Dispersion and dissipation in bore propagation in natural and artificial channels: what are we modelling ?

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Inria at University of Bordeaux

THANKS TO

P. Bonneton (CNRS)

R. Chassagne (U. Grenoble)

A.G. Filippini (BRGM)

S. Gavrilyuk (U. Aix-Marseille)

B. Jouy (EDF)

M. Kazolea (Inria)

M. Le (LHSV)

H. Ranocha (Mainz U.)

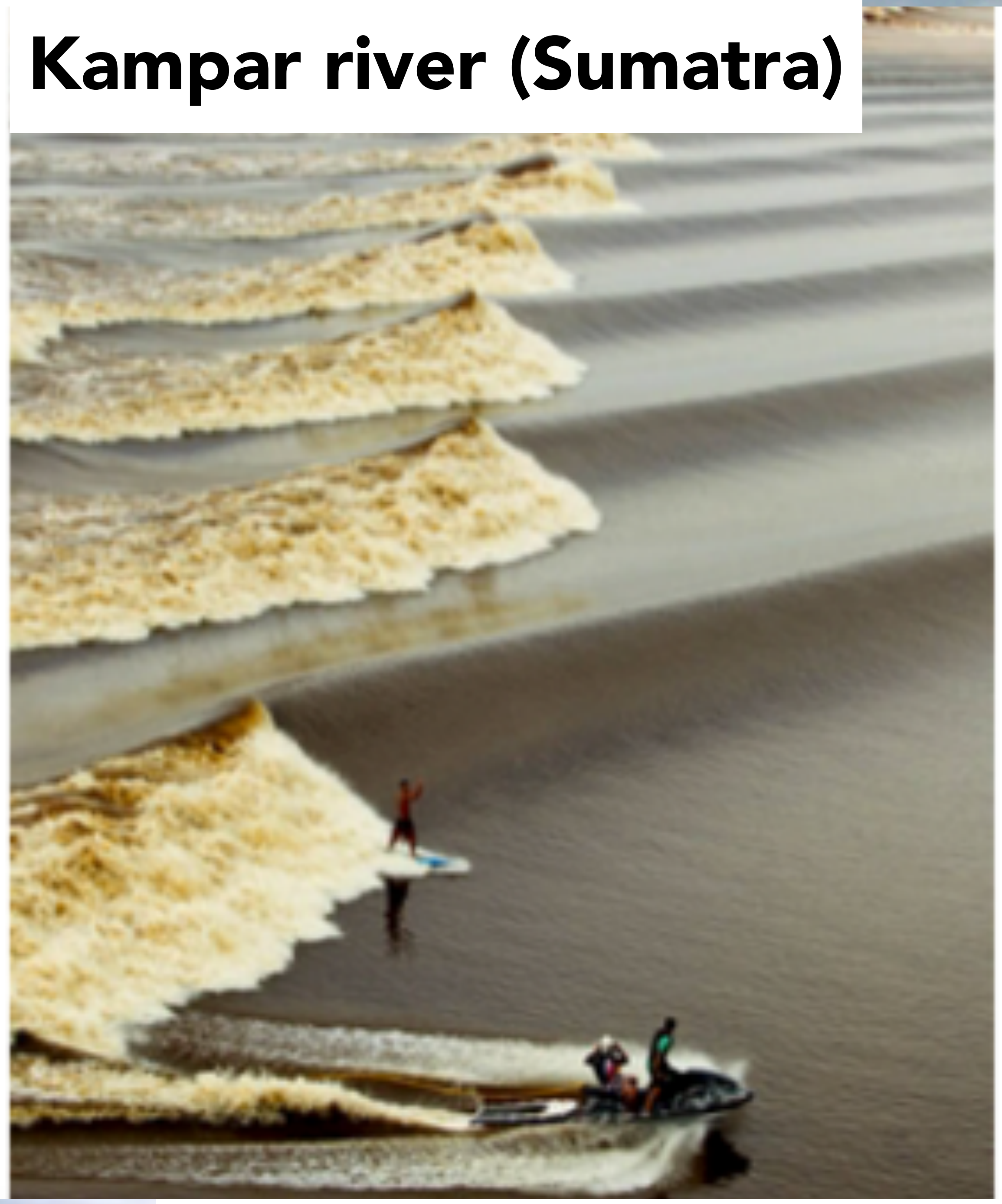
D. Violeau (EDF)

credit to them for the good stuff, blame me for the rest

Intro: two sides to every story

Side 1: real life


Severn river (UK)


Kampar river (Sumatra)


Qiantang river (China)


Amazon river (Brazil)


Dordogne river (France)


Garonne river (France)

Intro: two sides to every story

Side 1: real life

Sisteron plant, France
(Courtesy of EDF)

Manosque plant, France
(Courtesy of EDF)

Hydrodynamics

- strong acceleration at the front
- vertical and horizontal kinematics
- turbulent flow (breaking)

→

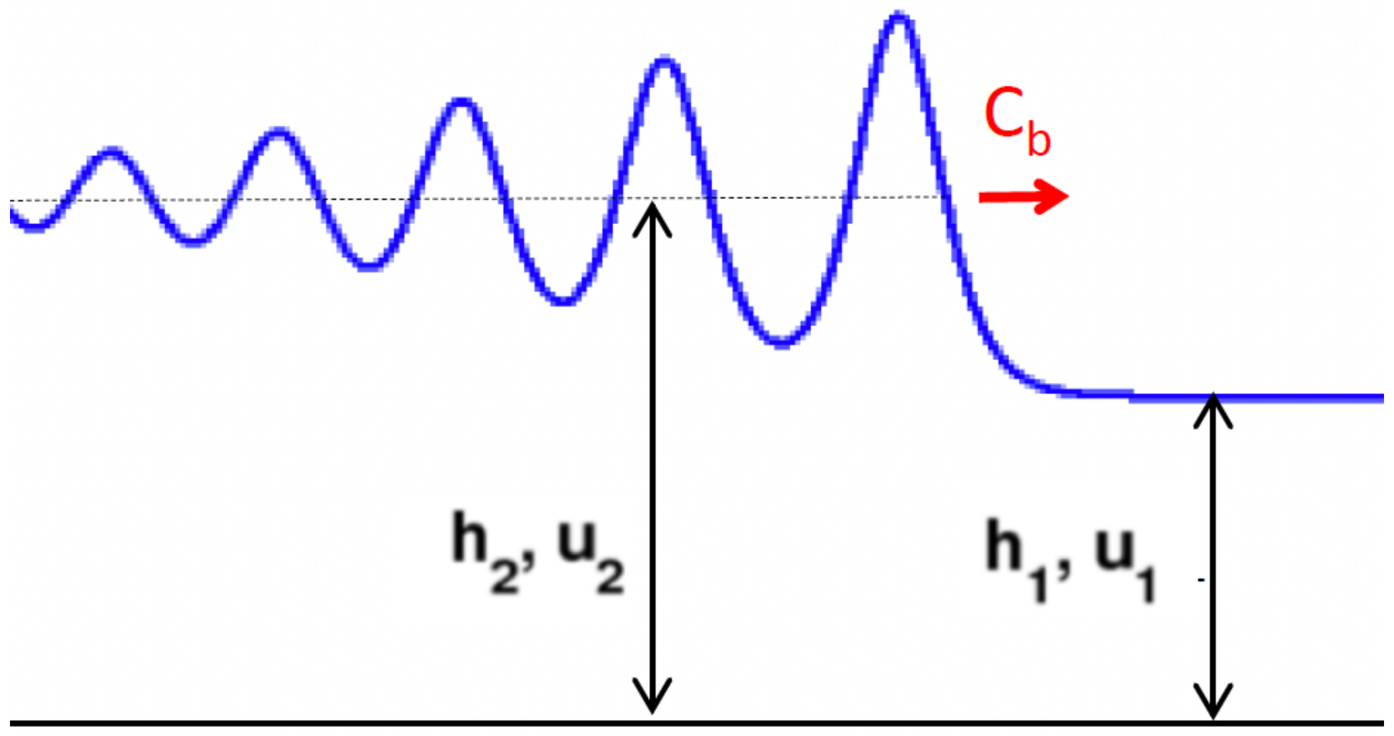
Hydro-sedimetary processes

- resuspension of bottom sediments
- diffusion across the water column
- diffusion across the section

Side 2: math

Delicate equilibrium between

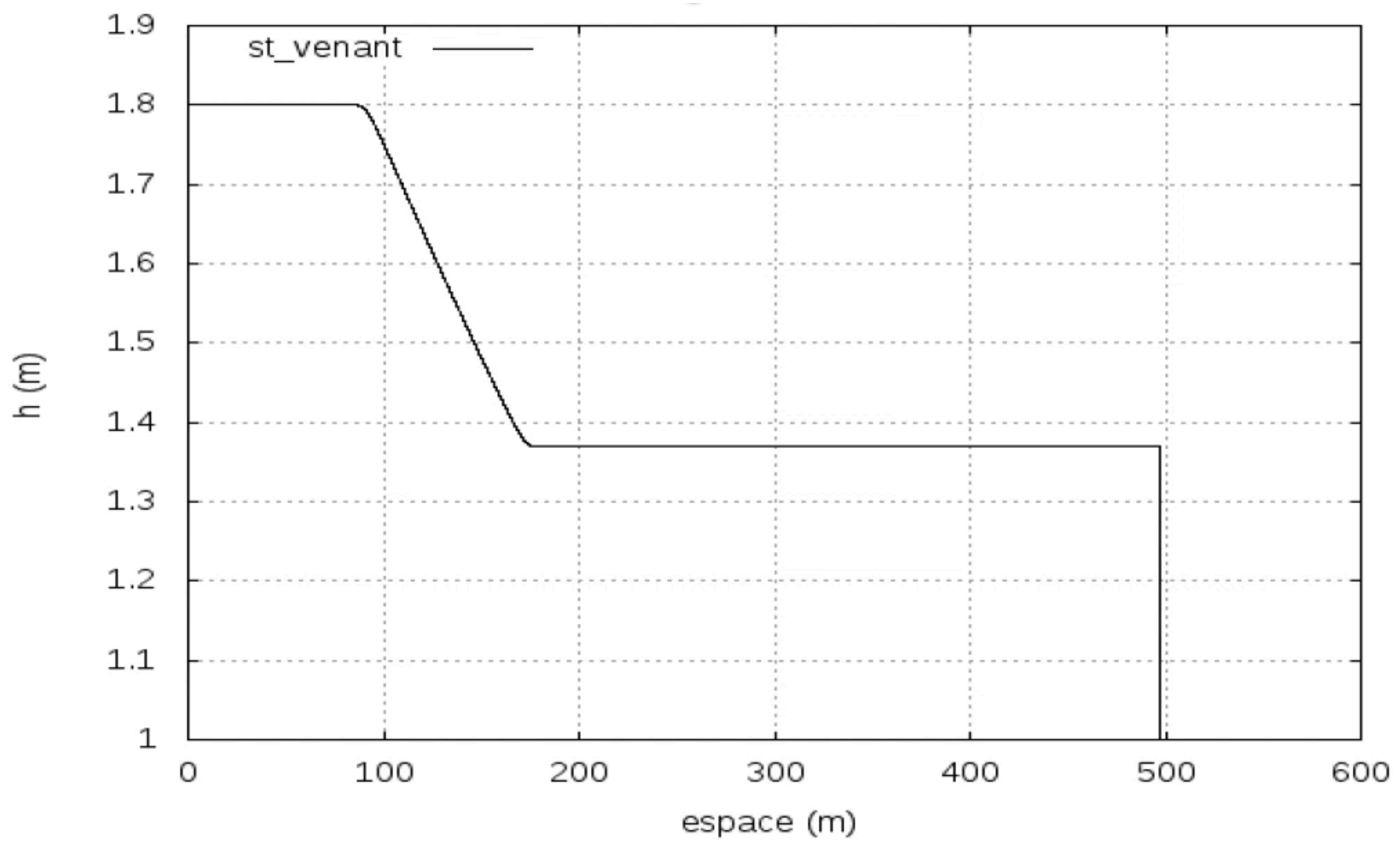
- nonlinearity
- dispersion
- dissipation

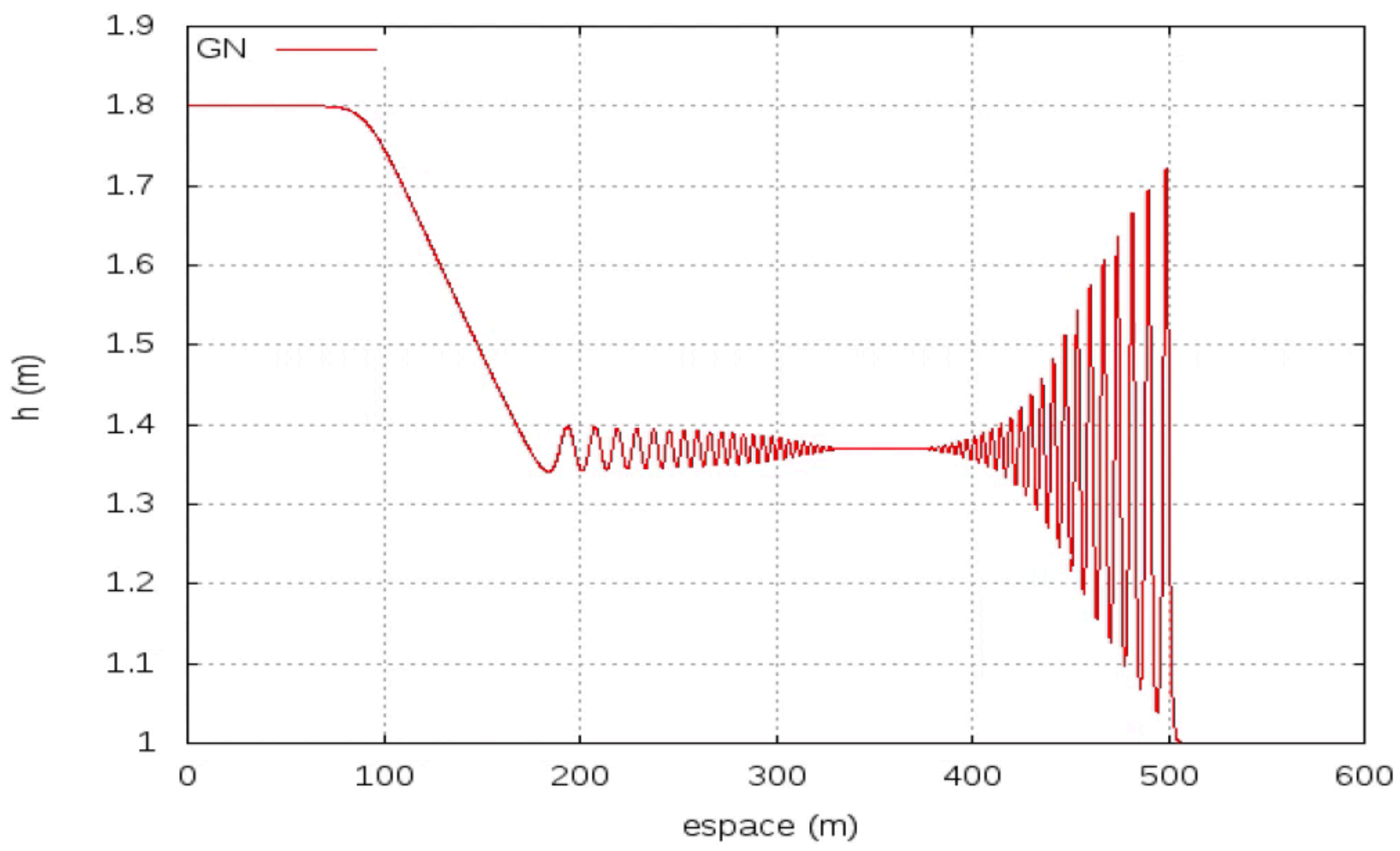


$$\partial_t u = -\partial_x f(u) + \alpha \partial_{txx} u + \beta \partial_{xxx} u + \partial_x(\mu \partial_x u) - \sigma u$$

Intro: two sides to every story

Side 2: math





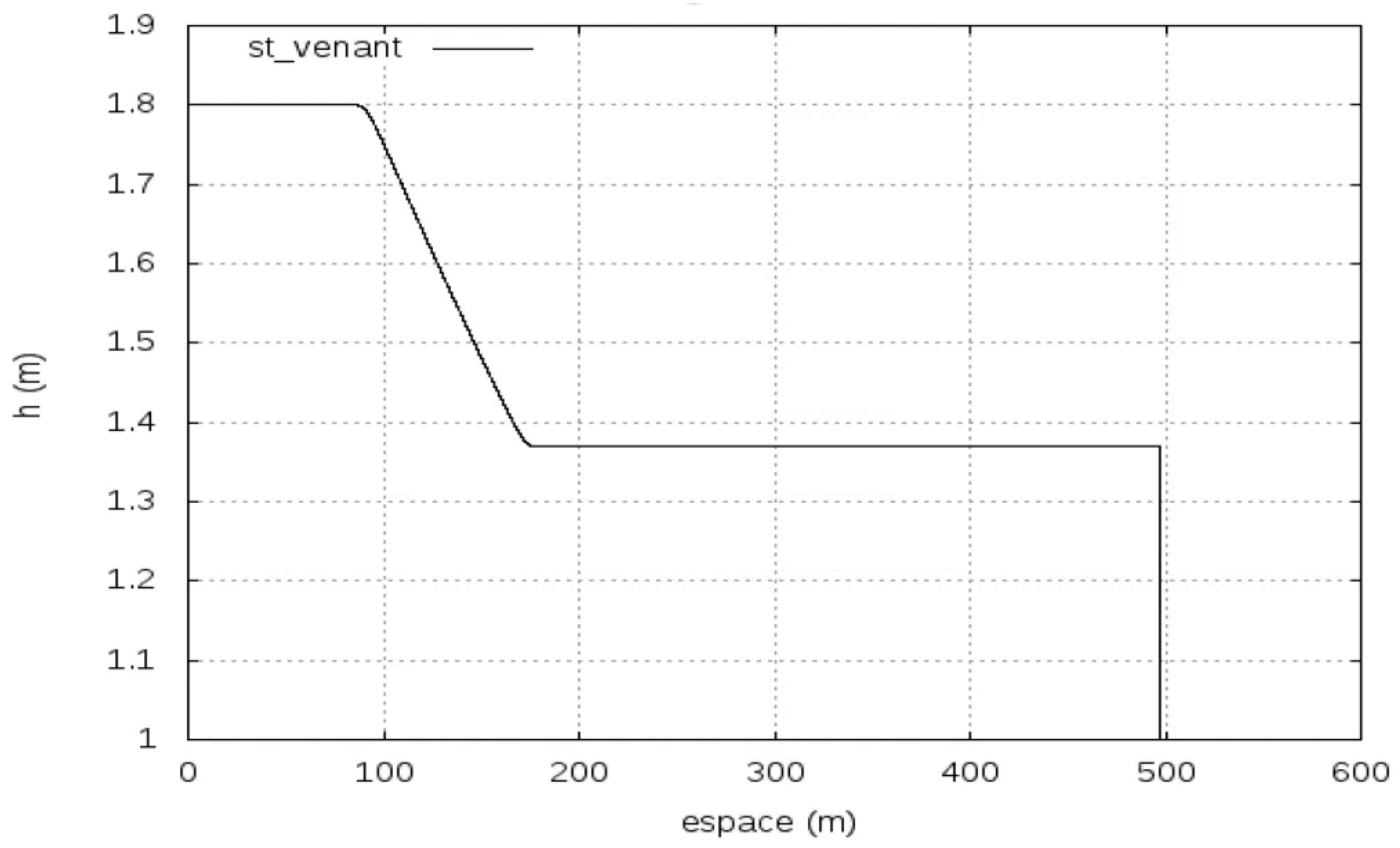
Hyperbolic shallow water

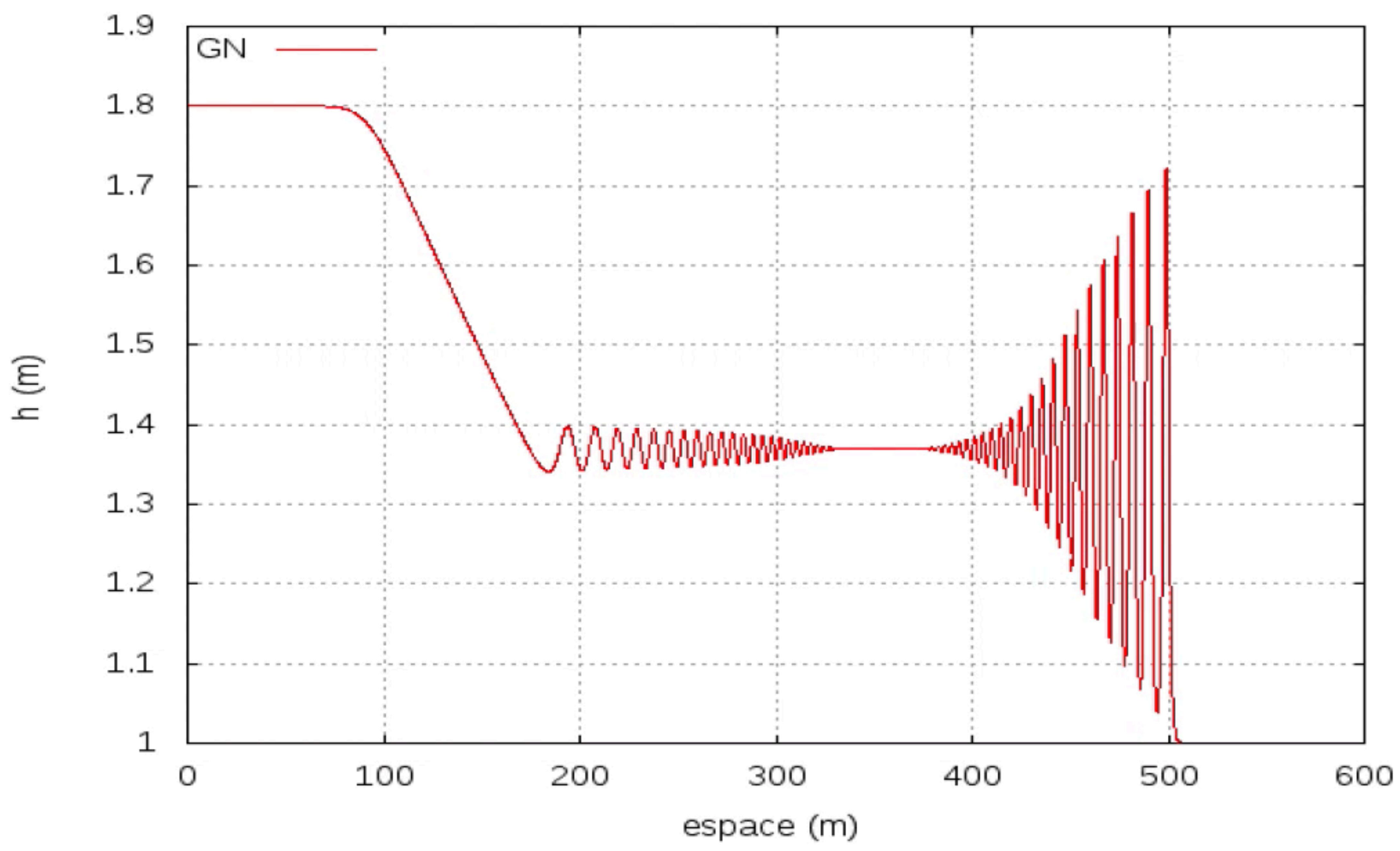
- discontinuous data -> genuine shocks
- discontinuous weak solutions
- admissible (viscosity) solutions !

