ discovered in this paper are, to our knowledge, the first known examples of non-standard spaces $L_{\lambda p^{n-1},n}$ with $p \neq 2$ that are not equivalent to étale pullbacks of standard spaces. Since $p = 3$ in these examples, their étale pullbacks generate examples of non-standard spaces $L_{36d,2}$ and $L_{45d,2}$ for every positive integer d . The spaces $L_{45d,2}$ when d is odd can not be equivalent to étale pullbacks of standard spaces, by a simple argument of pole counts, and therefore we have an infinite class of examples not arising from standard spaces. We don't know to what extent this generalizes to other values of p and n . More specifically, the following questions remain open:

- Are there spaces $L_{\lambda p,2}$ for $p \geq 5$, that are not étale pullbacks of standard spaces?
- Are there spaces $L_{\lambda p^{n-1},n}$ for $n \geq 3$, that are not étale pullbacks of standard spaces?

Answers to these questions would result in great progress in the understanding the role of étale pullbacks in generating examples of spaces $L_{\lambda p^{n-1},n}$, and more generally in the structure of these spaces when p, λ, n vary.

Structure of the paper. In Section 2, we present known results and useful constructions on $L_{\lambda p^{n-1},n}$ that are used in all the other sections. In Section 3, we recall an important characterization of spaces $L_{\lambda p,2}$, due to Pagot (Proposition 3.1), and we prove the non existence of spaces $L_{\lambda p,2}$ when $p > 3\lambda$. In Section 4, we generalize Proposition 3.1 to the case of spaces $L_{\lambda p^{n-1},n}$ and we apply this to the case $p = 2$ to construct our new examples of spaces $L_{\lambda 2^{n-1},n}$. In Section 5, we define and study *standard* $L_{\lambda p^{n-1},n}$ -spaces. In Section 6, we fix $p = 3$ and provide a complete classification of spaces $L_{12,2}$ and $L_{15,2}$. Finally, in Appendix A we collect all the results on Moore determinants that are used throughout the paper (mostly in Sections 4 and 5).

Notation and conventions. Let k be an algebraically closed field of characteristic $p > 0$. Recall that a differential form on \mathbb{P}_k^1 can be written as $\omega = f(X)dX$ for $f(X) \in k(X)$, the field of rational fractions in one variable over k . Such a differential form is called *logarithmic* if it is of the form $\omega = \frac{dF}{F}$ for some $F \in k(X)$. We usually denote by Ω a space $L_{m+1,n}$ and by $\{\omega_1, \dots, \omega_n\}$ a basis for this space. For a given subset $S \subset \Omega$, we denote by $\mathcal{P}(S)$ the subset of k consisting of elements that are poles of at least a non-zero differential form in S . If $\omega \in \Omega - \{0\}$, we write $\mathcal{P}(\omega)$ for the set of poles of ω . One deduces from the definition that the set $\mathcal{P}(\omega)$ consists of $m + 1$ simple poles. Finally, given a finite set of logarithmic differential forms $\{\omega_1, \dots, \omega_n\}$, we denote by $\langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ the \mathbb{F}_p -vector space that they generate. Most of the times, this will not be a space $L_{m+1,n}$, but it will be one under certain predetermined conditions.

2. PRELIMINARIES

In this section, we collect the preliminary results later used to show existence, non-existence and classification results of spaces $L_{m+1,n}$. We first recall known results on the number of poles in such spaces, as well as proving new lemmas on the combinatorics of the arrangements of such poles. Then we introduce useful constructions: Frobenius twists and étale pullbacks of spaces $L_{m+1,n}$. Finally, we briefly discuss known results in the case $n = 1$. The main result in this section that was not previously known is Corollary 2.10, stating that a space $L_{m+1,n}$ is characterized by its set of poles.

2.1. The Jacobson-Cartier condition. Let us recall here the *Jacobson-Cartier* condition for verifying that a meromorphic differential form on \mathbb{P}_k^1 is logarithmic. Let $\omega = f(X)dX \in \Omega(k(X))$ be such a form. Since k is perfect, we have that $k(X) = \bigoplus_{i=0}^{p-1} k(X)^p X^i$, and hence a unique writing

$$f(X) = \bigoplus_{i=0}^{p-1} f_i(X)^p X^i.$$

Note that the polynomial f_{p-1} is invariant by translation by any element $a \in k$. In fact, we can also write $f(X) = \bigoplus_{i=0}^{p-1} g_i(X)^p (X-a)^i$, and by comparing the coefficients we find that $f_{p-1} = g_{p-1}$. It is a classical result that ω is logarithmic if, and only if, $f(X) = f_{p-1}(X)$ ⁴. In our case, that of differential forms over the projective line, this fact has an elementary proof, that we provide below.

Proposition 2.1. *Let $\omega = f(X)dX \in \Omega(k(X))$ with $f(X) = \bigoplus_{i=0}^{p-1} f_i(X)^p X^i$ be a non-zero differential form. Then $\omega = \frac{dF}{F}$ for some $F \in k(X)$ if, and only if, $f(X) = f_{p-1}(X)$.*

Proof. Let $\{x_1, \dots, x_r\}$ be the set of poles of ω , which is non-empty because $\omega \neq 0$. To have $\omega = \frac{dF}{F}$ it is necessary and sufficient that the x_i 's are simple and their residues are in \mathbb{F}_p^\times . When this is the case, $\omega = \sum_{i=1}^r \frac{a_i}{X-x_i} dX$. So, assuming that ω is logarithmic (and hence $a_i \in \mathbb{F}_p^\times$), we find that

$$f_{p-1}(X) = \left(\sum_{i=1}^r \frac{a_i}{X-x_i} \right)_{p-1} = \sum_{i=1}^r \left(\frac{a_i}{X-x_i} \right)_{p-1} = \sum_{i=1}^r \frac{a_i}{X-x_i} = f(X).$$

Conversely, suppose that $f(X) = f_{p-1}(X)$. Then we can consider the pole expansion for the meromorphic function $f(X)$, namely:

$$f(X) = E(X) + \sum_{i=1}^r g_{x_i}(X), \text{ where } E(X) \in k[X] \text{ and } g_{x_i}(X) := \sum_{j>0} \frac{a_{ij}^p}{(X-x_i)^j} \text{ with } a_{ij} \in k.$$

Our assumption that $f(X) = f_{p-1}(X)$ can be rewritten as

$$E(X) = E_{p-1}(X) \text{ and } g_{x_i}(X) = (g_{x_i})_{p-1}(X).$$

So, if we write $E(X) = \sum_{i=0}^{p-1} E_i(X)^p X^i$ with $E_i(X) \in k[X]$ we have $\deg(E) = \max_i (p \deg E_i + i)$, so that $\deg(E) = \deg(E_{p-1})$ implies $E = 0$. Similarly, by writing $\frac{a_{ij}^p}{(X-x_i)^j} = \frac{a_{ij}^p}{(X-x_i)^{q+s}}$ for suitable q and $0 \leq s < p$, we have that $\frac{a_{ij}^p}{(X-x_i)^j} = \frac{a_{ij}^p (X-x_i)^{p-s}}{(X-x_i)^{(q+1)p}}$. If $s \neq 1$, then $\left(\frac{a_{ij}^p}{(X-x_i)^j} \right)_{p-1} = 0$, hence we

just need to consider the case $s = 1$, where we have $\left(\frac{a_{ij}^p}{(X-x_i)^j} \right)_{p-1} = \frac{a_{ij}}{(X-x_i)^{1+\frac{j-1}{p}}}$. The condition $g_{x_i}(X) = (g_{x_i})_{p-1}(X)$ implies then that $a_{ij}^p = a_{i(1+(j-1)p)}$ which is to say that $a_{i1} \in \mathbb{F}_p$ and $a_{ij} = 0$ for $j > 1$. Summarizing, we find that $E(X) = 0$ and $g_{x_i}(X) = \frac{a_{i1}}{X-x_i}$ with $a_{i1} \in \mathbb{F}_p$ for every i , which is to say that ω is a logarithmic differential form. \square

Corollary 2.2. *Let $\omega = f(X)dX \in \Omega(k(X))$ be such that $f(X) = \frac{1}{P(X)}$ with $P(X) \in k[X]$. Then $\omega = \frac{dF}{F}$ for some $F \in k(X)$ if, and only if, the $p-1$ -th derivative $(P^{p-1})^{(p-1)}$ of the polynomial P^{p-1} is equal to -1 .*

Proof. By Proposition 2.1, we have that $\omega = \frac{dF}{F}$ if, and only if, $f = f_{p-1}$. Since the $p-1$ -th derivative of f satisfies $f^{(p-1)} = (p-1)! f_{p-1}^p = -f_{p-1}^p$, then it is equivalent to require that $f^{(p-1)} = -f^p$. In the case where $f(X) = \frac{1}{P(X)}$, we find $f^{(p-1)} = \left(\frac{p^{p-1}}{P^p} \right)^{(p-1)} = \frac{(P^{p-1})^{(p-1)}}{P^p}$. On the other hand, $-f^p = -\frac{1}{P^p}$, and so the condition $f = f_{p-1}$ becomes $(P^{p-1})^{(p-1)} = -1$. \square

⁴In the case of curves this result is due to Jacobson. The study of the correspondence $f \rightarrow f_{p-1}$ as an operation over differential forms over curves is addressed in subsequent work of Tate, that was later generalized to higher dimensions by Cartier. Because of this, such correspondence is often known under the name of *Cartier operator*. We therefore deem it reasonable to refer to the result on curves as to the ‘‘Jacobson-Cartier’’ condition. We refer to [11, §10, §11] for a discussion of the topic that brings all these different perspectives together.

2.2. The combinatorics of poles of a space $L_{m+1,n}$. The condition that n logarithmic differential forms $\omega_1, \dots, \omega_n$ generate Ω a space $L_{m+1,n}$ put additional restrictions. Recall that $\mathcal{P}(\Omega)$ is the set of poles of at least a differential form in Ω . Pagot proved the following result [8, Lemme 5 and Lemme 6].

Lemma 2.3. *Let Ω be a space $L_{m+1,n}$. Then, there is an integer $\lambda > 0$ such that $m + 1 = \lambda p^{n-1}$. Moreover, $|\mathcal{P}(\Omega)| = \lambda \frac{p^n - 1}{p - 1}$. \square*

Thanks to this result, in the rest of the paper we can restrict to study spaces $L_{\lambda p^{n-1},n}$ for different values of p and λ , knowing that this hypothesis does not constitute a loss of generality.

Let Ω be a space $L_{\lambda p^{n-1},n}$, fix a basis $\{\omega_1, \dots, \omega_n\}$ of Ω and pick a pole $x \in \mathcal{P}(\Omega)$. We denote by h_i the residue of ω_i at x for $i = 1, \dots, n$, setting $h_i = 0$ if x is not a pole of ω_i . Let $\omega = a_1\omega_1 + \dots + a_n\omega_n \in \Omega$ with $a_i \in \mathbb{F}_p$. Then we have that $x \in \mathcal{P}(\omega)$ if, and only if, $a_1h_1 + \dots + a_nh_n \neq 0$. As a result, the set $\mathfrak{H}(x) = \{\sum_{i=1}^n a_i\omega_i \in \Omega : \sum_{i=1}^n a_ih_i = 0\}$ is a hyperplane of Ω consisting of all the differential forms that do not have x as a pole. Conversely, given a hyperplane $H \subset \Omega$, we define $X_H := \mathcal{P}(\Omega) - \mathcal{P}(H)$, the subset of $\mathcal{P}(\Omega)$ consisting of all poles that do not occur as poles of any element in H . If we denote by $\mathcal{H}(\Omega)$ the set of hyperplanes of Ω , the correspondence $x \mapsto \mathfrak{H}(x)$ defines a function $\mathfrak{H} : \mathcal{P}(\Omega) \rightarrow \mathcal{H}(\Omega)$ with the property that $\mathfrak{H}^{-1}(H) = X_H$ for every $H \in \mathcal{H}(\Omega)$.

Lemma 2.4. *We have that $|X_H| = \lambda$ for every $H \in \mathcal{H}(\Omega)$. As a result, the map \mathfrak{H} is surjective and it is injective if, and only if, $\lambda = 1$.*

Proof. Note that a hyperplane $H \in \mathcal{H}(\Omega)$ is a space $L_{(\lambda p)p^{n-2},n-1}$. By Lemma 2.3 we have that $|\mathcal{P}(\Omega)| = \lambda \frac{p^n - 1}{p - 1}$, $|\mathcal{P}(H)| = \lambda p \frac{p^{n-1} - 1}{p - 1}$, and we know that $\mathcal{P}(H) \subset \mathcal{P}(\Omega)$. From this it follows that

$$|X_H| = |\mathcal{P}(\Omega) - \mathcal{P}(H)| = |\mathcal{P}(\Omega)| - |\mathcal{P}(H)| = \lambda \frac{p^n - 1}{p - 1} - \lambda p \frac{p^{n-1} - 1}{p - 1} = \lambda \frac{p^n - 1 - p^n + p}{p - 1} = \lambda. \quad \square$$

The result of Lemma 2.4 can be rephrased by observing that $\mathcal{P}(\Omega) = \cup_H X_H$ is a union of sets of cardinality λ , indexed by the $\frac{p^n - 1}{p - 1}$ hyperplanes of Ω . In the light of 2.3, we see that this is in fact a disjoint union.

The following Corollary is essentially equivalent to [7, Lemme 2.2.2.]. We include here a proof that uses the notation above for the reader's convenience.

Corollary 2.5. *Let Ω be a space $L_{\lambda p^{n-1},n}$ and let $\{\omega_1, \dots, \omega_n\}$ be a basis of Ω . Then, for every $1 \leq r \leq n$ we have that $|\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r)| = \lambda(p - 1)^{r-1} p^{n-r}$.*

Proof. We begin by remarking that, for every $\omega \in \Omega - \{0\}$ and $H \in \mathcal{H}(\Omega)$ either $\mathfrak{H}^{-1}(H) \subset \mathcal{P}(\omega)$ or $\mathfrak{H}^{-1}(H) \cap \mathcal{P}(\omega) = \emptyset$. As a result, $\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r)$ is a union of sets of the form X_H , and therefore $\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r) = \mathfrak{H}^{-1}(\mathfrak{H}(\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r)))$. We then have

$$|\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r)| = \lambda |\mathfrak{H}(\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r))|.$$

We can then conclude with a hyperplane-counting argument: we have that $\mathfrak{H}(\mathcal{P}(\omega_1))$ is the set of hyperplanes of Ω not containing ω_1 . More in general, since $\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r)$ is a union of sets of the form X_H , we have that $\mathfrak{H}(\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r)) = \mathfrak{H}(\mathcal{P}(\omega_1)) \cap \dots \cap \mathfrak{H}(\mathcal{P}(\omega_r))$. This latter is the set of hyperplanes of Ω not containing ω_i for any $i \in \{1, \dots, r\}$ and so its cardinality is $\frac{p^{n-r}(p-1)^r}{p-1}$. As a result, we also have that

$$|\mathcal{P}(\omega_1) \cap \dots \cap \mathcal{P}(\omega_r)| = \lambda p^{n-r} (p - 1)^{r-1}. \quad \square$$

When $n = 2$, the result above specializes to the following.

Corollary 2.6. *Let Ω be a space $L_{\lambda p, 2}$, and let $\omega, \omega' \in \Omega - \{0\}$. Then,*

- (i) $\langle \omega \rangle_{\mathbb{F}_p} = \langle \omega' \rangle_{\mathbb{F}_p}$ if, and only if, $|\mathcal{P}(\omega) \cap \mathcal{P}(\omega')| = \lambda p$
- (ii) $\langle \omega, \omega' \rangle_{\mathbb{F}_p} = \Omega$ if, and only if, $|\mathcal{P}(\omega) \cap \mathcal{P}(\omega')| = \lambda(p - 1)$.

□

Let us now show that the set of poles of a space $L_{m+1, n}$ completely characterizes such a space, starting with the case $n = 1$.

Lemma 2.7. *Let ω, ω' be logarithmic differential forms on \mathbb{P}_k^1 with a unique zero at ∞ of order $m - 1$, and let $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega')$ be the respective set of poles. Then, $\mathcal{P}(\omega) = \mathcal{P}(\omega')$ if, and only if $\langle \omega \rangle_{\mathbb{F}_p} = \langle \omega' \rangle_{\mathbb{F}_p}$.*

Proof. The non-trivial part is to prove that $\mathcal{P}(\omega) = \mathcal{P}(\omega')$ implies that the two forms generate the same \mathbb{F}_p -vector space. Let $\mathcal{P}(\omega) = \mathcal{P}(\omega') = \{x_0, \dots, x_m\}$ and let us show that there exists $c \in \mathbb{F}_p^\times$ such that $\omega' = c\omega$. As these differential forms are logarithmic with no zeroes outside ∞ , we have the unique writings $\omega = \sum_{i=0}^m \frac{a_i}{X-x_i} dX$ and $\omega' = \sum_{i=0}^m \frac{b_i}{X-x_i} dX$, with $a_i, b_i \in \mathbb{F}_p^\times$.

Suppose by contradiction that there is no $c \in \mathbb{F}_p^\times$ such that $a_i = cb_i$ for every i . Then there is a $j \in \mathbb{F}_p^\times$ such that $a_i + jb_i = 0$ for some but not all i 's. Then, the form $\omega + j\omega'$ is a non-zero logarithmic differential form with a zero of order at least $m - 1$ at infinity, and at the same time it has at most m simple poles. This is not possible, since the degree of any (non-zero) meromorphic differential form on \mathbb{P}_k^1 is -2 . □

Proposition 2.8. *Let $n \geq 2$, let Ω be a space $L_{\lambda p^{n-1}, n}$ and let ω' be a logarithmic differential form having a unique zero at infinity of order $\lambda p^{n-1} - 2$ and such that $\mathcal{P}(\omega') \subset \mathcal{P}(\Omega)$. Then $\omega' \in \Omega$.*

Proof. Suppose by contradiction that $\omega' \notin \Omega$. Let $\omega \in \Omega - \{0\}$ be a non-zero differential form. We begin our argument with a proof by contradiction that

$$(2.9) \quad |\mathcal{P}(\omega) \cap \mathcal{P}(\omega')| \leq \lambda(p^{n-1} - p^{n-2}).$$

For $j = 1, \dots, p - 1$, we set $Y_j := \{x \in \mathcal{P}(\omega) \cap \mathcal{P}(\omega') : \text{res}_x(\omega') = j \cdot \text{res}_x(\omega)\}$ in such a way that

$$\mathcal{P}(\omega) \cap \mathcal{P}(\omega') = \bigcup_{j=1}^{p-1} Y_j$$

is a disjoint union. If we assume by contradiction that $|\mathcal{P}(\omega) \cap \mathcal{P}(\omega')| > \lambda(p^{n-1} - p^{n-2})$, then there exists at least a value of j for which $|Y_j| > \frac{\lambda(p^{n-1} - p^{n-2})}{p-1} = \lambda p^{n-2}$. For such a j , we set $\omega_j := \omega' - j\omega$.

Since $\omega' \notin \Omega$, we have that $\omega_j \neq 0$ and then that ω_j is a logarithmic differential form with $\mathcal{P}(\omega_j) \cap Y_j = \emptyset$. Moreover, since both ω and ω' have a zero of order $\lambda p^{n-1} - 2$ at ∞ , then the order of ∞ as a zero of ω_j is at least $\lambda p^{n-1} - 2$, resulting in $|\mathcal{P}(\omega_j)| \geq \lambda p^{n-1}$, because all the poles are simple. By construction, we also have that $\mathcal{P}(\omega_j) = \mathcal{P}(\omega) \cup \mathcal{P}(\omega') - Y_j$. Combining this information, we estimate the cardinality $|\mathcal{P}(\omega_j)|$ as

$$\begin{aligned} \lambda p^{n-1} &\leq |\mathcal{P}(\omega_j)| = |\mathcal{P}(\omega)| + |\mathcal{P}(\omega')| - |\mathcal{P}(\omega) \cap \mathcal{P}(\omega')| - |Y_j| < 2\lambda p^{n-1} - \lambda(p^{n-1} - p^{n-2}) - \lambda p^{n-2} \\ &= \lambda p^{n-1}, \end{aligned}$$

leading to a contradiction and proving the validity of the inequality (2.9).

We then show that inequality (2.9) can not hold for every $\omega \in \Omega$ using the hypothesis that $\omega' \notin \Omega$. First of all, we fix a basis $\omega_1, \dots, \omega_n$ of Ω , we consider the dual basis $\omega_1^*, \dots, \omega_n^*$ of Ω^* and we index

every element of Ω as $\omega_{\underline{a}} := \sum a_i \omega_i$ for every $\underline{a} \in \mathbb{F}_p^n$. In this way, we have that $\omega' \neq \omega_{\underline{a}}$ for every $\underline{a} \in \mathbb{F}_p^n$. For all $\underline{\epsilon} := \epsilon_1 \omega_1^* + \dots + \epsilon_n \omega_n^* \in \mathbb{P}(\Omega^*)$, we set

$$H_{\underline{\epsilon}} := \{\omega_{\underline{a}} \in \Omega : \underline{\epsilon}(\omega_{\underline{a}}) = 0\},$$

$$X_{\underline{\epsilon}} := X_{H_{\underline{\epsilon}}} = \{x \in \mathcal{P}(\Omega) : \text{res}_x(\omega) = 0 \forall \omega \in H_{\underline{\epsilon}}\}$$

and

$$N_{\underline{\epsilon}} := |\mathcal{P}(\omega') \cap X_{\underline{\epsilon}}|.$$

Since $\mathcal{P}(\omega') \subset \mathcal{P}(\Omega)$, we have that $\mathcal{P}(\omega') = \bigsqcup_{\underline{\epsilon}} (\mathcal{P}(\omega') \cap X_{\underline{\epsilon}})$ and hence that $\sum_{\underline{\epsilon}} N_{\underline{\epsilon}} = \lambda p^{n-1}$.

We note that, for every $\underline{a} \in \mathbb{P}(\mathbb{F}_p^n)$ the set $\mathcal{P}(\omega_{\underline{a}})$ is the union

$$\mathcal{P}(\omega_{\underline{a}}) = \bigcup_{\substack{\underline{\epsilon} \in \mathbb{P}(\Omega^*) \\ \underline{\epsilon}(\omega_{\underline{a}}) \neq 0}} X_{\underline{\epsilon}},$$

and in particular it depends only on the class $[\underline{a}] \in \mathbb{P}(\mathbb{F}_p^n)$. To conclude the proof, we count $\sum_{[\underline{a}] \in \mathbb{P}(\mathbb{F}_p^n)} |\mathcal{P}(\omega_{\underline{a}}) \cap \mathcal{P}(\omega')|$ in two different ways:

On the one hand, we have

$$\begin{aligned} \sum_{[\underline{a}] \in \mathbb{P}(\mathbb{F}_p^n)} |\mathcal{P}(\omega_{\underline{a}}) \cap \mathcal{P}(\omega')| &= \sum_{[\underline{a}] \in \mathbb{P}(\mathbb{F}_p^n)} \sum_{\substack{\underline{\epsilon} \in \mathbb{P}(\Omega^*) \\ \underline{\epsilon}(\omega_{\underline{a}}) \neq 0}} N_{\underline{\epsilon}} = \sum_{\underline{\epsilon} \in \mathbb{P}(\Omega^*)} \sum_{\substack{[\underline{a}] \in \mathbb{P}(\mathbb{F}_p^n) \\ \underline{\epsilon}(\omega_{\underline{a}}) \neq 0}} N_{\underline{\epsilon}} \\ &= \sum_{\underline{\epsilon} \in \mathbb{P}(\Omega^*)} p^{n-1} N_{\underline{\epsilon}} = \lambda p^{n-1} p^{n-1} = \lambda p^{2n-2}. \end{aligned}$$

On the other hand, since $\omega' \neq \omega_{\underline{a}}$ for every $\underline{a} \in \mathbb{P}(\mathbb{F}_p^n)$, we can apply the inequality (2.9) with $\omega = \omega_{\underline{a}}$ for every $\underline{a} \in \mathbb{P}(\mathbb{F}_p^n)$ to get that

$$\sum_{[\underline{a}] \in \mathbb{P}(\mathbb{F}_p^n)} |\mathcal{P}(\omega_{\underline{a}}) \cap \mathcal{P}(\omega')| \leq \lambda(p^{n-1} - p^{n-2})(p^{n-1} + \dots + 1) = \lambda(p^{2n-2} - p^{n-2}).$$

We obtain the contradiction $\lambda p^{2n-2} \leq \lambda(p^{2n-2} - p^{n-2})$ and we conclude that $\omega' \in \Omega$. □

From Lemma 2.7 and Proposition 2.8 we deduce the very useful corollary that a space $L_{\lambda p^{n-1}, n}$ is characterized by its set of poles.

Corollary 2.10. *Let $n \geq 1$ and let Ω and Ω' be spaces $L_{\lambda p^{n-1}, n}$. Then $\Omega = \Omega'$ if, and only if, $\mathcal{P}(\Omega) = \mathcal{P}(\Omega')$.*

We conclude this section by establishing a notion of equivalence between two spaces $L_{\lambda p^{n-1}, n}$, which will be employed in later sections.

Definition 2.11. If Ω, Ω' are two spaces $L_{\lambda p^{n-1}, n}$, we say that they are *equivalent* if there is an automorphism $\sigma \in \text{Aut}_k(\mathbb{P}_k^1)$ such that $\sigma(\infty) = \infty$ and $\Omega' = \sigma^* \Omega$.

An immediate consequence of Corollary 2.10 is that Ω is equivalent to Ω' if, and only if, there exist $a \in k^\times, b \in k$ such that $\mathcal{P}(\Omega') = a\mathcal{P}(\Omega) + b$.

2.3. Frobenius action and étale pullbacks. Given a space $L_{\lambda p^{n-1}, n}$, there are two constructions that we can apply to construct more spaces.

The first construction exploits the action of the relative Frobenius on a space $L_{\lambda p^{n-1}, n}$. Recall that $\Omega(k(X))$ denotes the k -algebra of meromorphic differential forms on \mathbb{P}_k^1 . Consider the relative Frobenius operator $\Phi : \Omega(k(X)) \rightarrow \Omega(k(X))$ acting on the coefficients of a form by raising them to the power p :

$$\Phi \left(\frac{\sum a_i X^i}{\sum b_i X^i} dX \right) = \frac{\sum a_i^p X^i}{\sum b_i^p X^i} dX.$$

Then we have the following result:

Lemma 2.12. *Let Ω be a space $L_{\lambda p^{n-1}, n}$. Then $\Phi(\Omega)$ is again a space $L_{\lambda p^{n-1}, n}$. Moreover, for every choice of p, λ , and n , Φ is bijective when restricted to the set of spaces $L_{\lambda p^{n-1}, n}$.*

Proof. Let $\omega = \sum_{i=1}^{\lambda p^{n-1}} \frac{a_i}{X-x_i} dX \in \Omega$ with $a_i \in \mathbb{F}_p^\times$. Then $\Phi(\omega) = \sum_{i=1}^{\lambda p^{n-1}} \frac{a_i^p}{X-x_i^p} dX$ is clearly logarithmic. Moreover, it has a unique zero at ∞ because this condition is equivalent to the first line of the equations (2.15). Since \mathbb{F}_p -linearly independent forms are sent to linearly independent forms, we have that $\Phi(\Omega)$ is a space $L_{\lambda p^{n-1}, n}$.

To show the bijectivity note that, since k is algebraically closed, the Frobenius is an automorphism of k . Its inverse induces the inverse of Φ , which restricts naturally to the set of spaces $L_{\lambda p^{n-1}, n}$ for the reasons above. \square

Concretely, the relative Frobenius acts on the points of \mathbb{P}_k^1 by raising them to the p -th power. If Ω is a space $L_{\lambda p^{n-1}, n}$, then one gets the poles of $\Phi(\Omega)$ by raising to the p -th power the poles of Ω . This condition determines uniquely the space $\Phi(\Omega)$ thanks to Corollary 2.10.