Dispersive Serre-Green-Naghdi

- discontinuous data -> dispersive shocks
- continuous (high frequency) solutions
- admissibility/uniqueness: no need to invoke viscosity

Intro: two sides to every story

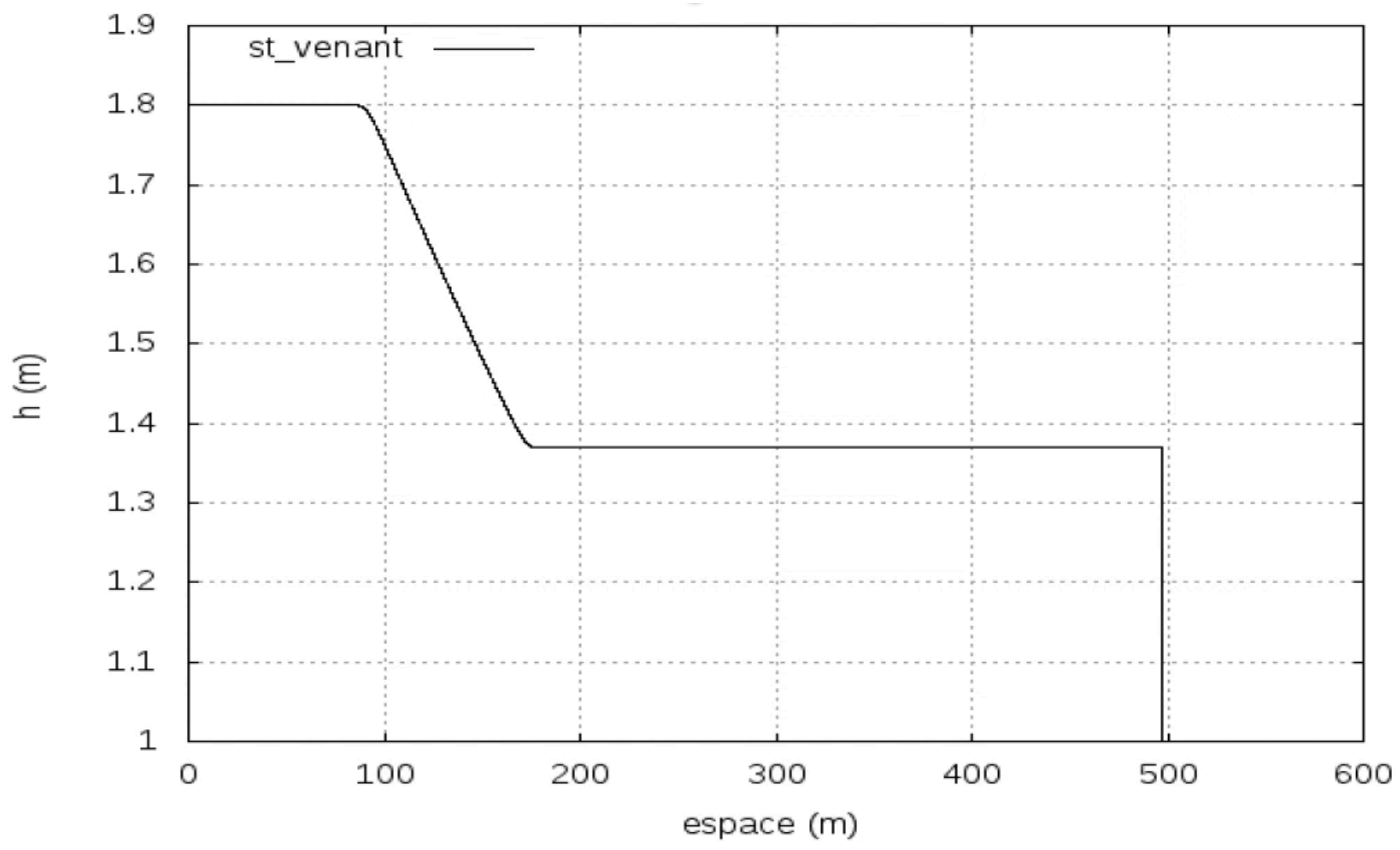


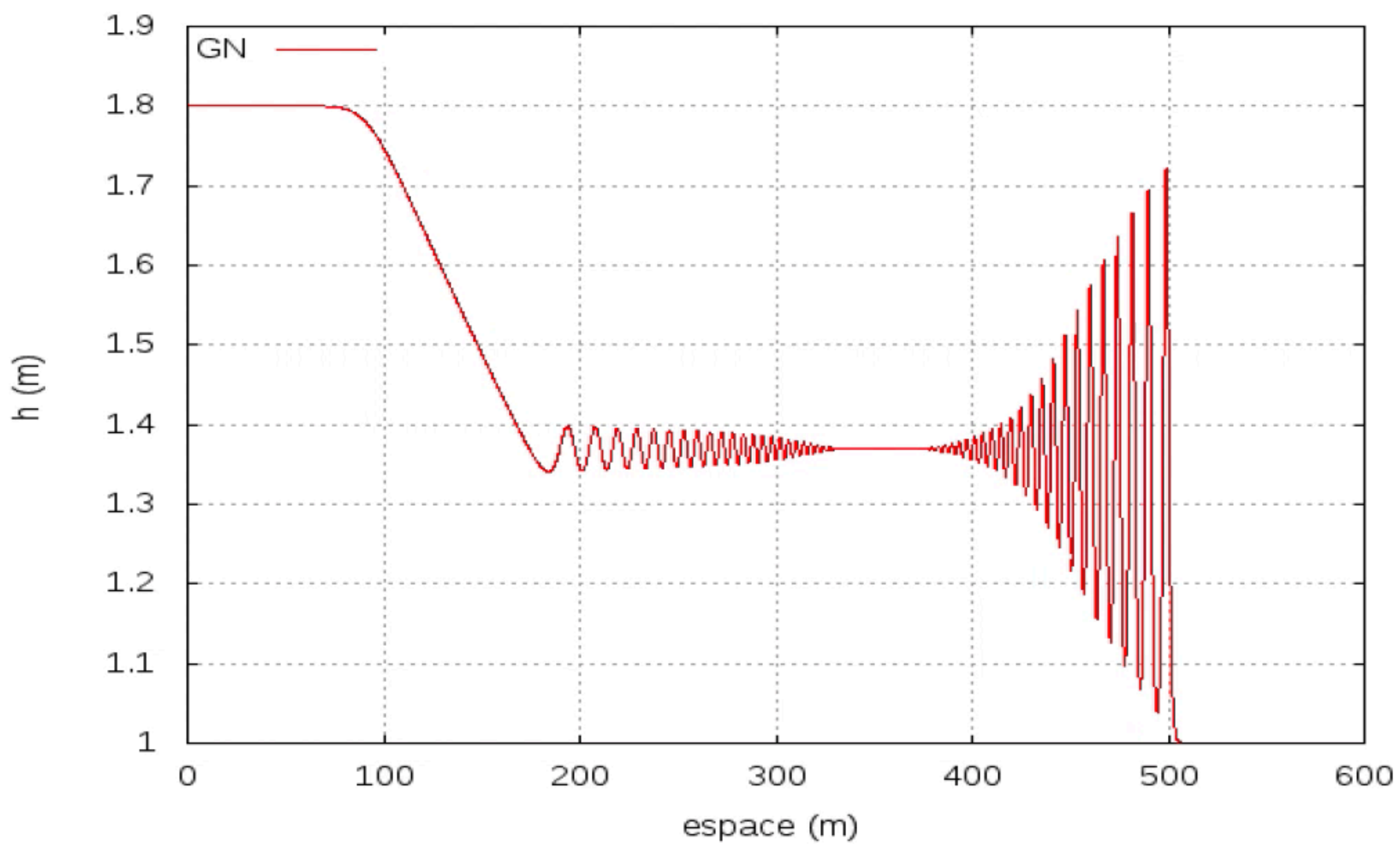


Talk Part I : can hyperbolic models also provide dispersive shocks ?

- **yes, whenever small scale heterogeneity is there. In multiD -> bathymetric variations**
- **these *dispersive-like* undular bores occur systematically fin real life for Froude numbers below ~ 1.15-1.17**
- **dispersive (transverse averaged) models are constructed**

Intro: two sides to every story





Talk Part II (only if time): quid of the notion of viscosity solution ?

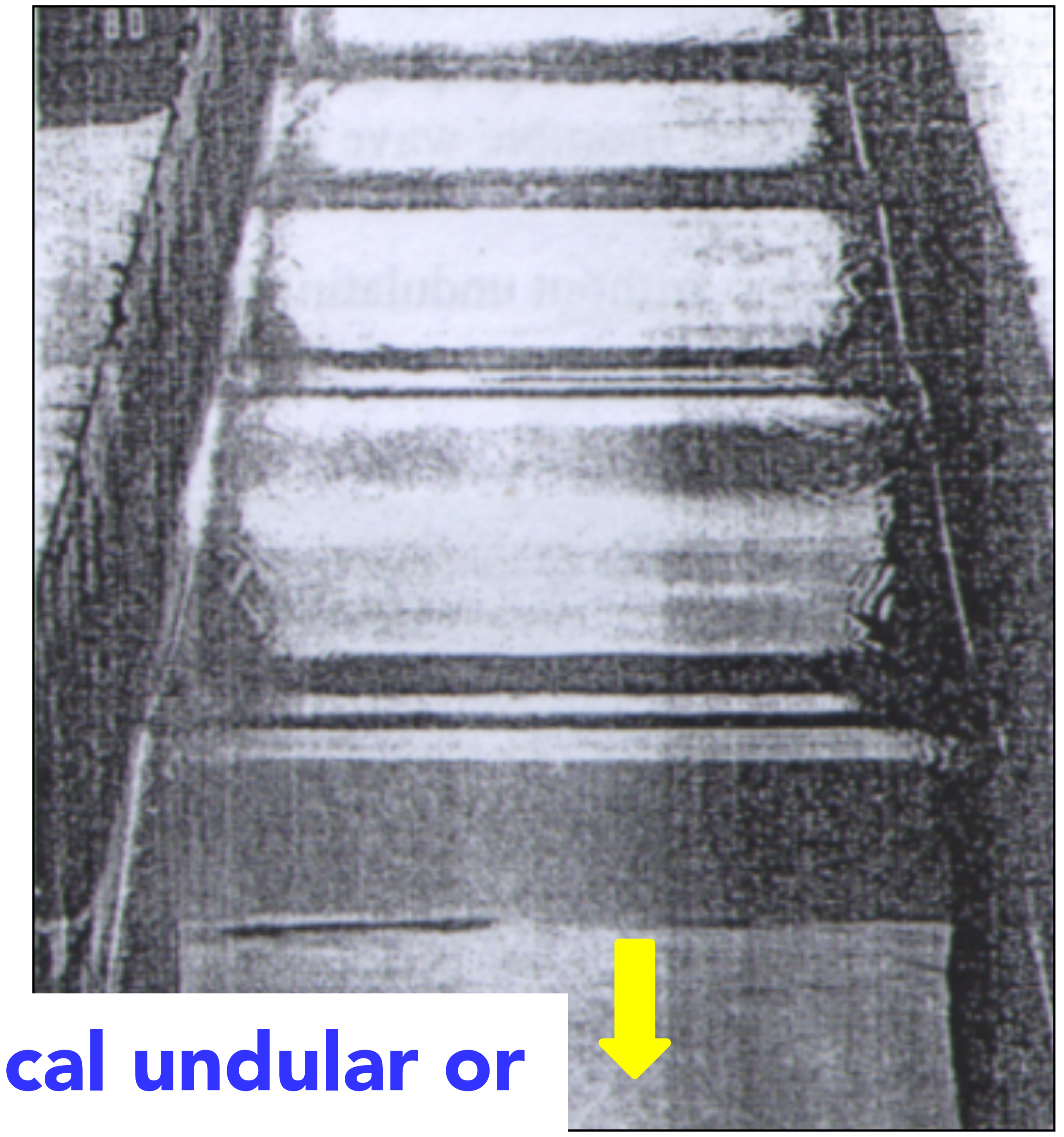
- **open question … energy is ALWAYS conserved when (physical) dispersion is active**
- **in practice: numerical dissipation significantly alters (negatively) simulation results, also in presence of physical dissipation**
- **examples are provided**

Undular bores:

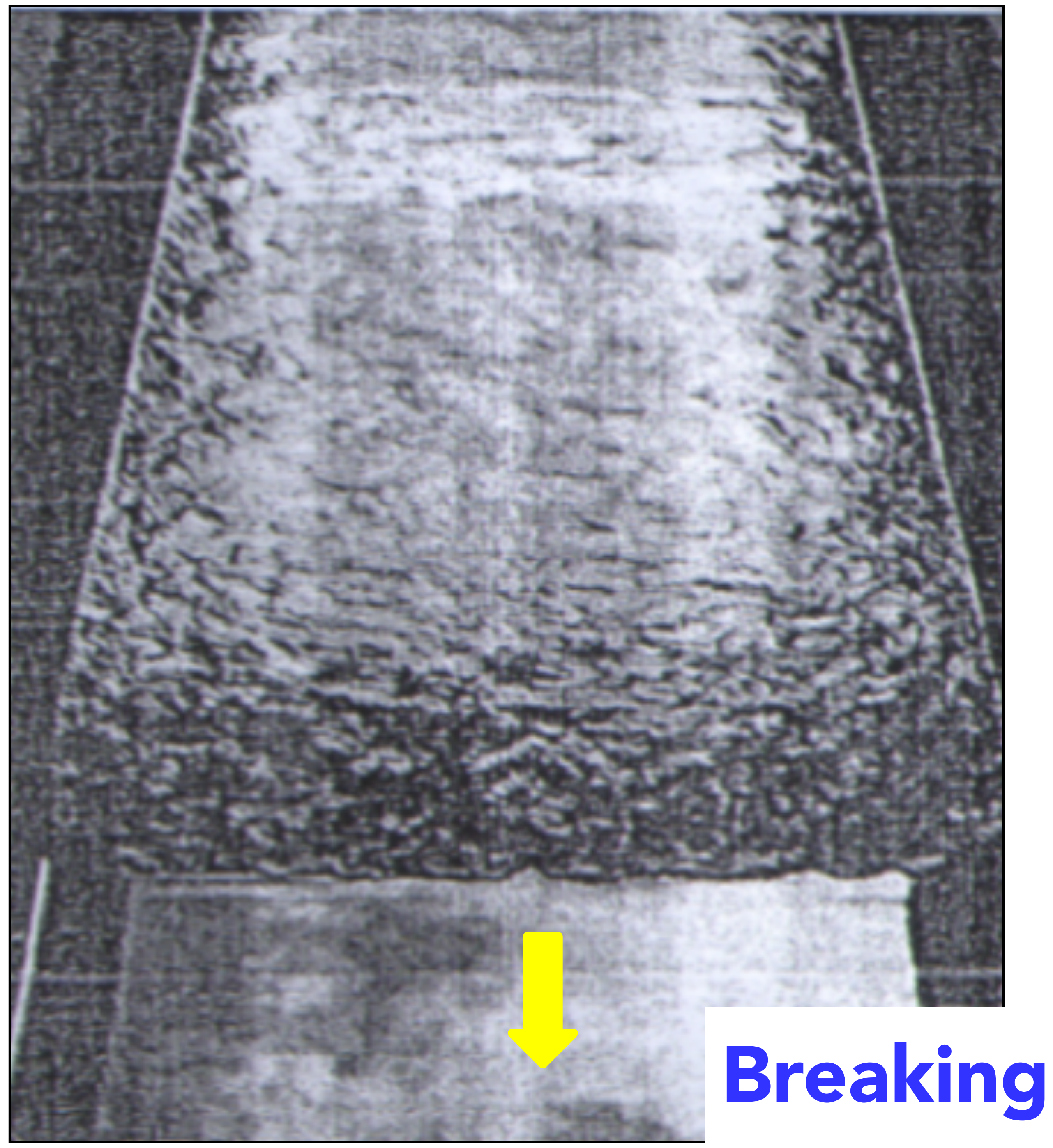
straight walled channels,

estuaries, and man made channels

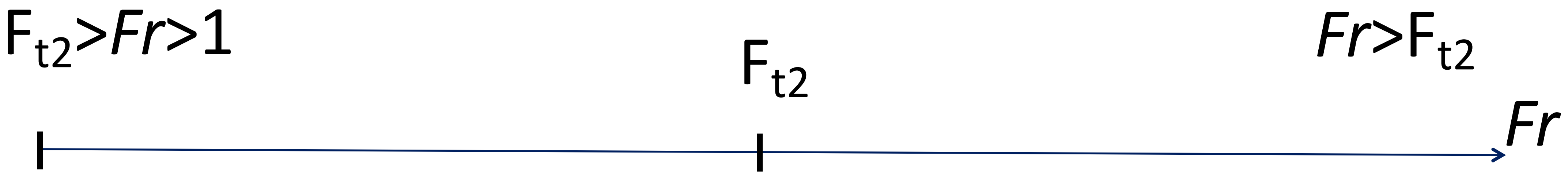
Experiments in rectangular channels (no banks)



classical undular or "dispersive bore" or "Favre wave"



Breaking bore



Favre, *Dunod, 1935*

Treske, *J. Hydraulic Research, 1994*

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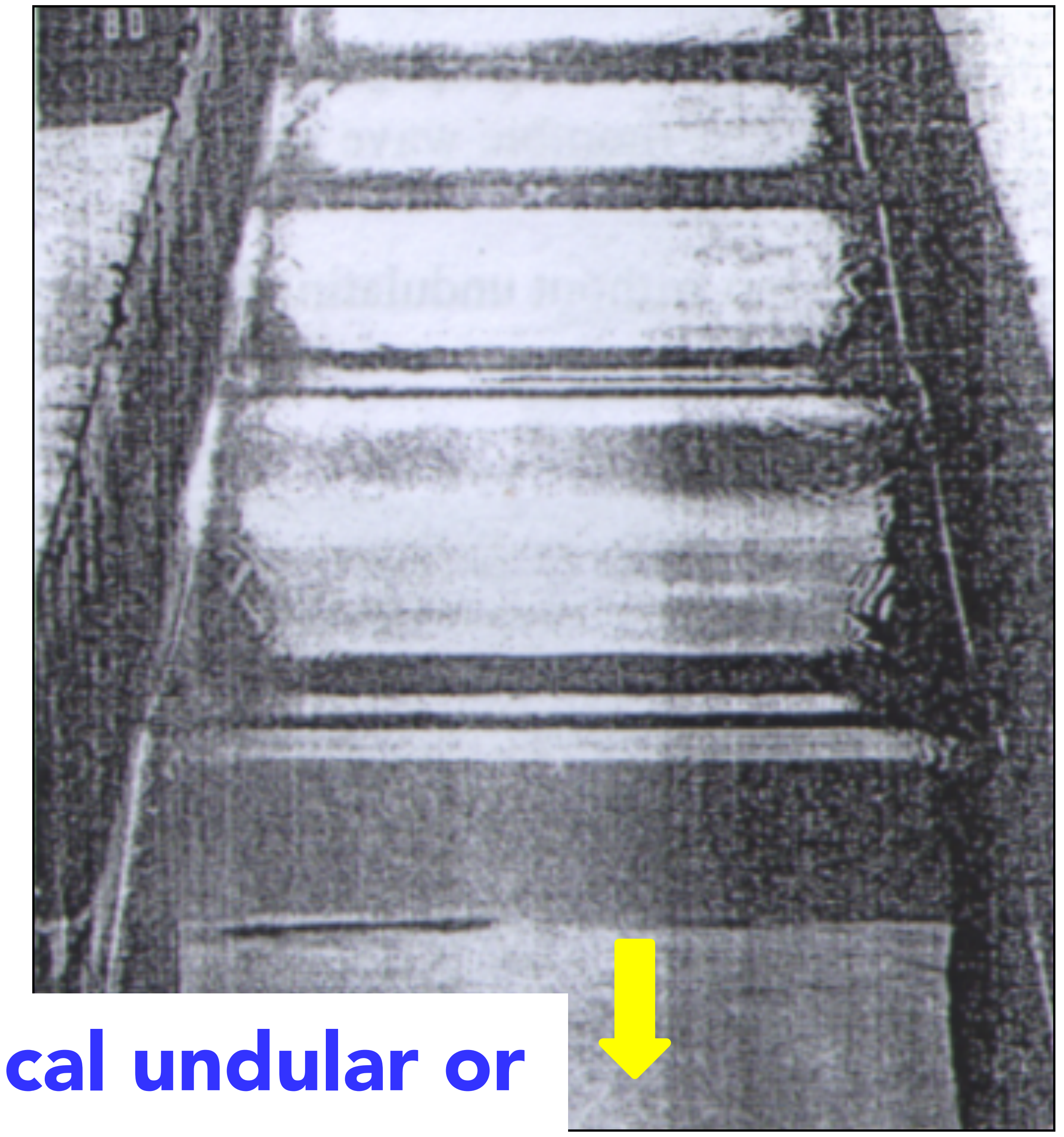

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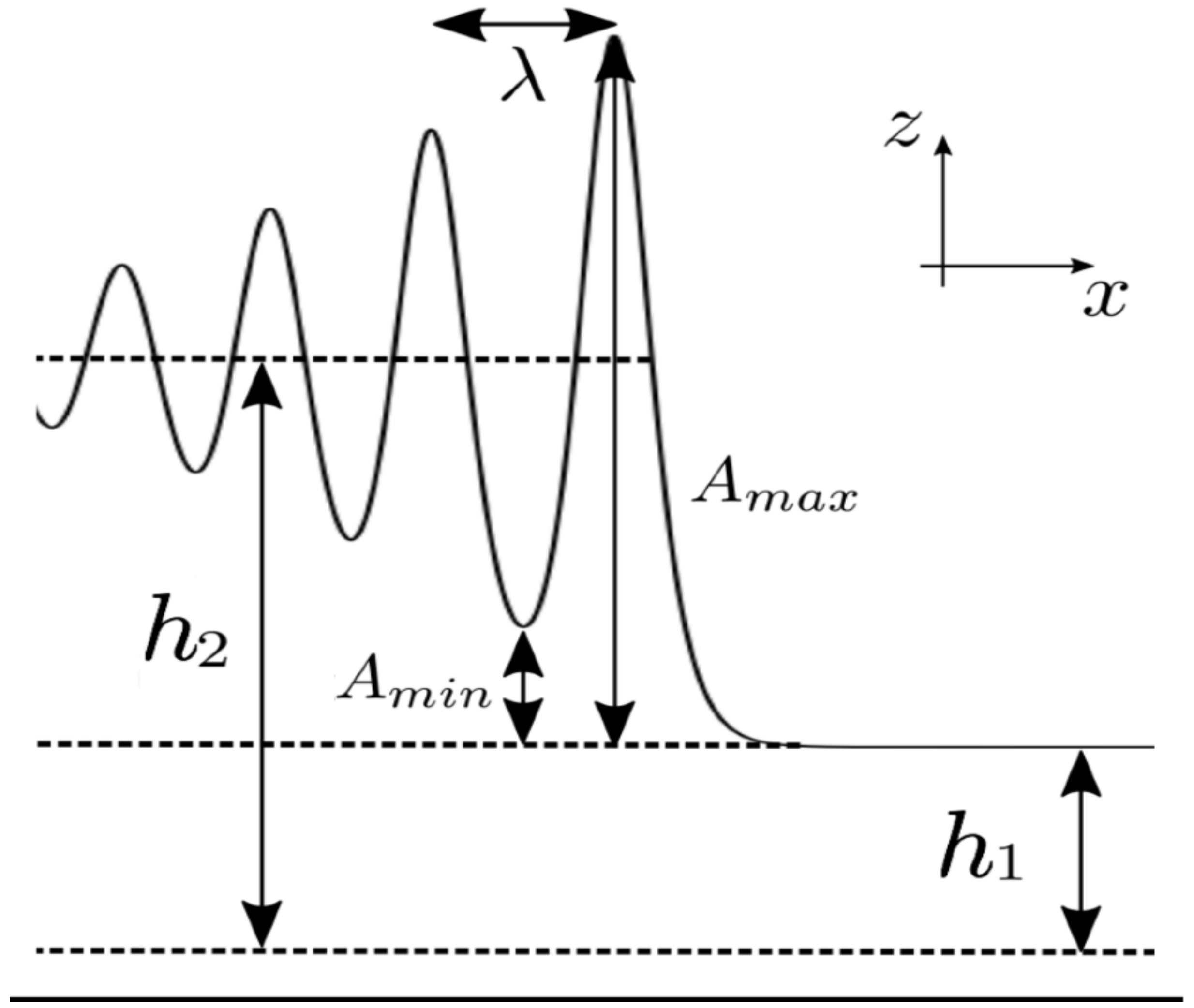


$F_{t2} > Fr > 1$

F_{t2}

$Fr > F_{t2}$

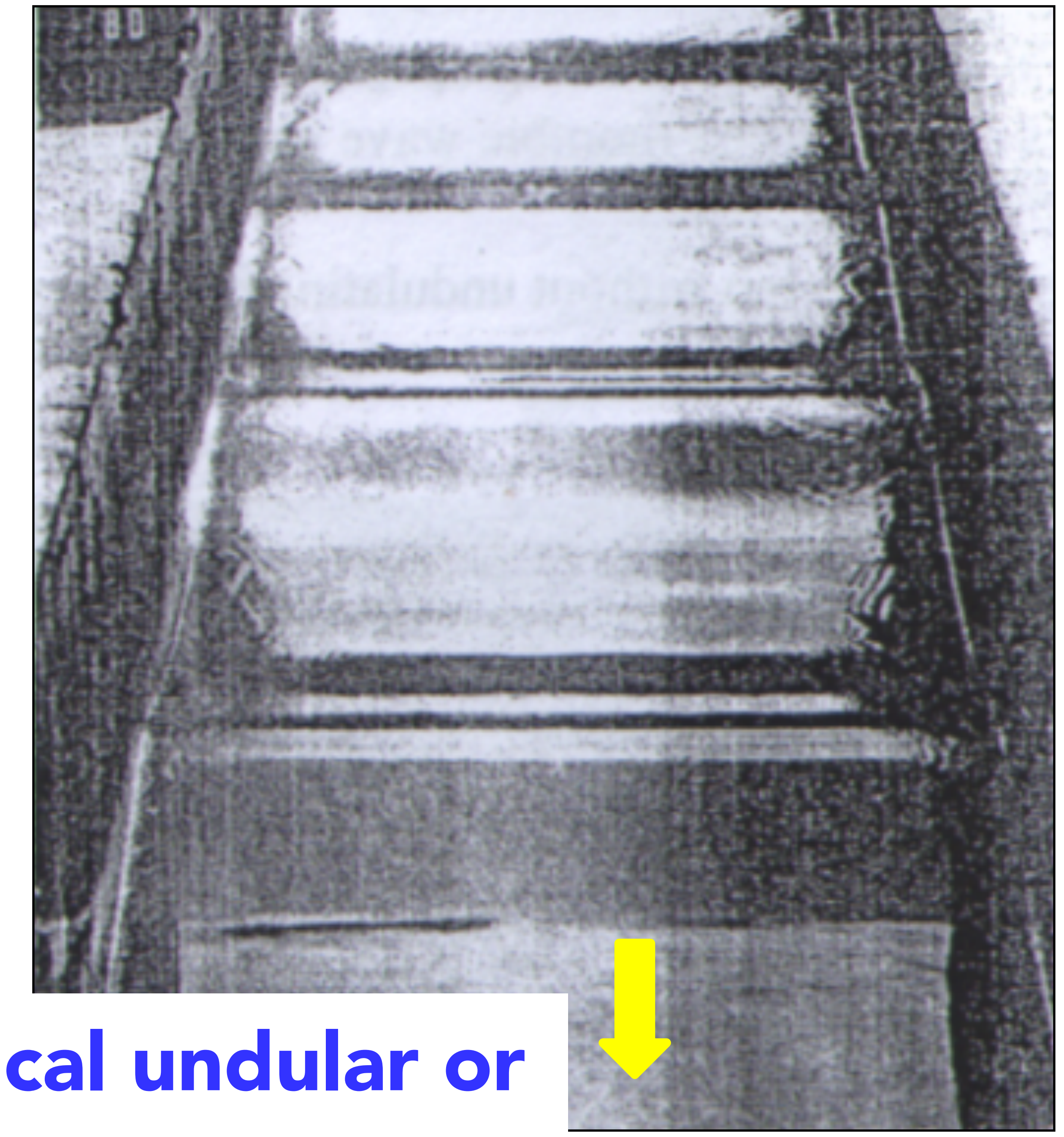
Fr

Favre, *Dunod, 1935*

Treske, *J. Hydraulic Research, 1994*

Bores (straight walled channels) | Experimental studies

Experiments in rectangular channels (no banks)



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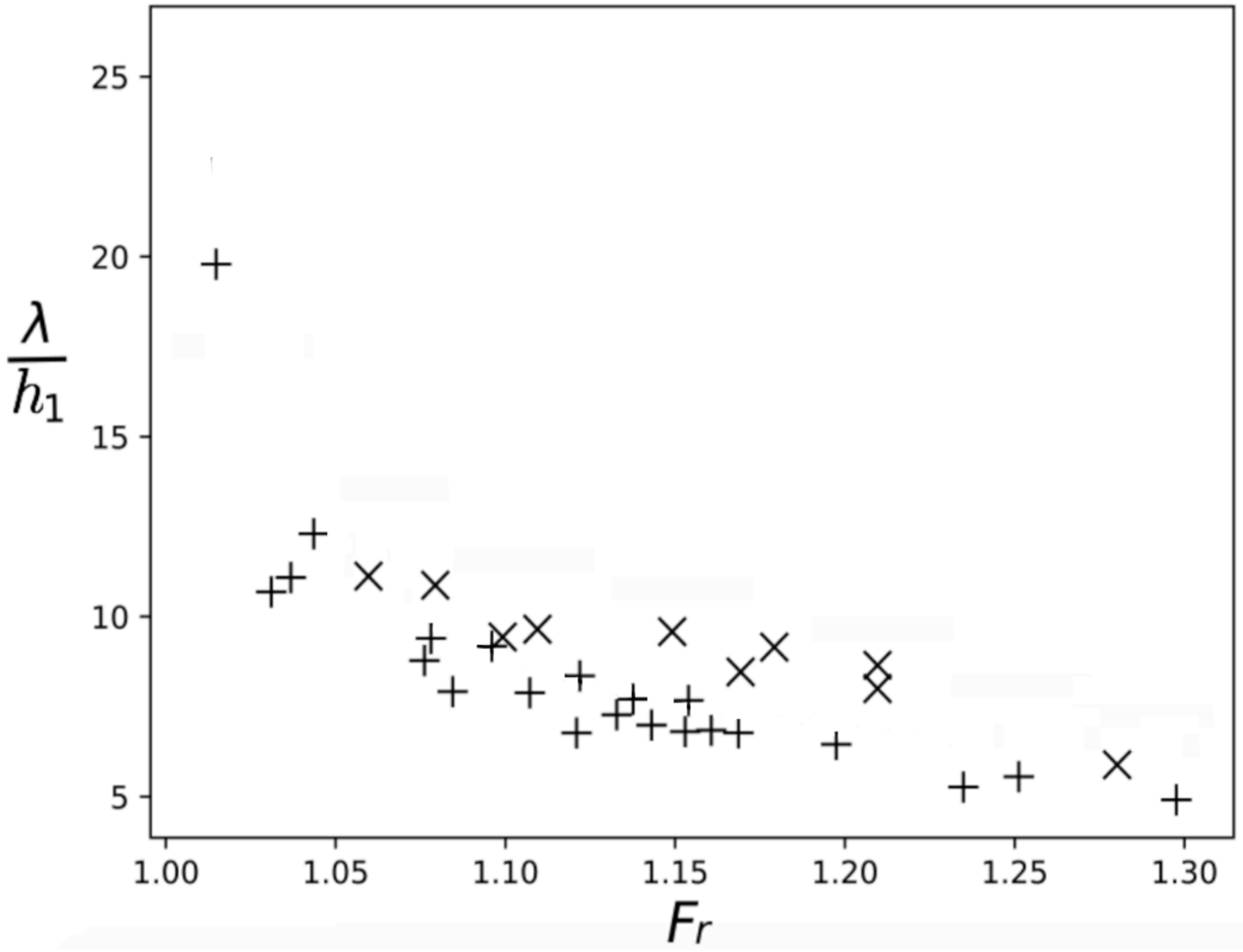

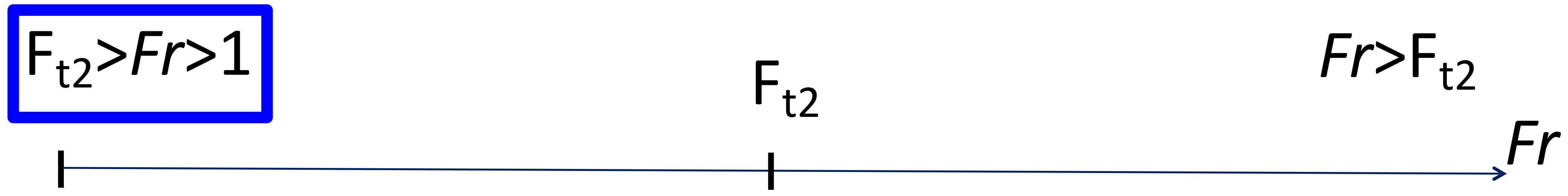

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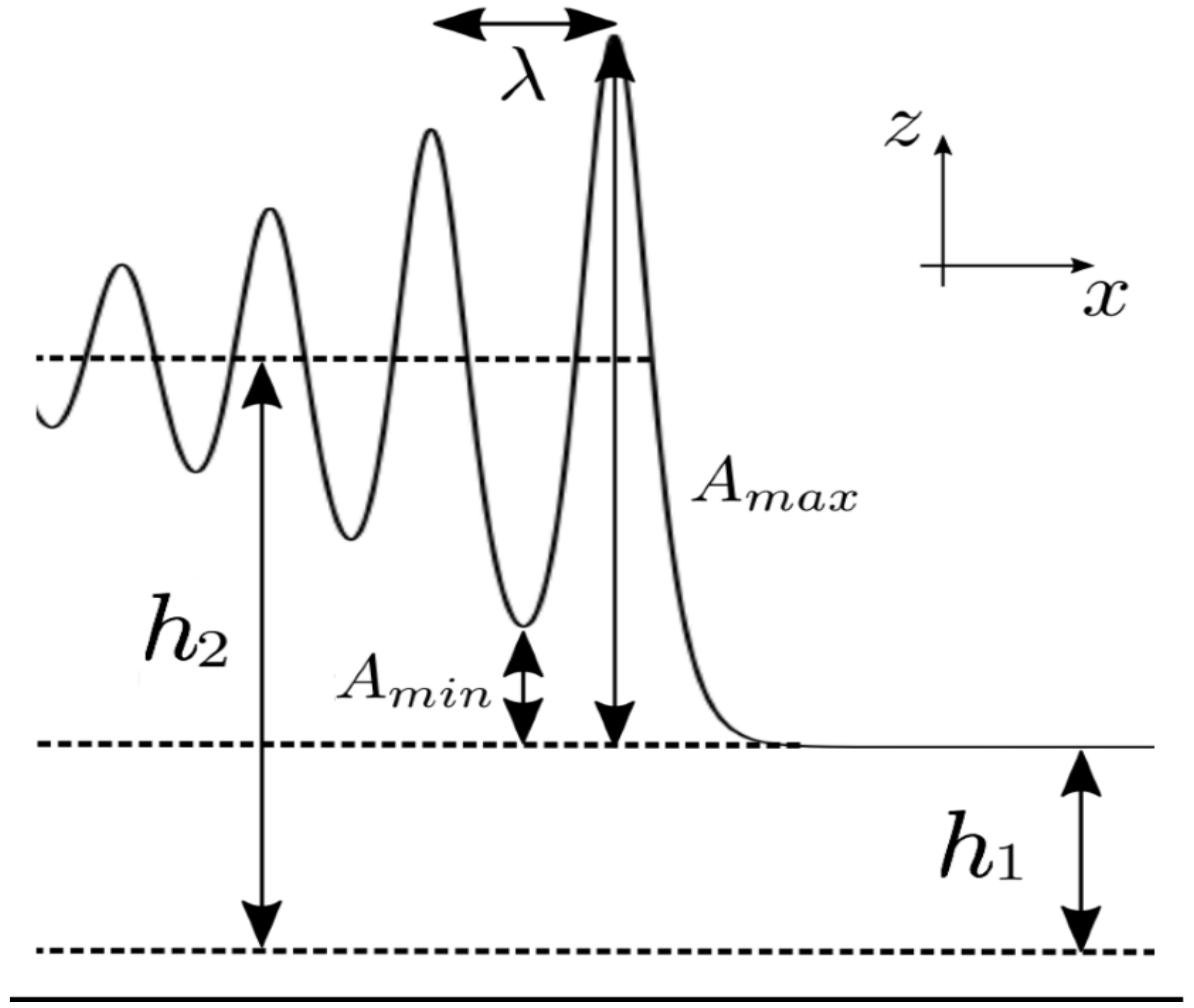
Lemoine analogy

Lemoine, La Houille Blanche, 1948



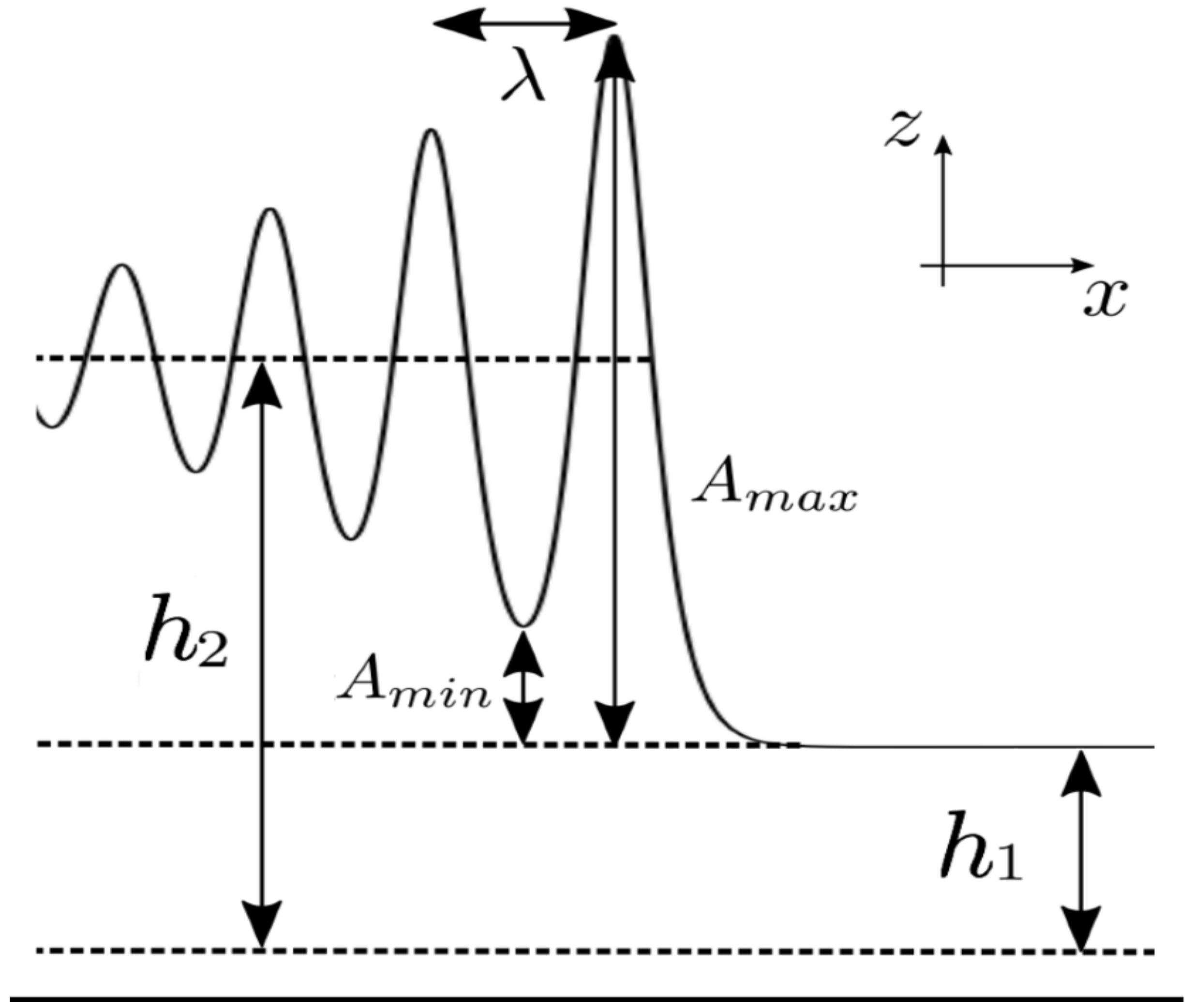
1. Secondary waves conserve mass/momentum

2. The undular front moves at the speed of the bore: $C_b = U_2 + C_\lambda$

3. No energy dissipation, energy goes into the secondary waves

Lemoine analogy

Lemoine, La Houille Blanche, 1948



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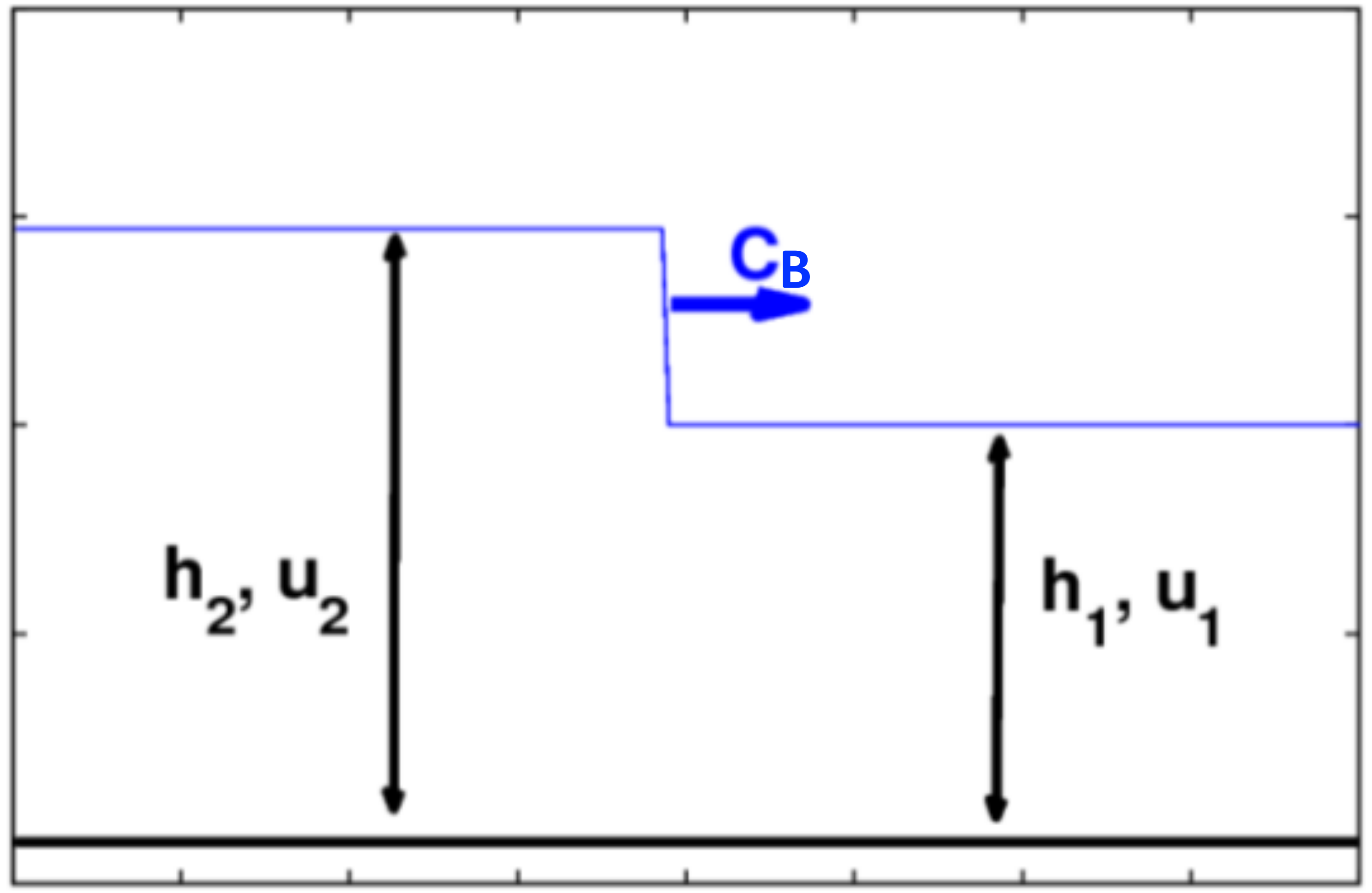
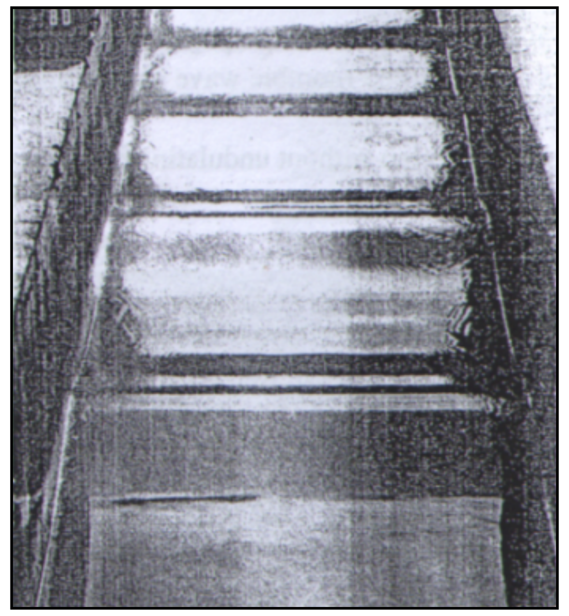
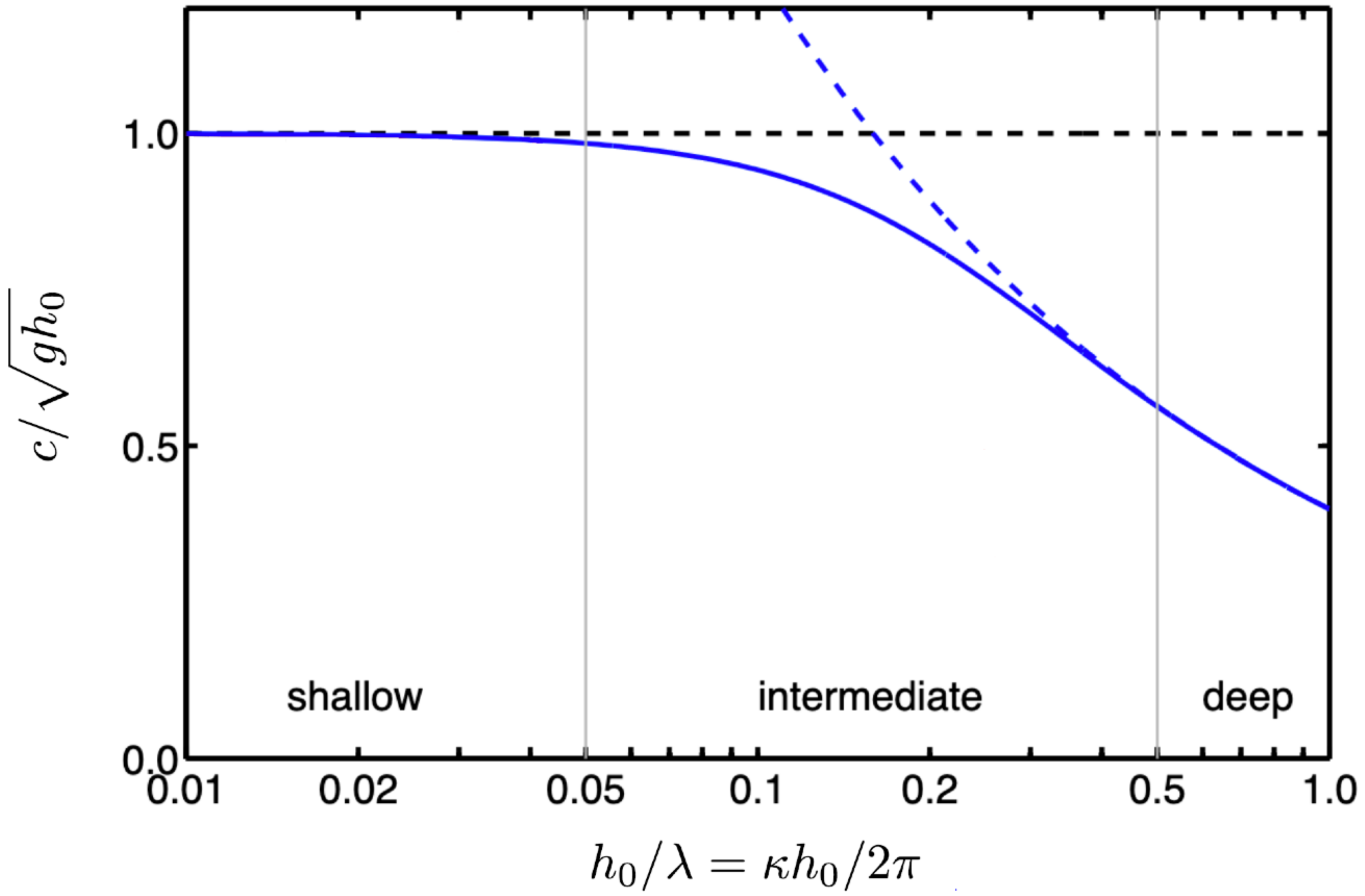
Bores (straight walled channels)

Lemoine analogy







Bore:

Shallow water Rankine-Hugoniot relation (no dispersion !!):

$$C_b - U_2 = \sqrt{\frac{h_1}{h_2} g \frac{h_1 + h_2}{2}}$$

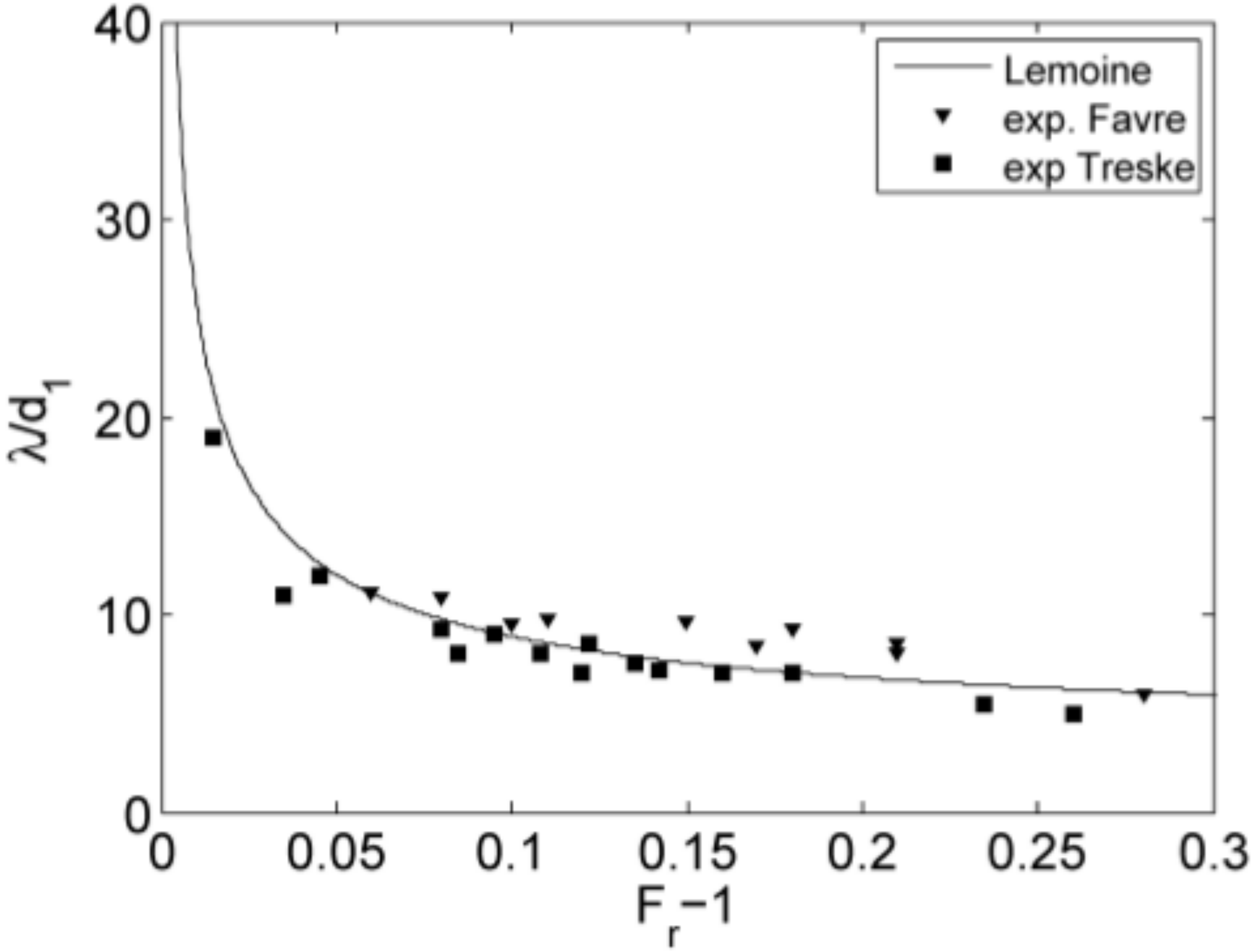
Water waves:

exact dispersion of Euler equations (Airy theory)

$$C_\lambda = \sqrt{g \frac{\lambda}{2\pi} \tanh(\frac{2\pi}{\lambda} h)}$$

$$C_b = U_2 + C_\lambda \Longrightarrow \lambda(Fr)$$

Bores (straight walled channels) | Lemoine analogy



$$C_b = U_2 + C_\lambda \Longrightarrow \lambda(Fr)$$

Bores (trapezoidal channels)

Treske's experiments

Fig. 8. Undular bore at Froude ~ 1.04.