The second construction exploits the properties of finite étale covers of the affine line in characteristic $p > 0$. More precisely, we fix $d > 0$ and we recall that in this setting a finite étale morphism $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ of degree dp is induced by a map $k[X] \rightarrow k[X]$ sending X to a polynomial of the form $\gamma X + T(X^p)$ with $\gamma \in k^\times$ and $T \in k[X]$ a polynomial of degree d . This is equivalent to ask that X is sent to a polynomial whose derivative is a non-zero constant. Such a morphism extends uniquely to a degree dp cover $\phi : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ branched only over ∞ , and we can consider the pullback map $\phi^* : \Omega(k(X)) \rightarrow \Omega(k(X))$.

Lemma 2.13. *Let $\phi : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ be the compactification of a finite étale morphism $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ of degree dp . Let $\Omega = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ be a space $L_{\lambda p^{n-1}, n}$. Then, the \mathbb{F}_p -vector space generated by the differential forms $\phi^*(\omega_1), \dots, \phi^*(\omega_n)$ is a space $L_{d\lambda p^n, n}$, called the étale pullback of Ω via ϕ .*

Proof. The restriction of ϕ to \mathbb{A}_k^1 is induced by a polynomial $S(X)$ such that $S'(X) = \gamma \in k^\times$. Set $Z = S(X)$ and $\omega_i = \frac{dX}{P_i(X)} = \frac{F_i'(X)}{F_i(X)} dX$. Then $\phi^*(\omega_i) = \frac{[F_i(S(X))]' dX}{F_i(S(X))}$ and hence is logarithmic. Moreover

$$\phi^*(\omega_i) = \frac{dZ}{P_i(Z)} = \frac{\gamma dX}{P_i(S(X))}$$

and hence it has a unique zero of order $d\lambda p^n - 2$ at infinity. Finally, ϕ^* is a linear operator, hence we have $\sum_i a_i \phi^*(\omega_i) = \phi^*(\sum_i a_i \omega_i)$. It follows that any \mathbb{F}_p -linear combination of $\phi^*(\omega_i)$ is also a logarithmic differential form with a unique zero of order $d\lambda p^n - 2$ at infinity. \square

2.4. Known results on spaces $L_{\lambda, 1}$. It is easy to verify that every ℓ -dimensional subspace of a space $L_{\lambda p^{n-1}, n}$ is a space $L_{\lambda p^{n-1}, \ell}$. It is therefore useful to have results in the case $n = 1$, as these can provide significant information for studying the higher dimensions too. In this short section, we recall the known results in dimension one that will be used in the rest of the paper.

Proposition 2.14. *Let $\omega \in \Omega(k(X))$ be a differential form on the projective line \mathbb{P}_k^1 . The following conditions are equivalent:*

- (i) *The \mathbb{F}_p -vector space $\langle \omega \rangle_{\mathbb{F}_p}$ is a space $L_{\lambda,1}$.*
- (ii) *The differential form ω has precisely λ distinct simple poles x_1, \dots, x_λ and corresponding residues a_1, \dots, a_λ in \mathbb{F}_p^\times (which is equivalent to being logarithmic). Moreover, these poles and residues satisfy the polynomial equations:*

$$(2.15) \quad \sum_{i=1}^{\lambda} a_i x_i^k = 0 \text{ for } 0 \leq k \leq \lambda - 2.$$

- (iii) *We can write $\omega = \frac{1}{P(X)} dX$ with $P(X) \in k[X]$ of degree λ in such a way that the coefficient of X^{p-1} in the polynomial P^{p-1} is 1 and that the coefficient of $X^{\mu p-1}$ in the polynomial P^{p-1} vanishes for all $2 \leq \mu \leq \lambda + \lfloor \frac{\lambda-1}{p} \rfloor$.*

Proof. Let us assume (i). Then ω is logarithmic and has a unique zero of order $\lambda - 2$ at infinity. We write $\omega = \sum_{i=1}^{\lambda} \frac{a_i}{X-x_i} dX$ and we introduce a change of variable $Z = \frac{1}{X}$. Then,

$$\omega = \sum_{i=1}^{\lambda} \frac{-a_i}{Z(1-x_i Z)} dZ = \sum_{i=1}^{\lambda} \left(\frac{-a_i}{Z} + \frac{-a_i x_i}{1-x_i Z} \right) dZ = \sum_{i=1}^{\lambda} \frac{-a_i x_i}{(1-x_i Z)} dZ,$$

the first equality resulting from partial fraction expansion, and the second arising from the fact that ω has no poles at infinity and then $\sum_{i=1}^{\lambda} a_i = 0$. The further condition that the zero at infinity of ω is of order $\lambda - 2$ results in the equality

$$\sum_{i=1}^{\lambda} \frac{a_i x_i}{1-x_i Z} dZ = \frac{u Z^{\lambda-2}}{\prod_{i=1}^{\lambda} (1-x_i Z)} dZ,$$

for some $u \in k^\times$. By developing the denominators in this equality in formal power series in the variable Z , one obtains the following equations:

$$(2.16) \quad \begin{cases} \sum_{i=1}^{\lambda} a_i x_i^k = 0 \text{ for } 0 \leq k \leq \lambda - 2, \\ \sum_{i=1}^{\lambda} a_i x_i^{\lambda-1} = u, \\ \sum_{i=1}^{\lambda} a_i x_i^{\lambda+k-1} = u \cdot c_k(x_1, \dots, x_\lambda) \text{ for } k \geq 1, \end{cases}$$

where c_k is the k -th complete homogeneous symmetric polynomial. The first line show that the system (2.15) is satisfied, while the second line ensures that the poles are distinct.

Conversely, assume (ii). Then, we have that ω is logarithmic thanks to the condition on the poles and residues. Moreover, from the development in formal power series it follows that the equations in (2.15) imply that the zero at infinity is of order $\geq \lambda - 2$. Since there are precisely λ distinct simple poles, this is enough to conclude that there are no other zeroes and that the order at infinity is precisely $\lambda - 2$.

The equivalence between (iii) and (i) follows from Corollary 2.2. □

Remark 2.17. The element $u \in k^\times$ appearing in Equation (2.16) can be expressed in terms of the polynomial P appearing in condition (iii) as $u = \frac{1}{\alpha}$, where α is the leading coefficient of P .

It is easy to construct logarithmic differential forms ω satisfying the conditions of Proposition 2.14, as the following examples show:

Example 2.18. Let $\lambda \in \mathbb{Z}$ with $\lambda > 1$ and $(\lambda - 1, p) = 1$. Consider $f(X) := \frac{X^{\lambda-1}-1}{X^{\lambda-1}}$ and the associated logarithmic differential form $\omega := \frac{df}{f} = (\lambda - 1) \frac{dX}{X^{\lambda}-X}$. Then, we have that $\omega = \frac{dX}{P(X)}$ for $P(X) = \frac{X^{\lambda}-X}{\lambda-1}$. There are λ simple poles and no zeroes outside ∞ , then the unique zero at ∞ is of order $\lambda - 2$ and $\langle \omega \rangle_{\mathbb{F}_p}$ is a space $L_{\lambda,1}$

It follows from Example 2.18 that spaces $L_{\lambda,1}$ exist for all $(\lambda - 1, p) = 1$. The converse is also true: if $p | (\lambda - 1)$ then by Proposition 2.14 (iii) the leading term of P^{p-1} vanishes, but this implies that $\deg(P) < \lambda$, which is not possible. The paper [4] by Green and Matignon contains more results on spaces $L_{\lambda,1}$, such as a description of all possible spaces $L_{\lambda,1}$ in the case $\lambda < p + 1$. A simple but fundamental example that fits in this case is the following:

Example 2.19. Let $p \geq 3$ and $f(X) := \prod_{i=1}^{p-1} (X - i1_k)^i \in k[X]$, where $1_k \in k$ is the unity of the field k . Then, $\Omega := \langle \frac{df}{f} \rangle_{\mathbb{F}_p}$ is a space $L_{p-1,1}$. In fact, by construction the non-zero forms in Ω are logarithmic and their set of (simple) poles is $\{1, 2, \dots, p-1\}$. At the pole i , the residue of ω is equal to i . We can then verify that the equations (2.15) are satisfied:

$$\begin{cases} \sum_{i=1}^{p-1} i^k \equiv 0 \pmod{p} & \text{for } 1 \leq k \leq p-2 \\ \sum_{i=1}^{p-1} i^{p-1} \equiv \sum_{i=1}^{p-1} 1 \equiv -1 \pmod{p} \end{cases}$$

By Proposition 2.14, Ω is a space $L_{p-1,1}$. We can obtain the same result by rewriting the differential form as $\omega = \frac{df}{f} = \frac{dX}{1-X^{p-1}}$, from which the computation of residues also follows.

Remark 2.20. Without the assumption $m < p$, a deeper overview of the possible $m + 1$ -uples of residues \underline{a} (called *Hurwitz data*) is contained in Henrio's Ph.D. thesis [5]. It is worth noting that several questions about Hurwitz data remain unanswered (see also [8, §1.1]).

Remark 2.21. Let $q = p^t$ with $t \geq 1$. In [10, Definition 3.2.] a generalization of spaces $L_{m+1,n}$ to the setting of \mathbb{F}_q -vector spaces is introduced. Namely, a space $L_{m+1,n}^q$ is defined as a \mathbb{F}_q -vector space of differential forms on \mathbb{P}_k^1 whose nonzero elements have simple poles, a unique zero of order $m - 1$ at ∞ and residues in \mathbb{F}_q . We remark that all the results proved in this section for spaces $L_{m+1,n}$ generalize to spaces $L_{m+1,n}^q$. In particular, the key Proposition 2.1 has a natural analogue, and if one introduces the operator ∇ acting on the field $k(X)$ by

$$\nabla : f(X) \mapsto \left(f(X)^{(p-1)} \right)^{1/p},$$

the Jacobson-Cartier condition (Corollary 2.2) is generalized over \mathbb{F}_q by saying that a differential form $\omega = \frac{dX}{P(X)}$ has simple poles and residues in \mathbb{F}_q if, and only if, $\nabla^t(P(X)^{q-1}) = (-1)^t$. As in Proposition 2.14, it is therefore possible to express the fact that ω generates a space $L_{m+1,1}^q$ as a condition on the coefficients of $P(X)^{q-1}$. More precisely, the condition states that the coefficient of X^{q-1} is $(-1)^t$ and that the coefficients of $X^{\mu q-1}$ vanish for every $\mu \geq 2$.

More generally, it is not difficult to adapt the majority of the results in the rest of the papers to spaces $L_{m+1,n}^q$. The main idea is to change the key definition of the Moore determinant of an n -tuple $\underline{a} := (a_1, \dots, a_n) \in k^n$ into

$$\Delta_n(\underline{a}) := \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^q & a_2^q & \dots & a_n^q \\ \vdots & \vdots & \dots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \dots & a_n^{q^{n-1}} \end{vmatrix}$$

as already considered in [10]. The vanishing of this Moore determinant is a necessary and sufficient condition for \mathbb{F}_q -linear dependence of the elements of \underline{a} and key Theorem 4.8 restated in terms of q -Moore determinants and spaces $L_{m+1,n}^q$ holds with a very similar proof.

However, we have decided not to state our results in this full generality and the case $q = p$ remains the focus of our study: in fact, only in this setting the elements of a space $L_{m+1,n}^q$ are logarithmic, while we do not have a similar interpretation otherwise. This property is crucial to solve a concrete question about lifting local actions of elementary abelian groups (see [8, Théorème 11]), while it remains unknown what geometric interpretation or other use might have the existence of a space $L_{m+1,n}^q$ for q not prime.

3. AN OBSTRUCTION TO THE EXISTENCE OF SPACES $L_{\lambda p,2}$

We now consider the case $n = 2$, where we recall results established by Pagot and show that, for $p > 3\lambda$ there are no spaces $L_{\lambda p,2}$.

3.1. Known results on spaces $L_{\lambda p,2}$. We recall the following fundamental result by Pagot ([8, Proposition 7]). A proof is included, which contains elements that are crucial for the main result of this section, as well as for the generalization that we propose in Section 4.

Proposition 3.1. *Let $\omega_1, \omega_2 \in \Omega(k(X))$. Then the \mathbb{F}_p -vector space Ω generated by ω_1 and ω_2 is a space $L_{\lambda p,2}$ if and only if there exist two polynomials $Q_1, Q_2 \in k[X]$ satisfying the following conditions:*

- (i) $\deg(iQ_1 + jQ_2) = \lambda$, for every $[i : j] \in \mathbb{P}^1(\mathbb{F}_p)$
- (ii) $\omega_1 = \frac{Q_2 dX}{Q_1 Q_2^p - Q_1^p Q_2}$ and $\omega_2 = \frac{-Q_1 dX}{Q_1 Q_2^p - Q_1^p Q_2}$
- (iii) The $p - 1$ -th derivative $\left((Q_1^p - Q_1 Q_2^{p-1})^{p-1} \right)^{(p-1)}$ of the polynomial $(Q_1^p - Q_1 Q_2^{p-1})^{p-1}$ is equal to -1 .

Proof. Suppose that Ω is a space $L_{\lambda p,2}$. Let us fix a basis (ω_1, ω_2) of Ω , and remark that the set of poles of an element $i\omega_1 + j\omega_2 \in \Omega$ depends only on the corresponding $[i : j] \in \mathbb{P}^1(\mathbb{F}_p)$. Let us then denote by $X_{[i:j]}$ the set of poles of differential forms in Ω that are not poles of $i\omega_1 + j\omega_2$. By the results of section 2.2, every $X_{[i:j]}$ consists of λ elements and the set consisting of the $X_{[i:j]}$ for all $[i : j] \in \mathbb{P}^1(\mathbb{F}_p)$ is a partition of the set of $\lambda(p + 1)$ poles of Ω . Let us consider the polynomials

$$P_{[i:j]}(X) = \prod_{x \in X_{[i:j]}} (X - x) \quad \text{and} \quad P(X) = \prod_{[i:j] \in \mathbb{P}^1(\mathbb{F}_p)} P_{[i:j]}(X).$$

Since $i\omega_1 + j\omega_2$ has a unique zero at infinity and poles outside $X_{[i:j]}$, we need that

$$i\omega_1 + j\omega_2 = \frac{c_{ij} P_{[i:j]}(X)}{P(X)} dX$$

for nonzero constants $c_{ij} \in k$ that satisfy the condition

$$c_{ij} P_{[i:j]}(X) = ic_{10} P_{[1:0]}(X) + jc_{01} P_{[0:1]}(X).$$

Since all the $P_{[i:j]}$ are monic polynomials, this means in particular that $c_{ij} = ic_{10} + jc_{01}$. Let us set $a = \frac{c_{10}}{c_{01}}$ and note that $a \notin \mathbb{F}_p$, otherwise we would have $c_{(p-1)a} = 0$, which is not possible, as ω_1 and ω_2 are assumed \mathbb{F}_p -linearly independent. As a result, since k is algebraically closed, there is an element $c \in k^\times$ satisfying $c^p = -\frac{1}{c_{01}(a^p - a)}$. Let us set

$$Q_1 := -cP_{[0:1]} \quad \text{and} \quad Q_2 := acP_{[1:0]}.$$

Then, $iQ_2 - jQ_1 = c(iaP_{[1:0]} + jP_{[0:1]}) = c \frac{c_{ij}}{c_{01}} P_{[i:j]}$, which is a polynomial of degree λ .

Moreover, we have $Q_1 Q_2^p - Q_1^p Q_2 = -c^{p+1}(a^p - a)P = \frac{c}{c_{01}} P$ and hence $\omega_1 = \frac{Q_2 dX}{Q_1 Q_2^p - Q_1^p Q_2}$ and

$\omega_2 = \frac{-Q_1 dX}{Q_1 Q_2^p - Q_1^p Q_2}$, as required by condition (ii). Finally, condition (iii) in the statement is also satisfied, by virtue of Corollary 2.2.

Conversely, let us start with Q_1 and Q_2 satisfying the three conditions of the proposition and show that they give rise to a space $L_{\lambda p, 2}$. First let us show that the differential form $\omega_1 = \frac{Q_2 dX}{Q_1 Q_2^p - Q_1^p Q_2}$ is logarithmic if, and only if $\omega_2 = \frac{-Q_1 dX}{Q_1 Q_2^p - Q_1^p Q_2}$ is logarithmic. We have that

$$\begin{aligned} (Q_1^p Q_2 - Q_1 Q_2^p)^p &= Q_1^p Q_2 (Q_1^p Q_2 - Q_1 Q_2^p)^{p-1} - Q_1 Q_2^p (Q_1^p Q_2 - Q_1 Q_2^p)^{p-1} \\ &= Q_1^p Q_2^p (Q_1^p - Q_1 Q_2^{p-1})^{p-1} - Q_1^p Q_2^p (Q_1^{p-1} Q_2 - Q_2^p)^{p-1}. \end{aligned}$$

Since the derivative of the left hand side is 0, we have that the $p - 1$ -th derivative of the right hand side also vanishes, and therefore that $\left((Q_1^p - Q_1 Q_2^{p-1})^{p-1} \right)^{(p-1)} = \left((Q_1^{p-1} Q_2 - Q_2^p)^{p-1} \right)^{(p-1)}$. This implies that we only need to check via Corollary 2.2 that ω_2 is logarithmic to ensure that ω_1 is logarithmic too. In fact, this stays true if we replace ω_1 with $i\omega_1 + j\omega_2$ since $(Q_1^p Q_2 - Q_1 Q_2^p) = Q_1^p (iQ_1 + Q_2) - Q_1 (iQ_1 + Q_2)^p$. Hence, it suffices to have condition (iii) for the differential forms $i\omega_1 + j\omega_2$ to be logarithmic for every i and j . In particular, these have simple poles. To prove that the \mathbb{F}_p -vector space generated by ω_1 and ω_2 is of dimension 2, it suffices to remark that, by (i), there are λ poles of ω_1 that are not poles of ω_2 , so that ω_2 can not be a multiple of ω_1 . Finally, condition (ii) is enough to ensure that both ω_1 and ω_2 have ∞ as their only zero. \square

Remark 3.2. The polynomial $(Q_1 Q_2^p - Q_1^p Q_2)$ appearing in the denominators of ω_1, ω_2 is called the *Moore determinant* and denoted by $\Delta_2(Q_1, Q_2)$. This is the determinant of the Moore matrix $\begin{pmatrix} Q_1 & Q_2 \\ Q_1^p & Q_2^p \end{pmatrix}$. Its appearance is far from a coincidence: as we will see in Section 4, Moore determinants of higher order have a fundamental role in the generalizations of Pagot's result for spaces $L_{\lambda p^{n-1}, n}$. Moreover, results on Moore determinants will be helpful to simplify some proofs, even in the case of dimension 2. For this reason, we have collected the results we need on Moore determinants in the Appendix A.

In light of the result of the Proposition 3.1, we introduce the following definition, which will be extensively used in the classification of spaces $L_{12, 2}$ and $L_{15, 2}$:

Definition 3.3. Let $Q_1, Q_2 \in k[X]$ be polynomials of degree λ such that $\deg(iQ_1 + jQ_2) = \lambda$, for every $[i : j] \in \mathbb{P}^1(\mathbb{F}_p)$. Define the associated differential forms

$$\omega_1 := \frac{dX}{(Q_1 Q_2^{p-1} - Q_1^p)} \text{ and } \omega_2 := \frac{dX}{(Q_2 Q_1^{p-1} - Q_2^p)}.$$

We say that the pair (Q_1, Q_2) gives rise to the pair (ω_1, ω_2) . Moreover, if there exists $c \in k^\times$ such that the pair (cQ_1, cQ_2) gives rise to $(\frac{\omega_1}{c^p}, \frac{\omega_2}{c^p})$ a basis of Ω a space $L_{\lambda p, 2}$, then we say that the pair (Q_1, Q_2) is a *prompt* for the space Ω .

Remark 3.4. By Proposition 3.1, a pair (Q_1, Q_2) of polynomials of degree λ is a prompt for a space $L_{\lambda p, 2}$ if, and only if, Q_1 and Q_2 have leading coefficients that are \mathbb{F}_p -independent, and are such that $\left((Q_1^p - Q_1 Q_2^{p-1})^{p-1} \right)^{(p-1)}$ is a non-zero constant $d \in k^\times$. The number c then needs to satisfy $d(c^p)^{p-1} = -1$, and hence it is uniquely determined up to multiplication by a $p - 1$ -th root of unity.

Convention 3.5. If we have polynomials Q_1 and Q_2 satisfying the conditions (i) and (ii) of Proposition 3.1, we can write $(Q_1^p - Q_1 Q_2^{p-1})^{p-1}(X) = \sum r_i X^i$, and apply Proposition 2.14 to deduce that condition (iii) is equivalent to the equalities $r_{p-1} = 1$ and $r_{kp-1} = 0$ for $k = 2, \dots, \lambda(p - 1)$. The r_i 's are polynomials in the coefficients of Q_1 and Q_2 , and the equalities above will be used in Section

6 to classify certain spaces $L_{\lambda p, 2}$. In order to simplify a frequently used notation, we set $R_k := r_{kp-1}$ for $k = 1, \dots, \lambda(p-1)$.

Lemma 3.6. *Let $Q_1, Q_2 \in k[X]$ be polynomials of degree λ with*

$$Q_1(X) = a \left(X^\lambda + \sum_{i=1}^{\lambda} (-1)^i s_i X^{\lambda-i} \right)$$

$$Q_2(X) = b \left(X^\lambda + \sum_{i=1}^{\lambda} (-1)^i t_i X^{\lambda-i} \right).$$

If the pair (Q_1, Q_2) is a prompt for a space $L_{\lambda p, 2}$, then we have $s_1 = t_1$.

Proof. We have that Q_1 and Q_2 satisfy condition (iii) of Proposition 3.1 and hence the polynomials R_k of Convention 3.5 vanish for $k > 1$. In particular, this is true for $R_{\lambda(p-1)}$, the coefficient of degree $\lambda(p-1)p - 1$ of the polynomial $(Q_1^p - Q_1 Q_2^{p-1})^{p-1}(X)$. To compute $R_{\lambda(p-1)}$, let us write $(Q_1^p - Q_1 Q_2^{p-1})^{p-1} = X^{\lambda(p-1)p} \left[\left(\frac{Q_1}{X^\lambda} \right)^{p-1} \frac{Q_2}{X^\lambda} - \left(\frac{Q_2}{X^\lambda} \right)^p \right]^{p-1}$, introduce the variable $Z = \frac{1}{X}$ and compute the coefficient of Z in the expression in brackets above. We have:

$$\begin{aligned} \left(\frac{Q_1}{X^\lambda} \right)^{p-1} \frac{Q_2}{X^\lambda} - \left(\frac{Q_2}{X^\lambda} \right)^p &\equiv a^{p-1} b (1 - s_1 Z)^{p-1} (1 - t_1 Z) - b^p (1 - t_1^p Z^p) \pmod{Z^2} \\ &\equiv b (a^{p-1} (1 - (t_1 - s_1) Z) - b^{p-1}) \pmod{Z^2}. \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \left(\left(\frac{Q_1}{X^\lambda} \right)^{p-1} \frac{Q_2}{X^\lambda} - \left(\frac{Q_2}{X^\lambda} \right)^p \right)^{p-1} &\equiv b^{p-1} (a^{p-1} (1 - (t_1 - s_1) Z) - b^{p-1})^{p-1} \pmod{Z^2} \\ &\equiv b^{p-1} ((a^{p-1} - b^{p-1})^{p-1} + a^{p-1} (a^{p-1} - b^{p-1})^{p-2} (t_1 - s_1) Z) \pmod{Z^2}. \end{aligned}$$

We see then that $R_{\lambda(p-1)} = b^{p-1} a^{p-1} (a^{p-1} - b^{p-1})^{p-2} (t_1 - s_1) = 0$. Since a, b are nonzero and \mathbb{F}_p -linearly independent, then we have that $s_1 = t_1$. \square

3.2. A new generic obstruction to the existence of spaces $L_{\lambda p, 2}$. It is a result of Pagot (cf. [8, Théorème 1 and Théorème 2]) that spaces $L_{p, 2}$ and $L_{3p, 2}$ exist only for $p = 2$ and spaces $L_{2p, 2}$ exist only for $p = 2, 3$. In this section, we show that there are no spaces $L_{\lambda p, 2}$ if p is large enough with respect to λ , vastly improving on the previously known situation. The genericity in the title of the section refers then to the fact that our result holds for all but finitely many primes once λ is fixed. This allows for the finite remaining cases to be checked with a computer, since it suffices to check the existence of solutions of a polynomial system (see Convention 3.5). For $p = 3$ and $\lambda = 4, 5$, this is done in Section 6.

Let us now state our result. To simplify the demonstration, we exclude from the statement the case $\lambda = 1$, which has a known short proof (see [8, Théorème 2, Part 1]). By contrast, the proof in the other known cases $\lambda = 2, 3$ consists of several pages and the argument below consistently simplifies it.

Theorem 3.7. *Let $p > 3\lambda$. Then there are no spaces $L_{\lambda p, 2}$.*

To prove the theorem, we need to recall some notation and establish two fundamental lemmas. If we have a space $L_{\lambda p, 2}$ generated by a basis (ω_1, ω_2) and we consider the polynomials Q_1 and Q_2 giving rise to (ω_1, ω_2) , we recall from the proof of Proposition 3.1 that, for every $[i : j] \in \mathbb{P}^1(\mathbb{F}_p)$, $P_{[i:j]}$ denotes the monic polynomial whose zeroes are those of $iQ_2 - jQ_1$, and that a denotes the quotient of the leading terms of Q_1 and Q_2 , which satisfies $a \notin \mathbb{F}_p$. We are now ready to establish our lemmas:

Lemma 3.8. For every $t \in k - \{-a\}$, let $P_t := \frac{aP_{[1:0]} + tP_{[0:1]}}{a+t}$ and denote by $\text{Disc}(P_t)$ its discriminant. Then there exists a polynomial $R(X) \in k[X]$ such that:

(i) We have $\text{Disc}(P_t) = \frac{R(t)}{(a+t)^{2\lambda-3}}$ and $\deg(R(X)) \leq 2\lambda - 3$.

(ii) Let $p > 3\lambda$. Then the element $-a^p \in k$ is a zero of order $\geq \lambda + 3$ of $R(X)$.

Proof. We will first prove item (i), and then use it as one of the ingredients for the proof of (ii).

Proof of (i): Let a_i be the coefficient of degree i in the polynomial P_t . The discriminant $\text{Disc}(P_t)$ is the determinant of the Sylvester matrix

$$\begin{pmatrix} a_\lambda & a_{\lambda-1} & a_{\lambda-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_\lambda & a_{\lambda-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_\lambda & a_{\lambda-1} & \cdots & a_0 \\ \lambda a_\lambda & (\lambda-1)a_{\lambda-1} & (\lambda-2)a_{\lambda-2} & \cdots & a_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda a_\lambda & (\lambda-1)a_{\lambda-1} & \cdots & 2a_2 & a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda a_\lambda & (\lambda-1)a_{\lambda-1} & (\lambda-2)a_{\lambda-2} & \cdots & a_1 \end{pmatrix}.$$

Since both a_λ and $a_{\lambda-1}$ are independent of t (the former being 1 and the latter as a result of Lemma 3.6), the first two columns of this matrix are always the same for every t . As a result, we are left with at most $2\lambda - 3$ rows that contain quantities of the form $\frac{p_{ij}(t)}{a+t}$, where $p_{ij}(t)$ is either zero or a polynomial of degree 1. Applying Leibniz formula for the determinant, we then obtain that $\text{Disc}(P_t) = \frac{R(t)}{(a+t)^{2\lambda-3}}$ for some $R(t)$ of degree at most $2\lambda - 3$ as desired.