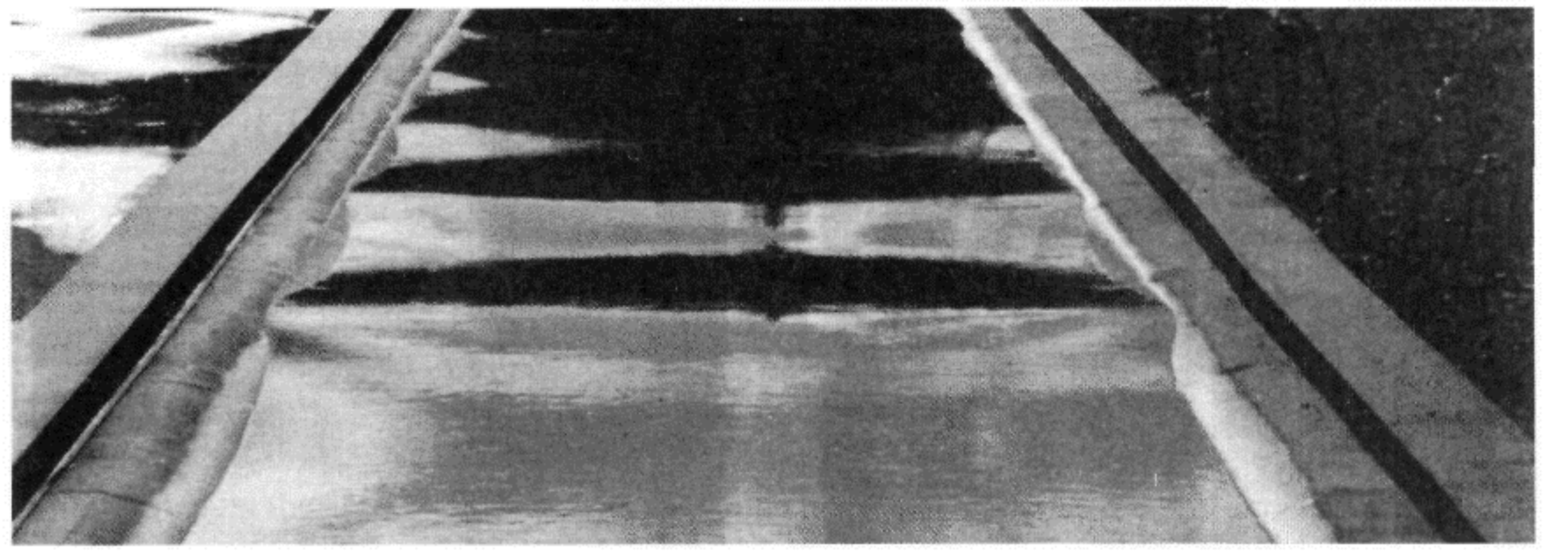


Fig. 9. Undular bore at Froude ~ 1.06.

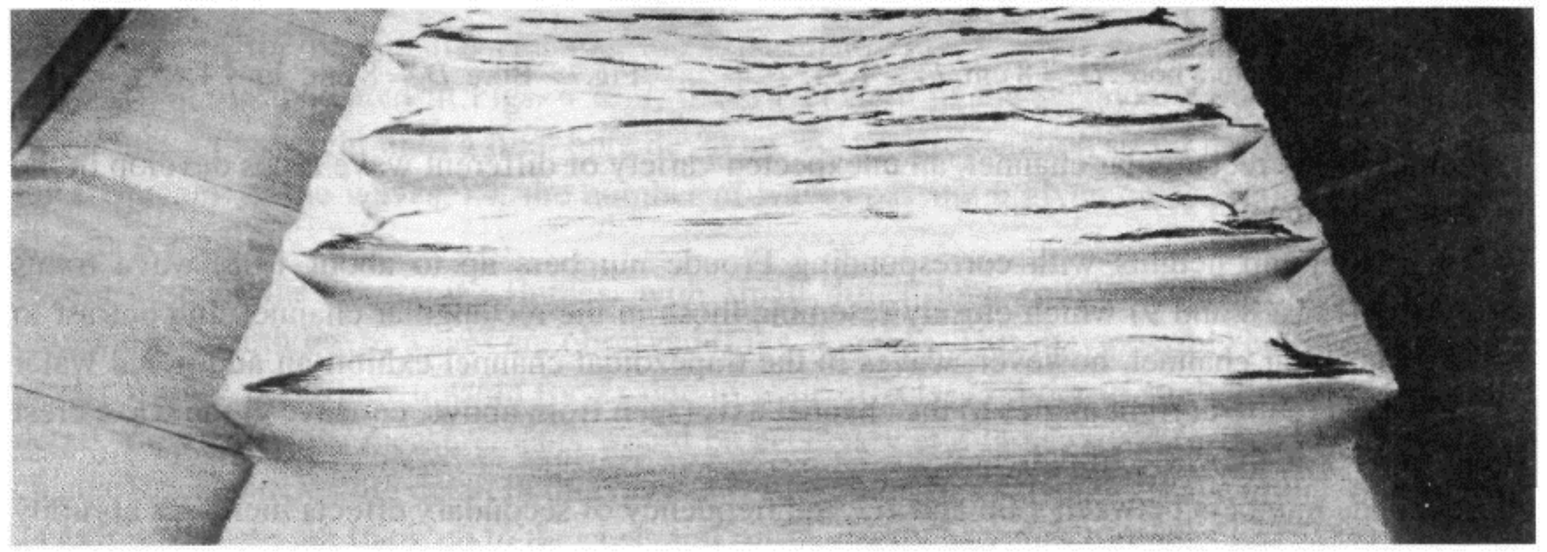


Fig. 10. Undular bore at Froude ~ 1.10.

Fr

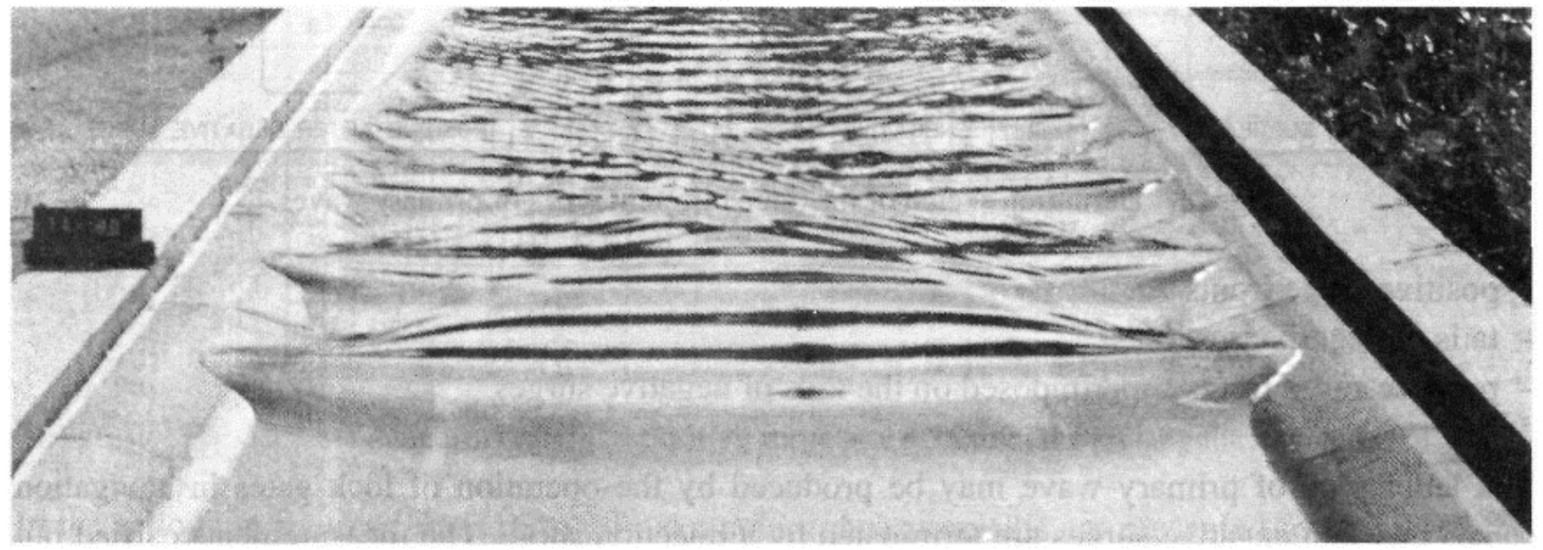


Fig. 11. Undular bore at Froude ~ 1.12.

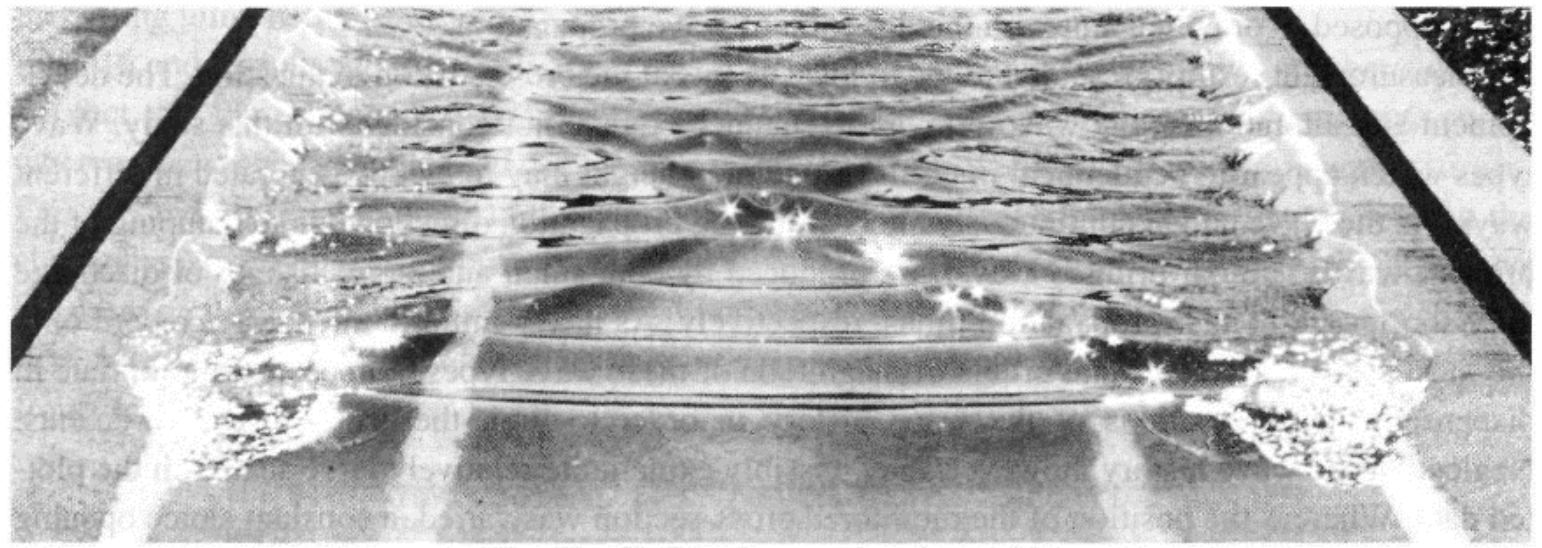


Fig. 12. Undular bore at Froude ~ 1.24.

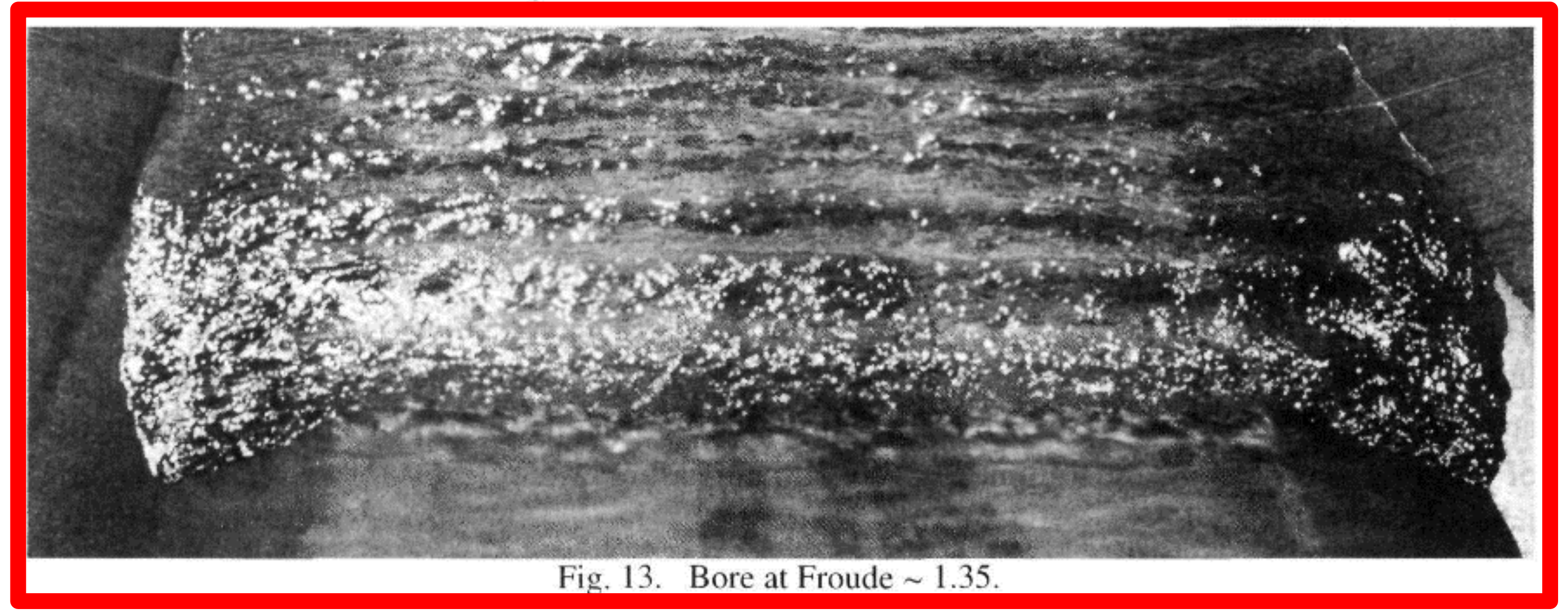
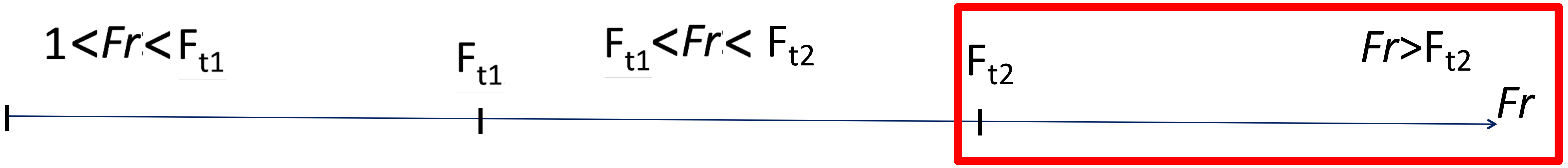


Fig. 13. Bore at Froude ~ 1.35.



$1 < Fr < F_{t1}$

F_{t1}

$F_{t1} < Fr < F_{t2}$

F_{t2}

$Fr > F_{t2}$

Fr

Treske, *J. Hydraulic Research, 1994*

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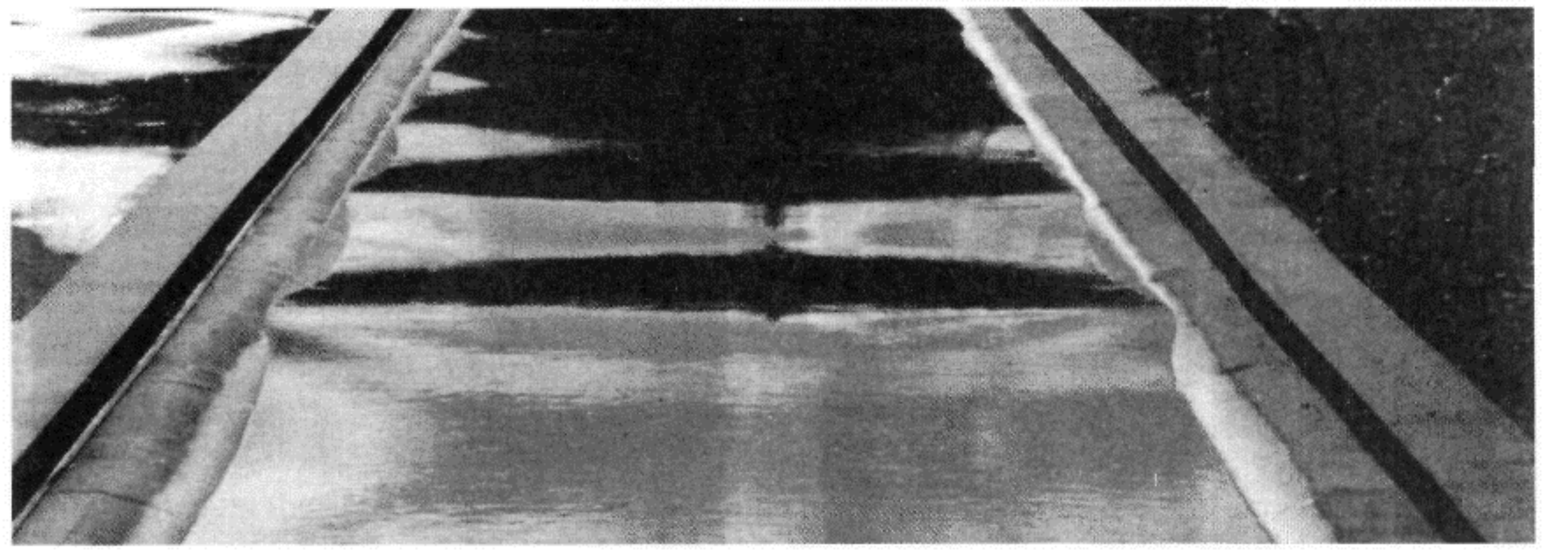

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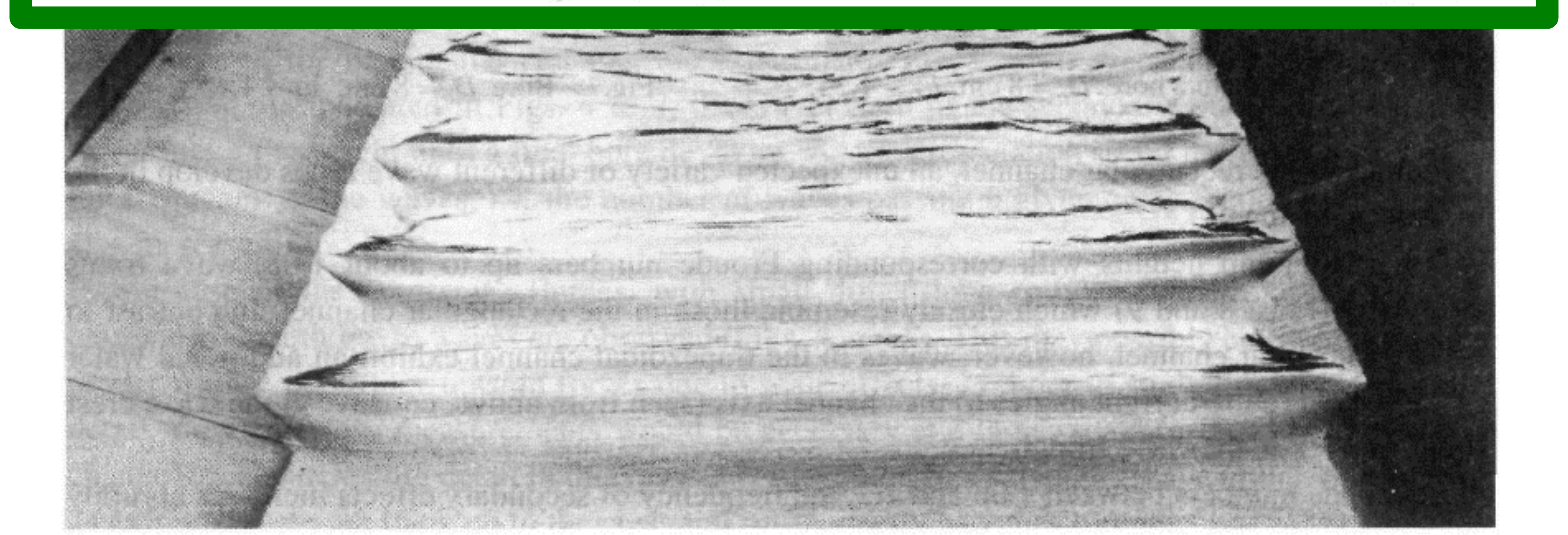

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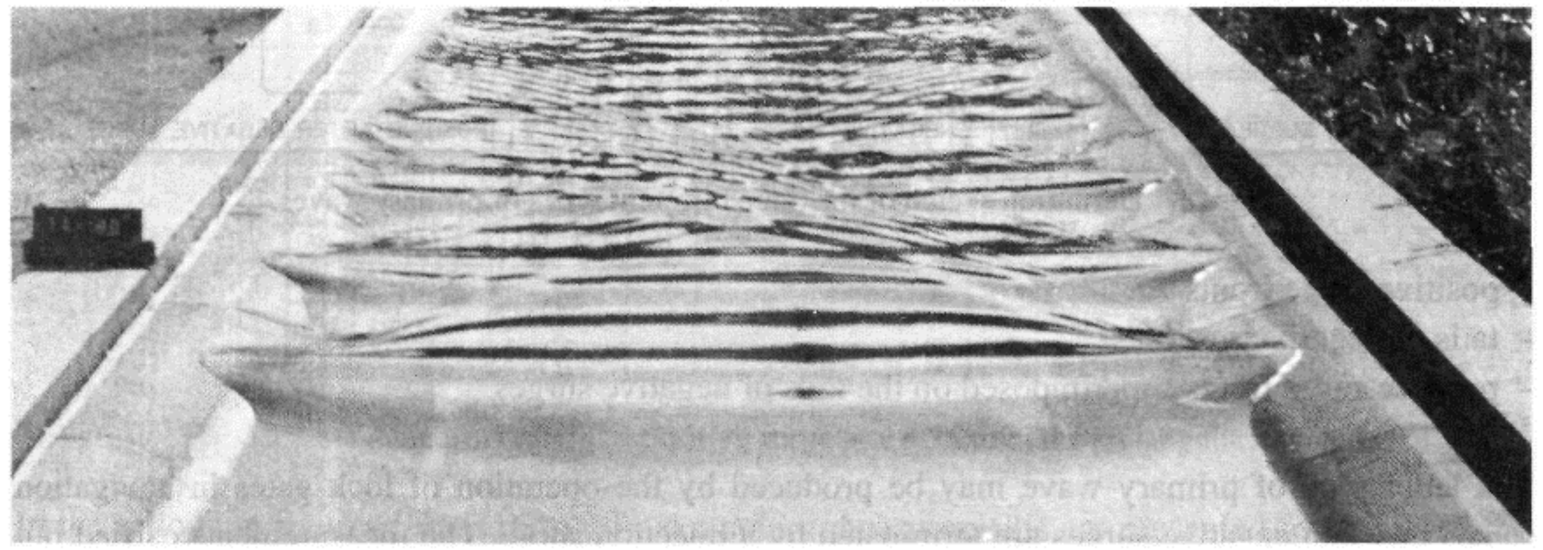

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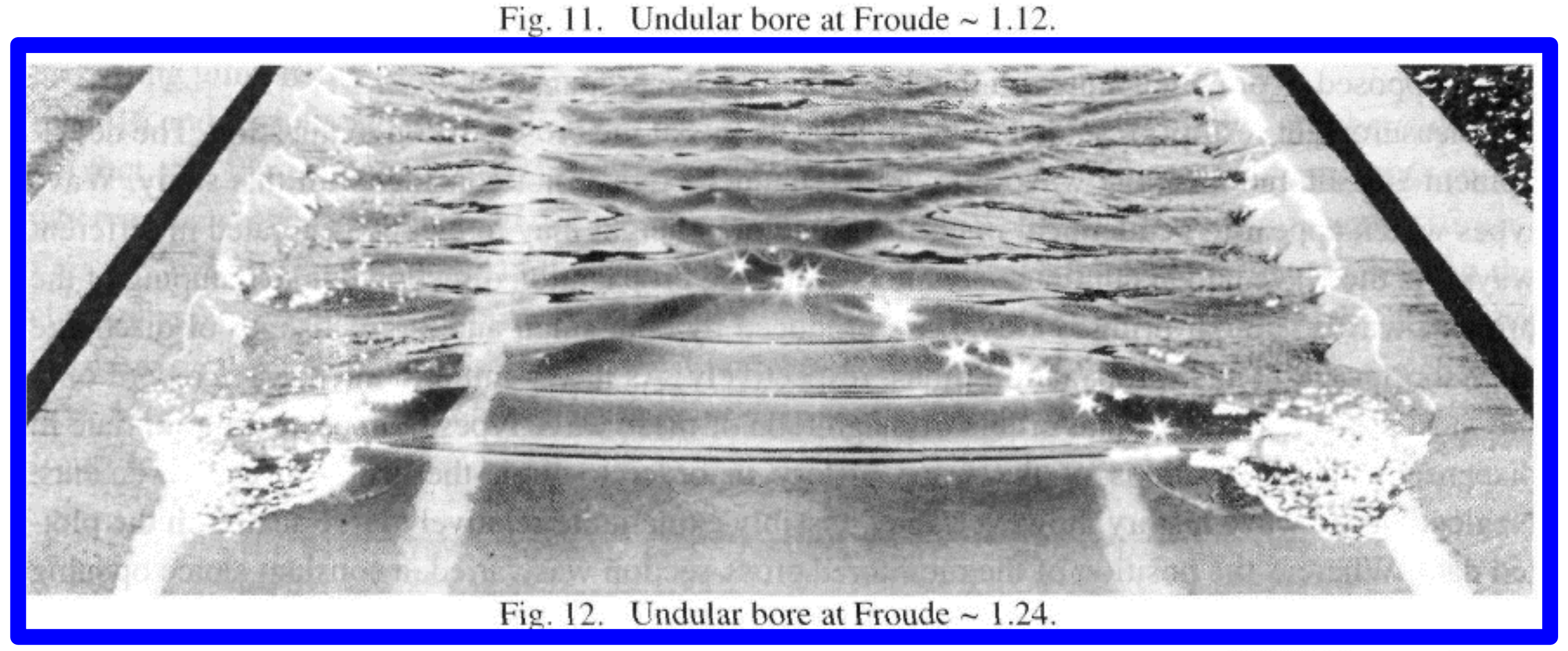