Proof of (ii): Consider the differential form $\omega_1 = \frac{Q_2 dX}{Q_1 Q_2^p - Q_1^p Q_2}$. Remark that the poles of ω_1 are the zeroes of the polynomial $Q_1^p - Q_2^{p-1} Q_1 = \prod_{j=0}^{p-1} (Q_1 - jQ_2)$. Hence, for every pole x of ω_1 , there exists a number $i \in \{0, \dots, p-1\}$ such that x is a zero of $iQ_2 - Q_1$. The residue of ω_1 at x can then be computed as follows:

$$\text{res}_{\omega_1}(x) = \frac{-1}{\left(Q_1^p - Q_2^{p-1} Q_1\right)'(x)} = \frac{1}{(iQ_2 - Q_1)'(x) \prod_{j \neq i} (jQ_2 - Q_1)(x)} = \frac{1}{(iQ_2 - Q_1)'(x) (-Q_2^{p-1}(x))}.$$

For every $i = 0, \dots, p-1$, we recall from the proof of Proposition 3.1 that $iQ_2 - Q_1 = c(a+i)P_{[i:1]}$, and we observe that $P_{[i:1]}$ coincides with P_i as in the statement of the Lemma. We can then rewrite the identity above as

$$\text{res}_{\omega_1}(x) = -\frac{1}{c(a+i)P_i'(x)Q_2(x)^{p-1}}.$$

We can then consider the product H_i of the residues at all the poles that are roots of P_i

$$H_i := \prod_{x \in Z(P_i)} \text{res}_{\omega_1}(x) = \frac{(-1)^\lambda}{c^\lambda (a+i)^\lambda \prod_{x \in Z(P_i)} P_i'(x) \prod_{x \in Z(P_i)} Q_2^{p-1}(x)}.$$

If we denote by $\text{Res}(\bullet, \bullet)$ the resultant of two polynomials, we can rewrite the above as

$$H_i = \frac{(c(a+i))^{\lambda(p-2)}}{(-1)^{\frac{\lambda(\lambda+1)}{2}} \text{Disc}(P_i) \text{Res}(iQ_2 - Q_1, Q_2)^{p-1}} = (-1)^{\frac{\lambda(\lambda+1)}{2}} \frac{(c(a+i))^{\lambda(p-2)}}{\text{Disc}(P_i) \text{Res}(-Q_1, Q_2)^{p-1}}.$$

By Lemma 3.8 (i), we can express the discriminant $\text{Disc}(P_i)$ in terms of the polynomial $R(X)$ to obtain that $H_i = \delta \frac{(a+i)^{\lambda p-3}}{R(i)}$, where we have set $\delta := (-1)^{\frac{\lambda(\lambda+1)}{2}} \frac{c^{\lambda(p-2)}}{\text{Res}(-Q_1, Q_2)^{p-1}}$ for ease of notation, since this is independent of i .

Since the differential form ω_1 is logarithmic, we have that $H_i \in \mathbb{F}_p^\times$ and in particular that $H_i^{p-1} = 1$. The following equations then hold for every $i \in \{0, \dots, p-1\}$:

$$\begin{aligned}\delta^{p-1}(a+i)^{(\lambda p-3)(p-1)} &= R(i)^{p-1} \\ \delta^{p-1}(a+i)^{(\lambda p^2-(\lambda+3)p+3)} R(i) &= R(i)^p \\ \delta^{p-1}(a+i)^{\lambda p^2} (a+i)^3 R(i) &= R(i)^p (a+i)^{(\lambda+3)p} \\ \delta^{p-1}(a^{p^2}+i)^\lambda (a+i)^3 R(i) &= R_p(i)(a^p+i)^{(\lambda+3)},\end{aligned}$$

where $R_p(X)$ denotes the polynomial obtained from $R(X)$ by raising its coefficients to the p -th power. We thus have obtained the equation

$$\delta^{p-1}(a^{p^2}+i)^\lambda (a+i)^3 R(i) - R_p(i)(a^p+i)^{(\lambda+3)} = 0,$$

which is a polynomial equation of degree at most 3λ in i that is satisfied for every $i = 0, \dots, p-1$. Since we have that $p > 3\lambda$, this is actually an equality of univariate polynomials, namely we have

$$\delta^{p-1}(a^{p^2}+X)^\lambda (a+X)^3 R(X) = R_p(X)(a^p+X)^{(\lambda+3)}$$

in the ring $k[X]$. The right hand side of the equation admits $-a^p$ as root of order at least $\lambda+3$. Since $a^p \neq a$ and $a^p \neq a^{p^2}$ we conclude that $R(X)$ has $-a^p$ as root of order at least $\lambda+3$, as well. \square

Remark 3.9. In spite of its fairly elementary proof, Lemma 3.8 (i) is already quite powerful. Combining the lower bound on the order of $-a^p$ as a zero of $R(X)$ given in (ii) and the upper bound on the degree of $R(X)$ given by (i) one gets that $2\lambda - 3 \geq \lambda + 3$, which gives $\lambda \geq 6$. It follows that no further argument is needed to prove Theorem 3.7 when $\lambda \leq 5$.

Lemma 3.10. *Let k be an algebraically closed field of characteristic $p > 0$. Let P, Q be coprime polynomials in $k[X]$ such that $0 \leq \deg(Q) < \deg(P) < p$ and consider the polynomial function*

$$z \mapsto D(z) := \text{Disc}(P - zQ) \in k[z].$$

Let Z_D be the set of zeroes of D and Z_δ be the set of zeroes of $\left(\frac{P}{Q}\right)'$ where in both sets zeroes are counted with multiplicity. Then the correspondence

$$\begin{aligned}F : Z_\delta &\rightarrow Z_D \\ x &\mapsto \frac{P(x)}{Q(x)}\end{aligned}$$

is a well defined bijective function.

Proof. For every $t \in k$, we denote by Z_t be the set of zeroes of $P - tQ$ (counted with multiplicity). The condition $\deg(Q) < \deg(P) < p$ ensures that Z_D , Z_δ and Z_t are all finite sets, as they are sets of roots of non-zero polynomials.

We have that $D(t) = c_0 \text{Res}(P - tQ, P' - tQ')$, where $c_0 \in k$ does not depend on t . and we can then write

$$D(t) = c_1 \prod_{x \in Z_t} (P' - tQ')(x),$$

for $c_1 \in k^\times$ not depending on t . We have that $\gcd(P, Q) = 1$, which guarantees that $\prod_{x \in Z_t} Q(x) \neq 0$, and we can then write

$$D(t) = c_1 \frac{1}{\prod_{x \in Z_t} Q(x)} \prod_{x \in Z_t} (P'(x)Q(x) - tQ'(x)Q(x)) = c_2 \frac{1}{\text{Res}(P - tQ, Q)} \text{Res}(P'Q - tQ'Q, P - tQ),$$

where $c_2 \in k^\times$ is independent of t .

We now set $N := P'Q - PQ'$ and denote by Z_N its set of zeroes (counted with multiplicity). By using this notation and properties of resultants, we transform the equation above into

$$D(t) = c_2 \frac{1}{\text{Res}(P, Q)} \text{Res}(P'Q - PQ', P - tQ) = c_3 \prod_{x \in Z_N} (P - tQ)(x),$$

where $c_3 \in k^\times$ is independent of t . Then, we define the following sets: the set $Z_{N,Q}$ of common zeroes of N and Q (counted with multiplicities) and the set ${}^c Z_{N,Q}$ of zeroes of N that are not zeroes of Q . We note that, if $x \in Z_{N,Q}$ then it is also a zero of Q' : as a result, its multiplicity as a zero of Q is at least two, and precisely one more than the multiplicity of x as a zero of N . From this, it follows that $Z_N = Z_{N,Q} \sqcup {}^c Z_{N,Q}$. On the other hand, if $x \in {}^c Z_{N,Q}$ we can use the fact that $\left(\frac{P}{Q}\right)' = \frac{N}{Q^2}$ to deduce that the multiplicity of x as a zero of N is the same as the multiplicity as a zero of $\left(\frac{P}{Q}\right)'$. It follows from this last consideration that $Z_\delta = {}^c Z_{N,Q}$. In particular, $Z_\delta \cap Z_Q = \emptyset$ and the function F is well defined.

The above equation then gets rewritten as

$$\begin{aligned} D(t) &= c_3 \prod_{x \in Z_{N,Q}} (P - tQ)(x) \prod_{x \in {}^c Z_{N,Q}} (P - tQ)(x) \\ &= c_3 \prod_{x \in Z_{N,Q}} P(x) \prod_{x \in {}^c Z_{N,Q}} (P - tQ)(x) \\ &= c_4 \prod_{x \in {}^c Z_{N,Q}} (P - tQ)(x) \\ &= c_4 \prod_{x \in {}^c Z_{N,Q}} Q(x) \prod_{x \in {}^c Z_{N,Q}} \left(\frac{P(x)}{Q(x)} - t\right) \\ &= c_5 \prod_{x \in {}^c Z_{N,Q}} \left(\frac{P(x)}{Q(x)} - t\right), \end{aligned}$$

where $c_4, c_5 \in k^\times$ are independent of t . It then follows that the function F is surjective onto Z_D . From the equation, it also follows that $|Z_D| = |{}^c Z_{N,Q}|$. Since $Z_\delta = {}^c Z_{N,Q}$, then Z_δ and Z_D have the same cardinality and F is bijective. \square

Proof of Theorem 3.7. Assume by contradiction that there is a space $L_{\lambda p, 2}$ with $p > 3\lambda$. We set $P := aP_{[1:0]} - a^p P_{[0:1]}$ and $Q := a(P_{[1:0]} - P_{[0:1]})$ and we remark that these satisfy the conditions of Lemma 3.10. As a result, the zeroes of $z \mapsto D(z) = \text{Disc}(P - zQ)$ are all of the form $\frac{P(x)}{Q(x)}$ for x a zero of $\left(\frac{P}{Q}\right)'$. In particular, $z = 0$ is a zero of D of order at most $\deg(P') = \lambda - 1$, as it corresponds to a zero x of $\left(\frac{P}{Q}\right)'$ that also satisfies $P(x) = 0$ (and therefore also $P'(x) = 0$).

Let us now give a lower bound to the order of 0 as a zero of D and see that it is incompatible with the one above. To do this, we apply the function D to a new variable $z = \frac{t+a^p}{t+a}$. In this way, we have

$$\begin{aligned} D(z) &= \text{Disc}\left(P - \frac{(t+a^p)}{(t+a)}Q\right) = \text{Disc}\left(\frac{(t+a)P - (t+a^p)Q}{(t+a)}\right) \\ &= \text{Disc}\left(\frac{(a-a^p)(aP_{[1:0]} + tP_{[0:1]})}{(t+a)}\right) = \text{Disc}((a-a^p)P_t) = (a-a^p)^{2\lambda-1} \text{Disc}(P_t). \end{aligned}$$

By Lemma 3.8 (i), this last expression can be written in terms of $R(t)$, giving

$$D(z) = (a - a^p)^{2\lambda-1} \frac{R(t)}{(a+t)^{2\lambda-3}}.$$

By Lemma 3.8 (ii), $-a^p$ is a zero of order at least $\lambda + 3$ of the polynomial function $t \mapsto R(t)$ and, since the expression of z in t is a linear fractional transformation, we have that 0 is a zero of the same order of the polynomial $z \mapsto D(z)$. This gives the desired contradiction and concludes the proof of the theorem. \square

4. CONDITIONS FOR THE EXISTENCE OF SPACES $L_{\lambda p^{n-1}, n}$

In this section, we prove a generalization of Proposition 3.1 that applies to spaces $L_{\lambda p^{n-1}, n}$ for any $n \geq 2$ and discuss some of its consequences. As anticipated in 3.2, our strategy makes a crucial use of Moore determinants. Definition and results about Moore determinants that we use in this section are recalled in Appendix A. For every n -tuple of the form $\underline{X} := (X_1, \dots, X_n)$, we denote by $\Delta_n(\underline{X})$ the associated Moore determinant. Moreover, we denote by \hat{X}_i the $n-1$ -uple obtained from \underline{X} by removing X_i and by $\Delta_{n-1}(\hat{X}_i)$ the associated Moore determinant.

We develop our results in the following setting: we let $Q_1, \dots, Q_n \in k[X]$ be polynomials of degree $\lambda \geq 1$ and denote by q_i the leading coefficient of Q_i for $1 \leq i \leq n$. We write $P := \Delta_n(\underline{Q})$, $P_i := (-1)^{i-1} \Delta_{n-1}(\hat{Q}_i)$ and $P_\epsilon := \sum_i \epsilon_i P_i$ for every $\epsilon \in \mathbb{F}_p^n - \{0\}$.

Lemma 4.1. *The polynomial P and the n -tuple $\underline{P} = (P_1, \dots, P_n)$ satisfy the relation*

$$\Delta_n(\underline{P}) = P^{1+p+\dots+p^{n-2}},$$

Proof. This is a direct corollary of Theorem A.3. \square

If we assume that $\Delta_n(\underline{q}) \neq 0$, then the q_i 's are \mathbb{F}_p -linearly independent. This entails that $\deg P = (1 + p + p^2 + \dots + p^{n-1})\lambda$ and $\deg P_\epsilon = (1 + p + p^2 + \dots + p^{n-2})\lambda$, since their leading coefficients are $\Delta_n(\underline{q})$ and $\sum_i \epsilon_i \Delta_{n-1}(\hat{q}_i)$ respectively. The first is nonzero by assumption, the second by virtue of [10, Corollary 2.1]

Proposition 4.2. *Assume that $\Delta_n(\underline{q}) \neq 0$. Then, we have that $P_\epsilon | P$ for every $\epsilon \in \mathbb{F}_p^n - \{0\}$.*

Proof. We first note that we have

$$P_\epsilon = \sum_{i=1}^n (-1)^{i-1} \epsilon_i \Delta_{n-1}(\hat{Q}_i) = \begin{vmatrix} \epsilon_1 & \epsilon_2 & \dots & \epsilon_n \\ Q_1 & Q_2 & \dots & Q_n \\ Q_1^p & Q_2^p & \dots & Q_n^p \\ \vdots & \vdots & \ddots & \vdots \\ Q_1^{p^{n-2}} & Q_2^{p^{n-2}} & \dots & Q_n^{p^{n-2}} \end{vmatrix},$$

a determinant that we denote by $\delta_\epsilon(\underline{Q})$ as in Appendix A.

Let $W := \langle Q_1, \dots, Q_n \rangle_{\mathbb{F}_p}$. Since $\Delta_n(\underline{q}) \neq 0$, then the Q_i 's are \mathbb{F}_p -linearly independent, hence $\dim W = n$. Let $\{Q_1^*, \dots, Q_n^*\}$ be the basis of W^* which is dual to $\{Q_1, \dots, Q_n\}$ and denote by $\varphi_\epsilon \in W^*$ the \mathbb{F}_p -linear form $\sum_{i=1}^n (-1)^{i-1} \epsilon_i Q_i^*$. Then by Formula (A.11) we have

$$\prod_{Q \in \ker \varphi_\epsilon - \{0\}} Q = (-1)^{n-1} \delta_\epsilon(\underline{Q})^{p-1} = (-1)^{n-1} P_\epsilon^{p-1}.$$

We choose the following system of representatives of the projectivization $\mathbb{P}(W)$ of W :

$$S(W) := \bigcup_{i=1}^n (Q_i + \mathbb{F}_p Q_{i-1} + \dots + \mathbb{F}_p Q_1).$$

and for every subspace $V \subset W$ we denote by $S(V)$ the intersection $V \cap S(W)$. It is a system of representatives of $\text{Proj}(V)$. A counting argument shows that $\prod_{Q \in V - \{0\}} Q = (-1)^{\dim V} (\prod_{Q \in S(V)} Q)^{p-1}$, which combined with the above gives the identity $P_{\underline{\epsilon}}^{p-1} = \left(\prod_{Q \in S(\ker \varphi_{\underline{\epsilon}})} Q \right)^{p-1}$. We have then that there exists $\mu \in \mathbb{F}_p^\times$ such that $P_{\underline{\epsilon}} = \mu \prod_{Q \in S(\ker \varphi_{\underline{\epsilon}})} Q$. By Equation (A.2), we have that $P = \Delta_n(\underline{Q}) = \prod_{Q \in S(W)} Q$. Since $S(\ker(\varphi_{\underline{\epsilon}})) \subset S(W)$ it follows that $P_{\underline{\epsilon}} | P$. \square

Remark 4.3. Proposition 4.2 shows that the expression $\frac{P}{P_{\underline{\epsilon}}}$ is a polynomial. We observe that it is an additive polynomial in the variable Q_i . Let us prove this for $i = n$, from which the other cases follow. First of all, from (A.2) we get that

$$P = \prod_{i=1}^n \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_1 \in \mathbb{F}_p} (Q_i + \epsilon_{i-1} Q_{i-1} + \cdots + \epsilon_1 Q_1)$$

and that

$$P_n = (-1)^{n-1} \prod_{i=1}^{n-1} \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_1 \in \mathbb{F}_p} (Q_i + \epsilon_{i-1} Q_{i-1} + \cdots + \epsilon_1 Q_1).$$

Putting these two equations together results in the formula

$$\frac{P}{P_n} = (-1)^{n-1} \prod_{\epsilon_{n-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_1 \in \mathbb{F}_p} (Q_n + \epsilon_{n-1} Q_{n-1} + \cdots + \epsilon_1 Q_1).$$

If we denote by \mathcal{Q}_{n-1} the \mathbb{F}_p -vector space $\langle Q_1, Q_2, \dots, Q_{n-1} \rangle_{\mathbb{F}_p}$ the formula above is rewritten as

$$\frac{P}{P_n} = (-1)^{n-1} P_{\mathcal{Q}_{n-1}}(Q_n),$$

where $P_{\mathcal{Q}_{n-1}}$ is the structural polynomial of \mathcal{Q}_{n-1} of Definition A.4, which is additive in the variable Q_n .

4.1. The main theorems. We now have all the tools to prove the two main results of this section (Theorems 4.7 and 4.8). Let us first establish some general results on the relationship between the Q_i 's and some spaces of differential forms that we can build from them:

Definition 4.4. Let $\underline{Q} := (Q_1, \dots, Q_n) \in k[X]^n$ be a n -tuple of polynomials of degree $\lambda \geq 1$ with leading coefficients q_i satisfying $\Delta_n(\underline{q}) \neq 0$. We write $P := \Delta_n(\underline{Q})$ and $P_i := (-1)^{i-1} \Delta_{n-1}(\hat{Q}_i)$. We define differential forms $\omega_i := \frac{P_i}{P} dX$ and the space $\Omega := \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$. We say that the n -tuple \underline{Q} gives rise to the basis $(\omega_1, \dots, \omega_n)$.

By Proposition 4.2 we have that for all $\underline{\epsilon} \in \mathbb{F}_p^n - \{0\}$ the polynomial $P_{\underline{\epsilon}}$ divides P and from Definition 4.4 we see that

$$\frac{P_{\underline{\epsilon}}}{P} dX = \epsilon_1 \omega_1 + \cdots + \epsilon_n \omega_n.$$

This entails that all the nonzero differential forms in Ω have a unique zero of order $\lambda p^{n-1} - 2$ at infinity. However, they are not in general logarithmic. In the following, we begin an investigation of conditions for Ω to be a space $L_{\lambda p^{n-1}, n}$ that culminates in Theorem 4.7.

For every $M \in GL_n(\mathbb{F}_p)$ we denote by $(QM)_1, \dots, (QM)_n$ the components of the vector \underline{QM} obtained by applying the matrix M to \underline{Q} . We note that all the entries $(QM)_i$'s are polynomials of degree λ with leading coefficients that are \mathbb{F}_p -independent. We associate with this n -tuple the differential forms $\omega_i^M := (-1)^{i-1} \frac{\Delta_{n-1}(\widehat{(QM)_i})}{\Delta_n(\underline{QM})} dX$ and the space $\Omega' := \langle \omega_1^M, \dots, \omega_n^M \rangle_{\mathbb{F}_p}$.

Proposition 4.5. *Assume the notation of Definition 4.4.*

4.5(i) *For every $M \in GL_n(\mathbb{F}_p)$ we have that*

$$(\omega_1^M, \dots, \omega_n^M) = (\omega_1, \dots, \omega_n)(M^{-1})^t,$$

where $(M^{-1})^t \in GL_n(\mathbb{F}_p)$ is the transpose of the inverse of M . In particular, $\Omega' = \Omega$.

4.5(ii) *Let $\underline{Q} := (Q_1, \dots, Q_n)$ and $\underline{T} := (T_1, \dots, T_n)$ be n -tuples of polynomials in $k[X]$ giving rise to the same basis $(\omega_1, \dots, \omega_n)$. Then $\underline{Q} = \underline{T}$.*

4.5(iii) *Let $\underline{Q} := (Q_1, \dots, Q_n)$ and $\underline{T} := (T_1, \dots, T_n)$ be n -tuples of polynomials in $k[X]$ giving rise to bases of the same space Ω . Then, there exists a matrix $M \in GL_n(\mathbb{F}_p)$ such that $\underline{T} = \underline{Q}M$.*

Proof. (i) For every $i, j \in \{1, \dots, n\}$ we denote by $M_{i,j}$ the (i, j) -th minor of the matrix M . We then have that $\Delta_{n-1}(\widehat{(\underline{Q}M)}_j) = \sum_{i=1}^n M_{i,j} \Delta_{n-1}(\hat{Q}_i)$ using \mathbb{F}_p -multilinearity and the alternating property of Δ_{n-1} . Then, using the fact that $\Delta_n(\underline{Q}M) = \Delta_n(\underline{Q}) \det(M)$ and that M is invertible, we get

$$\begin{aligned} \omega_j^M &:= \frac{(-1)^{j+1} \Delta_{n-1}(\widehat{(\underline{Q}M)}_j)}{\Delta_n(\underline{Q}M)} dX = \frac{(-1)^{j+1} \sum_{i=1}^n M_{i,j} \Delta_{n-1}(\hat{Q}_i)}{\Delta_n(\underline{Q}M)} dX = \frac{\sum_{i=1}^n (-1)^{i+j} M_{i,j} P_i}{\Delta_n(\underline{Q}M)} dX \\ &= \frac{1}{\det(M)} \sum_{i=1}^n (-1)^{i+j} \frac{M_{i,j} P_i}{P} dX = \sum_{i=1}^n \frac{(-1)^{i+j}}{\det(M)} M_{i,j} \omega_i. \end{aligned}$$

In other words, $(\omega_1^M, \dots, \omega_n^M) = (\omega_1, \dots, \omega_n)(M^{-1})^t$. We have then that $\omega_1^M, \dots, \omega_n^M$ is a basis of Ω for every invertible matrix $M \in GL_n(\mathbb{F}_p)$.

(ii) and (iii) Let $\omega_{i,Q} := \frac{(-1)^{i-1} \Delta_{n-1}(\hat{Q}_i)}{\Delta_n(\underline{Q})} dX$ and $\omega_{i,T} := \frac{(-1)^{i-1} \Delta_{n-1}(\hat{T}_i)}{\Delta_n(\underline{T})} dX$. Since \underline{T} and \underline{Q} both arise from the space Ω we have that there exists a matrix $N \in GL_n(\mathbb{F}_p)$ such that

$$(\omega_{1,Q}, \dots, \omega_{n,Q})N = (\omega_{1,T}, \dots, \omega_{n,T})$$

Then, by part (i), one has that the n -tuple $\underline{Q}(N^{-1})^t$ gives rise to the basis $(\omega_{1,T}, \dots, \omega_{n,T})$. Let $M := (N^{-1})^t$ and let us show that $\underline{Q}M = \underline{T}$: this will both prove (ii) (in which case $N = M = \mathbb{I}$) and (iii). From the fact that $\underline{Q}M$ and \underline{T} give rise to the same basis we get

$$\left(\frac{\Delta_{n-1}(\widehat{(\underline{Q}M)}_1)}{\Delta_n(\underline{Q}M)}, \dots, \frac{(-1)^{n-1} \Delta_{n-1}(\widehat{(\underline{Q}M)}_n)}{\Delta_n(\underline{Q}M)} \right) = \left(\frac{\Delta_{n-1}(\hat{T}_1)}{\Delta_n(\underline{T})}, \dots, \frac{(-1)^{n-1} \Delta_{n-1}(\hat{T}_n)}{\Delta_n(\underline{T})} \right).$$

We can then apply the Moore determinant to the terms of this equality and use Theorem A.3 to get that

$$\Delta_n(\underline{Q}M)^{p^{n-1}} = \Delta_n(\underline{T})^{p^{n-1}},$$

which implies that $\Delta_n(\underline{Q}M) = \Delta_n(\underline{T})$. Hence, we have that

$$(-1)^{i-1} \Delta_{n-1}(\widehat{(\underline{Q}M)}_i) = (-1)^{i-1} \Delta_{n-1}(\hat{T}_i) \quad \text{for every } i = 1, \dots, n$$

which we know by Proposition A.15 to be equivalent to the fact that $\underline{Q}M = \theta \underline{T}$ for some $\theta \in k(X)^{alg}$ with $\theta^{1+p+\dots+p^{n-2}} = 1$. But since $\Delta_n(\underline{Q}M) = \Delta_n(\underline{T})$, we have that $\theta^{1+\dots+p^{n-1}} = 1$ and hence $\theta^{p^{n-1}} = 1$, that is, $\theta = 1$. □

For a space $L_{\lambda p^{n-1}, n}$, we will show in Theorem 4.8 that we can always associate polynomials Q_1, \dots, Q_n giving rise to a basis as in Definition 4.4. In this context, Proposition 4.5(iii) says that

two choices of such a n -tuple are necessarily related by multiplication of an invertible matrix with entries in \mathbb{F}_p .

Let us now prove another useful proposition, first recalling from Definition A.4, that the *structural polynomial* of a \mathbb{F}_p -vector space V is defined as $P_V(X) := \prod_{v \in V} (X - v) \in k[X]$.

Proposition 4.6. *Let $\Omega = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ be as in definition 4.4. For every $1 \leq t \leq n$, let $\Omega_t \subset \Omega$ be the \mathbb{F}_p -subspace of Ω generated by $\{\omega_{n-t+1}, \dots, \omega_n\}$ and let $\mathcal{Q}_{n-t} = \langle Q_1, \dots, Q_{n-t} \rangle_{\mathbb{F}_p}$. Then, the t -tuple of polynomials $(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n))$ gives rise to the basis $(-1)^{n-t}(\omega_{n-t+1}, \dots, \omega_n)$ of Ω_t .*

Proof. If we specialize Corollary A.17 to the case $X_i = Q_i$, we get, for every $n - t + 1 \leq i \leq n$, that

$$\frac{\Delta_{n-1}(\widehat{Q_i})}{\Delta_n(\underline{Q})} = \frac{\Delta_{t-1}(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(\widehat{Q_i}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n))}{\Delta_t(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n))}.$$

As a result, the t -tuple $(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n))$ gives rise to the basis of Ω_t given by the t -tuple $((-1)^{n-t}\omega_{n-t+1}, \dots, (-1)^{n-t}\omega_n)$. \square

Theorem 4.7. *Let Ω be a space of differential forms constructed as in Definition 4.4. If there exists a non-zero $\omega \in \Omega$ that is a logarithmic differential form, then Ω is a space $L_{\lambda p^{n-1}, n}$.*

Proof. Lemma 4.1 ensures that $\Delta_n(\underline{P}) \neq 0$, hence the P_i 's are \mathbb{F}_p -linearly independent and then $\Omega = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ is a vector space of dimension n of differential forms that have a unique zero at ∞ (recall that by Proposition 4.2 we have that $P_{\underline{e}}|P$).