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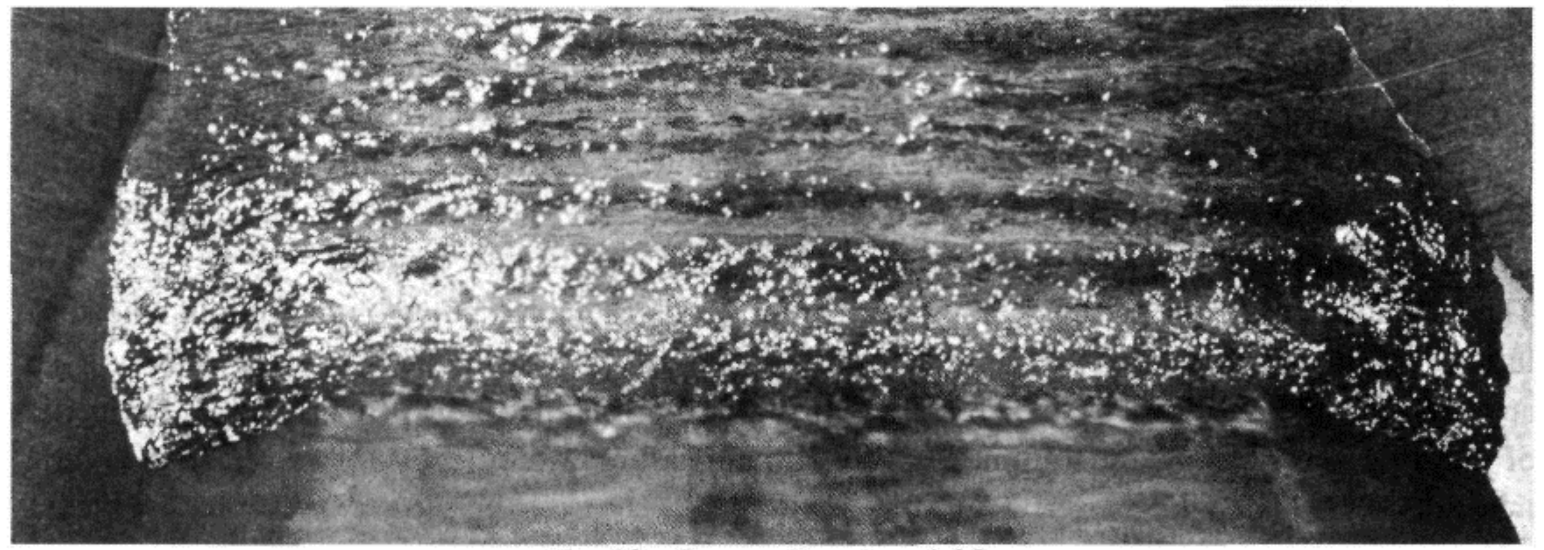

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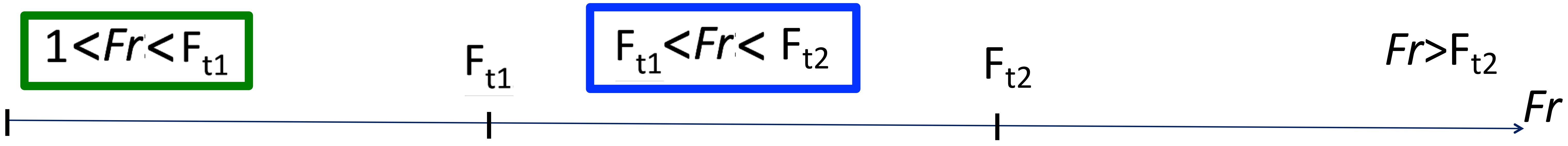


$1<Fr<F_{t1}$

F_{t1}

$F_{t1}<Fr<F_{t2}$

F_{t2}

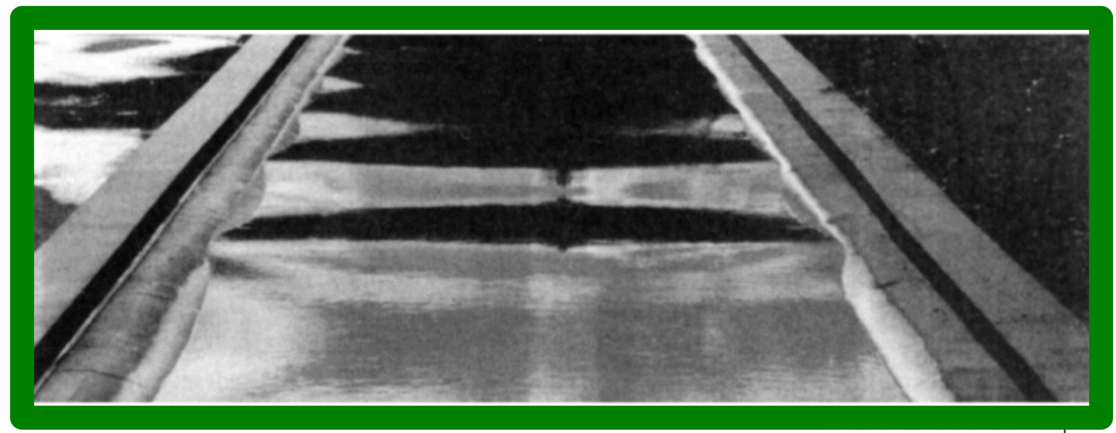
$Fr>F_{t2}$

Fr

Treske, *J. Hydraulic Research, 1994*

Bores (trapezoidal channels)

Treske's experiments



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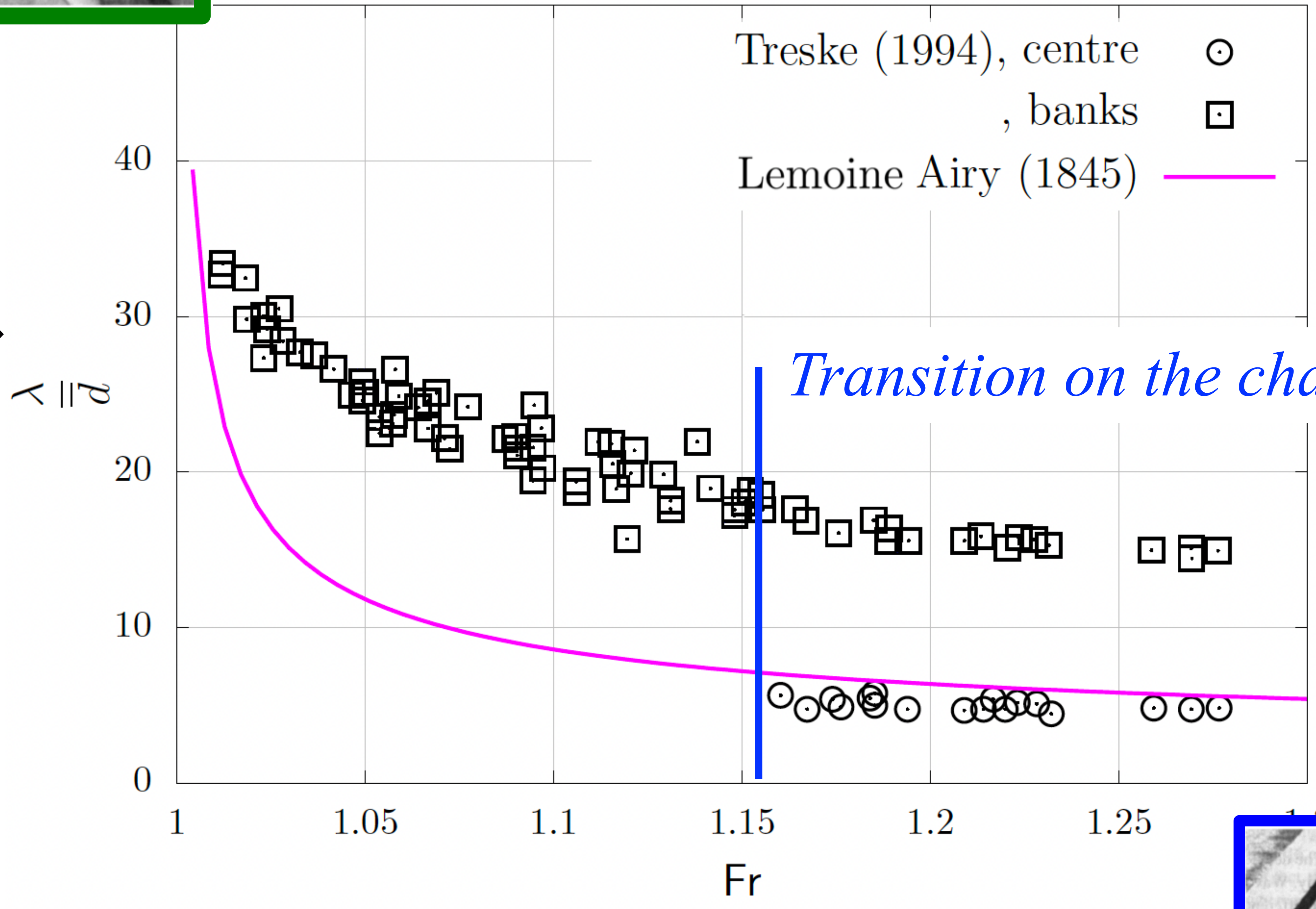
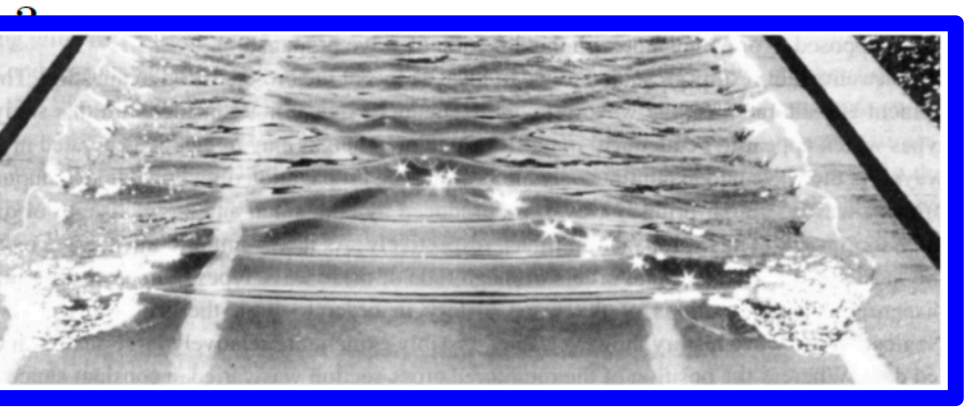


Transition on the channel axis

Favre waves

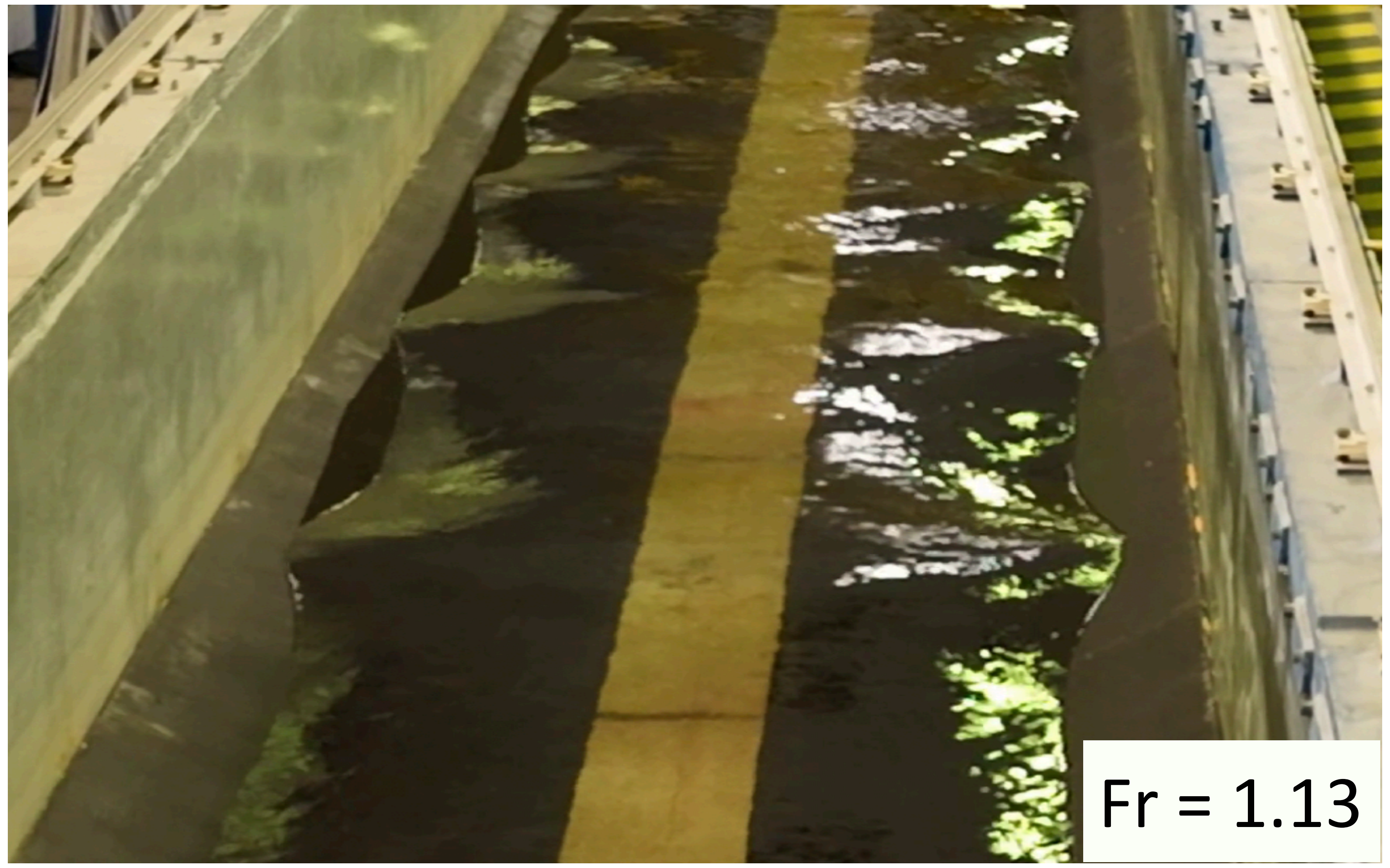


Treske, *J. Hydraulic Research, 1994*

Bores (trapezoidal channels)

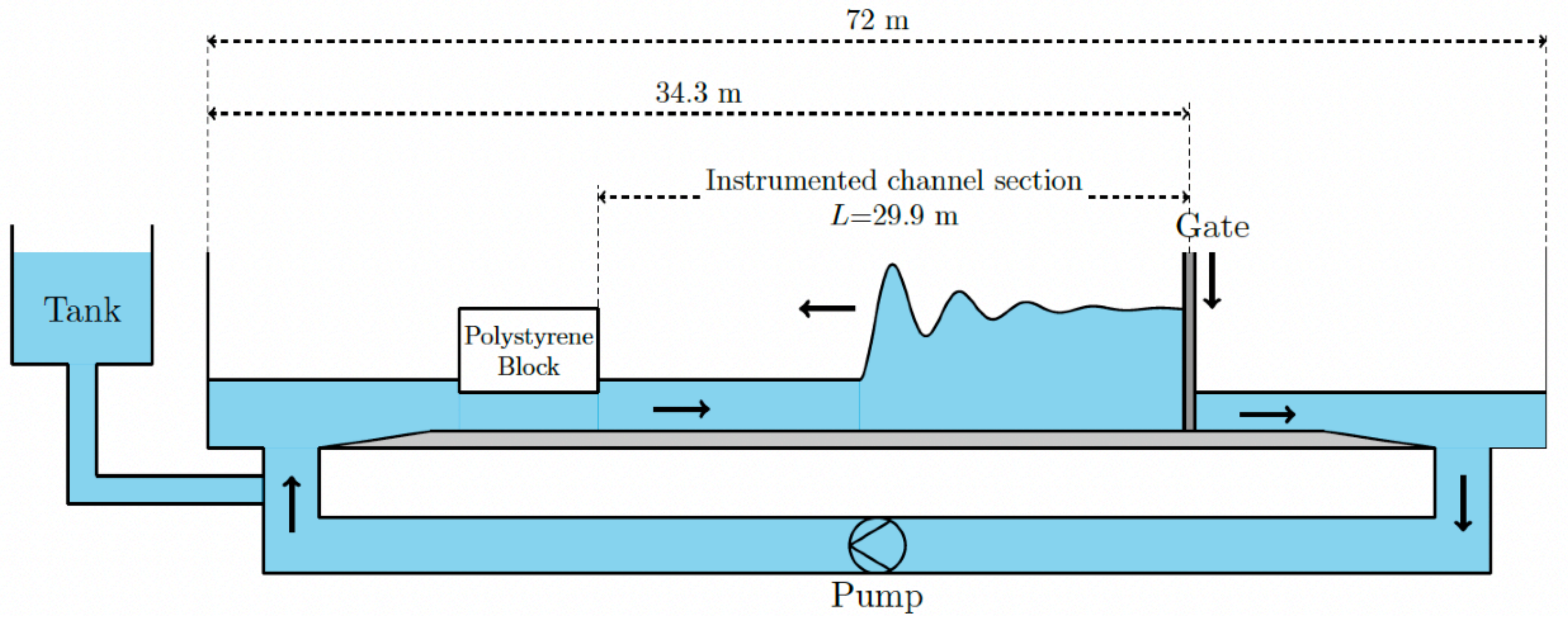
Jouy's experiments

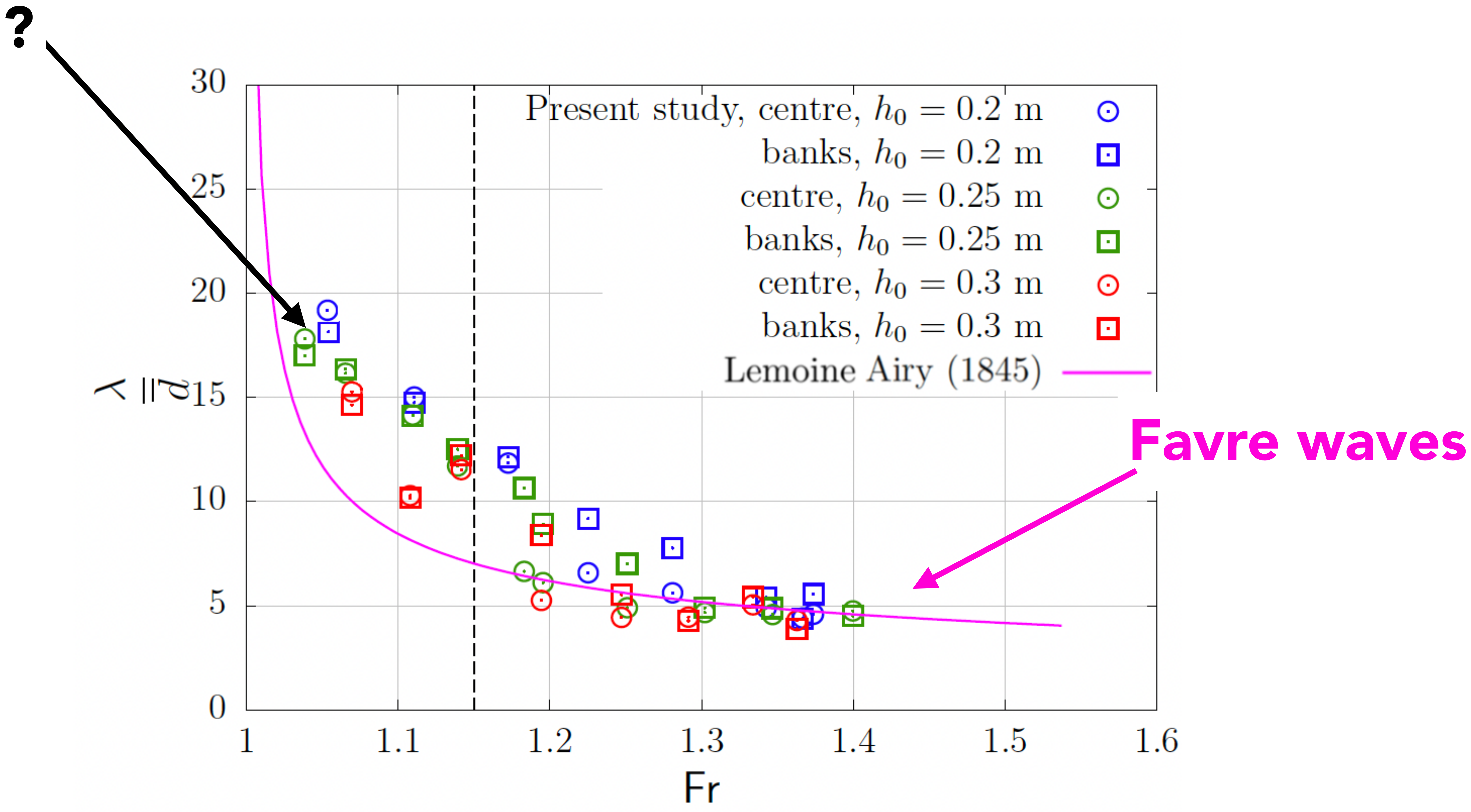
EDF Chatou's flume



Fr = 1.13

Fr = 1.18





Favre waves

Jouy et al,
IAHR-Int.Symp.Env.Hyd. 2024

Low Fr transition in Seine and Gironde: the invisible Mascaret

3 field campaigns :
a unique long-term high-frequency database

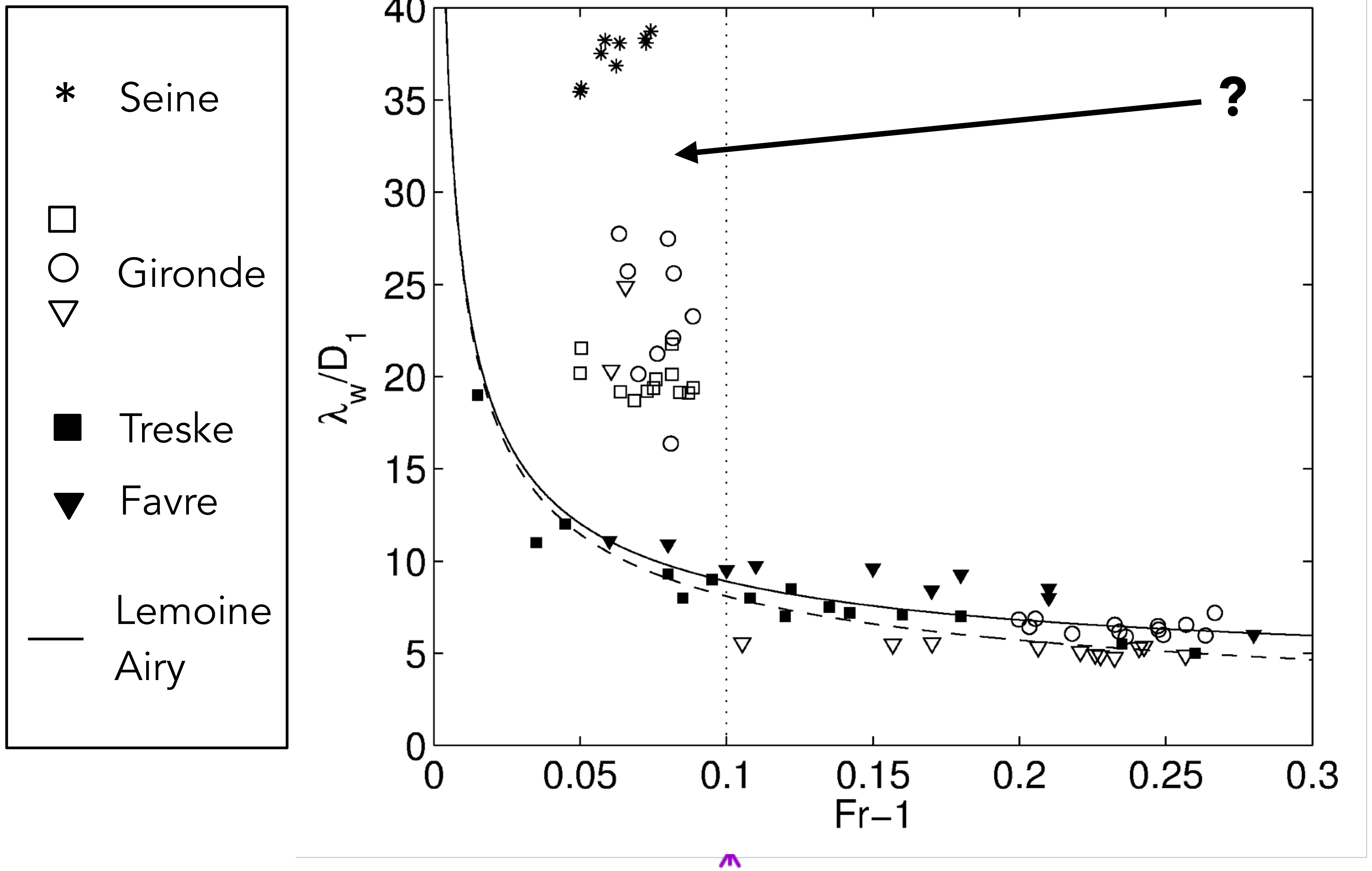
Bonneton et al, *Comptes Rendus Geoscience*, 2012