Up to a change of basis of Ω , we can assume that $\omega = \omega_n$. We then need to show that, if ω_n is a logarithmic differential form then all the forms in Ω are logarithmic. We start by claiming the following: if ω_n is logarithmic, then $\langle \omega_{n-1}, \omega_n \rangle_{\mathbb{F}_p}$ is a space $L_{\lambda p^{n-1}, 2}$. For this, we recall from the proof of Proposition 3.1 that it is sufficient to find polynomials R_1 and R_2 such that $\omega_n = \frac{R_1 dX}{R_1^p R_2 - R_1 R_2^p}$

and $\omega_{n-1} = \frac{R_2 dX}{R_1 R_2^p - R_1^p R_2}$. This is done by applying Proposition 4.6 to the case $t = 2$, and setting $R_1 = P_{\mathcal{Q}_{n-2}}(Q_{n-1})$ and $R_2 = P_{\mathcal{Q}_{n-2}}(Q_n)$.

It follows that ω_{n-1} is also a logarithmic differential form. Using the same argument, we can show that ω_i is logarithmic for every $i = 1, \dots, n - 1$ and then that every form in Ω is logarithmic, which entails that Ω is a space $L_{\lambda p^{n-1}, n}$.

To see that $P = \Delta_n(\underline{Q})$ has simple roots, note that its degree is precisely $\lambda \frac{p^n - 1}{p - 1}$, and that by construction its zeroes are the elements of $\mathcal{P}(\Omega)$, which we know by Lemma 2.3 to be a set of cardinality $\lambda \frac{p^n - 1}{p - 1}$. The roots then need to be simple. \square

The reciprocal of Theorem 4.7 also holds, completing the generalization of Proposition 3.1:

Theorem 4.8. *Let Ω be a space $L_{\lambda p^{n-1}, n}$ with $n \geq 2$. Then, there exist polynomials $Q_1, \dots, Q_n \in k[X]$ of degree λ such that, writing $P := \Delta_n(\underline{Q})$ and $P_i := (-1)^{i-1} \Delta_{n-1}(\widehat{Q_i})$, we have that $P_i|P$ and $\Omega = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ with $\omega_i = \frac{P_i}{P} dX$.*

Proof. The case $n = 2$ is provided by Proposition 3.1, so we can proceed by induction: we fix $n \geq 3$ and assume that we have the result of the theorem in dimension up to $n - 1$.

Let $(\omega_1, \dots, \omega_n)$ be a basis of Ω , and let $P(X) = \prod_{x \in \mathcal{P}(\Omega)} (X - x)$. Then we can find $P_1, \dots, P_n \in k[X]$

such that $\omega_i = \frac{P_i}{P} dX$. By Lemma 2.3 the degree of P is $\lambda(1 + p + \dots + p^{n-1})$ and then $\deg(P_i) = \lambda(1 + p + \dots + p^{n-2})$. By [10, Proposition 4.1] specialized at the case $q = p$ (cf. also the remark in

[8, p. 68]) we have that $\Delta_n(P_1, \dots, P_n) = \gamma P^{1+p+\dots+p^{n-2}}$ for some $\gamma \in k^\times$. By possibly multiplying P and the P_i 's by $\mu \in k^\times$ satisfying $\mu^{p^{n-1}} \gamma = 1$, we can assume that

$$(4.9) \quad \Delta_n(P_1, \dots, P_n) = P^{1+p+\dots+p^{n-2}}.$$

We now remark that the space $\langle \omega_1, \dots, \omega_{n-1} \rangle_{\mathbb{F}_p}$ is a space $L_{\lambda p^{n-1}, n-1}$ and then by inductive hypothesis there exist $S_1, \dots, S_{n-1} \in k[X]$ of degree λp such that $\frac{P_i}{P} = (-1)^{i-1} \frac{\Delta_{n-2}(\hat{S}_i)}{\Delta_{n-1}(\underline{S})}$, which means that $P_i = (-1)^{i-1} \Delta_{n-2}(\hat{S}_i) \frac{P}{\Delta_{n-1}(\underline{S})}$. We then have that

$$(4.10) \quad \Delta_{n-1}(P_1, \dots, P_{n-1}) = \Delta_{n-1}((-1)^{i-1} \Delta_{n-2}(\hat{S}_i)) \frac{P^{1+p+\dots+p^{n-2}}}{\Delta_{n-1}(\underline{S})^{1+p+\dots+p^{n-2}}} = \frac{P^{1+p+\dots+p^{n-2}}}{\Delta_{n-1}(\underline{S})^{p^{n-2}}},$$

where the last equality is obtained by applying Theorem A.3.

To conclude, we need to show that there exist $Q_1, Q_2, \dots, Q_n \in k[X]$ of degree λ such that $P = \Delta_n(Q)$ and $P_i = (-1)^{i-1} \Delta_{n-1}(\hat{Q}_i)$. Let $\varphi : k^n \rightarrow k^n$ be the map defined by $(\varphi(\underline{a}))_i = (-1)^{i-1} \Delta_{n-1}(\hat{a}_i)$. Then by Proposition A.15 (since $\Delta_n(\underline{P}) \neq 0$) there exist n -tuples $\underline{Q}, \underline{R}$ of elements of $k(X)^{\text{alg}}$ satisfying $\varphi(\underline{R}) = \underline{Q}$, $\varphi(\underline{Q}) = \underline{P}$, and

$$(4.11) \quad P_i = (-1)^{n-1} \Delta_n(\underline{R})^{1+p+\dots+p^{n-3}} R_i^{p^{n-2}}.$$

We ought to show that the entries of \underline{Q} are polynomials with coefficients in k . From equation (4.11) we can deduce the following identities

$$(4.12) \quad \Delta_{n-1}(P_1, \dots, P_{n-1}) = (-1)^{n-1} (\Delta_n(\underline{R}))^{(1+p+\dots+p^{n-3})(1+p+\dots+p^{n-2})} \Delta_{n-1}(R_1^{p^{n-2}}, R_2^{p^{n-2}}, \dots, R_{n-1}^{p^{n-2}})$$

$$(4.13) \quad \Delta_n(\underline{P}) = (-1)^{n(n-1)} \Delta_n(\underline{R})^{(1+p+\dots+p^{n-3})(1+p+\dots+p^{n-1})} \Delta_n(\underline{R})^{p^{n-2}} = \Delta_n(\underline{R})^{(1+p+\dots+p^{n-2})^2}.$$

Combining (4.13) with (4.9) gives

$$(4.14) \quad \Delta_n(\underline{R})^{1+p+\dots+p^{n-2}} = \theta P$$

for some $\theta \in k$ such that $\theta^{1+p+\dots+p^{n-2}} = 1$.

Moreover, we have that $Q_n = (-1)^{n-1} \Delta_{n-1}(R_1, \dots, R_{n-1})$, and then

$$\begin{aligned} Q_n^{p^{n-2}} &= (-1)^{n-1} \Delta_{n-1}(R_1^{p^{n-2}}, \dots, R_{n-1}^{p^{n-2}}) = \frac{\Delta_{n-1}(P_1, \dots, P_{n-1})}{\Delta_n(\underline{R})^{(1+p+\dots+p^{n-2})(1+p+\dots+p^{n-3})}} = \\ &= \frac{\Delta_{n-1}(P_1, \dots, P_{n-1})}{(\theta P)^{1+p+\dots+p^{n-3}}} = \frac{P^{1+p+\dots+p^{n-2}}}{(\theta P)^{1+p+\dots+p^{n-3}} \Delta_{n-1}(\underline{S})^{p^{n-2}}} = \frac{P^{p^{n-2}}}{\theta^{1+p+\dots+p^{n-3}} \Delta_{n-1}(\underline{S})^{p^{n-2}}}, \end{aligned}$$

where the equalities are obtained by applying equations (4.12), (4.14) and (4.10). Finally, using the fact that $\theta^{1+p+\dots+p^{n-2}} = 1$, we get that

$$Q_n = \frac{\theta P}{\Delta_{n-1}(\underline{S})},$$

which is a polynomial of degree λ thanks to the fact that the zeroes of $\Delta_{n-1}(\underline{S})$ are simple and correspond to the set of poles of the space $\langle \omega_1, \dots, \omega_{n-1} \rangle_{\mathbb{F}_p}$ (see Theorem 4.7). Moreover, by Lemma 2.4, we have that $\deg(Q_n) = \lambda$. In a completely analogous way, we can show that the Q_i 's are polynomials of degree λ also for $1 \leq i \leq n-1$. \square

Finally, we conclude the section with two results that relate the poles of a space $L_{\lambda p^{n-1}, n}$ and the zeroes of linear combinations of the polynomial Q_i 's.

Corollary 4.15. Let $\Omega = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ be a space $L_{\lambda p^{n-1}, n}$ and let Q_1, \dots, Q_n be the n -uple of polynomials arising from Theorem 4.8. Then, for every $1 \leq t \leq n$ the subspace $\Omega_t = \langle \omega_{n-t+1}, \dots, \omega_n \rangle_{\mathbb{F}_p}$ is such that

$$\mathcal{P}(\Omega_t) = \mathcal{P}(\Omega) - \bigcup_{\epsilon \in \mathbb{F}_p^{n-t} - \{0\}} Z \left(\sum_{i=1}^{n-t} \epsilon_i Q_i \right).$$

Proof. By Proposition 4.6, we have that $\mathcal{P}(\Omega_t) = Z(S_t)$ where

$$S_t = \Delta_t(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n)),$$

where $P_{\mathcal{Q}_{n-t}}$ is the structural polynomial of $\langle Q_1, \dots, Q_{n-t} \rangle_{\mathbb{F}_p}$. Since $P_{\mathcal{Q}_{n-t}}$ is an additive polynomial, we have that

$$\begin{aligned} S_t &= \prod_{i=n-t+1}^n \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_{n-t+1} \in \mathbb{F}_p} (P_{\mathcal{Q}_{n-t}}(Q_i + \epsilon_{i-1}Q_{i-1} + \cdots + \epsilon_{n-t+1}Q_{n-t+1})) = \\ &= \prod_{i=n-t+1}^n \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_{n-t+1} \in \mathbb{F}_p} \prod_{Q \in \mathcal{Q}_{n-t}} (Q + Q_i + \epsilon_{i-1}Q_{i-1} + \cdots + \epsilon_{n-t+1}Q_{n-t+1}), \end{aligned}$$

and the fact that the zeroes of $\Delta_n(Q)$ are simple (see Theorem 4.7) ensures that the zeroes of S_t are precisely those zeroes of $\Delta_n(Q)$ that are not zeroes of any $Q \in \mathcal{Q}_{n-t} - \{0\}$. This is equivalent to say that

$$\mathcal{P}(\Omega_t) = \mathcal{P}(\Omega) - \bigcup_{\epsilon \in \mathbb{F}_p^{n-t} - \{0\}} Z \left(\sum_{i=1}^{n-t} \epsilon_i Q_i \right).$$

□

Corollary 4.16. Let $\Omega = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ be a space $L_{\lambda p^{n-1}, n}$ and let Q_1, \dots, Q_n be the n -uple of polynomials arising from Theorem 4.8 for this basis.

Denote by q_i the leading coefficient of Q_i . Then we have the equality of Moore determinants

$$\Delta_n(Q_1, \dots, Q_n) = \alpha \Delta_t(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n)) \Delta_{n-t}(Q_1, \dots, Q_{n-t}),$$

where

$$\alpha = \frac{\Delta_n(q_1, \dots, q_n)}{\Delta_t(P_{\underline{q}}(q_{n-t+1}), \dots, P_{\underline{q}}(q_n)) \Delta_{n-t}(q_1, \dots, q_{n-t})} \in k^\times$$

and $P_{\underline{q}}$ the structural polynomial of the vector space $\langle q_1, \dots, q_{n-t} \rangle_{\mathbb{F}_p}$.

Proof. We know that the zeroes of $\Delta_n(Q)$ are simple and consist of the set $\mathcal{P}(\Omega)$, and by Proposition 4.6 the zeroes of $\Delta_t(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n))$ are simple and consist of the set $\mathcal{P}(\Omega_t)$ with $\Omega_t = \langle \omega_{n-t+1}, \dots, \omega_n \rangle_{\mathbb{F}_p}$. We may then apply Corollary 4.15 to see that the set of zeroes of the polynomials on both sides of the equation are equal, and that these zeroes are all simple. The corollary then follows from a comparison of the leading coefficients of these polynomials. □

We conclude this part with a result that describes the n -uples of polynomials giving rise to a basis of an étale pullback of a space $L_{\lambda p^{n-1}, n}$.

Proposition 4.17. Let $\Omega = \langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ be a space $L_{\lambda p^{n-1}, n}$ and let Q_1, \dots, Q_n be the n -uple of polynomials arising from Theorem 4.8. Let $S(X) \in k[X]$ with $S'(X) \in k^\times$ and let $\sigma^*(\Omega)$ be the pullback of Ω with respect to the morphism $\mathbb{P}_k^1 \xrightarrow{\sigma} \mathbb{P}_k^1$ induced by $X \mapsto S(X)$ (cf. Lemma 2.13). Then the polynomials arising from Theorem 4.8 for $\sigma^*(\Omega)$ are $(\eta Q_1(S), \dots, \eta Q_n(S))$ with $\eta^{p^{n-1}} = \frac{1}{S'(X)}$. In particular, we have that $\mathcal{P}(\sigma^*(\Omega)) = \{a \in k \mid S(a) \in \mathcal{P}(\Omega)\}$.

Proof. From the equation $\omega_i = \frac{(-1)^{i-1} \Delta_{n-1}(\widehat{Q}_i)}{\Delta_n(\widehat{Q})} dX$ it follows that

$$\sigma^*(\omega_i) = (-1)^{i-1} \frac{\Delta_{n-1}(\widehat{Q}_i(S))}{\Delta_n(\widehat{Q}(S))} S'(X) dX = (-1)^{i-1} \frac{\Delta_{n-1}(\widehat{\eta Q}_i(S))}{\Delta_n(\widehat{\eta Q}(S))} dX$$

where $\eta^{p^{n-1}} = \frac{1}{S'(X)}$. □

4.2. Polynomial conditions for the existence of spaces $L_{\lambda p^{n-1}, n}$. The results of Theorems 4.7 and 4.8 show that the existence of a space $L_{\lambda p^{n-1}, n}$ is equivalent to the existence of a n -uple $\underline{Q} \in k[X]^n$ satisfying certain conditions. In analogy with the case $n = 2$, Proposition 2.14 (iii) gives us a way to check this by solving the following system of polynomial equations (note that in this situation the number of poles is λp^{n-1} , and the number of equations is computed accordingly)

$$(4.18) \quad \begin{cases} \text{coeff} \left(\left(\frac{P}{P_n} \right)^{p-1}, X^{p-1} \right) = 1 \\ \text{coeff} \left(\left(\frac{P}{P_n} \right)^{p-1}, X^{\mu p-1} \right) = 0, \quad 2 \leq \mu \leq \lambda p^{n-2}(p-1). \end{cases}$$

In terms of the Q_i 's, we have from Remark 4.3 that

$$\frac{P}{P_n} = (-1)^{n-1} \prod_{\epsilon \in \mathbb{F}_p^{n-1}} (Q_n + \epsilon_{n-1} Q_{n-1} + \epsilon_{n-2} Q_{n-2} + \cdots + \epsilon_1 Q_1),$$

which in particular implies that the coefficients of $\left(\frac{P}{P_n} \right)^{p-1}$ are polynomial expressions in the coefficients of the Q_i 's. In what follows, we aim to describe these polynomial expressions more precisely. We begin by stating and proving the following proposition, which is independent of previous results obtained in this paper. This result is interesting also because it leads to a direct proof (i.e. not relying on the case $n = 2$) of Theorem 4.7.

Proposition 4.19. *Let $I_n := \mathbb{F}_p^n$. For every element $\epsilon \in I_n$, let $s(\epsilon) = \sum_{i=1}^n \epsilon_i$. If*

$$A := \prod_{\epsilon \in I_{n-1}} (X_n + \epsilon_{n-1} X_{n-1} + \epsilon_{n-2} X_{n-2} + \cdots + \epsilon_1 X_1)^{p-1}$$

and

$$B := \prod_{\substack{\epsilon \in I_n \\ s(\epsilon) \neq 0}} (\epsilon_n X_n + \epsilon_{n-1} X_{n-1} + \epsilon_{n-2} X_{n-2} + \cdots + \epsilon_1 X_1),$$

then

$$A + B = \prod_{\substack{\epsilon \in I_{n-1} \\ s(\epsilon) \neq -1}} (X_n + \epsilon_{n-1} X_{n-1} + \epsilon_{n-2} X_{n-2} + \cdots + \epsilon_1 X_1)^p.$$

In particular, we have that $A \equiv -B \pmod{k[X]^p}$.

Proof. Let us write $B = B_1 B_2$ with

$$B_1 := \prod_{\substack{\epsilon \in I_n \\ s(\epsilon) \neq 0, \epsilon_n \neq 0}} (\epsilon_n X_n + \epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1) \quad \text{and} \quad B_2 = \prod_{\substack{\epsilon \in I_{n-1} \\ s(\epsilon) \neq 0}} (\epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1).$$

Let us show that the polynomial B_1 contains the common factors between A and B : we can rewrite this polynomial as

$$\begin{aligned}
B_1 &= \prod_{\epsilon_n \in \mathbb{F}_p^\times} \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) \neq -\epsilon_n}} (\epsilon_n X_n + \epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1) \\
&= \prod_{\epsilon_n \in \mathbb{F}_p^\times} \epsilon_n^{(p-1)p^{n-2}} \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) \neq -\epsilon_n}} \left(X_n + \frac{\epsilon_{n-1}}{\epsilon_n} X_{n-1} + \cdots + \frac{\epsilon_1}{\epsilon_n} X_1 \right) \\
&= \prod_{\epsilon_n \in \mathbb{F}_p^\times} \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) \neq -1}} (X_n + \epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1) \\
&= \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) \neq -1}} (X_n + \epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1)^{p-1},
\end{aligned}$$

and this shows that $A + B = B_1 C$ with

$$C := \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) = -1}} (X_n + \epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1)^{p-1} + B_2.$$

In order to conclude, we need to show that

$$C = \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) \neq -1}} (X_n + \epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1),$$

which is equivalent to consider C as a monic univariate polynomial of degree $(p-1)p^{n-2}$ in the variable X_n and show that its set of roots is $\{\delta_{n-1} X_{n-1} + \cdots + \delta_1 X_1 : \underline{\delta} \in I_{n-1}, s(\underline{\delta}) \neq 1\}$. To verify this, for every $\underline{\delta} \in I_{n-1}$ with $s(\underline{\delta}) \neq 1$ we set $c := s(\underline{\delta}) - 1$ and we substitute X_n with $\delta_{n-1} X_{n-1} + \cdots + \delta_1 X_1$ in the expression of $C - B_2$. This gives

$$\begin{aligned}
&\prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) = -1}} (\delta_{n-1} X_{n-1} + \cdots + \delta_1 X_1 + \epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1)^{p-1} = \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) = c}} (\epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1)^{p-1} = \\
&= \prod_{\epsilon_n \in \mathbb{F}_p^\times} \left(\epsilon_n^{p^{n-2}} \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) = c}} \left(\frac{\epsilon_{n-1}}{\epsilon_n} X_{n-1} + \cdots + \frac{\epsilon_1}{\epsilon_n} X_1 \right) \right) = \prod_{\epsilon_n \in \mathbb{F}_p^\times} \left(\epsilon_n \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) = \frac{c}{\epsilon_n}}} (\epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1) \right) \\
&= - \prod_{\substack{\underline{\epsilon} \in I_{n-1} \\ s(\underline{\epsilon}) \neq 0}} (\epsilon_{n-1} X_{n-1} + \cdots + \epsilon_1 X_1) = -B_2.
\end{aligned}$$

As a result, $\delta_{n-1} X_{n-1} + \cdots + \delta_1 X_1$ is a zero of $C \in k[X_1, \dots, X_{n-1}][X_n]$ for every $\underline{\delta} \in I_{n-1}$ satisfying $s(\underline{\delta}) \neq 1$. \square

Remark 4.20. As a corollary of Proposition 4.19 we get a new proof of Theorem 4.7, which is of quite a different nature than the one given in Section 4.1. More precisely, employing the same notation as in the Theorem, the non-trivial part is to show that, if the form ω_n is logarithmic, then the forms ω_i are logarithmic for every $i = 1, \dots, n$. We then need to prove that, if $\left(\frac{P}{P_n}\right)^{p-1}$ satisfies the system of equations (4.18) then $\left(\frac{P}{P_i}\right)^{p-1}$ also satisfies (4.18) for every i . In fact, we can apply Remark 4.3

to show that $\left(\frac{P}{P_i}\right)^{p-1}$ is, up to a sign, equal to $A(Q_i, Q_1, \dots, \widehat{Q_i}, \dots, Q_n)$. From this, it follows that $\left(\frac{P}{P_i}\right)^{p-1}$ can be obtained from $\left(\frac{P}{P_n}\right)^{p-1}$ by applying a permutation of the Q_i 's.

On the other hand, Proposition 4.19 shows that the coefficients of A appearing in equations (4.18) are the same as those of $-B$, which is symmetric in the Q_i 's, hence we have that $\left(\frac{P}{P_n}\right)^{p-1}$ satisfies (4.18) if, and only if $\left(\frac{P}{P_i}\right)^{p-1}$ satisfies (4.18).

4.2.1. *Many new examples of spaces $L_{\lambda 2^{n-1}, n}$.* As an application of Proposition 4.19, we construct for all values of $n \geq 2$ new large classes of examples of spaces $L_{\lambda 2^{n-1}, n}$ in characteristic 2. In this case, we have that $p - 1 = 1$, and the equations (4.18) are equivalent to the condition

$$A(Q_1, \dots, Q_n) - X \in k[X]^2.$$

We can describe the coefficients of $A(Q_1, \dots, Q_n)$ appearing in (4.18) in more detail.

Proposition 4.21. *Let $p = 2$ and let \mathfrak{A}_n be the alternating group on n letters. Then, we have*

$$A(X_1, \dots, X_n) \equiv \sum_{\sigma \in \mathfrak{A}_n} X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)}^2 \cdots X_{\sigma(n)}^{2^{n-2}} \pmod{k[X]^2}.$$

Proof. The polynomial A is homogeneous of degree 2^{n-1} , and is additive in the variable X_n . By Proposition 4.19, its reduction modulo $k[X]^2$ is invariant with respect to any permutation of the X_i 's, and hence it is additive in X_i for every i . As a result, the monomials appearing in $A \pmod{k[X]^2}$ are all of the form $\prod_{i=1}^n \alpha_i X_i^{a_i}$ with $a_i \in \{2^j, 0 \leq j \leq n-2\}$ satisfying $\sum_{i=1}^n a_i = 2^{n-1}$. Since we are studying only the terms that are not squares, at least one of the a_i 's needs to be equal to 1. We claim that the conditions above uniquely determine the exponents a_i . More precisely, let us assume for simplicity that we have $a_1 \leq \dots \leq a_n$. We claim that then we have $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 2^2, \dots, a_n = 2^{n-2}$.

Proof of the claim. Since the a_i 's are powers of 2, in order to show the claim it is enough to show that $a_i \leq 2^{i-2}$ for all $i = 2, \dots, n$. We can do this by finite induction on i : we certainly need to have $a_2 = 1$ since the sum of all the terms is even. We then assume that, for every $j \leq i$, we have $a_j \leq 2^{j-2}$. It follows that

$$2^{n-1} - a_{i+1} - \dots - a_n = a_1 + \dots + a_i \leq 2^{i-1}.$$

Applying the 2-adic valuation v_2 , we have that

$$v_2(2^{n-1} - a_{i+1} - \dots - a_n) = v_2(a_1 + \dots + a_i) \leq i - 1.$$

Applying the non-archimedean triangular inequality gives

$$v_2(2^{n-1} - a_{i+1} - \dots - a_n) = v_2(a_{i+1} + \dots + a_n) \geq v_2(a_{i+1}),$$

which allows us to conclude that $v_2(a_{i+1}) \leq i - 1$, that is $a_{i+1} \leq 2^{i-1}$. \square

As a result of the claim, the monomials appearing in $A \pmod{k[X]^2}$ are precisely the required ones. We now have to show that the coefficients are all equal to 1. By symmetry, it is enough to show that the coefficient of $\left(\prod_{i=1}^{n-2} X_i^{2^{n-i-1}}\right) X_{n-1} X_n$ in the polynomial A is equal to 1. To produce such a term, we pick X_1 from all the 2^{n-2} factors where it appears (those with $\epsilon_1 = 1$), we pick X_2 from the 2^{n-3} factors where X_2 appears (those with $\epsilon_1 = 0, \epsilon_2 = 1$), and so on, until we remain with the expression $X_n(X_{n-1} + X_n)$ from which we need to pick the monomial $X_{n-1} X_n$. Since this is the only process that produces the monomial $\left(\prod_{i=1}^{n-2} X_i^{2^{n-i-1}}\right) X_{n-1} X_n$, its coefficient in A is equal to 1. \square

Corollary 4.22. *Let $p = 2$ and let $\underline{Q} \in k[X]^n$ be a n -uple giving rise to a basis of a space Ω as in Definition 4.4. Then Ω is a space $L_{\lambda 2^{n-1}, n}$ if, and only if*

$$\det(\underline{Q}', \underline{Q}, \underline{Q}^2, \dots, \underline{Q}^{2^{n-2}}) = 1.$$

Proof. As a direct consequence of Proposition 4.21, we have that the system of equations (4.18) is equivalent to $\left(\sum_{\sigma \in \mathfrak{A}_n} Q_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}}\right) - X \in k[X]^2$, which is equivalent to

$$\left(\sum_{\sigma \in \mathfrak{A}_n} Q_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}}\right)' = 1.$$

By denoting $\tau \in \mathfrak{S}_n$ the transposition exchanging 1 and 2, we can express the left hand side of this equation in the desired determinantal form. In fact, we have that

$$\begin{aligned} \left(\sum_{\sigma \in \mathfrak{A}_n} Q_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}}\right)' &= \sum_{\sigma \in \mathfrak{A}_n} Q'_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}} + \sum_{\sigma \in \mathfrak{A}_n} Q_{\sigma(1)} Q'_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}} \\ &= \sum_{\sigma \in \mathfrak{A}_n} Q'_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}} + \sum_{\sigma \in \mathfrak{A}_n} Q'_{\sigma\tau(1)} Q_{\sigma\tau(2)} Q_{\sigma\tau(3)}^2 \cdots Q_{\sigma\tau(n)}^{2^{n-2}} \\ &= \sum_{\sigma \in \mathfrak{A}_n} Q'_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}} + \sum_{\sigma \in \mathfrak{S}_n - \mathfrak{A}_n} Q'_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}} \\ &= \sum_{\sigma \in \mathfrak{S}_n} Q'_{\sigma(1)} Q_{\sigma(2)} Q_{\sigma(3)}^2 \cdots Q_{\sigma(n)}^{2^{n-2}} = \det(\underline{Q}', \underline{Q}, \underline{Q}^2, \dots, \underline{Q}^{2^{n-2}}). \end{aligned}$$

□

If $n = 2$, Corollary 4.22 leads to a complete classification of spaces $L_{2\lambda, 2}$ in characteristic 2. In fact, there are unique polynomials $U_1, U_2, V_1, V_2 \in k[X]$ such that $Q_1 = U_1^2 + XV_1^2$ and $Q_2 = U_2^2 + XV_2^2$. With these notations, we have $Q'_1 = V_1^2$ and $Q'_2 = V_2^2$, and hence

$$\det \begin{pmatrix} Q'_1 & Q_1 \\ Q'_2 & Q_2 \end{pmatrix} = \det \begin{pmatrix} V_1^2 & U_1^2 + XV_1^2 \\ V_2^2 & U_2^2 + XV_2^2 \end{pmatrix} = \det \begin{pmatrix} V_1^2 & U_1^2 \\ V_2^2 & U_2^2 \end{pmatrix} = (V_1 U_2 + U_1 V_2)^2.$$

The condition imposed by Corollary 4.22 then is equivalent to

$$V_1 U_2 + U_1 V_2 = 1.$$

If λ is odd, then the polynomials V_1, V_2 have degree $\frac{\lambda-1}{2}$. By Bézout's theorem, if V_1 and V_2 are coprime, then there exists a unique pair (U_1, U_2) with $\deg(U_i) < \frac{\lambda-1}{2}$ satisfying the above condition. If we pick (V_1, V_2) such that $V_1 + V_2$ has also degree $\frac{\lambda-1}{2}$, and we let (U_1, U_2) be the pair given by Bézout's theorem, then by Corollary 4.22 the polynomials $U_1^2 + XV_1^2$ and $U_2^2 + XV_2^2$ give rise to a basis of a space $L_{2\lambda, 2}$. If we relax the condition $\deg(U_i) < \frac{\lambda-1}{2}$, then we have other pairs that satisfy Bézout's theorem: these can be obtained from the minimal one (U_1, U_2) as $(U_1 + aV_1, U_2 + aV_2)$ for $a \in k$.