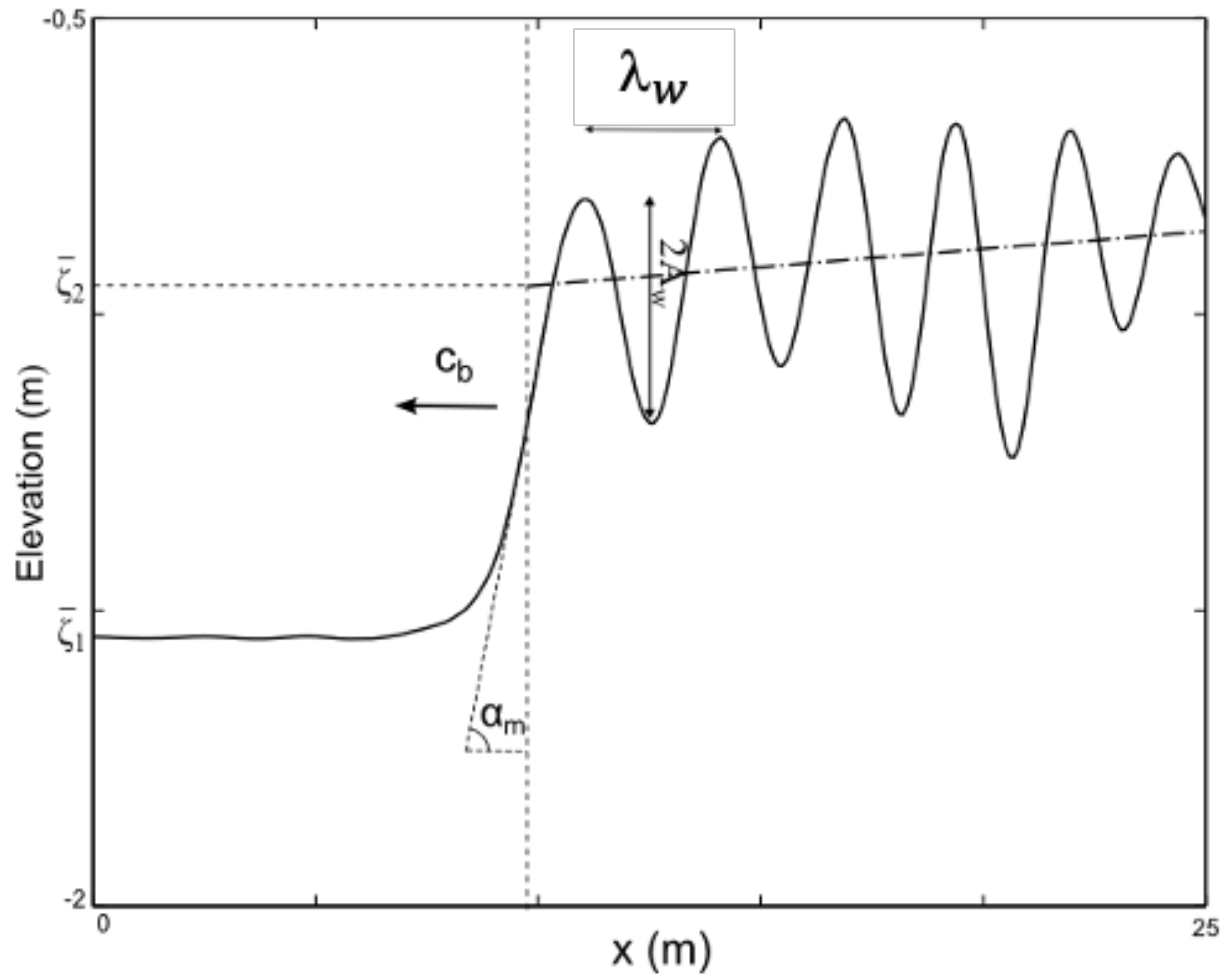
Bonneton et al, *J. Geophysical Research - Oceans,* 2015

Field studies: Gironde and Seine

Baptised *Ressaut de marée* (tidal jumps) in **Bonneton et al**, *C.R. Geosciences*, 2012

- not visible naked eyes
- mechanism not known





Transition for **Fr=F$_T$** (F$_T$≈1.1)

Common undular tidal bore (mascaret):Favre wave

Bores (trapezoidal channels)

Field studies: Gironde and Seine

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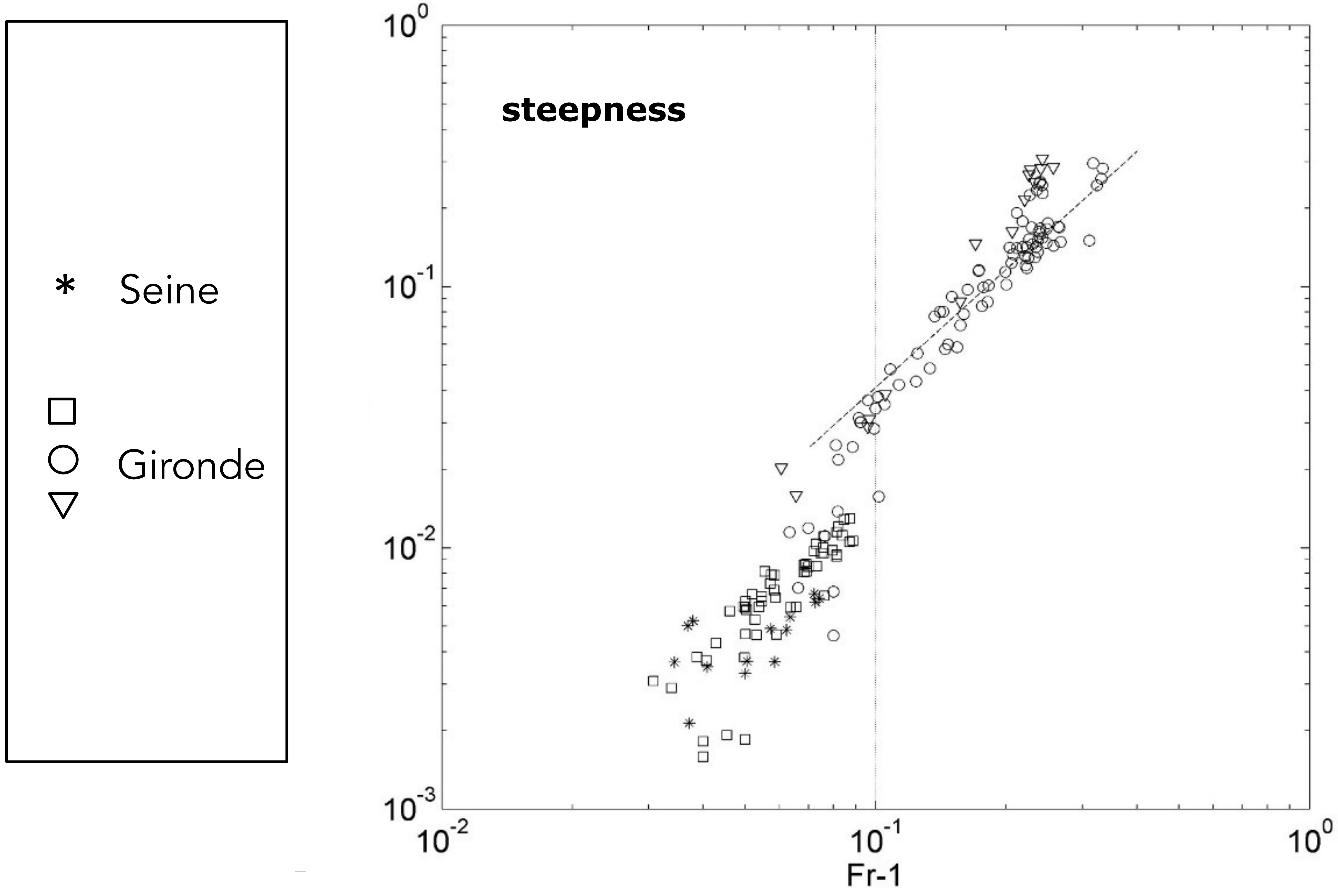
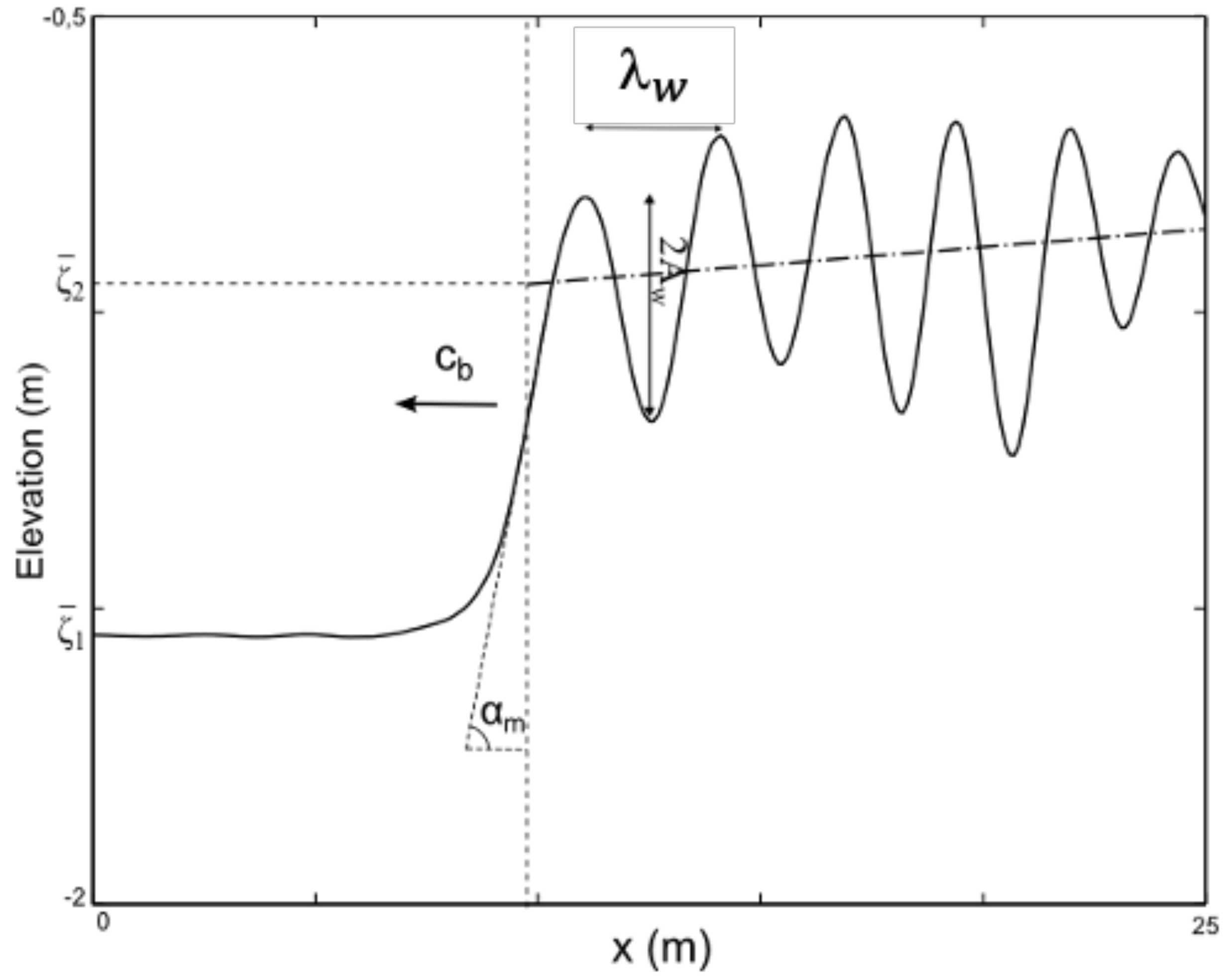




Low steepness secondary wave regime

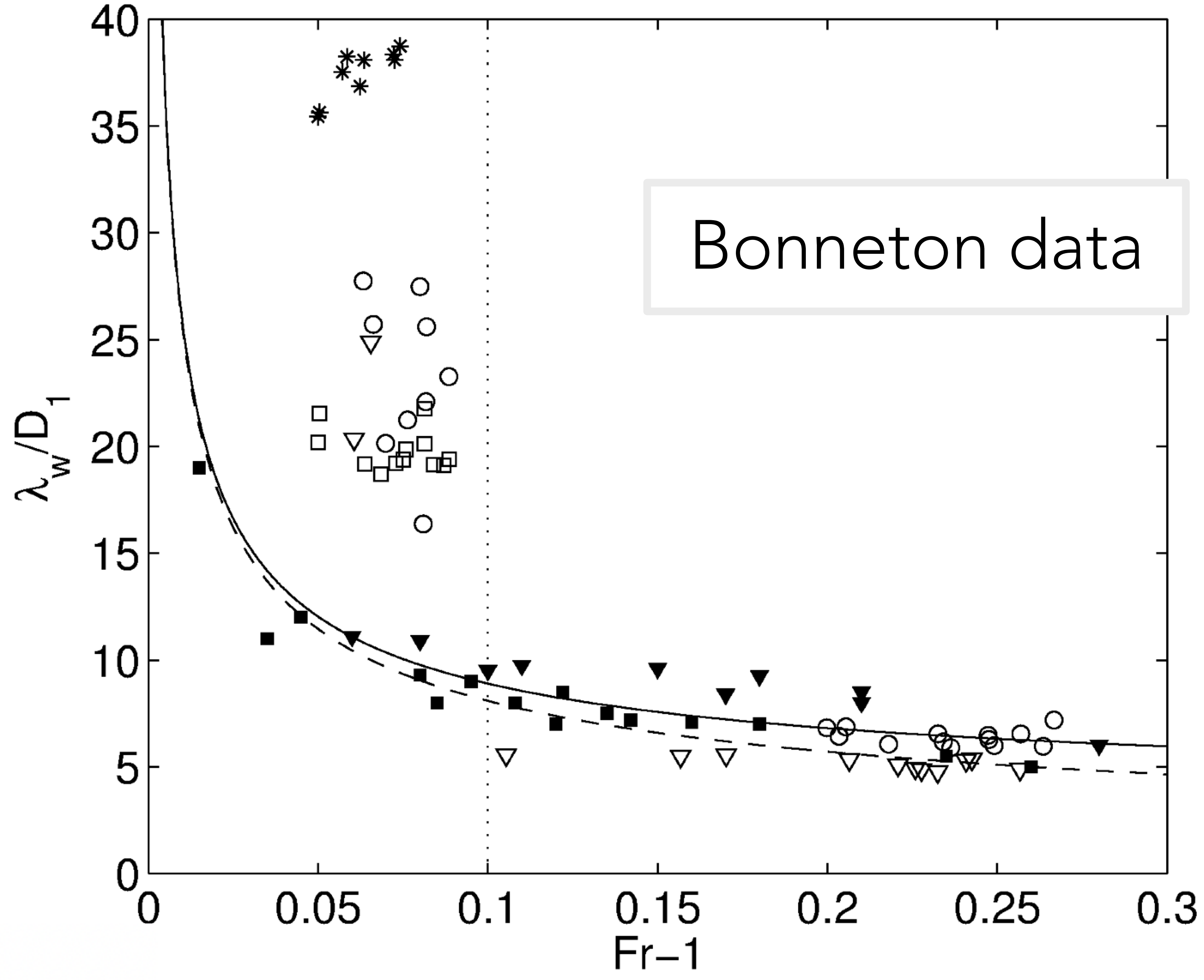
→ **not visually observable**

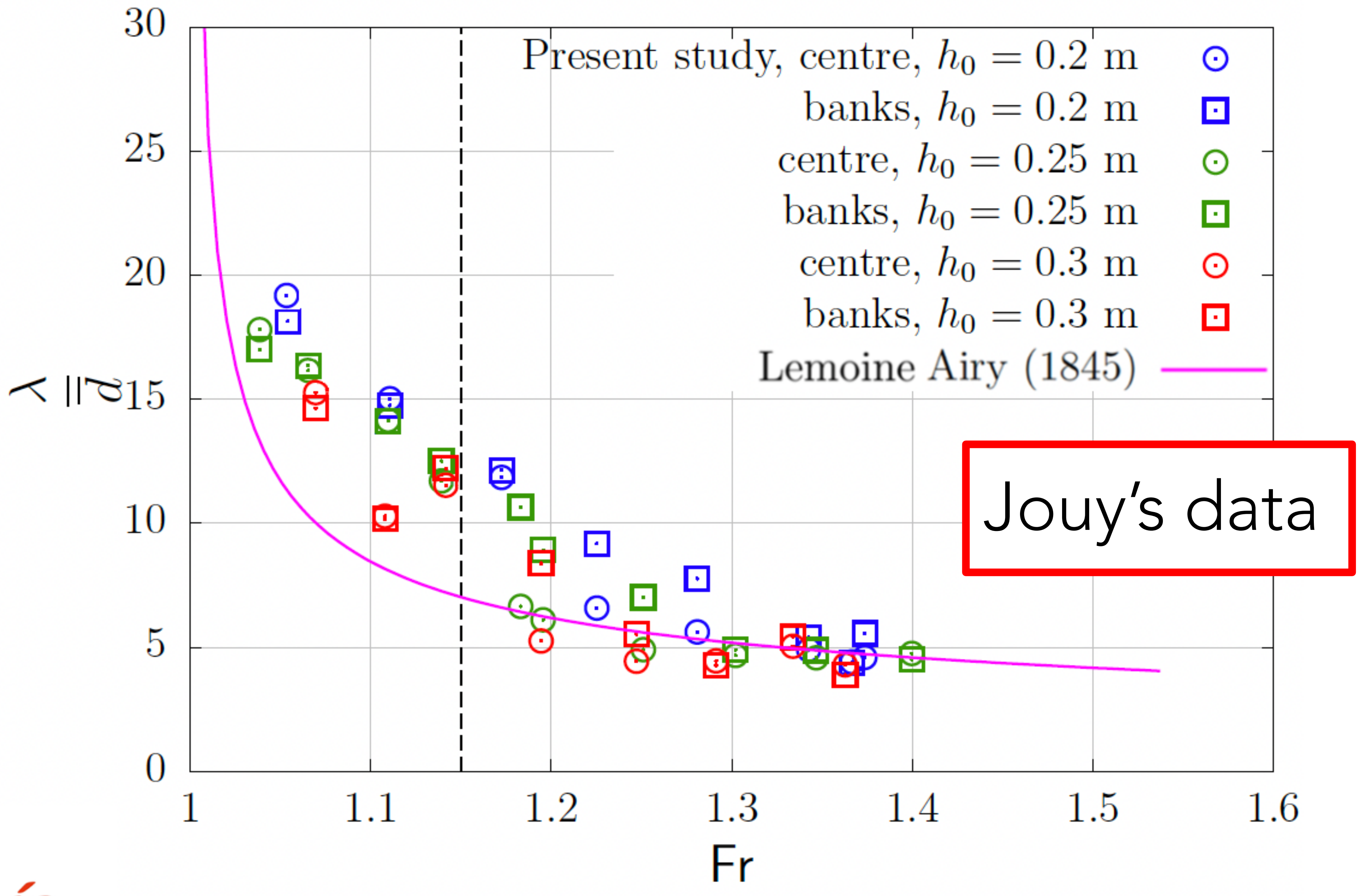
high steepness secondary wave regime

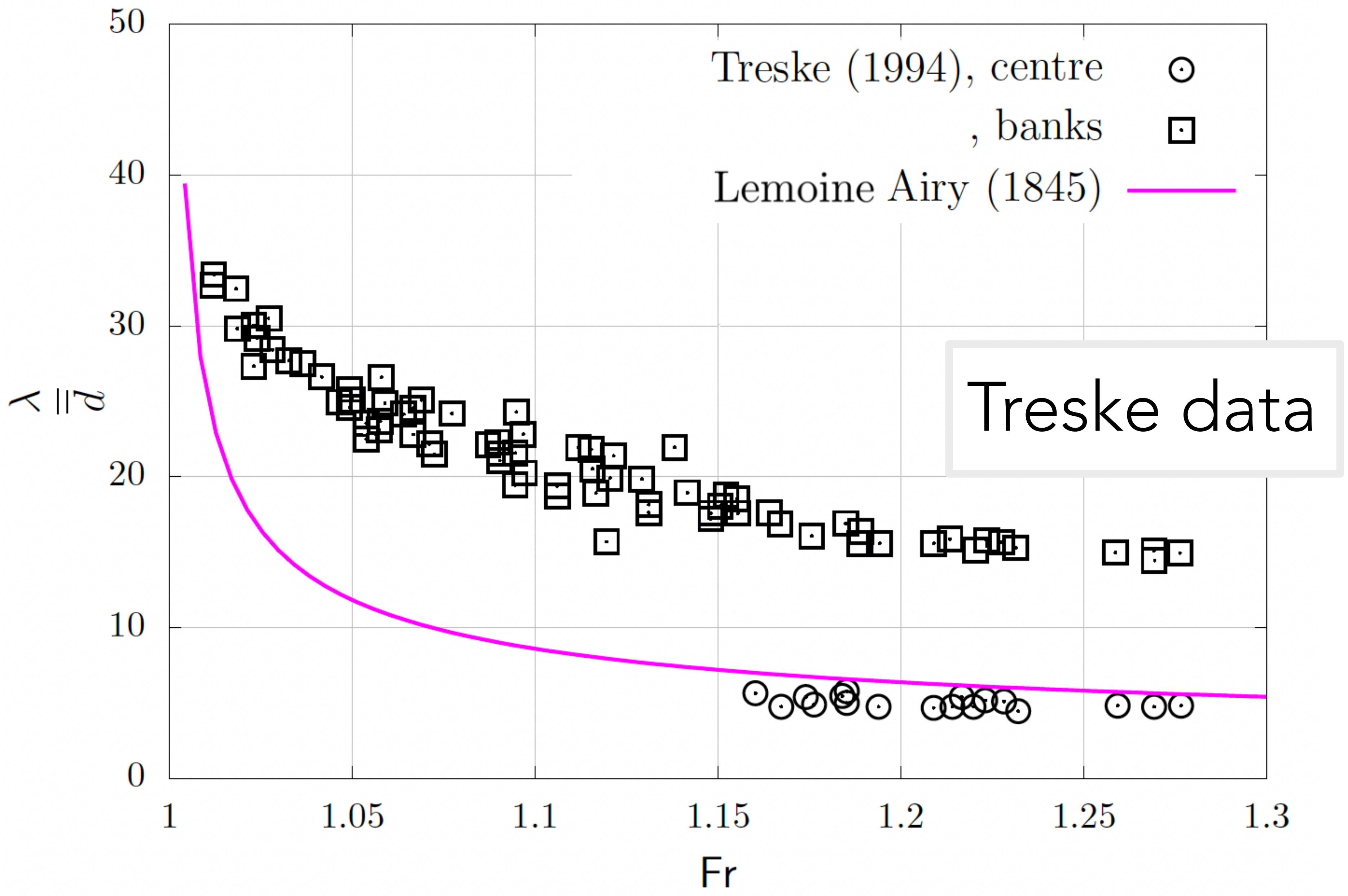
→ **mascaret**

Common undular tidal bore (mascaret):Favre wave

Bores (trapezoidal channels)







The occurrence of these tidal jumps is enormously underestimated.
According to **Bonneton et al**, *J.Coast.Res. 2011*

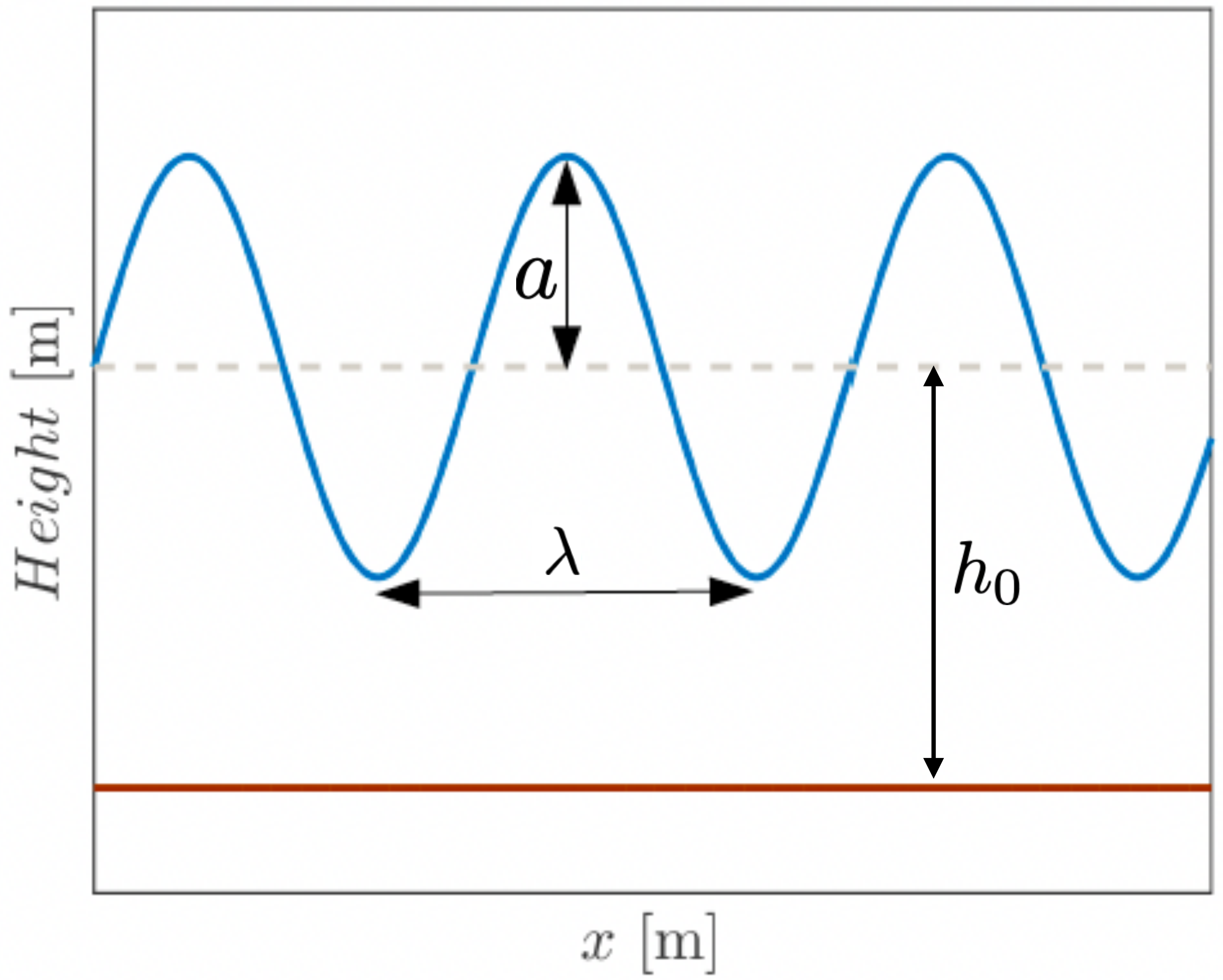
- In the Garonne river they may appear for 90% of tides during low flow period
- In the Seine river, bores were thought to have disappeared due to dredging

Tidal jumps still involve significant acceleration at the front and could have important impact on sediment dynamics.

These bores do not agree with the Lemoine analogy using the classical dispersive wave (Airy) theory associated to vertical kinematics. They involve other processes

Numerical modelling using

Serre-Green-Nagdhi and Shallow Water (!)