If λ is even, then the argument above works by choosing coprime $U_1, U_2 \in k[X]$ of degree $\frac{\lambda}{2}$ with $U_1 + U_2$ of degree $\frac{\lambda}{2}$ and applying Bézout's theorem to find V_1, V_2 . This shows that spaces $L_{2\lambda, 2}$ exist for every λ and in large abundance. While this is already known by work of Pagot [7, Théorème 2.2.4], the approach relying on Corollary 4.22 is of a different nature.

For $n > 2$, this approach is in principle not enough to give a complete classification, but we can extract sufficient conditions for the existence of spaces $L_{2^{n-1}\lambda, n}$ that lead to the discovery of new large classes of examples for every n .

Proposition 4.23. *Let $p = 2$ and $n \geq 3$. Let $\underline{Q} \in k[X]^n$ be a n -uple giving rise to a basis of a space Ω as in Definition 4.4. For every $i = 1, \dots, n$ let $U_i, V_i \in k[X]$ be such that $Q_i = U_i^2 + XV_i^2$ and that they satisfy the system of equations*

$$(4.24) \quad \begin{cases} \det(\underline{U}, \underline{V}, \underline{U}^2, \dots, \underline{U}^{2^{n-2}}) = 1 \\ \det(\underline{U}, \underline{V}, ((1 + \epsilon_3)\underline{U} + \epsilon_3\underline{V})^2, \dots, ((1 + \epsilon_n)\underline{U} + \epsilon_n\underline{V})^{2^{n-2}}) = 0, \end{cases}$$

where $(\epsilon_3, \dots, \epsilon_n)$ runs over all elements of $\mathbb{F}_2^{n-2} - \{0\}$. Then Ω is a space $L_{\lambda 2^{n-1}, n}$.

Proof. We need to show that, if \underline{U} and \underline{V} satisfy (4.24), then \underline{Q} satisfy the condition of Corollary 4.22. This latter is equivalent to $\det(\underline{V}^2, \underline{U}^2, \underline{Q}^2, \dots, \underline{Q}^{2^{n-2}}) = 1$, which is in turn equivalent to

$$\det(\underline{V}, \underline{U}, \underline{Q}, \dots, \underline{Q}^{2^{n-3}}) = \det(\underline{V}, \underline{U}, \underline{U}^2 + X\underline{V}^2, \dots, \underline{U}^{2^{n-2}} + X\underline{V}^{2^{n-2}}) = 1.$$

By linearity of determinants, the above condition can be expressed as a polynomial condition in X , namely

$$\sum_{(\epsilon_3, \dots, \epsilon_n) \in \mathbb{F}_2^{n-2}} \det(\underline{U}, \underline{V}, ((1 + \epsilon_3)\underline{U} + \epsilon_3\underline{V})^2, \dots, ((1 + \epsilon_n)\underline{U} + \epsilon_n\underline{V})^{2^{n-2}}) \cdot X^{(\sum_i \epsilon_i 2^{i-3})} = 1.$$

If the system (4.24) is satisfied, then the non-constant coefficients of the polynomial above vanish and the constant term is equal to 1. Hence we can apply Corollary 4.22 to get that Ω is a space $L_{\lambda 2^{n-1}, n}$. \square

Proposition 4.25. *Let $n \geq 3$. Let U_1, \dots, U_n and V_1, \dots, V_n be polynomials in $k[X]$ and $\alpha, \beta \in k(X)$ such that:*

- (i) $\underline{U} = \alpha\underline{V} + \beta\underline{V}^2$
- (ii) $\beta^{2^{n-1}-1} \cdot \Delta_n(\underline{V}) = 1$.

Then \underline{U} and \underline{V} satisfy the system of equations (4.24).

Proof. The proof consists of two steps:

- We first check that \underline{U} and \underline{V} satisfy the first line of (4.24): by repeatedly using condition (i) we find that

$$\begin{aligned} \det(\underline{U}, \underline{V}, \underline{U}^2, \underline{U}^4, \dots, \underline{U}^{2^{n-2}}) &= \det(\beta\underline{V}^2, \underline{V}, \underline{U}^2, \underline{U}^4, \dots, \underline{U}^{2^{n-2}}) \\ &= \beta \det(\underline{V}^2, \underline{V}, \beta^2\underline{V}^4, \underline{U}^4, \dots, \underline{U}^{2^{n-2}}) \\ &= \beta^3 \det(\underline{V}^2, \underline{V}, \underline{V}^4, \beta^4\underline{V}^8, \dots, \underline{U}^{2^{n-2}}) \\ &= \beta^7 \det(\underline{V}^2, \underline{V}, \underline{V}^4, \underline{V}^8, \dots, \underline{U}^{2^{n-2}}) \\ &= \dots \\ &= \beta^{2^{n-1}-1} \det(\underline{V}^2, \underline{V}, \underline{V}^4, \underline{V}^8, \dots, \underline{V}^{2^{n-2}}) = \beta^{2^{n-1}-1} \cdot \Delta_n(\underline{V}) = 1. \end{aligned}$$

- We then check the equations in the second line of (4.24): for each $(\epsilon_3, \dots, \epsilon_n) \in \mathbb{F}_2^{n-2} - \{0\}$ we let $k \in \{3, \dots, n\}$ be the smallest number such that $\epsilon_k \neq 0$. Since we need every ϵ_i with $i < k$ to be equal to 0, the vectors $(\underline{U}, \underline{V}, ((1 + \epsilon_3)\underline{U} + \epsilon_3\underline{V})^2, \dots, ((1 + \epsilon_k)\underline{U} + \epsilon_k\underline{V})^{2^{k-2}})$ forming the first k columns inside the determinant can be rewritten as $(\underline{U}, \underline{V}, \underline{U}^2, \dots, \underline{U}^{2^{k-1}}, \underline{V}^{2^{k-2}})$. By condition (i), we have that all these column vectors belong to the $k - 1$ dimensional space generated by $\underline{V}, \dots, \underline{V}^{2^{k-2}}$ and hence

$$\det(\underline{U}, \underline{V}, \underline{U}^2, \dots, \underline{U}^{2^{k-1}}, \underline{V}^{2^{k-2}}, \dots, ((1 + \epsilon_n)\underline{U} + \epsilon_n\underline{V})^{2^{n-2}}) = 0.$$

As this is true for every $(\epsilon_3, \dots, \epsilon_n) \in \mathbb{F}_2^{n-2} - \{0\}$, we have that all the equations in the second line of (4.24) are satisfied. \square

Theorem 4.26. *Let k be an algebraically closed field of characteristic 2 and $n \geq 3$.*

Let $\underline{W} := (W_1, \dots, W_n) \subset k[X]^n$ be a n -uple of polynomials such that all the nonzero elements of $\langle W_1, \dots, W_n \rangle_{\mathbb{F}_2}$ are pairwise coprime and of the same degree, denoted by d (in particular, we have that $\Delta_n(\underline{W}) \neq 0$). If we set $V_i := \Delta_{n-1}(\widehat{W}_i)$ for every $i = 1, \dots, n$, then there exists a rational function $\alpha \in \frac{1}{\Delta_n(\underline{W})}k[X]$ satisfying the following properties:

(i) *The rational function*

$$Q_i := \frac{V_i^4}{\Delta_n(\underline{W})^2} + (X + \alpha^2)V_i^2$$

is a polynomial for every $i = 1, \dots, n$.

(ii) *The n -uple $\underline{Q} = (Q_1, \dots, Q_n) \in k[X]^n$ gives rise to a basis of a space $L_{2^{n-1}\lambda, n}$.*

Moreover, given such an α , the set of all rational functions satisfying (i) and (ii) is $\{\alpha + R \mid R \in k[X]\}$.

Proof. We set $\beta = \frac{1}{\Delta_n(\underline{W})}$ and remark that the expression of Q_i given at the point (i) is equal to $U_i^2 + XV_i^2$ where $U_i := \alpha V_i + \beta V_i^2$. As a result, to prove the theorem we need to define $\alpha \in k(X)$ such that both the conditions of Proposition 4.25 are met. In fact, by virtue of that Proposition, it follows that \underline{Q} satisfies Proposition 4.23 and then gives rise to a basis of a space $L_{2^{n-1}\lambda, n}$.

We begin by observing that the hypothesis that $V_i = \Delta_{n-1}(\widehat{W}_i)$ for every i , combined with Theorem A.3 implies that

$$\Delta_n(\underline{V}) = \Delta_n(\underline{W})^{2^{n-1}-1}.$$

Hence condition (ii) of Proposition 4.25 is met. It then remains to find an appropriate $\alpha \in k(X)$ such that $\alpha V_i + \beta V_i^2$ is a polynomial for every $i = 1, \dots, n$.

We denote by \mathfrak{W} the space $\langle W_1, \dots, W_n \rangle_{\mathbb{F}_2}$ and for every subspace $\mathfrak{W}' \subset \mathfrak{W}$ we consider its structural polynomial $P_{\mathfrak{W}'}(Y) := \prod_{W \in \mathfrak{W}'} (Y - W) \in k(X)[Y]$ (cf. Definition A.4). We then consider the spaces

$$\mathfrak{W}_i := \langle W_1, \dots, \widehat{W}_i, \dots, W_n \rangle_{\mathbb{F}_2} \text{ for every } i = 1, \dots, n,$$

and note that every polynomial in $W_i + \mathfrak{W}_i$ divides $P_{\mathfrak{W}_i}(W_i)$ by definition. Conversely, if $W \in \mathfrak{W}$ divides $P_{\mathfrak{W}_i}(W_i)$ then $W \in W_i + \mathfrak{W}_i$. In fact, it is clear that W has a factor in common with a polynomial $W' \in W_i + \mathfrak{W}_i$ and since any two distinct elements in $\mathfrak{W} - \{0\}$ are coprime, then $W = W'$.

We then apply Lemma A.5 to get that $\beta V_i = \frac{\Delta_{n-1}(\widehat{W}_i)}{\Delta_n(\underline{W})} = \frac{1}{P_{\mathfrak{W}_i}(W_i)}$. If we write $\alpha = \beta\gamma$ with $\gamma \in k[X]$, we then have that

$$U_i = \alpha V_i + \beta V_i^2 = \beta V_i(\gamma + V_i) = \frac{\gamma + V_i}{P_{\mathfrak{W}_i}(W_i)}.$$

We now find the desired γ as a solution of a system of congruences in $k[X]$ with coprime moduli: for every $W \in \langle W_1, \dots, W_n \rangle - \{0\}$, we let $k_W := \min\{k \in \mathbb{N} : W \in \langle W_1, \dots, W_k \rangle\}$ and we consider the set of congruences

$$(4.27) \quad \{\gamma \equiv V_{k_W} \pmod{W} \mid W \in \mathfrak{W} - \{0\}\}.$$

We claim that this system is equivalent to the condition that $P_{\mathfrak{W}_i}(W_i)$ divides $\gamma + V_i$ for every $i \in \{1, \dots, n\}$. In fact, let $i \neq j$ and set $\mathfrak{W}_{ij} := \mathfrak{W}_i \cap \mathfrak{W}_j$. If $W \in \langle W_1, \dots, W_n \rangle - \{0\}$ is a common divisor of $P_{\mathfrak{W}_i}(W_i)$ and $P_{\mathfrak{W}_j}(W_j)$ for $i \neq j$, then by the observation above $W \in$

$(W_i + \mathfrak{W}_i) \cap (W_j + \mathfrak{W}_j) = W_i + W_j + \mathfrak{W}_{ij}$. Therefore, W divides also $P_{\mathfrak{W}_{ij}}(W_i + W_j)$. Combining the additivity of $P_{\mathfrak{W}_{ij}}$ with Lemma A.5, we have

$$P_{\mathfrak{W}_{ij}}(W_i + W_j) = P_{\mathfrak{W}_{ij}}(W_i) + P_{\mathfrak{W}_{ij}}(W_j) = \frac{V_i + V_j}{\Delta_{n-2}(W_1, \dots, \widehat{W}_i, \dots, \widehat{W}_j, \dots, W_n)},$$

which implies that W divides $V_i + V_j$. As a result, every solution γ to (4.27) satisfies

$$\gamma \equiv V_j \pmod{W} \text{ for all pairs } (j, W) \text{ with } W | P_{\mathfrak{W}_j}(W_j).$$

This ensures that $\gamma \equiv V_i \pmod{P_{\mathfrak{W}_i}(W_i)}$ for every $i \in \{1, \dots, n\}$.

Since the elements of $\mathfrak{W} - \{0\}$ are pairwise coprime, we can apply the Chinese remainder theorem to find a unique solution $\gamma \in k[X]$ to (4.27) such that $\deg(\gamma) < d(2^n - 1)$. Moreover, all the solutions to (4.27) are of the form $\gamma + R\Delta_n(\underline{W})$, for $R \in k[X]$. As a consequence, α satisfies (i) and (ii) if, and only if, $\alpha + R$ satisfies (i) and (ii).

In summary, for every choice of the polynomials W_1, \dots, W_n , and of a solution of (4.27), this construction gives rise to a unique n -uple of polynomials U_1, \dots, U_n satisfying Proposition 4.25. Since the nonzero elements of \mathfrak{W} are all of the same degree, and R is fixed, then also the resulting $Q_i = U_i^2 + XV_i^2$ are such that all the nonzero polynomials in $\langle Q_1, \dots, Q_n \rangle_{\mathbb{F}_2}$ are all of the same degree, denoted by λ . The n -uple \underline{Q} satisfies the conditions of Proposition 4.23 and then gives rise to a basis of a space $L_{\lambda 2^{n-1}, n}$. \square

We want to investigate the parameter space of spaces $L_{\lambda 2^{n-1}, n}$ arising from the construction of Theorem 4.26. For this, we use the notion of equivalence between spaces $L_{\lambda 2^{n-1}, n}$ introduced in Definition 2.11. For every $d \geq 0$, let $\mathcal{W}_d \subset (k[X])^n$ be the quasi-affine variety consisting of elements (W_1, \dots, W_n) of the same degree d such that all the non-zero elements of $\mathcal{W} := \langle W_1, \dots, W_n \rangle_{\mathbb{F}_2}$ are of degree d and pairwise coprime. This is defined inside the space of coefficients $k^{(d+1)n}$ by the inequation

$$\Delta_n(w_1, \dots, w_n) \prod_{W, W' \in \mathcal{W} - \{0\}} \text{Res}(W, W') \neq 0,$$

where, for every $i = 1, \dots, n$, w_i is the leading term of W_i . Then the construction of Theorem 4.26 associates with every element of $(W_1, \dots, W_n, R) \in \mathcal{W}_d \times k[X]$ a n -uple of polynomials (Q_1, \dots, Q_n) giving rise to a basis of a space $L_{\lambda 2^{n-1}, n}$. More precisely, the construction yields $Q_i = U_i^2 + XV_i^2$ with $V_i = \Delta_{n-1}(\widehat{W}_i)$ and $U_i = (\alpha + R)V_i + \frac{V_i^2}{\Delta_n(\underline{W})}$, where α denotes the unique proper rational function (i.e. of the form $\frac{\gamma}{\Delta_n(\underline{W})}$ with $\gamma \in k[X]$ such that $\deg(\gamma) < \deg(\Delta_n(\underline{W}))$) satisfying conditions (i) and (ii) in Theorem 4.26.

Corollary 4.28. *Let $n \geq 3$, $d \geq 0$ and let Ω and Ω' be the spaces $L_{\lambda 2^{n-1}, n}$ arising respectively from elements (W_1, \dots, W_n, R) and $(W'_1, \dots, W'_n, R') \in \mathcal{W}_d \times k[X]$ as in Theorem 4.26. Then the following hold:*

- (i) *We have $\Omega = \Omega'$ if, and only if, $R = R'$ and $\langle W'_1, \dots, W'_n \rangle_{\mathbb{F}_2} = \langle W_1, \dots, W_n \rangle_{\mathbb{F}_2}$.*
- (ii) *We have that Ω is equivalent to Ω' if, and only if, there exists $b \in k$ such that*

$$\langle W'_1, \dots, W'_n \rangle_{\mathbb{F}_2} = \langle W_1(X + b^2), \dots, W_n(X + b^2) \rangle_{\mathbb{F}_2} \text{ and } R' = R + b.$$

- (iii) *If $\lambda \equiv 1 \pmod{(2^n - 2)}$, then there exist infinitely many equivalence classes of spaces $L_{\lambda 2^{n-1}, n}$. They arise from elements $(W_1, \dots, W_n, R) \in \mathcal{W}_d \times k[X]$ such that R is constant.*
- (iv) *If λ is even, then there exist infinitely many equivalence classes of spaces $L_{\lambda 2^{n-1}, n}$. They arise from elements $(W_1, \dots, W_n, R) \in \mathcal{W}_d \times k[X]$ such that $\deg(R) \geq 1$.*

Proof. Let $\underline{Q} = (Q_1, \dots, Q_n)$ be the n -uple arising from (W_1, \dots, W_n, R) and $\underline{Q}' = (Q'_1, \dots, Q'_n)$ be the n -uple arising from (W'_1, \dots, W'_n, R') . We recall that the writings

$$(4.29) \quad Q_i = U_i^2 + XV_i^2 \text{ and } Q'_i = U_i'^2 + XV_i'^2$$

are unique, and call $\underline{U}, \underline{U}', \underline{V}, \underline{V}'$ the n -uples arising from these. We now prove separately the statements of the corollary:

- (i) We first show that to have $\Omega = \Omega'$ it is necessary and sufficient to find a matrix $M \in GL_n(\mathbb{F}_2)$ such that $\underline{V}M = \underline{V}'$ and $\underline{U}M = \underline{U}'$. In fact, by Proposition 4.5(i) and 4.5(iii) $\Omega = \Omega'$ if, and only if, there exists $M \in GL_n(\mathbb{F}_2)$ such that $\underline{Q}' = \underline{Q}M$. By uniqueness of 4.29, this is equivalent to have $(\underline{U}')^2 = (\underline{U}^2)M$ and $(\underline{V}')^2 = (\underline{V}^2)M$, and, since the entries of M are elements of \mathbb{F}_2 , it is equivalent to have that $\underline{U}' = \underline{U}M$ and $\underline{V}' = \underline{V}M$.

We then prove the two implications stated above:

- Let $R = R'$ and $\langle W'_1, \dots, W'_n \rangle_{\mathbb{F}_2} = \langle W_1, \dots, W_n \rangle_{\mathbb{F}_2}$, so that there is $M \in GL_n(\mathbb{F}_2)$ such that $\underline{W}' = \underline{W}M$. Then by Lemma A.18, we have that $\underline{V}' = \underline{V}M^c$. Since $R = R'$, this implies that we also have that $\underline{U}' = \underline{U}M^c$ and hence $\Omega = \Omega'$.
- Conversely, suppose that $\underline{U}' = \underline{U}M$ and $\underline{V}' = \underline{V}M$ for some $M \in GL_n(\mathbb{F}_2)$. Then, applying M^{-1} on both sides of the equation

$$\underline{U}' = (\alpha' + R')\underline{V}' + \frac{(\underline{V}')^2}{\Delta_n(\underline{W}')}$$

results in $\underline{U} = (\alpha' + R')\underline{V} + \frac{\underline{V}^2}{\Delta_n(\underline{W})}$. We also have that $\underline{U} = (\alpha + R)\underline{V} + \frac{\underline{V}^2}{\Delta_n(\underline{W})}$, so that

$$(\alpha + R) + \frac{\underline{V}}{\Delta_n(\underline{W})} = (\alpha' + R') + \frac{\underline{V}}{\Delta_n(\underline{W}')}.$$

By rearranging the terms, we get that

$$\left(\frac{1}{\Delta_n(\underline{W})} - \frac{1}{\Delta_n(\underline{W}')} \right) \cdot V_i = (\alpha' + R') - (\alpha + R) \quad \forall i = 1, \dots, n$$

and since $V_i \neq V_j$ if $i \neq j$, and $n \geq 3$, we deduce that $\Delta_n(\underline{W}) = \Delta_n(\underline{W}')$ and $\alpha + R = \alpha' + R'$.

Note that $\underline{V} = \varphi(\underline{W})$ and $\underline{V}' = \varphi(\underline{W}')$, where φ is the map defined in Proposition A.15. Then, by applying this proposition, we have that

$$\varphi(\underline{V}) = \Delta_n(\underline{W})^{2^{n-2}-1} \underline{W}^{2^{n-2}} \quad \text{and} \quad \varphi(\underline{V}') = \Delta_n(\underline{W}')^{2^{n-2}-1} \underline{W}'^{2^{n-2}}.$$

Since $\underline{V}' = \underline{V}M$, then we can apply Lemma A.18 and get $\varphi(\underline{V}') = \varphi(\underline{V})M^c$, which, combined with the equality $\Delta_n(\underline{W}) = \Delta_n(\underline{W}')$ results in

$$\underline{W}'^{2^{n-2}} = \underline{W}^{2^{n-2}} M^c,$$

proving that $\langle W'_1, \dots, W'_n \rangle_{\mathbb{F}_2} = \langle W_1, \dots, W_n \rangle_{\mathbb{F}_2}$. Finally, from $\alpha + R = \alpha' + R'$ and the fact that there is a unique proper rational function in the set $\{\alpha + R | R \in k[X]\}$, we obtain that $\alpha = \alpha'$ and $R = R'$.

- (ii) If \underline{W}' and R' are as in the statement, then it is easy to see that they give rise to Ω' equivalent to Ω , as we can see by applying the construction that $Q'_i(X) = Q_i(X + b^2)$ for all $i = 1, \dots, n$.

Conversely, assume that there exist $a \in k^\times$ and $b \in k$ such that $Q_i(aX + b^2) = Q'_i(X)$. We can see that

$$\begin{aligned} Q_i(aX + b^2) &= U_i(aX + b^2)^2 + b^2 V_i(aX + b^2)^2 + aX V_i(aX + b^2)^2 \\ &= [U_i(aX + b^2) + bV_i(aX + b^2)]^2 + X[\sqrt{a}V_i(aX + b^2)]^2, \end{aligned}$$

resulting in the relations

$$\begin{cases} U'_i(X) = U_i(aX + b^2) + bV_i(aX + b^2) \\ V'_i(X) = \sqrt{a}V_i(aX + b^2). \end{cases}$$

From the latter of these, combined with Proposition A.15, we get that

$$W'_i(X) = \theta a^{\frac{1}{2(2^{n-1}-1)}} W_i(aX + b^2)$$

for $\theta \in k$ such that $\theta^{2^{n-1}-1} = 1$. In particular, we have that

$$\Delta_n(\underline{W}') = \theta^{2^n-1} a^{\frac{2^n-1}{2(2^{n-1}-1)}} \Delta_n(W(aX + b^2)).$$

We now compute U'_i in two different ways. On the one hand, we have

$$\begin{aligned} U'_i &= U_i(aX + b^2) + bV_i(aX + b^2) = (\alpha + R + b)V_i(aX + b^2) + \frac{V_i(aX + b^2)^2}{\Delta_n(W(aX + b^2))} \\ &= \frac{(\alpha + R + b)}{\sqrt{a}} V'_i(X) + \frac{a^{-1} V'_i(X)^2}{\theta^{1-2^n} a^{\frac{1-2^n}{2(2^{n-1}-1)}} \Delta_n(\underline{W}')} = \frac{(\alpha + R + b)}{\sqrt{a}} V'_i(X) + \theta^{2^n-1} a^{\frac{1}{2^n-2}} \frac{V'_i(X)^2}{\Delta_n(\underline{W}')}. \end{aligned}$$

On the other hand, we have

$$U'_i = (\alpha' + R')V'_i(X) + \frac{V'_i(X)^2}{\Delta_n(\underline{W}')}$$

and since this both computations are true for all i , an argument analogue to the one used to prove (i) shows that $\theta^{2^n-1} a^{\frac{1}{2^n-2}} = \theta a^{\frac{1}{2^n-2}} = 1$. But we have that $\theta^{2^n-2} = \theta^{2(2^{n-1}-1)} = 1$ and hence $a = 1$. As a result, $\alpha + R + b = \alpha' + R'$ and hence $\alpha = \alpha'$ and $R' = R + b$.

We have shown so far that applying the construction of Theorem 4.26 to the $n + 1$ -uple $(W_1(X + b^2), \dots, W_n(X + b^2), R + b)$ produces (Q'_1, \dots, Q'_n) giving rise to a basis of Ω' equivalent to Ω . By applying part (i) we conclude that the same Ω' can only arise from (W'_1, \dots, W'_n, R') with

$$\langle W'_1, \dots, W'_n \rangle_{\mathbb{F}_2} = \langle W_1(X + b^2), \dots, W_n(X + b^2) \rangle_{\mathbb{F}_2} \text{ and } R' = R + b.$$

(iii) If R is constant, we have

$$\lambda = 1 + 2 \deg V_i = 1 + 2d(2^{n-1} - 1) = 1 + d(2^n - 2).$$

For every $\lambda \equiv 1 \pmod{2^n - 2}$ we can then construct infinite equivalence classes of spaces $L_{\lambda 2^{n-1}, n}$ by picking $d = \frac{\lambda-1}{2^n-2}$ and all possible n -uples $(W_1, \dots, W_n) \in \mathcal{W}_d$.

(iv) If R is not constant, then $\deg(U_i) > \deg(V_i)$ and λ is even. In fact, any even value of λ can be achieved in this way, simply by choosing W_1, \dots, W_n to be constant and R to be of degree $\frac{\lambda}{2}$. □

Remark 4.30. Even though in Theorem 4.26 we assumed $n \geq 3$, the construction of the n -uple \underline{Q} in its proof makes sense also for $n = 2$. Namely, for every pair $(W_1, W_2) \in \mathcal{W}_d$ one can set $V_1 = \overline{W_2}$, $V_2 = W_1$ and fix a solution γ to the congruences

$$\begin{cases} \gamma \equiv V_1 \pmod{V_2} \\ \gamma \equiv V_1 \pmod{V_1 + V_2} \\ \gamma \equiv V_2 \pmod{V_1}. \end{cases}$$

By considering

$$U_1 = \frac{(\gamma + V_1)}{V_2(V_1 + V_2)} \text{ and } U_2 = \frac{(\gamma + V_2)}{V_1(V_1 + V_2)},$$

one produces a pair $(U_1^2 + XV_1^2, U_2^2 + XV_2^2)$ giving rise to a space $L_{2\lambda, 2}$. We can show that this construction recovers all the spaces $L_{2\lambda, 2}$. In fact, starting from pairs $\underline{U}, \underline{V}$ satisfying Bezout's

identity $U_1V_2 + V_1U_2 = 1$ (cf. the discussion following Corollary 4.22 to see that these define all possible spaces $L_{2\lambda,2}$), we have that

$$\begin{cases} U_1V_2 \equiv 1 \pmod{V_1} \\ U_2V_1 \equiv 1 \pmod{V_2}. \end{cases}$$

Using these, we verify that the polynomial

$$\gamma = V_1^2U_2 + V_2^2U_1 = V_1 + V_2U_1(V_1 + V_2) = V_2 + U_2V_1(V_1 + V_2)$$

solves the congruences (4.27) and yields

$$U_1 = \frac{(\gamma + V_1)}{V_2(V_1 + V_2)} \quad \text{and} \quad U_2 = \frac{(\gamma + V_2)}{V_1(V_1 + V_2)}$$

as above. For every choice of \underline{U} and \underline{V} we can find such a γ : hence we can recover in this way all the spaces $L_{2\lambda,2}$.