Dimensionless parameters

- dispersion: $\mu = \dfrac{h_0}{\lambda} = \dfrac{\kappa h_0}{2\pi}$
- non-linearity: $\epsilon = \dfrac{a}{h_0}$

Physical hypotheses

Long waves : small μ

Weakly dispersive waves : $\mu^2 \ll 1\,,\quad \mu^4$ negligible

Weak/full non-linearity : $\epsilon = \mathcal{O}(\mu^2)$ and $\epsilon = \mathcal{O}(1)$ respectively

Dispersive wave models | Depth averaged asymptotic models

Boussinesq, J.Math. Pures Appl., 1872

Dingemans, World Scientific, 1997

Lannes, AMS, 2013

Lannes, Nonlinearity, 2020

Asymptotic expansion, depth averaging

1. Starting point : nonlinear wave equations

$$\begin{aligned} \Delta\Phi &= 0 \\ \partial_t\Phi + \frac{1}{2}\|\nabla\Phi\|^2 + \mathrm{g}\zeta &= 0 \\ \partial_t\zeta + \partial_x\Phi\partial_x\zeta &= \partial_z\Phi \\ \partial_z\Phi &= 0 \end{aligned}$$

2. Asymptotic dev. wrt :

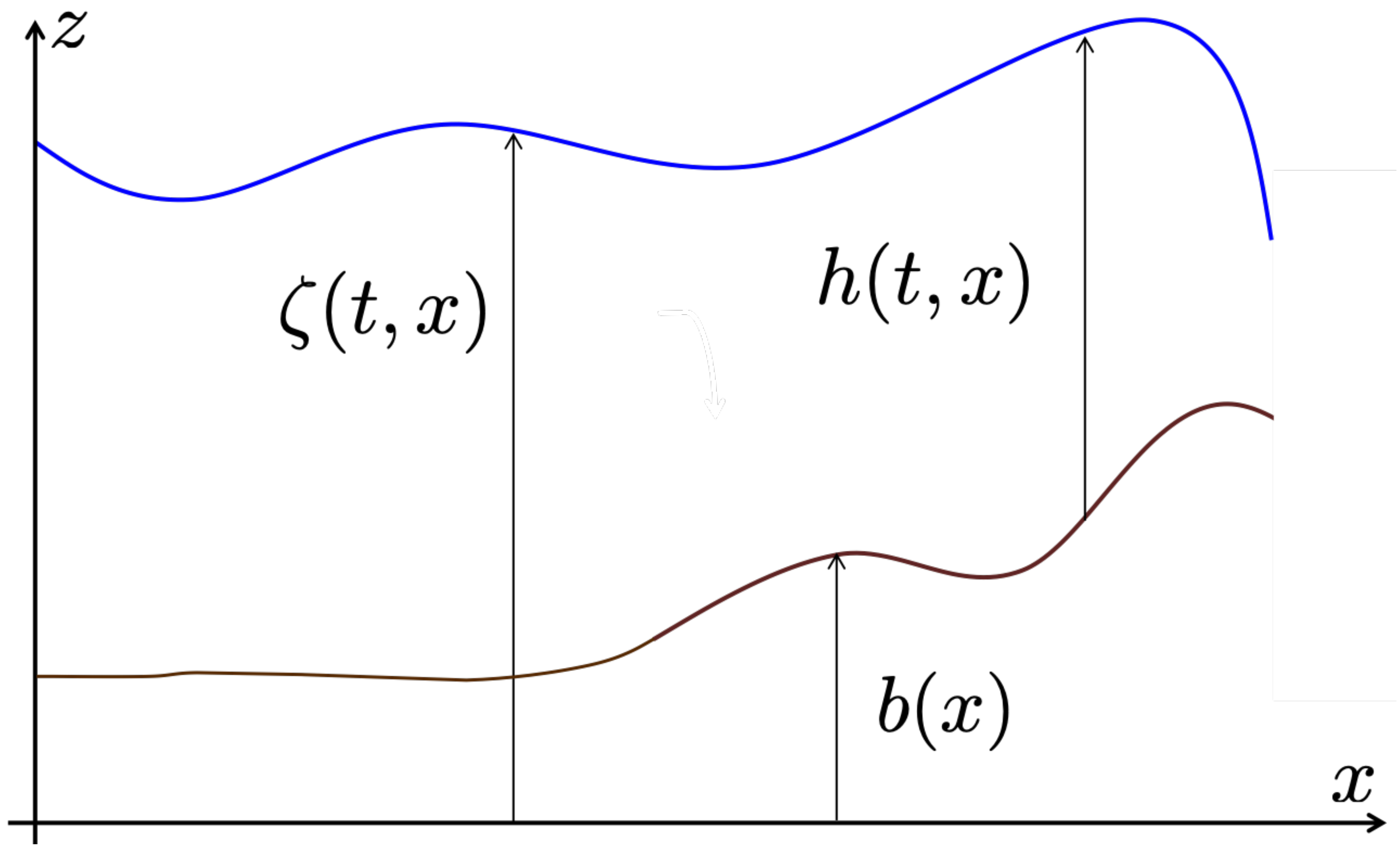
$$\mu^2 \quad \Phi = \Phi_0 + \mu^2\Phi_1 + \mu^4\Phi_2 + \dots$$

3. **VERTICAL** averaging :

$$\int_0^{h_0+\zeta} (\cdot)\, dz \quad \Longrightarrow \quad h\vec{\mathrm{u}} = \int_b^{\zeta} \vec{\mathrm{v}}\, dz$$

Shallow water equations in 1D

$$\partial_t h + \partial_x (h\mathsf{u}) = 0$$
$$\partial_t (h\mathsf{u}) + \partial_x (h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$



$\mathcal{D} = 0$: usual shallow water eqs.

$\mu = 0$ limit, or equivalently

zero-th order term in the expansion

Serre Green-Naghdi equations in 1D

$$\partial_t h + \partial_x(h\mathsf{u}) = 0$$

$$\partial_t(h\mathsf{u}) + \partial_x(h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = \partial_x\left(\frac{h^3}{3}\partial_x\dot{\mathsf{u}}\right) - \partial_x\left(\frac{\partial_x b}{2}h^2\dot{\mathsf{u}}\right) - h(\partial_x b)^2\dot{\mathsf{u}}$$

$$-\frac{2}{3}\partial_x(h^2(\partial_x\mathsf{u})^2 + \frac{3}{4}h^2\mathsf{u}^2\partial_{xx}b) - \frac{h\mathsf{u}^2}{2}\partial_x(\partial_x b)^2$$

μ^2 correction, with

$$\dot{\mathsf{u}} = \partial_t\mathsf{u} + \mathsf{u}\partial_x\mathsf{u}$$

Green & Naghdi, J.Fluid Mech, 1976

Chazel et al, J.Sci.Comp. 2011

Lannes, Nonlinearity 2020

Serre Green-Naghdi equations in 1D

$$\partial_t h + \partial_x(h\mathsf{u}) = 0$$

$$\partial_t(h\mathsf{u}) + \partial_x(h\mathsf{u}^2 + gh^2/2) + gh\partial_x b = \mathcal{D}$$

$$\mathcal{D} = -\partial_x\left(\frac{h^2}{3}\ddot{h}\right) - \partial_x\left(\frac{h^2}{2}\dot{\kappa}\right) - h\left(\frac{\ddot{h}}{2} + \dot{\kappa}\right)\partial_x b$$

μ^2 correction, with

$$\dot{\mathsf{u}} = \partial_t \mathsf{u} + \mathsf{u}\partial_x \mathsf{u}$$

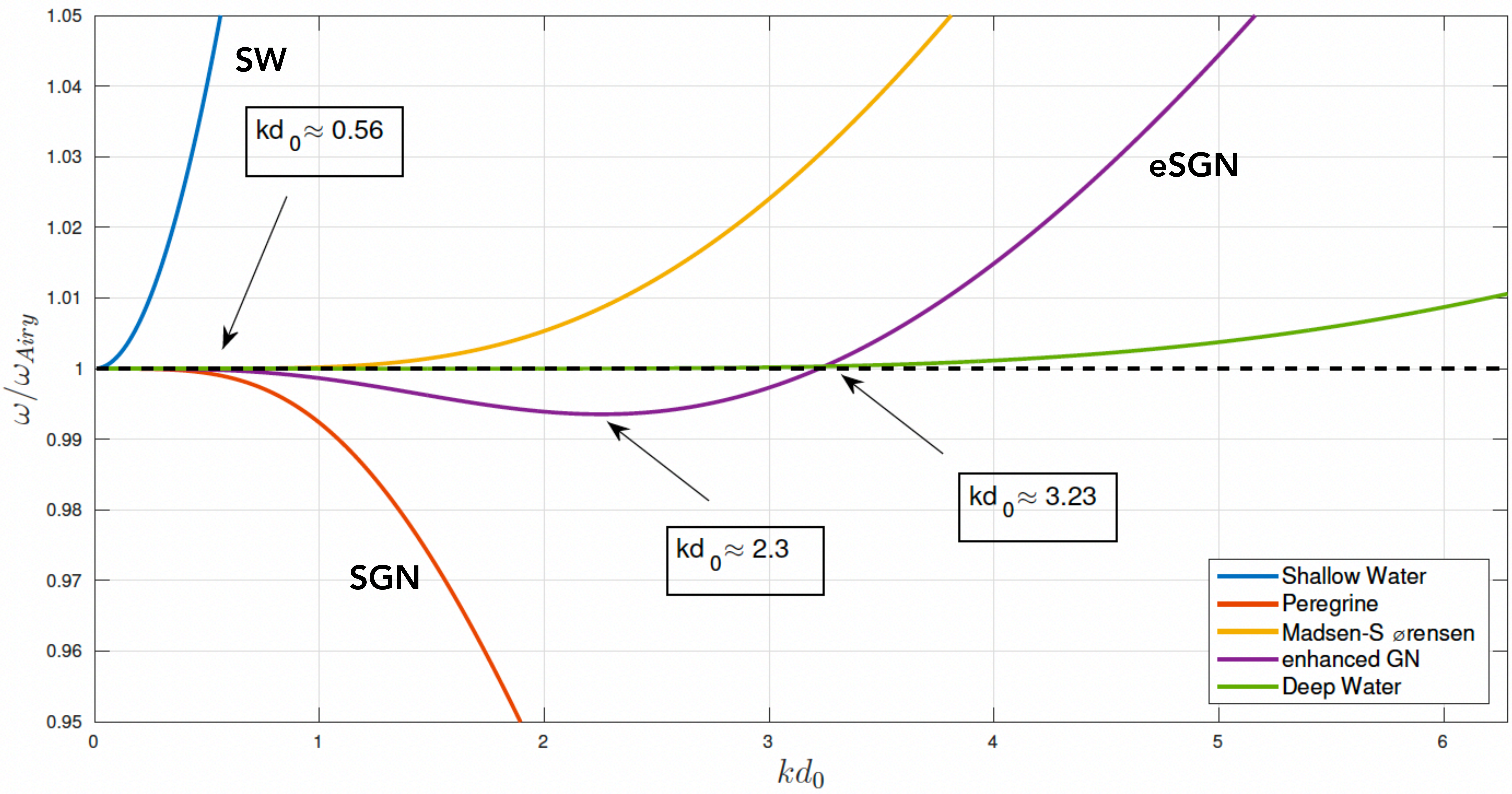
with

$$\kappa = u\partial_x b$$

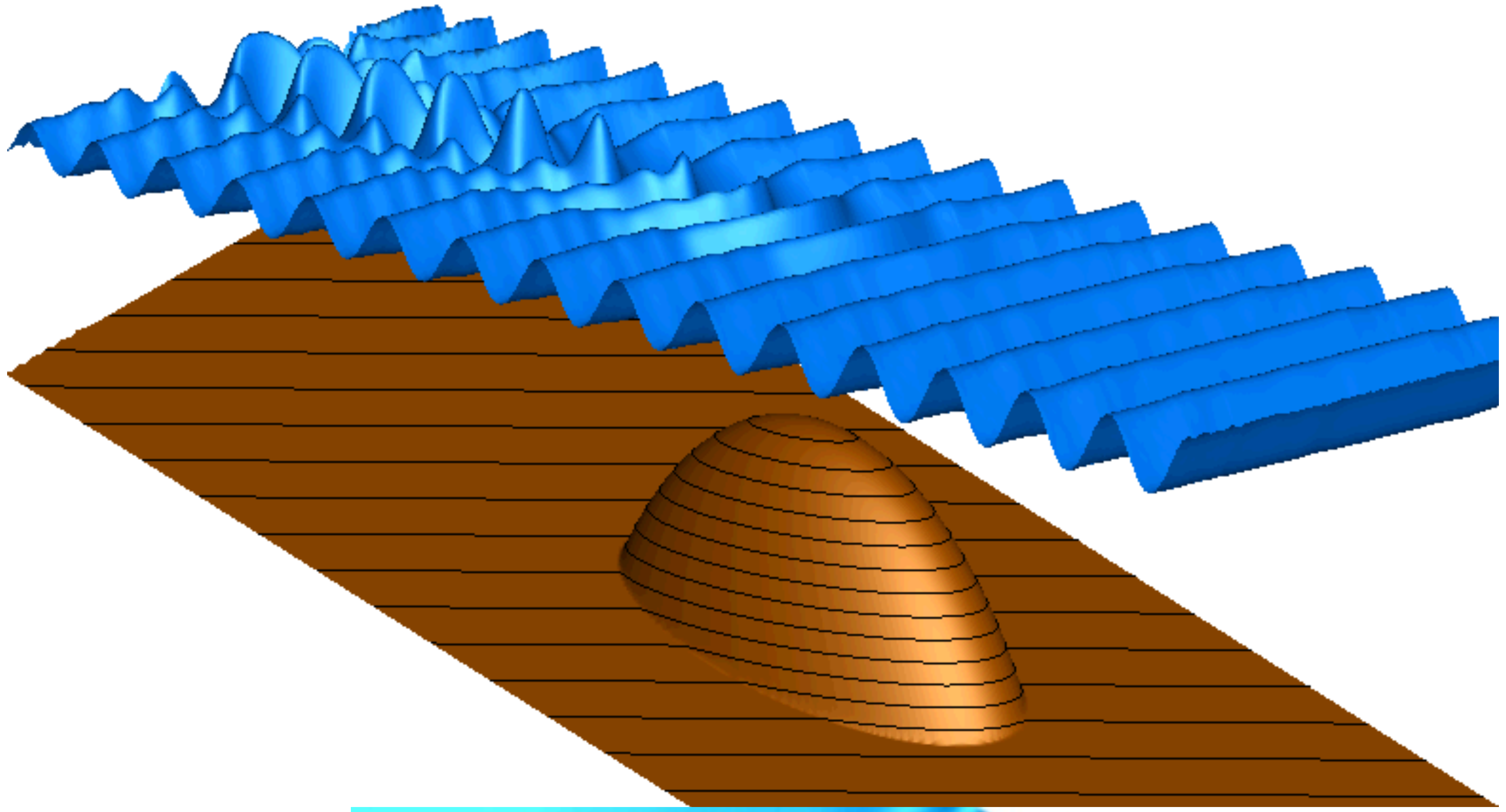
Green & Naghdi, J.Fluid Mech, 1976

Chazel et al, J.Sci.Comp. 2011

Lannes, Nonlinearity 2020


SW
$kd_0 \approx 0.56$
eSGN
$kd_0 \approx 3.23$
$kd_0 \approx 2.3$
SGN
Shallow Water
Peregrine
Madsen-S ørensen
enhanced GN
Deep Water
ω/ω_{Airy}
1.05
1.04
1.03
1.02
1.01
1
0.99
0.98
0.97
0.96
0.95
0
1
2
3
4
5
6
kd_0

Multi dimensional (and other) extension



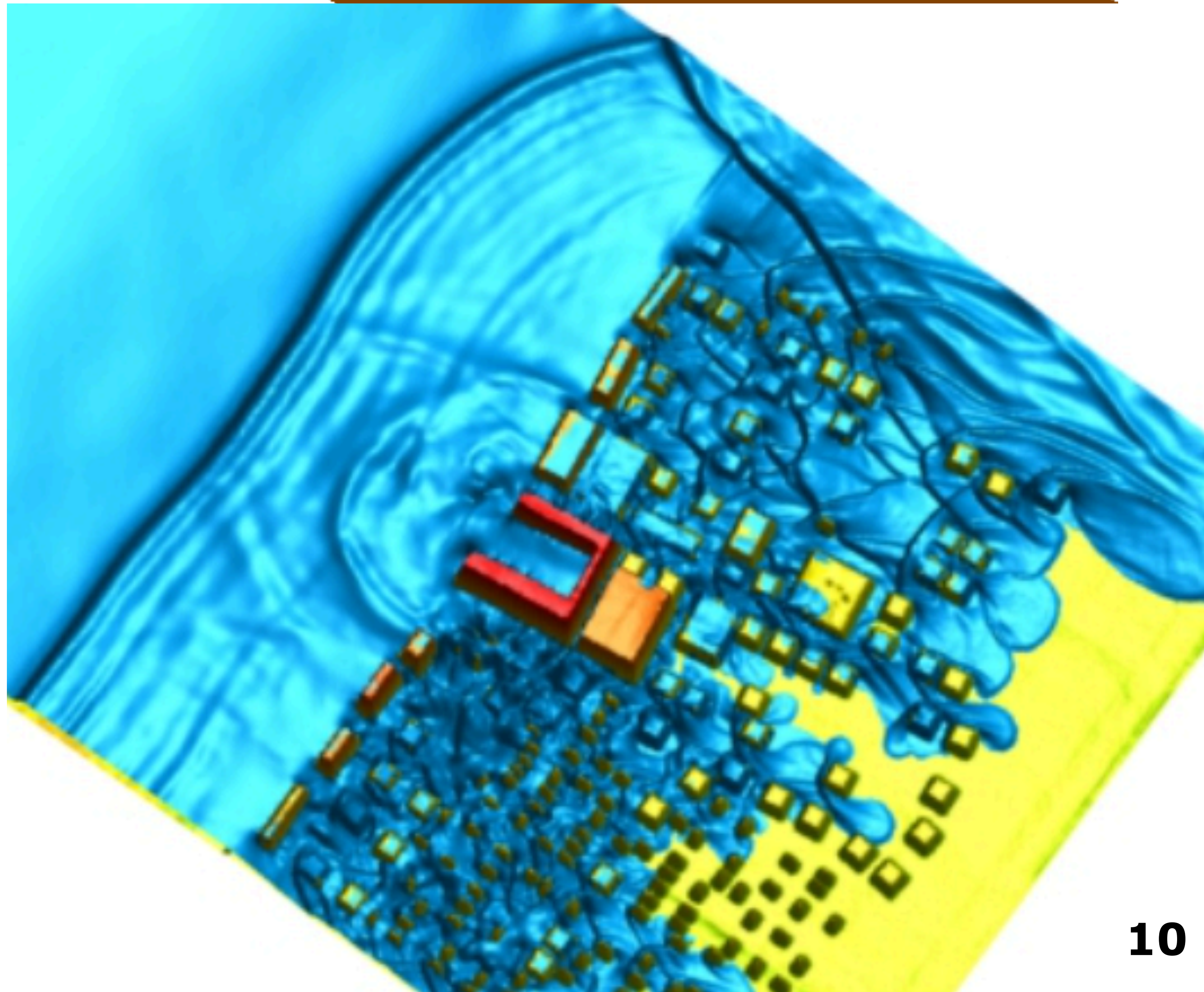
See e.g. the book by D. Lannes (AMS 2013) or

Shi et al, Ocean Mod 2012

Lannes and Marche, J.Comput.Phys., 2015

Lannes, Nonlinearity, 2020

Gavrilyuk and Shyue, J.Hyd.Res., 2023



We use the formulations discussed in

Filippini et al, J.Comput.Physi. 2016

Kazolea et al, Ocean Mod. 2023

2D simulations (trapezoidal)

Treske's experiments

$$\partial_t A(h) + \partial_x (A(h)U) = 0$$

$$\partial_t (A(h)U) + \partial_x (A(h)U^2 + K(h)) = 0$$

$$\partial K = gh\partial A$$

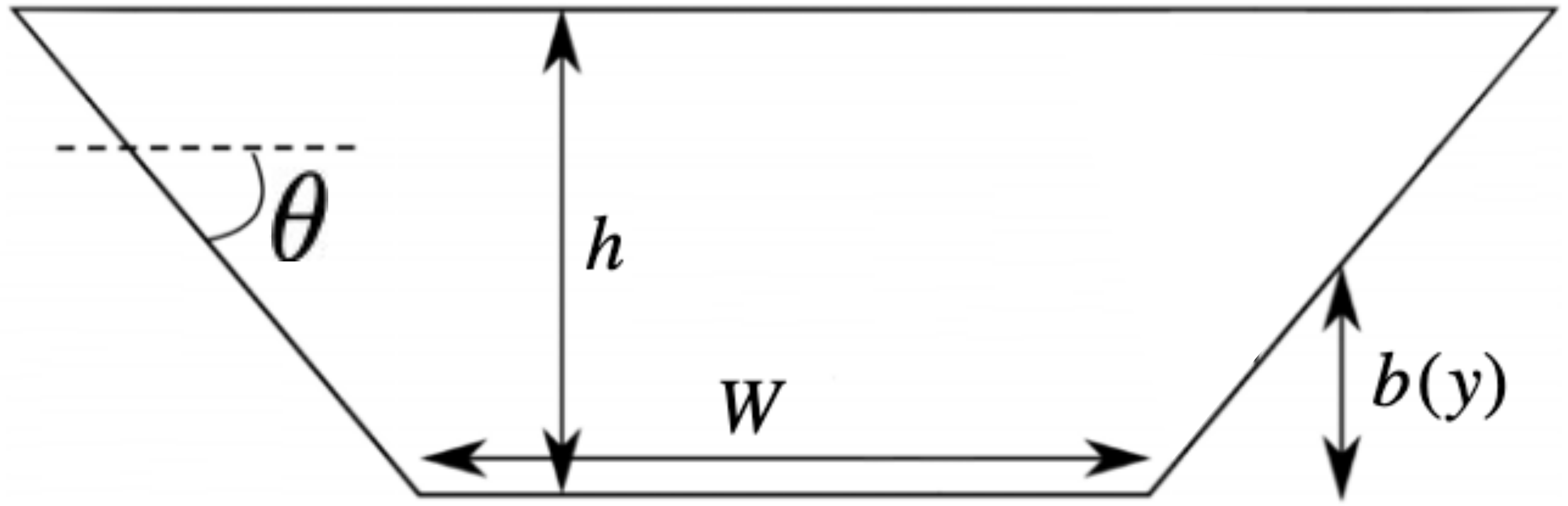
for a trapezium

$$A(h) = Wh + \frac{h^2}{\tan\theta}$$

$$K(h) = Wg\frac{h^2}{2} + \frac{g}{\tan\theta}\frac{h^3}{3}$$

Smoothed initial discontinuous state from Rankine-Hugoniot condition of classical section averaged shallow water system for different Froude numbers

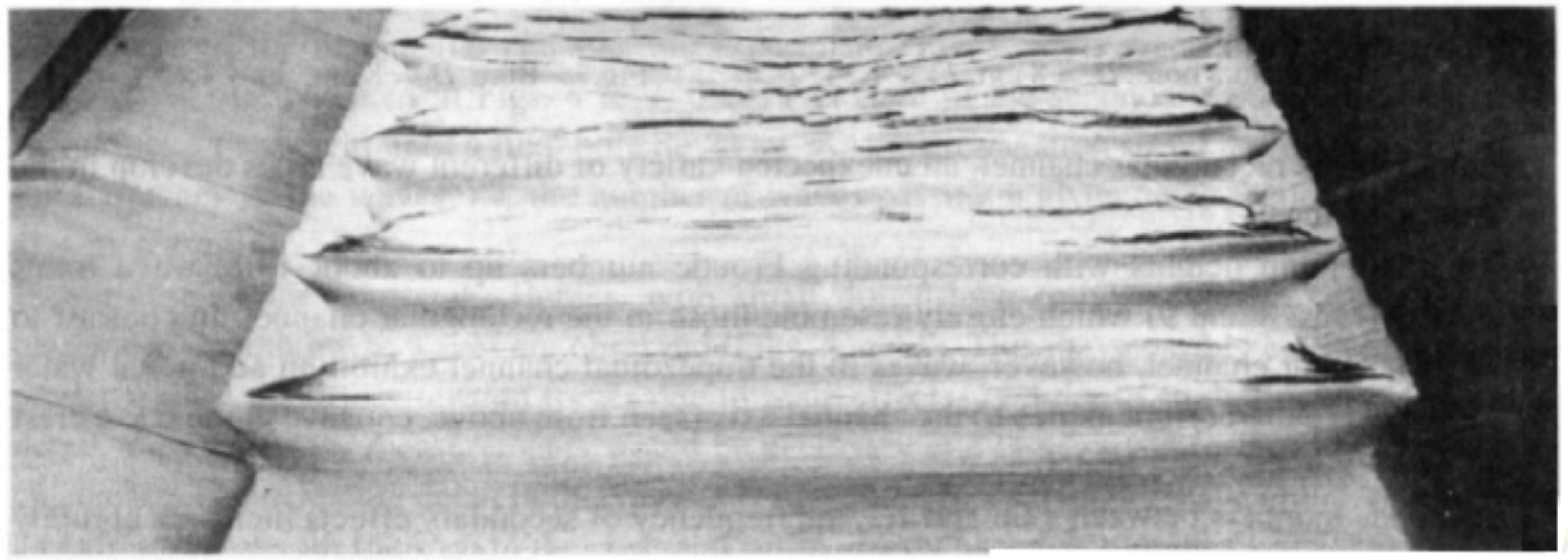
Chanson, Elsevier, 2024



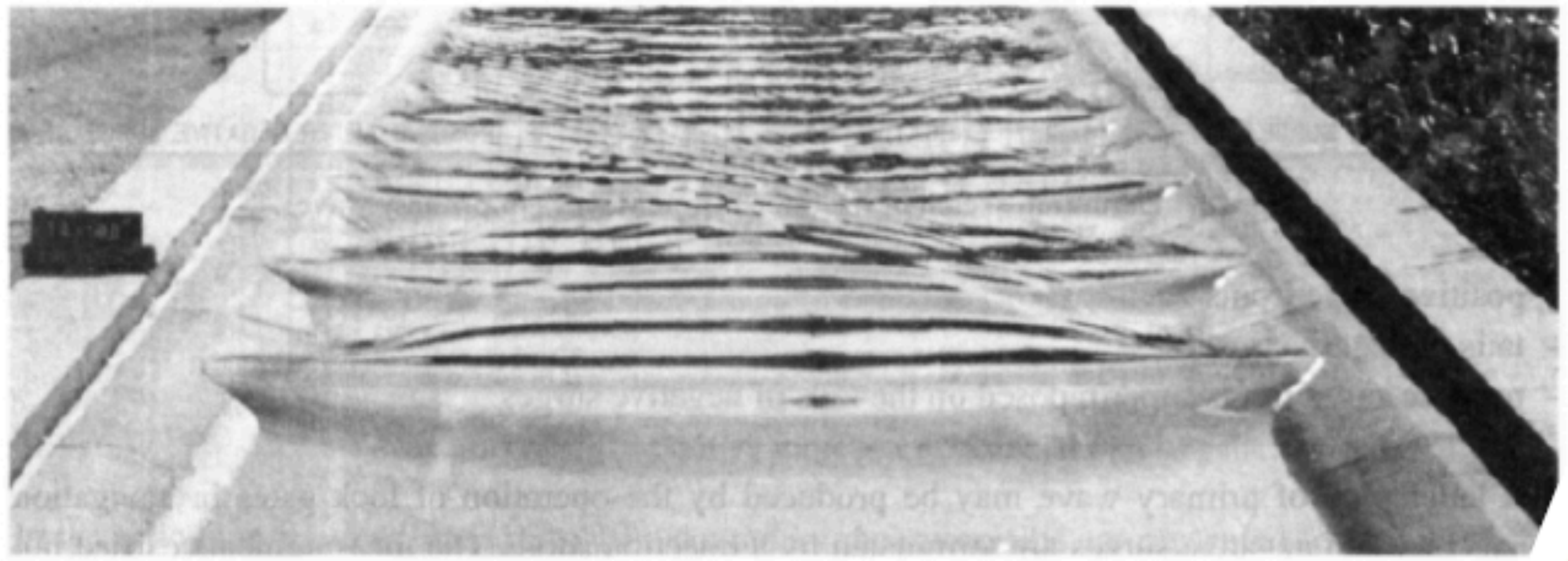
2D simulations (trapezoidal)