Remark 4.31. We remark that, if Ω a space $L_{2^{n-1}\lambda,n}$ is arising from an element $(W_1, \dots, W_n, R) \in \mathcal{W}_d \times k[X]$ as in Theorem 4.26, then every étale pullback of Ω also arise from this construction. More precisely, let $S(X) \in k[X]$ with $S'(X) = 1$ and let $\sigma^*(\Omega)$ be the pullback of Ω with respect to the morphism $\sigma \in \text{End}(\mathbb{P}_k^1)$ induced by $X \mapsto S(X)$ (cf. Lemma 2.13). Then, we know by Proposition 4.17 that $(Q_1(S(X)), \dots, Q_n(S(X)))$ is the n -uple arising from Theorem 4.8 for $\sigma^*(\Omega)$. One verifies that $Q_i(S(X)) = (U_i(S(X)) + RV_i(S(X))^2 + XV_i(S(X))^2)$, and as a result one proves that $\sigma^*(\Omega)$ is the space arising from $(W_1 \circ S, \dots, W_n \circ S, R + S - X)$.

5. STANDARD $L_{(p-1)p^{n-1},n}$ SPACES AND THEIR SUBSPACES

In this section, we analyze certain spaces $L_{\lambda p^{n-1},n}$ for $\lambda = p - 1$ that appear implicitly in [6]. In that paper, they are instrumental to lift to characteristic zero certain local actions of elementary abelian groups and they have since been made explicit in [8] as well as in [10], in a generalized form. Due to their relevance, and for the fact that they were among the first examples of spaces $L_{\lambda p^{n-1},n}$ to be discovered, we choose to give these spaces a name and we call them *standard $L_{(p-1)p^{n-1},n}$ spaces*. In the first subsection, we explain how to construct these spaces using Theorem 4.8. Namely, we find sufficient conditions for a set of polynomial Q_1, \dots, Q_n as in the theorem to produce a standard space. In the second subsection, we show that we can construct all the subspaces of a given standard space using étale pullbacks of other standard spaces.

5.1. Standard spaces. We begin by giving the definition of a standard $L_{p^{n-1}(p-1),n}$ space. We recall from Corollary 2.10 that the set of poles of a space $L_{\lambda p^{n-1},n}$ characterizes the space itself.

Definition 5.1. A *standard $L_{p^{n-1}(p-1),n}$ space* Ω is a space $L_{p^{n-1}(p-1),n}$ such that there exists a n -uple $\underline{a} := (a_1, a_2, \dots, a_n) \in k^n$ with $\Delta_n(\underline{a}) \neq 0$ for which $\mathcal{P}(\Omega) = \langle a_1, a_2, \dots, a_n \rangle_{\mathbb{F}_p} - \{0\}$.

For $i \in \{1, \dots, n\}$ and \underline{a} as above, let

$$(5.2) \quad \omega_i := \sum_{(\epsilon_1, \dots, \epsilon_n) \in \mathbb{F}_p^n} \frac{\epsilon_i}{X - \sum_{j=1}^n \epsilon_j a_j} dX.$$

Then $\langle \omega_1, \dots, \omega_n \rangle_{\mathbb{F}_p}$ is a standard space by [10, Proposition 3.1]. This shows that, for every $\underline{a} \in k^n$ whose entries are \mathbb{F}_p -linearly independent, there exists a standard space whose set of poles is $\langle a_1, a_2, \dots, a_n \rangle_{\mathbb{F}_p} - \{0\}$.

We can recover a description of standard spaces using the polynomials Q_i 's of Theorem 4.8, as we establish in the following proposition

Proposition 5.3. Let $\omega_1, \dots, \omega_n$ be defined by the identities (5.2), and let

$$Q_i := \mu \frac{\Delta_2(a_i, X)}{X} = \mu(a_i X^{p-1} - a_i^p)$$

with $\mu^{p^{n-1}} = \frac{-1}{\Delta_n(\underline{a})^{p-1}}$. Then we have $\omega_i = \frac{P_i}{P} dX$ where $P := \Delta_n(Q)$ and $P_i := (-1)^{i-1} \Delta_{n-1}(\widehat{Q}_i)$.

Proof. First, we compute

$$\frac{P_i}{P} = (-1)^{i-1} \left(\frac{X}{\mu} \right)^{p^{n-1}} \frac{\Delta_{n-1}(\widehat{\Delta_2(a_i, X)})}{\Delta_n(\Delta_2(a_1, X), \dots, \Delta_2(a_n, X))},$$

which combined with Lemma A.6 (for $m = 1$) gives

$$\frac{P_i}{P} = \frac{1}{\mu^{p^{n-1}}} \frac{(-1)^{i-1} \Delta_n(\hat{a}_i, X)}{\Delta_{n+1}(\underline{a}, X)}.$$

Using [10, Proposition 3.1], we can write

$$\omega_i = -\Delta_n(\underline{a})^{p-1} \frac{(-1)^{i-1} \Delta_n(\hat{a}_i, X)}{\Delta_{n+1}(\underline{a}, X)} dX \quad \forall i = 1, \dots, n,$$

and this proves the proposition. \square

Example 5.4. Let $n = 2$ and choose $a_1 = 1, a_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$.

Then we have

$$\begin{cases} \omega_1 &= \sum_{i=0}^{p-1} \left(\frac{1}{X - (1 + ia_2)} + \frac{2}{X - (2 + ia_2)} + \dots + \frac{p-1}{X - (p-1 + ia_2)} \right) dX \\ \omega_2 &= \sum_{i=0}^{p-1} \left(\frac{1}{X - (i + a_2)} + \frac{2}{X - (i + 2a_2)} + \dots + \frac{p-1}{X - (i + (p-1)a_2)} \right) dX. \end{cases}$$

Each ω_i has $p(p-1)$ poles. The poles in $\mathcal{P}(\Omega)$ are all the elements of the multiplicative group $\mathbb{F}_{p^2}^\times$ and those in common between ω_1 and ω_2 are those elements of \mathbb{F}_{p^2} that neither belong to the one-dimensional \mathbb{F}_p -vector space generated by 1 nor to the one generated by a_2 .

Remark 5.5. Let us fix a n -tuple $\underline{a} \in k^n$ such that $\Delta_n(\underline{a}) \neq 0$ and let us denote by $A := \langle a_1, \dots, a_n \rangle_{\mathbb{F}_p}$ the \mathbb{F}_p -vector space generated by \underline{a} and by A^* its dual. We can make several remarks:

(i) If Ω is the standard space with $\mathcal{P}(\Omega) = A - \{0\}$, there is a natural pairing

$$\begin{aligned} \psi : \Omega \times A &\longrightarrow \mathbb{F}_p \\ (\omega, x) &\longmapsto \begin{cases} \text{res}_x(\omega) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \end{aligned}$$

We remark that this is a perfect pairing.

In fact, writing $x = \epsilon_1 a_1 + \dots + \epsilon_n a_n$, we deduce from equation (5.2) that

$$\text{res}_x \left(\sum_i \alpha_i \omega_i \right) = \sum_i \alpha_i \epsilon_i,$$

which shows that ψ is bilinear and that $\psi(\omega_i, a_j) = \delta_{ij}$, the Kronecker symbol. Hence ψ is perfect and, as a result, the homomorphism ω to $x \mapsto \psi(\omega, x)$ defines an isomorphism

$\iota : \Omega \cong A^*$. We can explicitly describe the inverse of ι as follows: for all $\varphi \in A^* - \{0\}$ define the differential form $\omega_\varphi := \sum_{a \in A} \frac{\varphi(a)dX}{X-a}$. Then the isomorphism

$$\begin{aligned} \iota' : A^* &\longrightarrow \Omega \\ \varphi &\longmapsto \begin{cases} \omega_\varphi & \text{if } \varphi \neq 0, \\ 0 & \text{if } \varphi = 0. \end{cases} \end{aligned}$$

satisfies $\iota \circ \iota' = \text{id}$ and $\iota' \circ \iota = \text{id}$.

- (ii) Using Definition 5.1 it is easy to show that the Frobenius twist of the standard $L_{\lambda p^{n-1}, n}$ space associated to the vector space $A = \langle a_1, \dots, a_n \rangle_{\mathbb{F}_p}$ is the standard space associated to $\Phi(A) = \langle a_1^p, \dots, a_n^p \rangle_{\mathbb{F}_p}$.
- (iii) We can apply a translation to a standard space Ω to get a space $L_{p^{n-1}(p-1), n}$ equivalent to Ω , in the sense of the definition at the beginning of Section 6. This will not be in general a standard space.

5.2. Subspaces of standard spaces via étale pullbacks. Let A be a n -dimensional \mathbb{F}_p -vector subspace of k . The construction of Section 5.1 results in a map

$$A \rightarrow \Omega(A)$$

associating with it the standard $L_{p^{n-1}(p-1), n}$ space whose set of poles is $A - \{0\}$. It is easy to see that every subspace of $\Omega(A)$ of dimension $t < n$ is a space $L_{p^{n-1}(p-1), t}$, and that it can not be equivalent to a standard space.

However, we have seen in Lemma 2.13 how to find spaces $L_{\lambda dp^n, n}$ starting from spaces $L_{\lambda p^{n-1}, n}$ via the pullback of the differential forms under suitable morphisms of degree dp . In this section, we characterize the subspaces of the standard spaces constructed in §5.1 as étale pullbacks of standard spaces of lower dimension. More precisely, let $n > 1$, $1 \leq t \leq n$ and $A := \langle a_1, \dots, a_n \rangle_{\mathbb{F}_p} \subset k$ with $\underline{a} := (a_1, a_2, \dots, a_n)$ such that $\Delta_n(\underline{a}) \neq 0$. Let $A_{n-t} = \langle a_1, \dots, a_{n-t} \rangle_{\mathbb{F}_p}$, $A_{n-t}^s = \langle a_{n-t+1}, \dots, a_n \rangle_{\mathbb{F}_p}$ and let $P_{A_{n-t}}$ be the structural polynomial of A_{n-t} (cf. Definition A.4). For every $n-t+1 \leq i \leq n$, we set $\tilde{a}_i := P_{A_{n-t}}(a_i)$. Then, by Lemma A.16 we have that $\Delta_t(\tilde{\underline{a}}) \neq 0$ and hence $\widetilde{A_{n-t}^s} = \langle \tilde{a}_{n-t+1}, \dots, \tilde{a}_n \rangle$ is a \mathbb{F}_p vector space of dimension t . Let $\Omega(A)$ be the standard space associated with A and let

$$\omega_i = \sum_{(\epsilon_1, \dots, \epsilon_n) \in \mathbb{F}_p^n} \frac{\epsilon_i}{X - \sum_{j=1}^n \epsilon_j a_j} dX, \quad 1 \leq i \leq n$$

be the elements of its usual basis. Let Q_1, \dots, Q_n be the n -uple of polynomials arising from Theorem 4.8, which we know by proposition 5.3 to satisfy $Q_i = \mu \frac{\Delta_2(a_i, X)}{X}$ with $\mu^{p^n-1} = \frac{-1}{\Delta_n(\underline{a})^{p-1}}$. Similarly, we let

$$\tilde{\omega}_i = \sum_{(\epsilon_{n-t+1}, \dots, \epsilon_n) \in \mathbb{F}_p^t} \frac{\epsilon_i}{X - \sum_{j=n-t+1}^n \epsilon_j \tilde{a}_j} dX, \quad n-t+1 \leq i \leq n$$

be the elements of the usual basis of the standard $L_{p^{t-1}(p-1), t}$ -space $\Omega(\widetilde{A_{n-t}^s})$. Finally, we let $\widetilde{Q_{n-t+1}}, \dots, \widetilde{Q_n}$ be the t -uple of polynomials arising from Theorem 4.8, which satisfy $\widetilde{Q}_i = \tilde{\mu} \frac{\Delta_2(\tilde{a}_i, X)}{X}$ with $\tilde{\mu}^{p^{t-1}-1} = \frac{-1}{\Delta_t(\tilde{a}_{n-t+1}, \dots, \tilde{a}_n)^{p-1}}$.

Proposition 5.6. *Consider the subspace $\Omega^s = \langle \omega_{n-t+1}, \dots, \omega_n \rangle_{\mathbb{F}_p} \subset \Omega(A)$.*

- (i) *We have that $\Omega^s = \sigma^*(\Omega(\widetilde{A_{n-t}^s}))$. More precisely, for every $n-t+1 \leq i \leq n$, ω_i is equal to $\sigma^*(\tilde{\omega}_i)$, the pullback of $\tilde{\omega}_i$ with respect to the morphism $\mathbb{P}_k^1 \xrightarrow{\sigma} \mathbb{P}_k^1$ induced by $X \mapsto P_{A_{n-t}}(X)$.*

(ii) The t -uple of polynomials arising from Theorem 4.8 for the basis $(\sigma^*\tilde{\omega}_{n-t+1}, \dots, \sigma^*\tilde{\omega}_n)$ of $\Omega^s = \sigma^*(\Omega(\widetilde{A_{n-t}^s}))$ is given by

$$\eta\widetilde{Q_{n-t+1}}(P_{A_{n-t}}), \dots, \eta\widetilde{Q_n}(P_{A_{n-t}}),$$

with η satisfying $\eta^{p^{t-1}} = \frac{1}{P'_{A_{n-t}}} = \frac{(-1)^{n-t}}{\Delta_{n-t}(a_1, \dots, a_{n-t})^{p-1}}$.

These polynomials satisfy the condition

$$(-1)^{n-t}\eta\widetilde{Q_i}(P_{A_{n-t}}) = P_{\mathcal{Q}_{n-t}}(Q_i) \quad \forall n-t+1 \leq i \leq n,$$

where $P_{\mathcal{Q}_{n-t}}$ is the structural polynomial of the \mathbb{F}_p -vector space $\mathcal{Q}_{n-t} := \langle Q_1, \dots, Q_{n-t} \rangle_{\mathbb{F}_p}$.

Proof. (i) By Lemma A.5, we have that $P'_{A_{n-t}}(X) = (-1)^{n-t}\Delta_{n-t}(a_1, \dots, a_{n-t})^{p-1} \in k^\times$, so that the morphism σ gives rise to an étale pullback as in Section 2.3. Since the degree of $P_{A_{n-t}}$ is p^{n-t} , it follows that $\sigma^*(\Omega(\widetilde{A_{n-t}^s}))$ is a space $L_{(p-1)p^{n-t}}$. For every $n-t+1 \leq i \leq n$, we remark that $\tilde{\omega}_i = \frac{dF_i}{F_i}$ with

$$F_i(X) = \prod_{(\epsilon_{n-t+1}, \dots, \epsilon_n) \in \mathbb{F}_p^t} \left(X - \sum_{j=n-t+1}^n \epsilon_j \tilde{a}_j \right)^{\epsilon_i} = \prod_{(\epsilon_{n-t+1}, \dots, \epsilon_n) \in \mathbb{F}_p^t} \left(X - \sum_{j=n-t+1}^n \epsilon_j P_{A_{n-t}}(a_j) \right)^{\epsilon_i}.$$

By using additivity of $P_{A_{n-t}}$, we have that

$$\begin{aligned} F_i(P_{A_{n-t}}(X)) &= \prod_{(\epsilon_{n-t+1}, \dots, \epsilon_n) \in \mathbb{F}_p^t} \left(P_{A_{n-t}} \left(X - \sum_{j=n-t+1}^n \epsilon_j a_j \right) \right)^{\epsilon_i} \\ &= \prod_{a \in A_{n-t}} \prod_{(\epsilon_{n-t+1}, \dots, \epsilon_n) \in \mathbb{F}_p^t} \left(X - \left(a + \sum_{j=n-t+1}^n \epsilon_j a_j \right) \right)^{\epsilon_i} \\ &= \prod_{(\epsilon_1, \dots, \epsilon_n) \in \mathbb{F}_p^n} \left(X - \left(a + \sum_{j=n-t+1}^n \epsilon_j a_j \right) \right)^{\epsilon_i}. \end{aligned}$$

As a result, $\sigma^*(\tilde{\omega}_i) = \frac{dF_i(P_{A_{n-t}})}{F_i(P_{A_{n-t}})}$ has the same set of poles as ω_i , and each pole has the same residue for both forms. By applying Lemma 2.7, we conclude that $\sigma^*(\tilde{\omega}_i) = \omega_i$.

(ii) By Proposition 4.17, the t -uple of polynomials arising from Theorem 4.8 for the basis $(\sigma^*\tilde{\omega}_{n-t+1}, \dots, \sigma^*\tilde{\omega}_n)$ of $\sigma^*(\Omega(\widetilde{A_{n-t}^s}))$ is $\eta\widetilde{Q_{n-t+1}}(P_{A_{n-t}}), \dots, \eta\widetilde{Q_n}(P_{A_{n-t}})$, with $\eta^{p^{t-1}} = \frac{1}{P'_{A_{n-t}}} = \frac{(-1)^{n-t}}{\Delta_{n-t}(a_1, \dots, a_{n-t})^{p-1}}$, as stated.

From Proposition 4.6, we have that the t -uple of polynomials arising from Theorem 4.8 for the basis $((-1)^{n-t}\omega_{n-t+1}, \dots, (-1)^{n-t}\omega_n)$ of Ω^s is $(P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, P_{\mathcal{Q}_{n-t}}(Q_n))$. We can apply Proposition 4.5(i) to find that the t -uple of polynomials arising from Theorem 4.8 for the basis $(\omega_{n-t+1}, \dots, \omega_n)$ of Ω^s is $((-1)^{n-t}P_{\mathcal{Q}_{n-t}}(Q_{n-t+1}), \dots, (-1)^{n-t}P_{\mathcal{Q}_{n-t}}(Q_n))$. Since by (i) we know that $\sigma^*(\tilde{\omega}_i) = \omega_i$, and Proposition 4.5(ii) ensures the uniqueness of the t -uples of polynomials arising from Theorem 4.8 for the same basis, we have that

$$(-1)^{n-t}\eta\widetilde{Q_i}(P_{A_{n-t}}) = P_{\mathcal{Q}_{n-t}}(Q_i) \quad \forall n-t+1 \leq i \leq n.$$

□

Example 5.7. Let $n = 3$, and $t = 2$. Choose $A' := \langle \alpha, \beta \rangle_{\mathbb{F}_p} \subset k$ of dimension 2 and consider the corresponding standard $L_{p(p-1), 2}$ space $\Omega(A')$. Let $a_1 \in k^\times$, and define $S(X) := X^p - a_1^{p-1}X$.

Since k is algebraically closed, we can choose a solution a_2 to the equation $S(X) = \alpha$, as well as a solution a_3 to the equation $S(X) = \beta$. We remark that

$$S(f_1 a_1 + f_2 a_2 + f_3 a_3) = f_2 S(a_2) + f_3 S(a_3) = f_2 \alpha + f_3 \beta \quad \forall f_1, f_2, f_3 \in \mathbb{F}_p,$$

and this vanishes only when $f_2 = f_3 = 0$. In particular, if $(f_2, f_3) \neq (0, 0)$ then $f_1 a_1 + f_2 a_2 + f_3 a_3 \neq 0$. We then have that $\Delta_3(a_1, a_2, a_3) \neq 0$, so we can consider the standard space $\Omega(A)$ with $A := \langle a_1, a_2, a_3 \rangle_{\mathbb{F}_p}$ and Proposition 5.6 tells us that the pullback of $\Omega(A')$ with respect to the morphism induced by $X \mapsto S(X)$ is a two dimensional subspace of $\Omega(A)$.

Note that, by making different choices of a_2, a_3 we still end up with the same vector space A . Hence, the datum of the space A' together with the element $a_1 \in k^\times$ is enough to determine the space A .

Proposition 5.6 proves that subspaces of standard spaces can be realized as étale pullbacks of standard spaces. While a complete characterization of the spaces that can be obtained as étale pullbacks of standard spaces is outside the scope of this paper, we include here a lemma that gives a necessary condition for a space $L_{\lambda p^{n-1}, n}$ to be equivalent to such a pullback.

Lemma 5.8. *Let Ω be a space $L_{\lambda p^{n-1}, n}$ obtained as in Lemma 2.13 via an étale pullback of Ω_1 a space $L_{\lambda_1 p^{n-1}, n}$. If Ω can also be obtained via an étale pullback of Ω_2 , a $L_{\lambda_2 p^{n-1}, n}$ space with $\lambda_1 < \lambda_2$ then $p \mid \lambda_2$.*

Proof. For $i = 1, 2$ let S_i be the polynomial defining the étale pullback of Ω_i . We write

$$S_1(X) = a_{d_1} X^{d_1 p} + a_{d_1-1} X^{(d_1-1)p} + \dots + \gamma_1 X + a_0$$

$$S_2(X) = b_{d_2} X^{d_2 p} + b_{d_2-1} X^{(d_2-1)p} + \dots + \gamma_2 X + b_0$$

with non-zero leading coefficients. Let Q_{11}, \dots, Q_{n1} be a n -uple of polynomials giving rise to a basis of Ω_1 . Then by Proposition 4.17, up to multiplying by a constant, $Q_{11}(S_1(X)), \dots, Q_{n1}(S_1(X))$ give rise to a basis of Ω . Similarly, if Q_{12}, \dots, Q_{n2} give rise to a basis of Ω_2 then (up to multiplying by a constant) $Q_{12}(S_2(X)), \dots, Q_{n2}(S_2(X))$ give rise to a basis of Ω , as well. Up to changing the basis of Ω_2 , we can assume that the bases of Ω produced in the two cases are the same, and then apply Proposition 4.5(ii) to get that $Q_{j1}(S_1(X)) = \eta Q_{j2}(S_2(X))$ for every $j = 1, \dots, n$ and $\eta \in k^\times$. By definition we have $\deg(Q_{j1}) = \lambda_1$ and $\deg(Q_{j2}) = \lambda_2$, and the condition $\lambda_1 < \lambda_2$ implies that $d_1 > d_2$, since we ought to have $\lambda_1 d_1 = \lambda_2 d_2$.

Assume by contradiction that $\gcd(p, \lambda_2) = 1$. Then $\deg(Q'_{j2}) = \lambda_2 - 1$, and since $S_2(X)'$ is a constant we have that

$$\deg([Q_{j2}(S_2(X))])' = \deg(Q'_{j2}(S_2(X))) = (\lambda_2 - 1)d_2 p.$$

At the same time,

$$\deg([Q_{j1}(S_1(X))])' = \deg(Q'_{j1}(S_1(X))) \leq (\lambda_1 - 1)d_1 p < (\lambda_2 - 1)d_2 p.$$

But $\deg(Q_{j1}(S_1(X))) = \deg(Q_{j2}(S_2(X)))$ and hence a contradiction arises. \square

A direct consequence of Lemma 5.8 is that a space obtained as a pullback of a space $L_{\lambda p^{n-1}, n}$ with $(p, \lambda) = 1$ and $\lambda > p - 1$ will never be equivalent to the pullback of a standard space. For example, the spaces $L_{12,2}$ and $L_{15,2}$ constructed in Section 6 give rise by étale pullbacks to certain spaces $L_{36d,2}$ and $L_{45d,2}$ for every positive integer d , and the lemma ensures that these spaces are never equivalent to étale pullbacks of standard spaces.

5.2.1. *Standard spaces for $p = 2$.* When $p = 2$, the standard $L_{2^{n-1}, n}$ -spaces can be also obtained using the techniques of section 4.2.1: by Proposition 5.3 we have that any such space is generated by a basis arising from Q_1, \dots, Q_n with

$$Q_i := \mu a_i X + \mu a_i^2 = V_i^2 X + U_i^2$$

for $V_i = (\mu a_i)^{1/2}$ and $U_i = \mu^{1/2} a_i$. We then have that $U_i = \mu^{-1/2} V_i^2$ and then the condition (i) of Proposition 4.25 is met by setting $\alpha = 0$ and $\beta = \mu^{-1/2}$. Recall that $\mu^{2^{n-1}} = \frac{1}{\Delta_n(\underline{a})}$. We then have that

$$\beta^{2^{n-1}-1} \Delta_n(\underline{V}) = \frac{\Delta_n((\mu a_1)^{1/2}, \dots, (\mu a_n)^{1/2})}{(\mu^{1/2})^{2^{n-1}-1}} = (\mu^{1/2})^{2^{n-1}} \Delta_n(\underline{a})^{1/2} = 1,$$

and hence condition (ii) of Proposition 4.25 is also met. By using Proposition A.15, we know that there exist elements W_1, \dots, W_n with $\Delta_n(\underline{W}) \neq 0$ giving rise to this space via the construction of Theorem 4.26.

Conversely, for every n -uple of elements $W_1, \dots, W_n \in k^\times$ such that $\Delta_n(\underline{W}) \neq 0$ one can set $V_i = \Delta_{n-1}(\hat{W}_i) \in k^\times$ and $U_i = \frac{V_i^2}{\Delta_n(\underline{W})} \in k^\times$, and show that the resulting

$$Q_i = U_i^2 + X V_i^2 = V_i^2 X + \left(\frac{V_i^2}{\Delta_n(\underline{W})} \right)^2$$

give rise to a basis of a space $L_{2^{n-1}, n}$ whose set of poles is the set of non-zero vectors in $\langle a_1^2, \dots, a_n^2 \rangle_{\mathbb{F}_p}$ with $a_i := \frac{V_i}{\Delta_n(\underline{W})}$. By definition, this is a standard space.

Finally, we observe that we can also obtain étale pullbacks of standard spaces using the techniques of section 4.2.1. In fact, by Remark 4.31, we can realise étale pullbacks of standard spaces with respect to $X \mapsto S(X)$ with $S'(X) = 1$ using the techniques of section 4.2.1. All étale pullbacks of standard spaces are equivalent to an étale pullback with $S'(X) = 1$, so we can realise in this way at least one member from each equivalence class.

6. CLASSIFICATION OF SPACES $L_{12,2}$ AND $L_{15,2}$ OVER \mathbb{F}_3

The aim of this section is to completely classify spaces $L_{12,2}$ and $L_{15,2}$ up to equivalence and Frobenius equivalence in the case where $p = 3$. We recall that the spaces $L_{3,2}, L_{6,2}$ and $L_{9,2}$ ($p = 3$) are classified by Pagot in [8], so the results of this section are a natural prosecution of that work. By exhibiting the existence of a space $L_{15,2}$, we provide the first known example of a space $L_{\lambda p, 2}$ where $p - 1$ does not divide λ . This section relies on computations of Gröbner bases to solve polynomial systems in characteristic 3. The supporting *Macaulay2* code can be found in a public repository⁵, in a form that can be easily replicated using other computer algebra systems.

6.1. **Classification of spaces $L_{12,2}$.** Let $p = 3$ and $\lambda = 4$. We set

$$\begin{aligned} Q_1 &:= a(X^4 - s_1 X^3 + s_2 X^2 - s_3 X + s_4) \\ Q_2 &:= X^4 - t_1 X^3 + t_2 X^2 - t_3 X + t_4, \end{aligned}$$

and we look for conditions such that the pair (Q_1, Q_2) is a prompt for a space $L_{12,2}$. To this aim, we consider the expressions R_k 's of Convention 3.5 as polynomials with coefficients in $\{a, s_1, \dots, s_4, t_1, \dots, t_4\}$ and look for a solution to the system of equations

$$(6.1) \quad \begin{cases} R_1(a, s_i, t_i) \neq 0, \\ R_k(a, s_i, t_i) = 0 \quad \text{for } k = 2, \dots, 8. \end{cases}$$

The main result of this section is the following:

⁵available at the url <https://github.com/DanieleTurchetti/equidistant>

Theorem 6.2. *Let k be an algebraically closed field containing \mathbb{F}_3 and let $a \in k$ be such that $a^2 \notin \mathbb{F}_3$. Then, the pair $(Q_{1,a}, Q_{2,a})$ with*

$$Q_{1,a} := a(X^4 + (a^4 - a^2 - 1)X^2 + a^8)$$

$$Q_{2,a} := X^4 - (a^4 + a^2 - 1)X^2 + 1,$$

is a prompt for a space $L_{12,2}$ in $\Omega(k(X))$ denoted by Ω_a . Conversely, if $\Omega \subset \Omega(k(X))$ is a space $L_{12,2}$ then there exists $a \in k$ with $a^2 \notin \mathbb{F}_3$ such that Ω is equivalent to Ω_a .

Proof. We prove the two statements separately.

- Let $(Q_{1,a}, Q_{2,a})$ be as in the statement. We can verify that

$$((Q_{1,a}^3 - Q_{1,a}Q_{2,a}^2)^2)'' = -(a^3 - a)^{10}(a^2 + 1)^5.$$

Since we assumed that $a^2 \notin \mathbb{F}_3$, this is non-zero. We can then apply Proposition 3.1 and Remark 3.4 to get that the pair $(Q_{1,a}, Q_{2,a})$ is a prompt for a space $L_{12,2}$. In the rest of the proof, we will denote this space by Ω_a .

- Let $\Omega \subset \Omega(k(X))$ be a space $L_{12,2}$, and let (A, B) be a pair of polynomials in $k[X]$ of the form

$$Q_1 := a(X^4 - s_1X^3 + s_2X^2 - s_3X + s_4)$$

$$Q_2 := X^4 - t_1X^3 + t_2X^2 - t_3X + t_4,$$

such that (Q_1, Q_2) is a prompt for Ω . Then, we can apply the following successive reductions to get a situation where the coefficients s_i 's and t_i 's can be retrieved computationally :

- ‘ $s_1 = t_1 = 0$ ’: We know by Lemma 3.6 that $s_1 = t_1$ and applying the translation $X \rightarrow X + s_1$ allows us to suppose $s_1 = t_1 = 0$.
- ‘ $s_3 = t_3 = 0$ ’: Suppose that this is not the case. Then, up to replacing Q_1 with aQ_2 and Q_2 with $\frac{1}{a}Q_1$, we can suppose that $s_3 \neq 0$. Let $\alpha \in k$ such that $\alpha^3 = s_3^{-1}$ and apply the transformation $X \rightarrow \alpha X$ to get that $s_3 = 1$. Then one finds that the system (6.1) has no solution (see [12, Program 6.1]), giving rise to a contradiction.
- ‘ $s_4 = a^8, t_4 = 1$ ’: A computation of Gröbner basis under the reductions above (see [12, Program 6.2]) returns the condition $s_4 = a^8 t_4$. Since $t_3 = 0$ we know that $t_4 \neq 0$ otherwise the polynomial B would have multiple roots. We can then pick $\beta \in k$ such that $\beta^4 = t_4^{-1}$ and apply the transformation $X \rightarrow \beta X$ to get that $t_4 = 1$.

The reductions above leave us with polynomials of the form

$$Q_1 := a(X^4 + s_2X^2 + a^8)$$

$$Q_2 := X^4 + t_2X^2 + 1.$$

Let us simplify the notation by setting $s := s_2$ and $t := t_2$, and compute these coefficients. Now, [12, Program 6.3] can compute a Grobner basis of the system (6.1). By inspecting the first element of the basis, we conclude that we have at most six possible expressions for s :

$$s = \begin{cases} \pm(a^4 + a^2 - a + 1) \\ \pm(a^4 - a^2 - 1) \\ \pm(a^4 + a^2 + a + 1) \end{cases}$$

Let us show how the situation can be further simplified: first, we note that the transformation $X \rightarrow iX$ with $i^2 = -1$ produces a new pair (S_1, S_2) with

$$S_1 := a(X^4 - s_2X^2 + a^8)$$

$$S_2 := X^4 - t_2X^2 + 1.$$

Then, we note that the pair $(-Q_1, Q_2)$ produces the same space $L_{12,2}$. Hence, if we can find a solution to the system (6.1) for a pair (a, s) we also have a solution for pairs $(-a, s), (a, -s)$, and $(-a, -s)$, which reduces our search to the following two situations:

Case 1: $s = a^4 - a^2 - 1$. In this case, Program [12, Program 6.4] returns $t = -(a^4 + a^2 - 1)$ (under the condition that $s \neq 0$), and then we find that $Q_1 = Q_{1,a}$ and $Q_2 = Q_{2,a}$, as required. If $s = 0$, then the program returns $t^2 = a^2 + 1 = a^4$, which gives the two values $t = \pm a^2 = \pm(a^4 + a^2 - 1)$. Note that these two values correspond to equivalent spaces, under the transformation $X \mapsto iX$. We conclude that, in Case 1, the pair (Q_1, Q_2) is a prompt for a space equivalent to Ω_a .

Case 2: $s = a^4 + a^2 + a + 1$. In this case, Program [12, Program 6.5] returns $t = -(a^4 - a^3 + a^2 + 1)$ (under the condition that $s \neq 0$).

We now compare the pair (Q_1, Q_2) obtained with the pair $(Q_{1,a-1}, Q_{2,a-1})$ which is a prompt for the space Ω_{a-1} , namely

$$Q_{1,a-1} = (a-1)(X^4 + (a^4 - a^3 - a^2 + a - 1)X^2 + (a-1)^8)$$

$$Q_{2,a-1} = X^4 - (a^4 - a^3 + a^2 + 1)X^2 + 1.$$

One verifies that $Q_1^3 Q_2 - Q_1 Q_2^3 = Q_{1,a-1}^3 Q_{2,a-1} - Q_{1,a-1} Q_{2,a-1}^3$. Hence the pair (Q_1, Q_2) is a prompt for a space $L_{12,2}$ with the same set of poles as Ω_{a-1} . By Corollary 2.10, we have then that (Q_1, Q_2) is a prompt for Ω_{a-1} . If $s = 0$, the program returns $t^2 = -a^3 - a^2 - a - 1$, giving the two values $t = \pm(a^3 + a) = \pm(a^4 - a^3 + a^2 + 1)$, that correspond to equivalent spaces under the transformation $X \rightarrow iX$. Hence in Case 2 the pair (Q_1, Q_2) is a prompt for a space equivalent to Ω_{a-1} . □

We can be more explicit about the spaces Ω_a classified in the theorem, and compute their residues and their writing in logarithmic form. This is the content of the following result.

Corollary 6.3. *Let k be an algebraically closed field containing \mathbb{F}_3 . Let $a \in k$ be such that $(a^3 - a)(a^2 + 1) \neq 0$ and fix $i, j \in k$ such that $i^2 = -1$ and $j^2 = a^2 + 1$. Then the space Ω_a is generated by $\frac{df_1}{f_1}$ and $\frac{df_2}{f_2}$ with*

$$f_1 = \prod_{\substack{\epsilon_1 = \pm 1 \\ \epsilon_2 = \pm 1}} (X - (\epsilon_1(a-1)j + \epsilon_2 ai))^{\epsilon_1} \prod_{\substack{\epsilon_1 = \pm 1 \\ \epsilon_2 = \pm 1}} (X - (\epsilon_1(a+1)j + \epsilon_2 ai))^{\epsilon_1} \prod_{\substack{\epsilon_1 = \pm 1 \\ \epsilon_2 = \pm 1}} (X - (\epsilon_1 aj + \epsilon_2(a^2 - 1)))^{\epsilon_1}$$

and

$$f_2 = \prod_{\substack{\epsilon_1 = \pm 1 \\ \epsilon_2 = \pm 1}} (X - (\epsilon_1(a-1)j + \epsilon_2 ai))^{-\epsilon_1} \prod_{\substack{\epsilon_1 = \pm 1 \\ \epsilon_2 = \pm 1}} (X - (\epsilon_1(a+1)j + \epsilon_2 ai))^{\epsilon_1} \prod_{\substack{\epsilon_1 = \pm 1 \\ \epsilon_2 = \pm 1}} (X - (\epsilon_1 j + \epsilon_2(a^2 - 1)))^{\epsilon_1}.$$

In particular, the set of poles and residues of these generators are:

x	$res_1(x)$	$res_2(x)$	x	$res_1(x)$	$res_2(x)$
$j + a^2 - 1$	0	1	$(a+1)j + ai$	1	1
$j - a^2 + 1$	0	1	$(a+1)j - ai$	1	1
$-j + a^2 - 1$	0	-1	$-(a+1)j + ai$	-1	-1
$-j - a^2 + 1$	0	-1	$-(a+1)j - ai$	-1	-1
$(a-1)j + ai$	1	-1	$aj + (a^2 - 1)$	1	0
$(a-1)j - ai$	1	-1	$aj - (a^2 - 1)$	1	0
$-(a-1)j + ai$	-1	1	$-aj + (a^2 - 1)$	-1	0
$-(a-1)j - ai$	-1	1	$-aj - (a^2 - 1)$	-1	0

Proof. We start by computing the zeroes of the polynomials $Q_{1,a}$, $Q_{1,a} + Q_{2,a}$, $Q_{1,a} - Q_{2,a}$, $Q_{2,a}$, which can be easily done by applying the formulas for solving biquadratic equations. We find that

$$Z(Q_{1,a}) = \{j + a^2 - 1, j - a^2 + 1, -j + a^2 - 1, -j - a^2 + 1\},$$

$$Z(Q_{1,a} + Q_{2,a}) = \{(a-1)j + ai, (a-1)j - ai, -(a-1)j + ai, -(a-1)j - ai\}$$

$$Z(Q_{1,a} - Q_{2,a}) = \{(a+1)j + ai, (a+1)j - ai, -(a-1)j + ai, -(a-1)j - ai\}, \text{ and}$$

$$Z(Q_{2,a}) = \{aj + (a^2 - 1), aj - (a^2 - 1), -aj + (a^2 - 1), -aj - (a^2 - 1)\},$$

as expected. By Theorem 6.2 and Remark 3.4 we know that Ω_a is generated by the differential forms $\omega_1 := \frac{Q_{1,a}dX}{c^3(Q_{1,a}^3Q_{2,a} - Q_{1,a}Q_{2,a}^3)}$ and $\omega_2 := \frac{Q_{2,a}dX}{c^3(Q_{1,a}^3Q_{2,a} - Q_{1,a}Q_{2,a}^3)}$ with $c^6(a^2+1)^5(a^3-a)^{10} = 1$. To compute the residues of these forms at their poles, we remark that $c^3 = \pm \frac{1}{(a^2+1)^{5/2}(a^3-a)^5} = \pm \frac{1}{j^5(a^3-a)^5}$. For each $x \in Z(Q_{1,a}) \cup Z(Q_{1,a} + Q_{2,a}) \cup Z(Q_{1,a} - Q_{2,a}) \cup Z(Q_{2,a})$ we can compute the quantities $(Q_{1,a}^2Q_{2,a} - Q_{2,a}^3)'(x)$ and $(Q_{1,a}^3 - Q_{1,a}Q_{2,a}^3)'(x)$ and check when they are equal to $\pm j^5(a^3-a)^5$. Up to possibly replacing j with $-j$, and using the formulas $res_1(x) = \frac{1}{c^3(Q_{1,a}^2Q_{2,a} - Q_{2,a}^3)'(x)}$ and $res_2(x) = \frac{1}{c^3(Q_{1,a}^3 - Q_{1,a}Q_{2,a}^3)'(x)}$, we get the table of residues above, or equivalently that $\omega_1 = \frac{df_1}{f_1}$ and $\omega_2 = \frac{df_2}{f_2}$ with f_1, f_2 as in the statement. \square

Remark 6.4. In the light of the results of Section 5.1, it is natural to wonder if the description of subspaces of standard spaces given by Proposition 5.6 has an analogue in the case of spaces that are not standard. Namely, given a space $L_{\lambda p^{n-1}, n}$ that is not standard, and assuming that $p|\lambda$, one can ask whether its t -dimensional subspaces for $t < n$ can be obtained as étale pullbacks of some space $L_{\mu p^{t-1}, t}$.

The classification of spaces $L_{12,2}$ achieved in this section shows that the answer to this question is negative, at least at the level of generality stated above. More precisely, for every value of a no one dimensional subspace of Ω_a can be obtained as a non-trivial étale pullback of some other space. First, note that we only need to check this for degree 6 pullbacks of spaces $L_{2,1}$, as there are no spaces $L_{4,1}$. Then, we remark that any space $L_{2,1}$ is equivalent to a standard space, whose set of poles is of the form $\{x, -x\}$.

The étale pullback $\phi^*(\omega)$ is then up to a constant of the form $\frac{dX}{S(X)^2 - x^2}$ for $S(X)$ of degree 6 such that $S'(X) \in k^\times$. In particular, $S(X)^2 - x^2$ has always a non zero term of degree 7. If we fix a differential form $\omega \in \Omega_a$, and we consider the polynomial whose zeroes are the poles of ω , we see that it has only terms of even degree. As a result, it is not possible to obtain ω via an étale pullback of a differential form in a space $L_{2,1}$.

6.2. Classification of spaces $L_{15,2}$. Let $p = 3$ and $\lambda = 5$. In the first part of the section, we exhibit explicitly two vector spaces $L_{15,2}$, one whose poles are all in \mathbb{F}_{27} and another one whose poles are all in \mathbb{F}_{81} .

Example 6.5. Let \mathbb{F}_{27} be the finite field with 27 elements, and let us write $\mathbb{F}_{27} = \mathbb{F}_3[\mu]$, with $\mu^3 - \mu + 1 = 0$. We have that $\mu^{13} = -1$, and then μ is a generator of the cyclic multiplicative group \mathbb{F}_{27}^\times . Following the notation used in the proof of Proposition 3.1, we define a subset of \mathbb{F}_{27} indexed by the elements of $\mathbb{P}^1(\mathbb{F}_3) = \{0, 1, 2, \infty\}$:

$$\begin{cases} X_0 = & \{\mu^2 - \mu, \mu + 1, -\mu^2 - \mu - 1, \mu, 0\} = \{\mu^4, \mu^9, \mu^{19}, \mu, 0\} \\ X_1 = & \{-\mu^2 + \mu - 1, -\mu^2, -\mu^2 - \mu + 1, -\mu^2 - 1, \mu^2 + 1\} = \{\mu^5, \mu^{15}, \mu^{24}, \mu^8, \mu^{21}\} \\ X_2 = & \{-\mu^2 + 1, -1, \mu^2 - \mu + 1, \mu^2 - 1, -\mu^2 + \mu\} = \{\mu^{25}, \mu^{13}, \mu^{18}, \mu^{12}, \mu^{17}\} \\ X_\infty = & \{\mu^2 + \mu, \mu - 1, \mu^2, \mu^2 - \mu - 1, -\mu - 1\} = \{\mu^{10}, \mu^3, \mu^2, \mu^7, \mu^{22}\}. \end{cases}$$

To these sets, we associate the corresponding polynomials of $\mathbb{F}_{27}[X]$:

$$\begin{cases} P_0(X) = \prod_{x \in X_0} (X - x) & = X^5 - (\mu^2 + \mu + 1)X^3 - X^2 + (\mu^2 - \mu - 1)X \\ P_1(X) = \prod_{x \in X_1} (X - x) & = X^5 - (\mu^2 - 1)X^3 - (\mu^2 - \mu + 1)X^2 - (\mu + 1)X - (\mu^2 + 1) \\ P_2(X) = \prod_{x \in X_2} (X - x) & = X^5 + (\mu^2 - \mu - 1)X^3 + (\mu + 1)X^2 + (\mu^2 - 1)X - (\mu^2 - \mu - 1) \\ P_\infty(X) = \prod_{x \in X_\infty} (X - x) & = X^5 - (\mu^2 + \mu)X^3 - (\mu + 1)X^2 - (\mu^2 + \mu - 1)X - (\mu^2 - \mu + 1). \end{cases}$$

Now, let

$$Q_1(X) := (\mu^2 - \mu - 1)P_0(X) \text{ and } Q_2(X) := -\mu P_\infty(X).$$

One verifies that the second derivative of the polynomial $(Q_1^3 - Q_1Q_2^2)^2$ is equal to -1 . Moreover, since $(\mu^2 - \mu - 1) \pm (-\mu) \notin \mathbb{F}_3$, the degree of $iQ_1 + jQ_2$ is equal to 4 for every $[i : j] \in \mathbb{P}^1(\mathbb{F}_3)$. Then, we can set $\omega_1 = \frac{Q_1}{Q_1^3Q_2 - Q_1Q_2^3}dX$ and $\omega_2 = \frac{Q_2}{Q_1^3Q_2 - Q_1Q_2^3}dX$ and apply Proposition 3.1 to show that $\Omega_1 := \langle \omega_1, \omega_2 \rangle$ is a \mathbb{F}_3 -vector space $L_{15,2}$.

To write the generators of this spaces in logarithmic form, we need to compute their residues at the poles. In order to do this, we denote by Z_1 the set of zeroes of $Q_1^2Q_2 - Q_2^3$, by Z_2 the set of zeroes of $Q_1^3 - Q_1Q_2^2$, and we introduce the functions $res_1 : Z_1 \rightarrow \mathbb{F}_3^\times$ and $res_2(x) : Z_2 \rightarrow \mathbb{F}_3^\times$ defined by $res_i(x) := res_x(\omega_i)$. Then, we compute them thanks to the formulas $res_1(x) = \frac{1}{(Q_1^2Q_2 - Q_2^3)'(x)}$ and $res_2(x) = \frac{1}{(Q_1^3 - Q_1Q_2^2)'(x)}$. We find the following table of values:

x	$res_1(x)$	$res_2(x)$	x	$res_1(x)$	$res_2(x)$
μ^4	0	1	μ^{25}	-1	-1
μ^9	0	1	μ^{13}	1	1
μ^{19}	0	1	μ^{18}	-1	-1
μ	0	-1	μ^{12}	1	1
0	0	-1	μ^{17}	-1	-1
μ^5	-1	1	μ^{10}	1	0
μ^{15}	1	-1	μ^3	1	0
μ^{24}	1	-1	μ^2	-1	0
μ^8	1	-1	μ^7	-1	0
μ^{21}	1	-1	μ^{22}	1	0

Then, $\omega_1 = \frac{dF_1}{F_1}$ and $\omega_2 = \frac{dF_2}{F_2}$ with

$$F_1(X) = \prod_{x \in Z_1} (X - x)^{res_1(x)} \text{ and } F_2(X) = \prod_{x \in Z_2} (X - x)^{res_2(x)}.$$

Example 6.6. Let \mathbb{F}_{81} be the finite field with 81 elements, and let us write $\mathbb{F}_{81} = \mathbb{F}_3[\mu]$, with $\mu^4 + \mu^3 - \mu^2 - \mu - 1 = 0$.⁶ We have that $\mu^{40} = -1$, and then μ is a generator of the cyclic multiplicative group \mathbb{F}_{81}^\times . In analogy with Example 6.5, we define for every element of $\mathbb{P}^1(\mathbb{F}_3)$ a subset of \mathbb{F}_{81} as follows:

$$\begin{cases} X_0 = \{\mu^7, \mu^{30}, \mu^{51}, \mu^{59}, \mu^{63}\} \\ X_1 = \{\mu^{26}, \mu^{50}, \mu^{52}, \mu^{68}, \mu^{74}\} \\ X_2 = \{\mu^{34}, \mu^{60}, \mu^{66}, \mu^{70}, 0\} \\ X_\infty = \{\mu^{10}, \mu^{11}, \mu^{19}, \mu^{20}, \mu^{40}\} \end{cases}.$$

Then, we associate to these sets the corresponding monic polynomials. In this case, it turns out that they have coefficients in \mathbb{F}_9 . More specifically, we set $a = -\mu^{20} \in \mathbb{F}_9$ and one can verify that we

⁶This is equivalent to write $\mathbb{F}_{81} = \mathbb{F}_3[\alpha]$ for a choice of α satisfying $\alpha^4 + \alpha^2 - 1 = 0$ and set $\mu = \alpha^3 + \alpha - 1$.

have:

$$\begin{cases} P_0 = \prod_{x \in X_0} (X - x) & = X^5 - X^3 - X^2 + aX - (a + 1) \\ P_1 = \prod_{x \in X_1} (X - x) & = X^5 - aX^3 + (a - 1)X - (a - 1) \\ P_2 = \prod_{x \in X_2} (X - x) & = X^5 - (a - 1)X^2 - (a - 1)X \\ P_\infty = \prod_{x \in X_\infty} (X - x) & = X^5 - (a + 1)X^3 + (a + 1)X^2 + X + a \end{cases}$$

Then, we set

$$Q_1(X) := aP_0(X) \quad \text{and} \quad Q_2(X) := -(a + 1)P_\infty(X),$$

and we can verify that the second derivative of the polynomial $(Q_1^3 - Q_1Q_2^2)^2$ is equal to -1 . Moreover, the degree of $iQ_1 + jQ_2$ is equal to 5 for every $[i : j] \in \mathbb{P}^1(\mathbb{F}_3)$. Hence, if we set $\omega_1 = \frac{Q_1}{Q_1^3Q_2 - Q_1Q_2^3}dX$ and $\omega_2 = \frac{Q_2}{Q_1^3Q_2 - Q_1Q_2^3}dX$ and apply Proposition 3.1 this shows that $\Omega_2 := \langle \omega_1, \omega_2 \rangle$ is a space $L_{15,2}$.

As in Example 6.5, we can easily compute the residues of ω_1 and ω_2 at their poles, and get the following table.

x	$res_1(x)$	$res_2(x)$	x	$res_1(x)$	$res_2(x)$
μ^7	0	-1	μ^{34}	-1	-1
μ^{30}	0	-1	μ^{60}	1	1
μ^{51}	0	-1	μ^{66}	-1	-1
μ^{59}	0	-1	μ^{70}	1	1
μ^{63}	0	-1	0	1	1
μ^{26}	1	-1	μ^{10}	1	0
μ^{50}	-1	1	μ^{11}	1	0
μ^{52}	-1	1	μ^{19}	1	0
μ^{68}	-1	1	μ^{20}	1	0
μ^{74}	1	-1	μ^{40}	-1	0

As in the previous example we have $\omega_1 = \frac{dF_1}{F_1}$ and $\omega_2 = \frac{dF_2}{F_2}$ with

$$F_1(X) = \prod_{x \in Z_1} (X - x)^{res_1(x)} \quad \text{and} \quad F_2(X) = \prod_{x \in Z_2} (X - x)^{res_2(x)}.$$

Our main result of this section says that the examples above are essentially the only spaces $L_{15,2}$.

Theorem 6.7. *Let k be an algebraically closed field containing \mathbb{F}_3 and let $\Phi : \Omega(k(X)) \rightarrow \Omega(k(X))$ be the relative Frobenius operator. Every $\Omega \subset \Omega(k(X))$ vector space $L_{15,2}$ is equivalent to one of the following spaces, each representing a distinct equivalence class: Ω_1 (defined in Example 6.5), $\Phi(\Omega_1)$, $\Phi^2(\Omega_1)$, Ω_2 (defined in Example 6.6), or $\Phi(\Omega_2)$.*

To prove the theorem, we set

$$Q_1 := a(X^5 - s_1X^4 + s_2X^3 - s_3X^2 + s_4X - s_5)$$

and

$$Q_2 := X^5 - t_1X^4 + t_2X^3 - t_3X^2 + t_4X - t_5.$$

Then, we consider the expressions R_k 's of Convention 3.5 as polynomials with coefficients in $\{a, s_1, \dots, s_5, t_1, \dots, t_5\}$ and we aim to find a solution to the system of equations

$$(6.8) \quad \begin{cases} R_1(a, s_i, t_i) \neq 0, \\ R_k(a, s_i, t_i) = 0 \quad \text{for } k = 2, \dots, 10. \end{cases}$$

The existence of a space $L_{15,2}$ gives rise to a solution to the system (6.8) (see Convention 3.5). Conversely a solution to (6.8) produces a space $L_{15,2}$. In fact, such a solution would imply that the second derivative of $(Q_1^3 - Q_1Q_2^2)^2$ is equal to the nonzero constant $-R_1(a, s_i, t_i)$. Then, Proposition 3.1 and Remark 3.4 tell us that we can build a space $L_{15,2}$ from Q_1 and Q_2 by setting

$$\omega_1 = \frac{Q_1}{Q_1^3Q_2 - Q_1Q_2^3}dX \text{ and } \omega_2 = \frac{Q_2}{Q_1^3Q_2 - Q_1Q_2^3}dX, \text{ and considering the space } \langle \omega_1, \omega_2 \rangle.$$

Unfortunately, solving (6.8) is not an easy task even when assisted by a computer (a brute-force calculation of Gröbner basis turns out to be hopeless without further assumptions), so we need to simplify the equations before we go further with our strategy. To this aim, we apply Lemma 3.6, we set $s_1 = t_1$ and we reparametrize \mathbb{P}_k^1 in such a way to have $s_1 = t_1 = 0$. Then, we obtain the following result.

Lemma 6.9. *Let (Q_1, Q_2) be a pair of polynomials as above, and suppose that they yield a solution of the system (6.8). Then, either $s_3 \neq 0$ or $t_3 \neq 0$.*

Proof. We can show this using a Gröbner basis computation: Program [12, Program 6.6] computes the ideal generated by all the relations in (6.8) assuming $s_1 = t_1 = 0$ and $s_3 = t_3 = 0$, and checks that this is the whole ring. As a result, the system has no solution, and we either have $s_3 \neq 0$ or $t_3 \neq 0$. \square

Thanks to Lemma 6.9, we can suppose that $s_3 \neq 0$, and use our last degree of freedom to reparametrize \mathbb{P}_k^1 in such a way that $s_3 = 1$. This is the content of our next Lemma.

Lemma 6.10. *Let (Q_1, Q_2) be a pair of polynomials as above and suppose that they yield a solution of the system (6.8), hence being a prompt for a space $L_{15,2}$ denoted by Ω . Then, there exists a pair (Q_1^\sharp, Q_2^\sharp) whose coefficients satisfy $s_1 = t_1 = 0$, $s_3 = 1$ and (6.8). In particular, (Q_1^\sharp, Q_2^\sharp) is a prompt for a space equivalent to Ω .*

Proof. We know by Lemma 3.6 that $s_1 = t_1$ and applying the translation $X \rightarrow X + s_1$ allows us to suppose $s_1 = t_1 = 0$. To get to $s_3 = 1$, we apply Lemma 6.9 to get to the situation where $s_3 \neq 0$, which we can do up to possibly swapping Q_1 and Q_2 . Then, we consider $\alpha \in k$ such that $\alpha^3 = s_3^{-1}$ and we apply the transformation $X \rightarrow \alpha X$. Under this transformation we have

$$P_0^\sharp(X) = \prod_{x \in X_{[1:0]}} (X - \alpha x) = X^5 + \alpha^2 s_2 X^3 - \alpha^3 s_3 X^2 + \alpha^4 s_4 X - \alpha^5 s_5.$$

and

$$P_\infty^\sharp(X) = \prod_{x \in X_{[0:1]}} (X - \alpha x) = X^5 + \alpha^2 t_2 X^3 - \alpha^3 t_3 X^2 + \alpha^4 t_4 X - \alpha^5 t_5,$$

so that the coefficient of X^2 in $P_0^\sharp(X)$ is -1 as needed. Since (Q_1, Q_2) is a prompt for a space $L_{15,2}$ and we applied a homothety to the zeroes of $Q_1^3Q_2 - Q_1Q_2^3$, then by setting $Q_1^\sharp = a^\sharp P_0^\sharp$ for a suitable nonzero constant a^\sharp and $Q_2^\sharp = P_\infty^\sharp$ we have that (Q_1^\sharp, Q_2^\sharp) is a prompt for a space $L_{15,2}$ that is equivalent to Ω . In particular, the coefficients of Q_1^\sharp and Q_2^\sharp satisfy the system (6.8). \square

Lemma 6.10 allows us then to consider without loss of generality the situation where $s_1 = t_1 = 0$ and $s_3 = 1$. Recall that we denoted by a the leading coefficient of the polynomial Q_1 . The following lemma shows that one can obtain the same space by applying a suitable substitution only to the parameter a .

Lemma 6.11. *Let (Q_1, Q_2) be a pair of polynomials with*

$$Q_1 := a(X^5 + s_2X^3 - X^2 + s_4X - s_5)$$

and

$$Q_2 := X^5 + t_2X^3 - t_3X^2 + t_4X - t_5$$

which is a prompt for a space $L_{15,2}$, denoted by Ω . For every $\alpha \in \{-1, \frac{1}{a+1}, -\frac{1}{a+1}, \frac{1}{a-1}, -\frac{1}{a-1}\}$ there exists a pair (Q_1^\sharp, Q_2^\sharp) which is a prompt for the same space Ω such that Q_2^\sharp is monic and $Q_1^\sharp = \alpha Q_1$.

Proof. By Definition 3.3, there exists a constant $c \in k^\times$ such that the space Ω is generated by ω_1 and ω_2 with $\omega_1 := \frac{dX}{c^3(Q_1^2Q_2 - Q_2^3)}$ and $\omega_2 := \frac{dX}{c^3(Q_1^3 - Q_1Q_2^2)}$. We consider the following situations:

- (i) Let $(\omega_1, -\omega_2)$ be another basis of Ω and pick a pair (Q_1^\sharp, Q_2^\sharp) associated with it. Since $-\omega_2$ has the same poles as ω_2 , then $Q_2^\sharp = Q_2$. We see then that $Q_1^\sharp = -Q_1$.
- (ii) Let $(\omega_1, \omega_1 + \omega_2)$ be another basis of Ω and pick a pair (Q_1^\sharp, Q_2^\sharp) associated with it. By looking at the poles of $\omega_1 + \omega_2$ we see that $Q_2^\sharp = \frac{Q_1 + Q_2}{a+1}$. The leading coefficient of Q_1^\sharp is computed as the ratio of leading coefficient of Q_1 by the one of $Q_1 + Q_2$. In other words, $Q_1^\sharp = \frac{Q_1}{a+1}$.

The coefficients $-\frac{1}{a+1}$, $\frac{1}{a-1}$, and $-\frac{1}{a-1}$ can be obtained by composing the changes of basis of (i) and (ii). \square

Remark 6.12. The transformations of the leading coefficients described in Lemma 6.11 are the unique possible if we want to keep the simplifications made previously. In fact, we are not allowed to change the zeroes of A as this would change the parameter s_3 in the general case.

Proof of Theorem 6.7. Using the simplifications above, we are now ready to prove the main theorem. More precisely, we use Lemma 6.10 to assume that $s_1 = t_1 = 0$ and $s_3 = 1$. Then, the program [12, Program 6.7] tells us that any solution to the system (6.8) needs to satisfy the following condition on the parameter a :

$$(a^3 - a^2 - a - 1)(a^3 - a + 1)(a^2 + 1)(a^3 - a^2 + 1)(a^3 + a^2 - a + 1)(a^3 - a^2 + a + 1)(a^3 - a - 1)(a^2 + a - 1)(a^3 + a^2 - 1)(a^3 + a^2 + a - 1)(a^2 - a - 1) = 0.$$

By applying Lemma 6.11 we can consider only an irreducible polynomial for each orbit under the group action on (Q_1, Q_2) generated by $a \mapsto -a$ and $a \mapsto \frac{a}{a+1}$. Namely, we are left with studying the following cases:

- Case 1: $a^2 + 1 = 0$
- Case 2: $a^3 - a^2 + 1 = 0$
- Case 3: $a^3 - a + 1 = 0$.

In all these cases, the successive elements of a Gröbner basis for the system (6.8) can be computed, and show us all the possibilities for the coefficients of Q_1 and Q_2 . Let $\Delta_2(Q_1, Q_2) = Q_1^3Q_2 - Q_1Q_2^3$ be the Moore determinant of (Q_1, Q_2) . We remark that, if (Q_1, Q_2) and (S_1, S_2) are prompts for spaces $L_{15,2}$ and $\Delta_2(Q_1, Q_2) = \Delta_2(S_1, S_2)$, then they actually are prompts for the same space by virtue of Corollary 2.10. Let us denote by $\Delta^{[1]}$ the Moore determinant of the pair (Q_1, Q_2) appearing in Example 6.5, by $\Delta^{[2]}$ the Moore determinant of the pair (Q_1, Q_2) appearing in Example 6.6, and by Δ^\bullet with an appropriate superscript the Moore determinants appearing in the cases below (e.g. $\Delta^{1.A}$ is the Moore determinant of the polynomials given by the program in case 1.A). We show that we can get a complete classification up to equivalence by studying the cases outlined above and comparing the respective Moore determinants.

In **Case 1**, we let a be a square root of -1 . For simplicity and consistency, we assume that $a = -\mu^{20}$ where μ is the generator of \mathbb{F}_{81}^\times appearing in Example 6.6. Then, the program [12, Program 6.8] returns the following subcases:

Case 1.A	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	0	1	$a - 1$	0	$-a + 1$	a	-1	$a + 1$

Case 1.A'	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	0	1	$-a-1$	0	$a+1$	$-a$	-1	$-a+1$

In Case 1.A we get a space that is equivalent to Ω_2 . In fact, one can verify that, after applying the homothety $X \mapsto -(a+1)X$ the Moore determinant $\Delta^{1.A}$ is equal to $\Delta^{[2]}$.

Since Case 1.A' is obtained from Case 1.A by applying the transformation $a \mapsto -a$, and we have that $a^3 = -a$, then it results that the polynomials appearing in Case 1.A' are prompts for the space $\Phi(\Omega_2)$.

Case 1.B	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	-1	1	a	$1+a$	$-a$	0	$a-1$	$a-1$

Case 1.B'	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	-1	1	$-a$	$1-a$	a	0	$-(a+1)$	$-(a+1)$

In Case 1.B, one gets the space Ω_2 without the need to reparametrize. In fact, the zeroes of A (resp. of B) in this case are the same as those of P_0 (resp. P_1) in the example. If we apply the transformation $a \mapsto -a$, we land in Case 1.B' which corresponds to the space $\Phi(\Omega_2)$.

Case 1.C	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	$a-1$	1	-1	$a-1$	0	$-a$	$a-1$	0

Case 1.C'	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	$-(a+1)$	1	-1	$-(a+1)$	0	a	$-(a+1)$	0

In Case 1.C, one gets a space that is equivalent to Ω_2 , as one can see by applying the transformation $X \mapsto (a-1)X$. If we apply the transformation $a \mapsto -a$, we land in Case 1.C' which then corresponds to the space $\Phi(\Omega_2)$, as above.

In **Case 2**, let a be a root of $X^3 - X^2 + 1$. For simplicity and consistency, we assume that $a = \mu^{-1}$ where μ is the generator of \mathbb{F}_{27}^\times appearing in Example 6.5. Then we have that $\Phi(a) = \frac{1}{\mu+1} = -a^2 - a - 1$ and $\Phi^2(a) = \frac{1}{\mu-1} = a^2 - 1$. Program [12, Program 6.9] returns the following subcases:

Case 2.A	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	-1	1	a^2	$-(a^2 + a + 1)$	$-(a+1)$	$a^2 + a + 1$	$a^2 - a$	$-(a^2 - a + 1)$

Case 2.B	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	-1	1	$-(a^2 + a)$	a	$-(a+1)$	a^2	$-(a^2 - a)$	0

Case 2.C	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	-1	1	$a+1$	$a^2 - 1$	$-(a+1)$	$a^2 - 1$	1	$a^2 - a - 1$

In Case 2.A we get a vector space equivalent to Ω_1 . In fact, one can verify that, after applying the homothety $X \mapsto (\mu^2 + \mu + 1)X = -(a^2 + 1)X$, the Moore determinant $\Delta^{2.A}$ is equal to $\Delta^{[1]}$. We can also verify that $\Phi(\Delta^{2.A})$ has the same zeroes of $\Delta^{2.C}$ and $\Phi^2(\Delta^{2.A})$ has the same zeroes of $\Delta^{2.B}$. Hence, Case 2.C corresponds to $\Phi(\Omega_1)$ and Case 2.B corresponds to $\Phi^2(\Omega_1)$.

Finally, in **Case 3** let a be a root of $X^3 - X + 1$. For simplicity and consistency, we assume that $a = \mu$, the generator of \mathbb{F}_{27}^\times appearing in Example 6.5. Then we have that $\Phi(a) = a - 1$ and $\Phi^2(a) = a + 1$. Program [12, Program 6.10] returns the following subcases:

Case 3.A	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	a	1	1	$-(a^2 - a + 1)$	$-(a^2 + a + 1)$	$-(a+1)$	$a^2 + a$	$-(a^2 + a + 1)$

Case 3.B	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	$-(a^2 + a)$	1	$a^2 - a$	a	$-a^2$	$a - 1$	$a^2 - 1$	a^2

Case 3.C	s_2	s_3	s_4	s_5	t_2	t_3	t_4	t_5
	$-(a^2 - a + 1)$	1	$a^2 + 1$	0	$-(a^2 + a)$	a^2	$-a$	$a^2 + a - 1$

In Case 3.A we get a vector space equivalent to Ω_1 under the homothety $X \mapsto (1 - a)X$.

In Case 3.B we get a vector space equivalent to $\Phi^2(\Omega_1)$. We can see this by applying Φ and then the homothety $X \mapsto (a - 1)X$ to the coefficients of our polynomials, and noticing that the resulting pair is a prompt for the space Ω_1 .

Finally, if we apply the transformation Φ to the coefficients of P_0 and P_1 of Example 6.5, we see that we get the coefficients of the table of Case 3.C. The pair (Q_1, Q_2) in this latter case then is a prompt for the space $\Phi(\Omega_1)$.

This exhausts all possible cases, and since the spaces Ω_1 , $\Phi(\Omega_1)$, $\Phi^2(\Omega_1)$, Ω_2 , and $\Phi(\Omega_2)$ make up distinct equivalence classes of spaces $L_{15,2}$, this concludes our proof of the classification theorem. \square

Remark 6.13. We believe that the classification of spaces $L_{12,2}$ and $L_{15,2}$ in this section is interesting in itself, as the nature of these examples is quite different from anything previously known, for example these spaces can not be obtained from standard spaces. Moreover, we remark that, by applying étale pullbacks to these examples, we can generate spaces $L_{36d,2}$ and $L_{45d,2}$ for every integer $d \geq 1$, resulting in large classes of examples useful for future investigation.

APPENDIX A. MOORE DETERMINANTS

In this section, we collect several results on Moore determinants. With the exception of Lemma A.6, for which we provide a proof, these results are already known (from work of Elkies in [13] and Fresnel-Matignon in [10]), and are therefore recalled without proof.

Let k be a field of characteristic $p > 0$, and denote by $F : k \rightarrow k$ the Frobenius automorphism $x \mapsto x^p$. The *Moore determinant* of an n -tuple $\underline{a} := (a_1, \dots, a_n) \in k^n$ is defined as

$$\Delta_n(\underline{a}) := \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^p & a_2^p & \dots & a_n^p \\ \vdots & \vdots & \dots & \vdots \\ a_1^{p^{n-1}} & a_2^{p^{n-1}} & \dots & a_n^{p^{n-1}} \end{vmatrix}.$$

Remark A.1. Here we list the first elementary results on Moore determinants. By multilinearity of determinants, for every $\alpha \in k$, we have that

$$\Delta_n(\alpha \underline{a}) = \alpha^{1+p+\dots+p^{n-1}} \Delta_n(\underline{a}).$$

Moreover, given an invertible matrix $M \in GL_n(\mathbb{F}_p)$, we have ([13, p. 80]) that

$$\Delta_n(\underline{a}M) = \Delta_n(\underline{a}) \det(M).$$

These relations are used in the proof of Proposition 4.5(iii). Finally, Moore shows the following identity:

$$(A.2) \quad \Delta_n(\underline{a}) = \prod_{i=1}^n \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \dots \prod_{\epsilon_1 \in \mathbb{F}_p} (a_i + \epsilon_{i-1}a_{i-1} + \dots + \epsilon_1 a_1).$$

As a result, we have that $\Delta_n(\underline{a}) \neq 0$ if, and only if, the a_i 's are \mathbb{F}_p -linearly independent.

Theorem A.3. *For every n -tuple $\underline{a} \in k^n$, and for every $1 \leq i \leq n$, we define the $n - 1$ -tuple $\hat{a}_i := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. Then, we have the formula*

$$\Delta_n(\Delta_{n-1}(\hat{a}_1), \dots, (-1)^{i-1} \Delta_{i-1}(\hat{a}_i), \dots, (-1)^{n-1} \Delta_{n-1}(\hat{a}_n)) = \Delta_n(\underline{a})^{1+p+\dots+p^{n-2}}.$$

Proof. This is the special case $m = 0$ of [10, Theorem 4.1]. \square

Definition A.4. For every $V \subset k$ is a \mathbb{F}_p -vector space of dimension n , the *structural polynomial* of V is

$$P_V(X) := \prod_{v \in V} (X - v) \in k[X].$$

It is the unique monic polynomial of degree p^n such that V is the set of zeroes of P_V .

Lemma A.5. (cf. Proposition 2.2. of [10]) Let $V \subset k$ be a \mathbb{F}_p -vector space of dimension n . The structural polynomial P_V is additive and for every choice of basis $\underline{v} = (v_1, \dots, v_n)$ of V it satisfies the identity

$$P_V(X) = \frac{\Delta_{n+1}(\underline{v}, X)}{\Delta_n(\underline{v})} = X^{p^n} + \dots + (-1)^n \Delta_n(\underline{v})^{p-1} X.$$

Proof. Consider $\Delta_{n+1}(\underline{v}, X)$ as a polynomial in $k[X]$. The development of the determinant along the last column gives

$$\Delta_{n+1}(\underline{v}, X) = \Delta_n(\underline{v}) X^{p^n} + \dots + (-1)^n \Delta_n(\underline{v})^p X,$$

which results in the second equality of the lemma. On the other hand, we have by definition that $\Delta_{n+1}(\underline{v}, v) = 0$ for every $v \in V$, which proves the first equality. Additivity then follows from the fundamental theorem of additive polynomials, as V is an additive subgroup of k . \square

Lemma A.6. Let $n \geq 1$, $m \geq 1$ and $Y_1, \dots, Y_n, X_1, \dots, X_m$ be variables over \mathbb{F}_p . Then

$$(A.7) \quad \begin{aligned} & \Delta_n(\Delta_{m+1}(Y_1, X_1, X_2, \dots, X_m), \dots, \Delta_{m+1}(Y_n, X_1, X_2, \dots, X_m)) = \\ & = \Delta_m(X_1, \dots, X_m)^{p+p^2+\dots+p^{n-1}} \Delta_{n+m}(Y_1, \dots, Y_n, X_1, \dots, X_m). \end{aligned}$$

Proof. We proceed by induction on n . If $n = 1$, we interpret the expression $p + p^2 + \dots + p^{n-1}$ as 0. If $n = 2$, then the Lemma is a special case of Theorem A.3. Let us then assume $n > 2$ and that the Lemma is satisfied when replacing n with $n - 1$.

We denote by \mathcal{X} , the \mathbb{F}_p -vector space $\langle X_1, X_2, \dots, X_m \rangle_{\mathbb{F}_p}$, and by $P_{\mathcal{X}}$ the structural polynomial of \mathcal{X} . It then follows from Lemma A.5 that

$$\begin{aligned} & \Delta_n(\Delta_{m+1}(Y_1, X_1, X_2, \dots, X_m), \dots, \Delta_{m+1}(Y_n, X_1, X_2, \dots, X_m)) = \\ & \Delta_n((-1)^m \Delta_m(\underline{X}) P_{\mathcal{X}}(Y_1), \dots, (-1)^m \Delta_m(\underline{X}) P_{\mathcal{X}}(Y_n)) = \\ & (-1)^{mn} \Delta_m(\underline{X})^{1+p+\dots+p^{n-1}} \Delta_n(P_{\mathcal{X}}(Y_1), \dots, P_{\mathcal{X}}(Y_n)). \end{aligned}$$

From this, we deduce that the identity (A.7) is equivalent to

$$(A.8) \quad \Delta_{n+m}(Y_1, \dots, Y_n, X_1, \dots, X_m) = (-1)^{mn} \Delta_m(\underline{X}) \Delta_n(P_{\mathcal{X}}(Y_1), \dots, P_{\mathcal{X}}(Y_n)).$$

Moreover, we remark that

$$\begin{aligned} \Delta_{n+m}(Y_1, \dots, Y_n, X_1, \dots, X_m) &= (-1)^m \Delta_{n+m}(Y_1, \dots, Y_{n-1}, X_1, \dots, X_m, Y_n) = \\ & (-1)^m \Delta_{n-1+m}(Y_1, \dots, Y_{n-1}, X_1, \dots, X_m) \prod_{v \in \langle Y_1, \dots, Y_{n-1}, X_1, \dots, X_m \rangle_{\mathbb{F}_p}} (Y_n + v), \end{aligned}$$

and then we can apply the identity (A.8) replacing n with $n - 1$ (satisfied by the inductive hypothesis) to show that (A.7) is equivalent to

$$\Delta_n(P_{\mathcal{X}}(Y_1), \dots, P_{\mathcal{X}}(Y_n)) = \Delta_{n-1}(P_{\mathcal{X}}(Y_1), \dots, P_{\mathcal{X}}(Y_{n-1})) \prod_{v \in \langle Y_1, \dots, Y_{n-1}, X_1, \dots, X_m \rangle} (Y_n + v)$$

which, by applying again Lemma A.5 is equivalent to

$$\prod_{w \in \langle P_{\mathcal{X}}(Y_1), \dots, P_{\mathcal{X}}(Y_{n-1}) \rangle_{\mathbb{F}_p}} (P_{\mathcal{X}}(Y_n) + w) = \prod_{v \in \langle Y_1, \dots, Y_{n-1}, X_1, \dots, X_m \rangle_{\mathbb{F}_p}} (Y_n + v).$$

\square

Remark A.9. When $m = 1$ the identity of Lemma A.6 is already known. It appears for example in work of Elkies [13, p.81] and it is used in the proof of Proposition 5.3.

To produce certain formulas that we need in Section 4, let us introduce the following determinants. For every non-zero n -tuple $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{F}_p^n - \underline{0}$ we define

$$\Delta_{\underline{\epsilon}}(\underline{a}, X) := \begin{vmatrix} \epsilon_1 & \epsilon_2 & \dots & \epsilon_n & 0 \\ a_1 & a_2 & \dots & a_n & X \\ a_1^p & a_2^p & \dots & a_n^p & X^p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1^{p^{n-1}} & a_2^{p^{n-1}} & \dots & a_n^{p^{n-1}} & X^{p^{n-1}} \end{vmatrix}$$

and

$$\delta_{\underline{\epsilon}}(\underline{a}) := \begin{vmatrix} \epsilon_1 & \epsilon_2 & \dots & \epsilon_n \\ a_1 & a_2 & \dots & a_n \\ a_1^p & a_2^p & \dots & a_n^p \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{p^{n-2}} & a_2^{p^{n-2}} & \dots & a_n^{p^{n-2}} \end{vmatrix}$$

Proposition A.10. (Proposition 2.3. of [10]) *Let $W \subset k$ be a \mathbb{F}_p vector space of dimension n . Let \underline{a} be an \mathbb{F}_p -basis of W and $\underline{a}^* = (a_1^*, \dots, a_n^*)$ be its dual basis. For every non-zero n -tuple $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{F}_p^n - \underline{0}$, we denote by $\varphi_{\underline{\epsilon}}$ the linear homomorphism $W \rightarrow \mathbb{F}_p$ given by $\varphi_{\underline{\epsilon}} := \sum_{i=1}^n \epsilon_i a_i^*$. Recall that we have defined $P_{\ker \varphi_{\underline{\epsilon}}}(X) := \prod_{w \in \ker \varphi_{\underline{\epsilon}}} (X - w)$.*

Then $\delta_{\underline{\epsilon}}(\underline{a}) \neq 0$ and we have the identity

$$(A.11) \quad P_{\ker \varphi_{\underline{\epsilon}}}(X) = \frac{\Delta_{\underline{\epsilon}}(\underline{a}, X)}{\delta_{\underline{\epsilon}}(\underline{a})} = X^{p^{n-1}} + \dots + (-1)^{n-1} \delta_{\underline{\epsilon}}(\underline{a})^{p-1} X.$$

Moreover, denoting by \underline{e}_i the i -th element of the standard basis of \mathbb{F}_p^n , we have the following formulas:

$$(A.12) \quad \Delta_{\underline{e}_i}(\underline{a}, X) = (-1)^{i-1} \Delta_n(\underline{\hat{a}}_i, X) \quad \forall i = 1, \dots, n$$

$$(A.13) \quad \Delta_{\underline{\epsilon}}(\underline{a}, X) = \sum_{i=1}^n \epsilon_i \Delta_{\underline{e}_i}(\underline{a}, X) = \sum_{i=1}^n (-1)^{i-1} \epsilon_i \Delta_n(\underline{\hat{a}}_i, X)$$

$$(A.14) \quad \delta_{\underline{\epsilon}}(\underline{a}) = \sum_{i=1}^n (-1)^{i-1} \epsilon_i \Delta_{n-1}(\underline{\hat{a}}_i).$$

Proposition A.15 (Proposition 5.1. of [10]). *Let k be an algebraically closed field containing \mathbb{F}_p . Let $V(\Delta_n) := \{(a_1, \dots, a_n) \mid \Delta_n(\underline{a}) = 0\}$. Then the map*

$$\begin{aligned} \varphi : \quad k^n &\longrightarrow k^n \\ \underline{a} &\longmapsto ((-1)^{i-1} \Delta_{n-1}(\underline{\hat{a}}_i))_{i=1, \dots, n} \end{aligned}$$

induces a surjective function $k^n - V(\Delta_n) \rightarrow k^n - V(\Delta_n)$. This satisfies the property that

$$\varphi^2(\underline{a}) = (-1)^{n-1} \Delta_n(\underline{a})^{1+p+\dots+p^{n-3}} (\underline{a})^{p^{n-2}}.$$

Moreover, $\underline{a}, \underline{a}'$ are such that $\varphi(\underline{a}) = \varphi(\underline{a}')$ if, and only if, $\underline{a} = \theta \underline{a}'$ for some θ satisfying

$$\theta^{1+p+\dots+p^{n-2}} = 1.$$

Finally, we prove two results that are not in [10] and that are used in Section 4.

Lemma A.16. Let X_1, \dots, X_n be variables over \mathbb{F}_p , and for every $0 \leq t \leq n$ let $V_{n-t} = \langle X_1, \dots, X_{n-t} \rangle_{\mathbb{F}_p}$. Then, we have that

$$\Delta_n(X_1, \dots, X_n) = \Delta_{n-t}(X_1, \dots, X_{n-t}) \Delta_t(P_{V_{n-t}}(X_{n-t+1}), \dots, P_{V_{n-t}}(X_n)),$$

where $P_{V_{n-t}}(X)$ is the structural polynomial of V_{n-t} (see Definition A.4).

Proof. From Moore's formula (A.2) we have

$$\Delta_n(\underline{X}) = \prod_{i=1}^n \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_1 \in \mathbb{F}_p} (X_i + \epsilon_{i-1} X_{i-1} + \cdots + \epsilon_1 X_1) = A \cdot B,$$

where

$$A := \prod_{i=1}^{n-t} \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_1 \in \mathbb{F}_p} (X_i + \epsilon_{i-1} X_{i-1} + \cdots + \epsilon_1 X_1)$$

and

$$B := \prod_{i=n-t+1}^n \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_1 \in \mathbb{F}_p} (X_i + \epsilon_{i-1} X_{i-1} + \cdots + \epsilon_1 X_1).$$

Moore's formula ensures that $A = \Delta_{n-t}(X_1, \dots, X_{n-t})$, while the definition of the structural polynomial gives that

$$B = \prod_{i=n-t+1}^n \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_{n-t+1} \in \mathbb{F}_p} P_{V_{n-t}}(X_i + \epsilon_{i-1} X_{i-1} + \cdots + \epsilon_{n-t+1} X_{n-t+1}).$$

By Lemma A.5, $P_{V_{n-t}}$ is an additive polynomial and therefore the above is also equal (after reindexing) to

$$B = \prod_{i=1}^t \prod_{\epsilon_{i-1} \in \mathbb{F}_p} \cdots \prod_{\epsilon_1 \in \mathbb{F}_p} (P_{V_{n-t}}(X_{n-t+i}) + \epsilon_{i-1} P_{V_{n-t}}(X_{n-t+i-1}) + \cdots + \epsilon_1 P_{V_{n-t}}(X_{n-t+1})),$$

and we can see from Moore's formula that this is precisely equal to

$$\Delta_t(P_{V_{n-t}}(X_{n-t+1}), \dots, P_{V_{n-t}}(X_n)),$$

which completes the proof of the claim. \square

Corollary A.17. Under the hypotheses of Lemma A.16, for every $n-t+1 \leq i \leq n$ we have that

$$\frac{\Delta_{n-1}(X_1, \dots, \widehat{X}_i, \dots, X_n)}{\Delta_n(X_1, \dots, X_n)} = \frac{\Delta_{t-1}(P_{V_{n-t}}(X_{n-t+1}), \dots, P_{V_{n-t}}(\widehat{X}_i), \dots, P_{V_{n-t}}(X_n))}{\Delta_t(P_{V_{n-t}}(X_{n-t+1}), \dots, P_{V_{n-t}}(X_n))}.$$

Lemma A.18. Let $n \geq 1$, let $M \in \text{GL}_n(\mathbb{F}_p)$ and let X_1, \dots, X_n be variables over \mathbb{F}_p . Denote by $Y_i = \Delta_{n-1}(\widehat{X}_i)$ and by $Y_i^M = \Delta_{n-1}(\widehat{(XM)}_i)$. Then, we have

$$(Y_1^M, \dots, Y_n^M) = \underline{Y} M^c,$$

where $M^c \in \text{GL}_n(\mathbb{F}_p)$ is the cofactor matrix of M

Proof. This is proved with a direct computation. We write $M = (m_{ij})_{ij}$ and, for simplicity, we show only that $Y_1^M = \underline{Y} M_{\bullet,1}^c$ where $M_{\bullet,1}^c$ is the first column of M^c , the proof that $Y_j^M = \underline{Y} M_{\bullet,j}^c$ being completely analogous. By definition, we have

$$Y_1^M = \Delta_{n-1} \left(\sum_{i=1}^n m_{i,2} X_2, \dots, \sum_{i=1}^n m_{i,n} X_n \right)$$

and, using the multi-linear properties of Moore determinants, this can be rewritten as

$$\begin{aligned} Y_1^M &= \sum_{i_2, \dots, i_n} m_{i_2, 2} \cdots m_{i_n, n} \Delta_{n-1}(X_{i_2}, \dots, X_{i_n}) = \sum_{\sigma \in \mathfrak{S}_n} m_{\sigma(2), 2} \cdots m_{\sigma(n), n} \Delta_{n-1}(X_{\sigma(2)}, \dots, X_{\sigma(n)}) \\ &= \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{sgn}(\sigma)} m_{\sigma(2), 2} \cdots m_{\sigma(n), n} Y_{\sigma(1)}, \end{aligned}$$

where \mathfrak{S}_n is the symmetric group over the set with n elements (note that we used that $m_{i,j}^p = m_{i,j}$ for getting rid of the exponents). Since for every i , the coefficient of Y_i is the minor M_{i1} of the matrix M , it results that $Y_1^M = \underline{Y}M_{\bullet, 1}^c$. \square

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