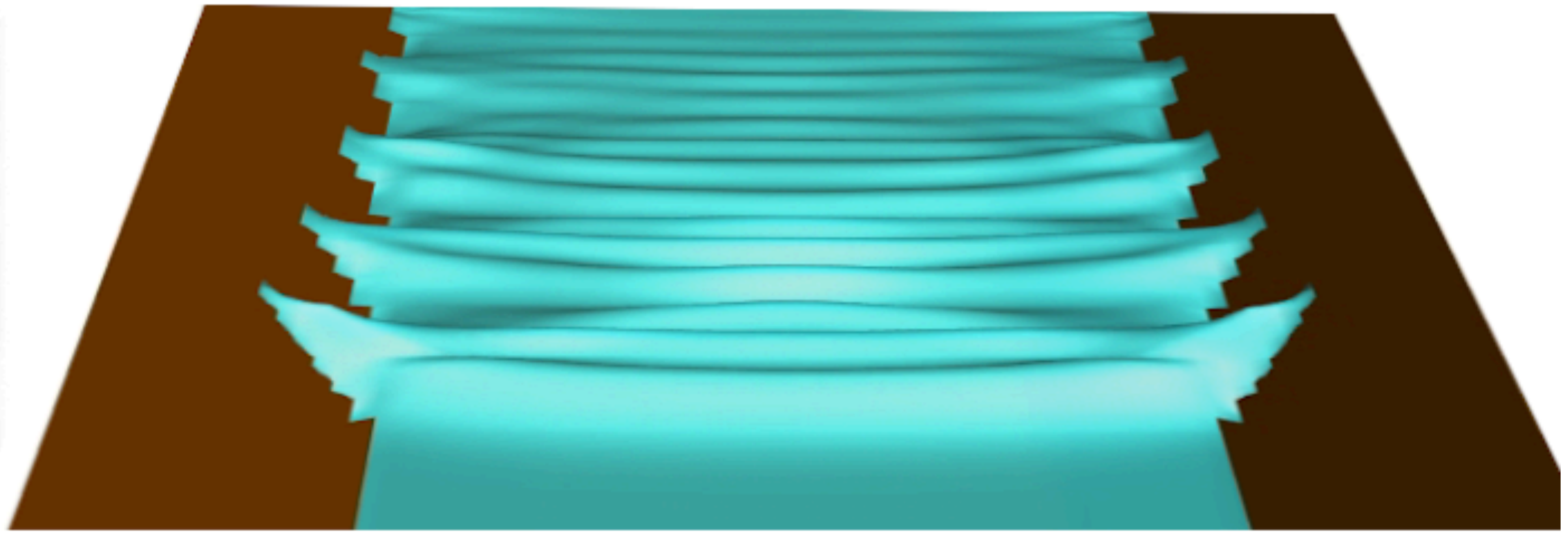
Treske’s experiments

$Fr = 1.10$



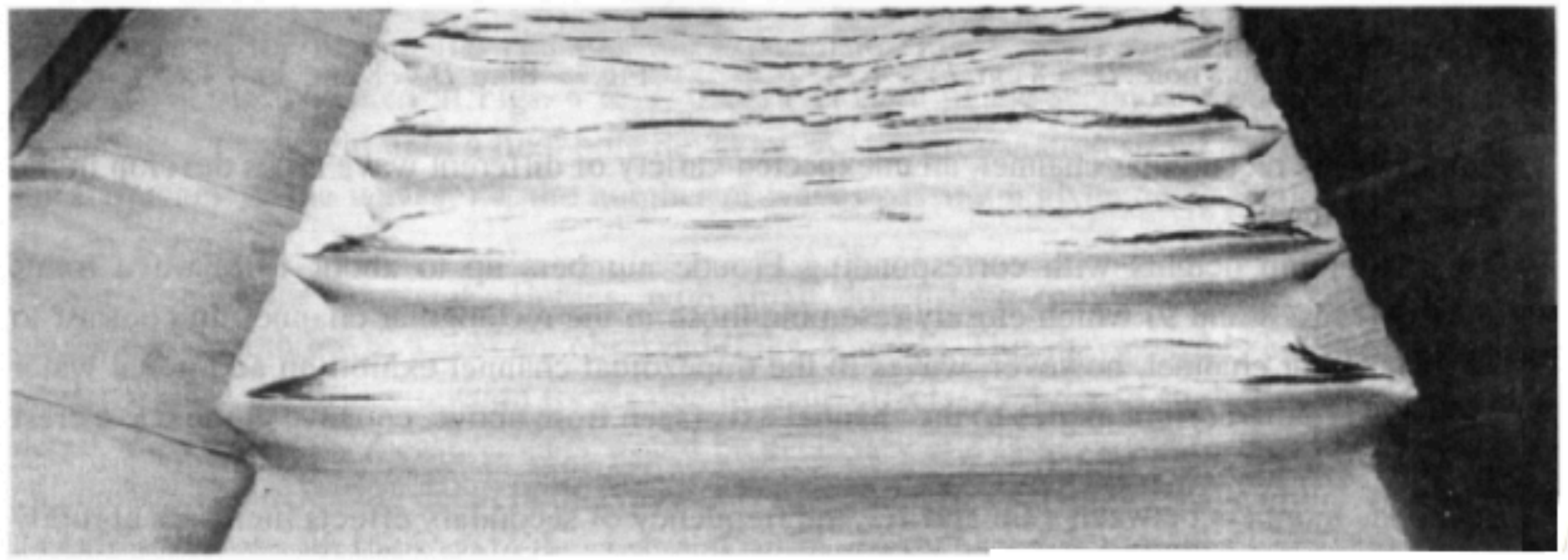
$Fr = 1.17$



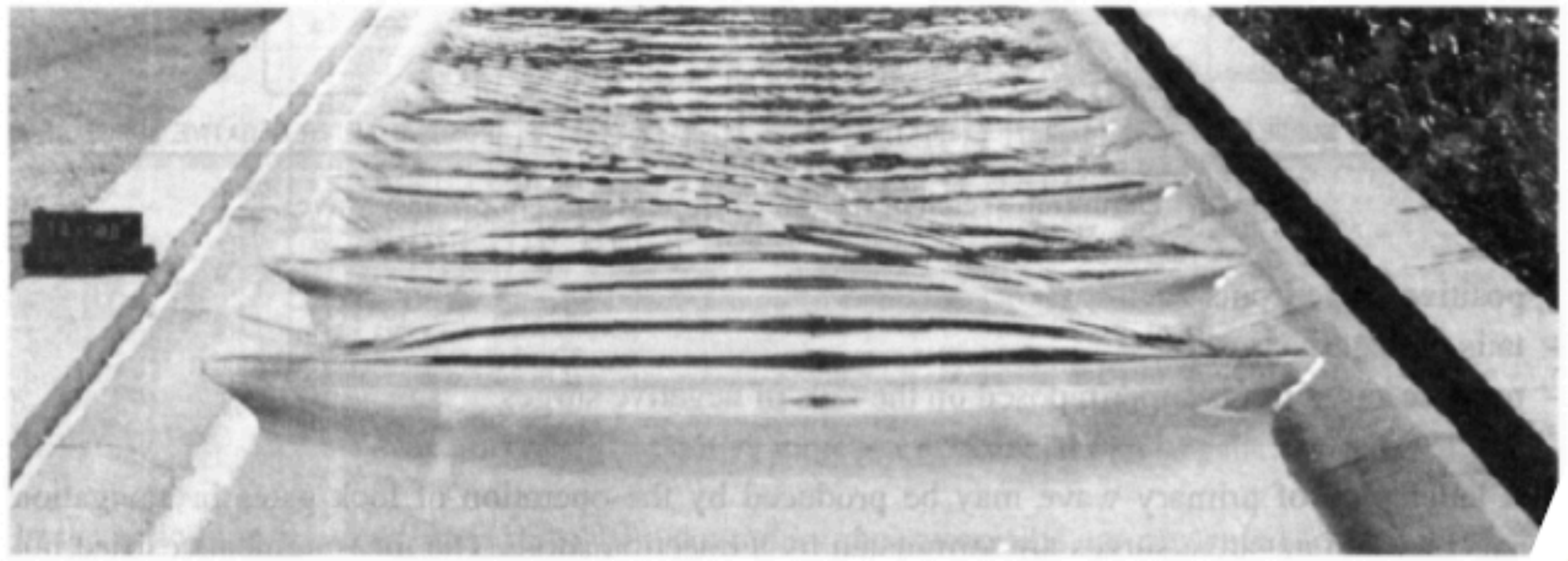


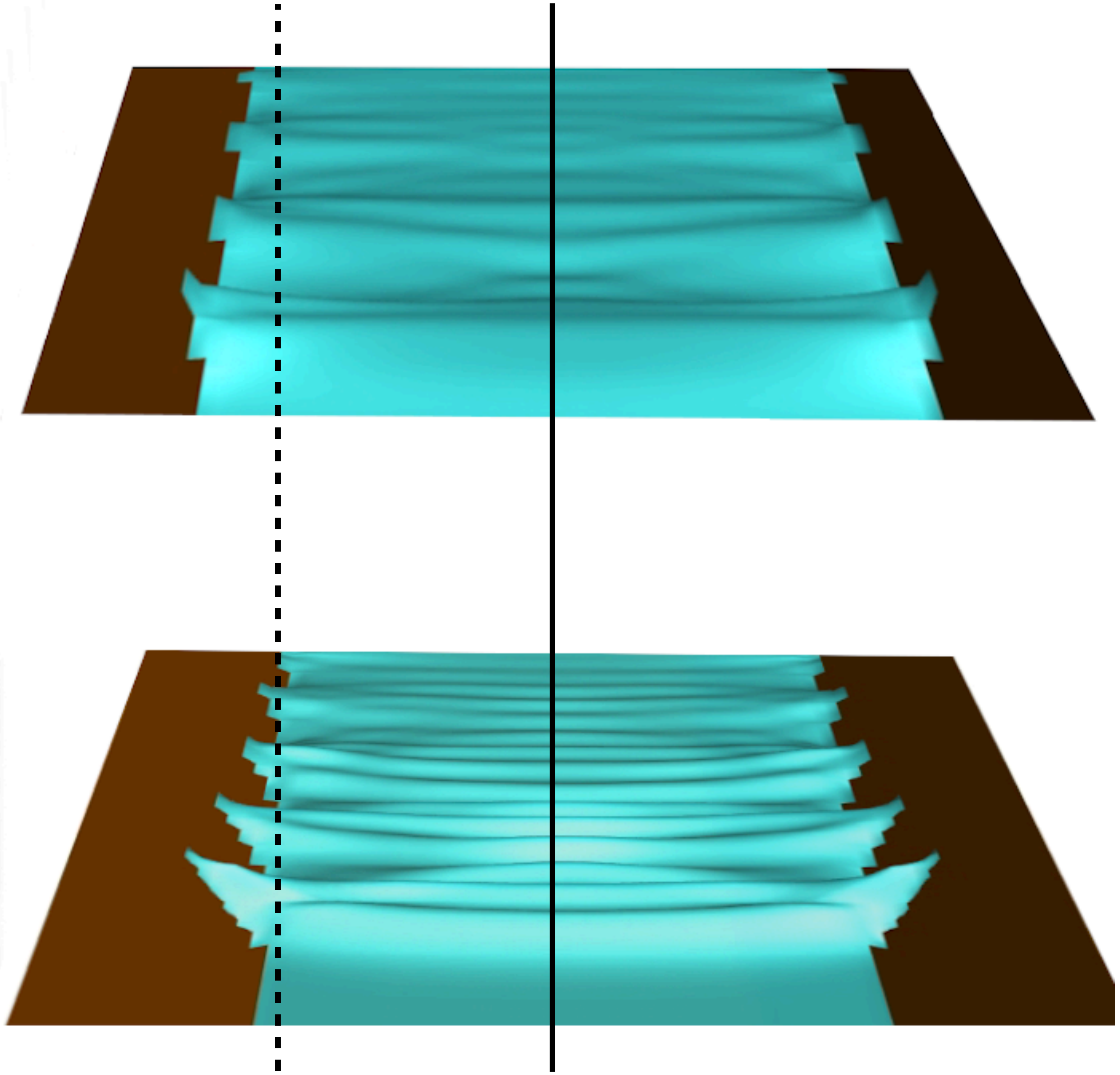
2D simulations (trapezoidal)	Treske’s experiments

$Fr = 1.10$



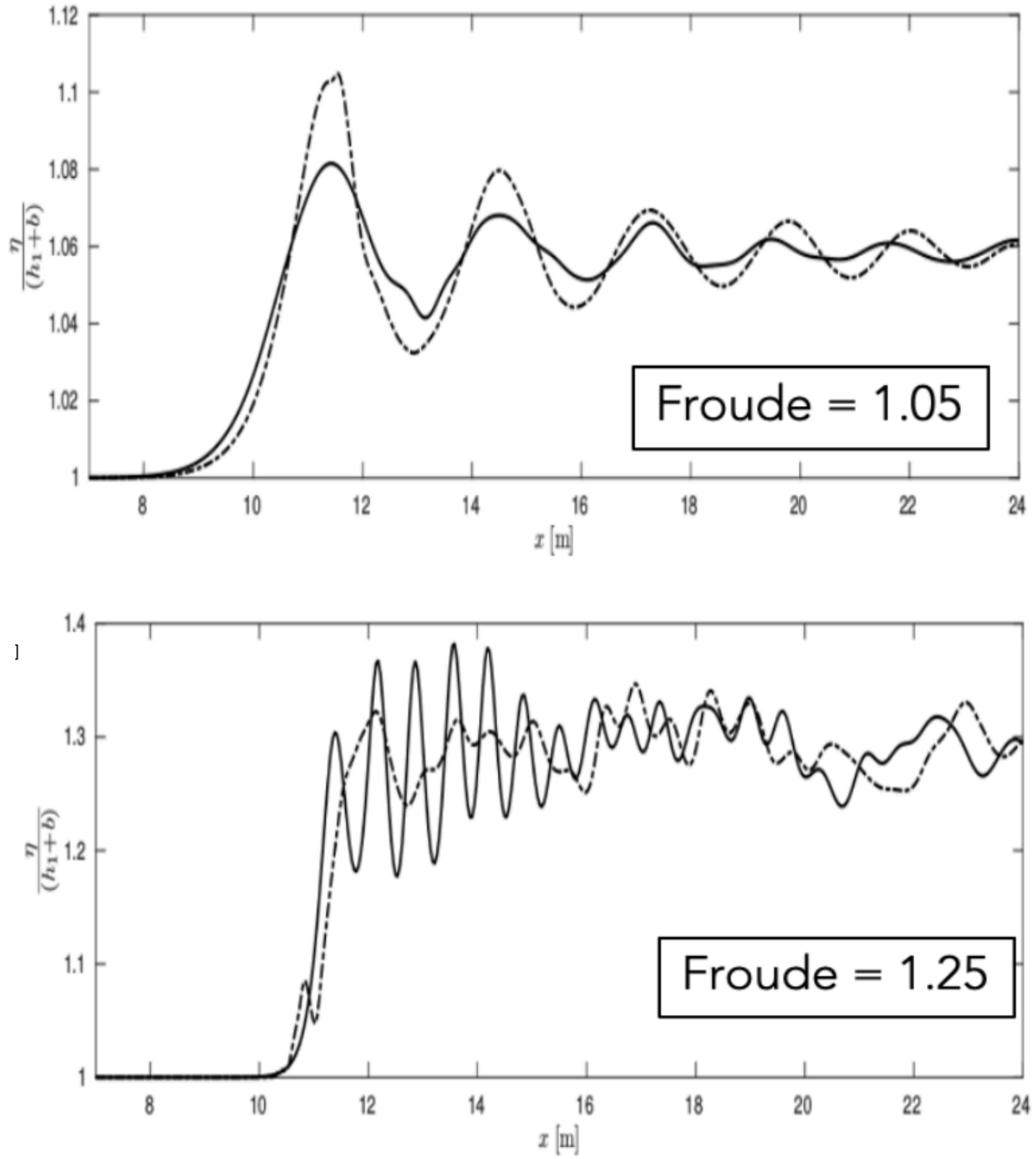
$Fr = 1.17$





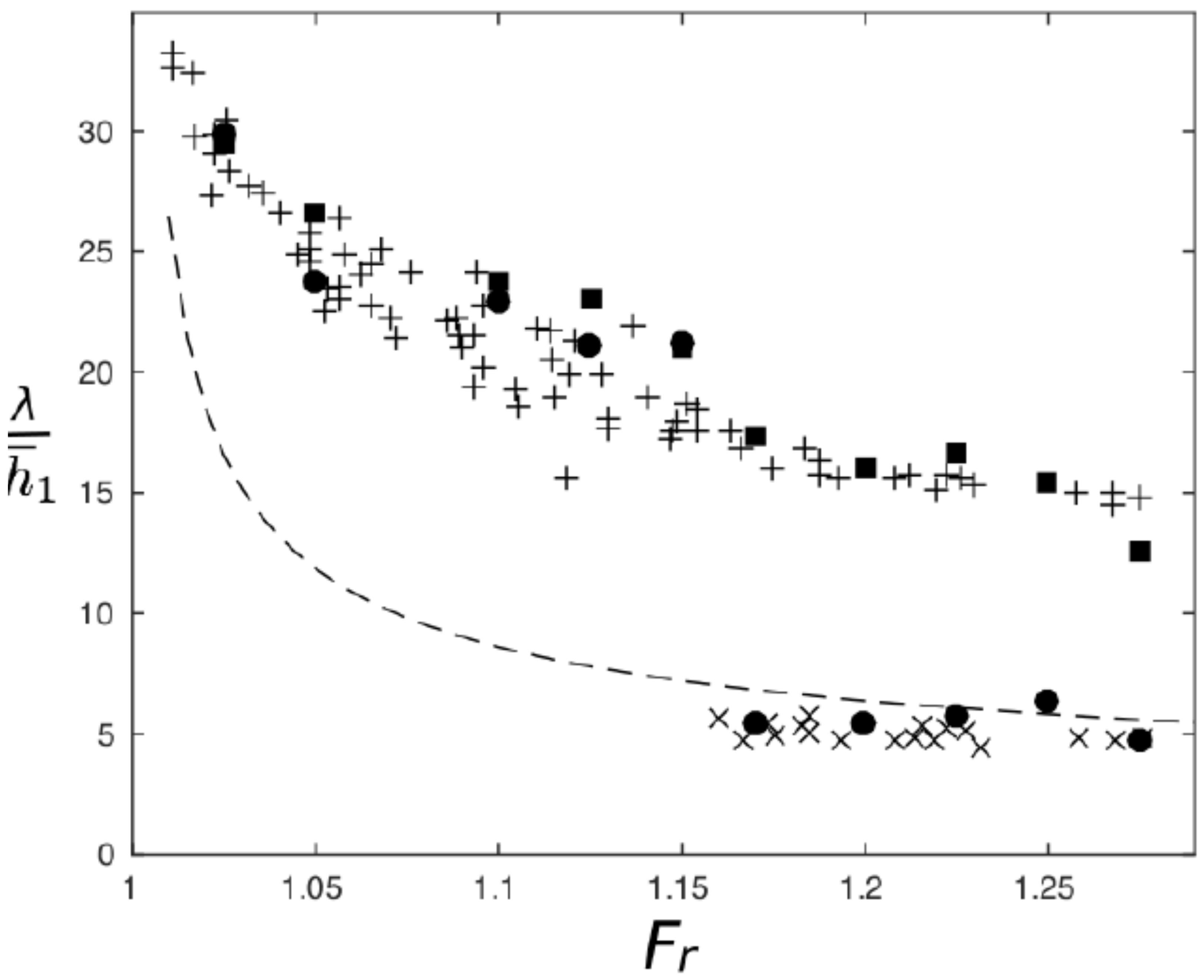
2D simulations (trapezoidal)

Treske's experiments



- - - - - - - - - - - banks

——————— axis



– – – – Lemoine theory (SGN)

■ SGN banks + Treske banks

● SGN axis x Treske axis

2D simulations (trapezoidal) | Dispersive-like waves

Several elements hint that it may be an hydrostatic process:

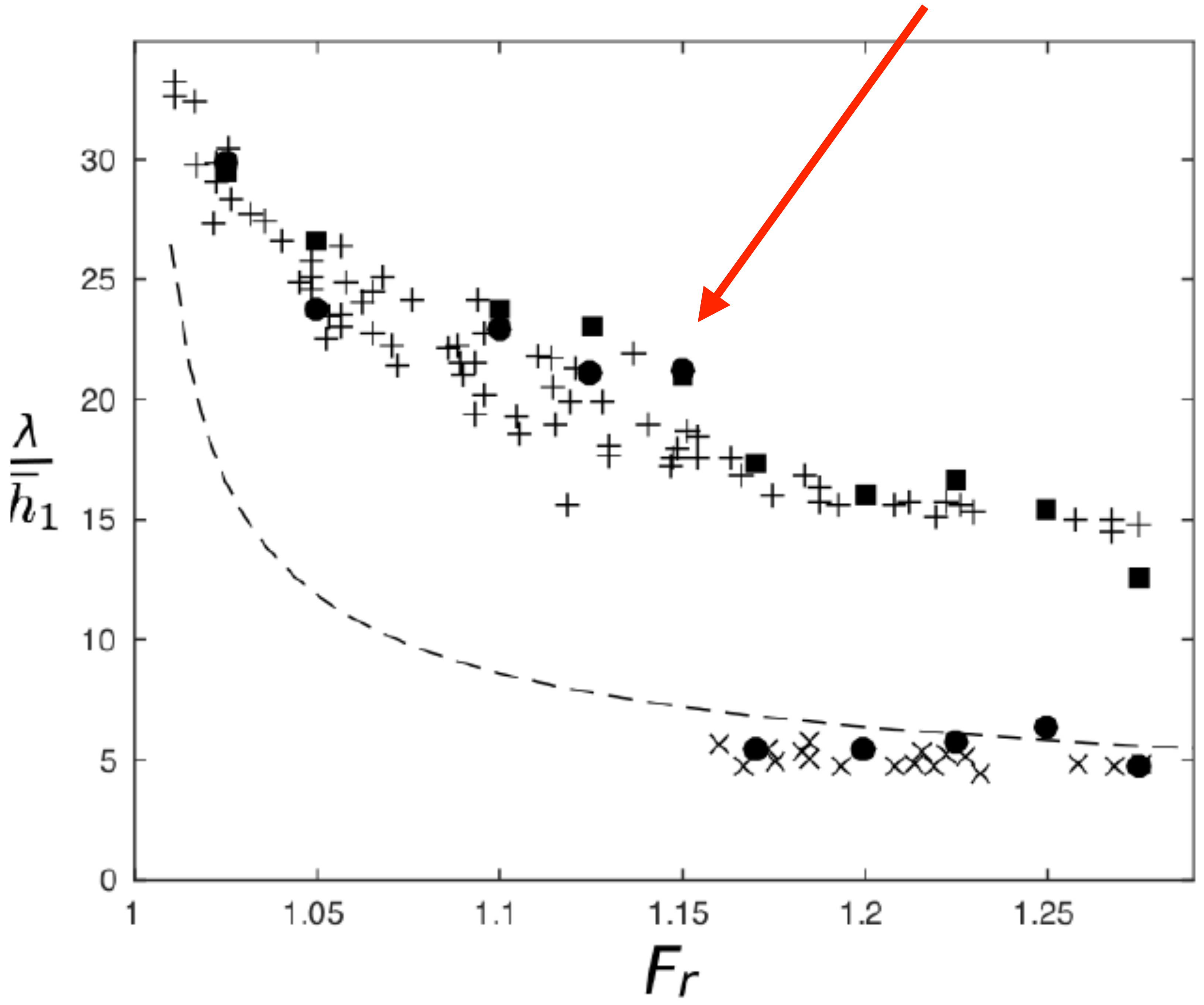
- It is predominant on the banks (very shallow limit)
- It involves long(er) waves
- Previous work on dispersion in wave propagation in heterogenous media:

Berezovski et al, Acta Mechanica 2011

Berezovski et al, Int.J.Solid and Structures 2013

Ketcheson & Quessada de Luna, SIAM Multiscale Mod. Simul., 2015

So, what are these ?



– – – – Lemoine theory (SGN)

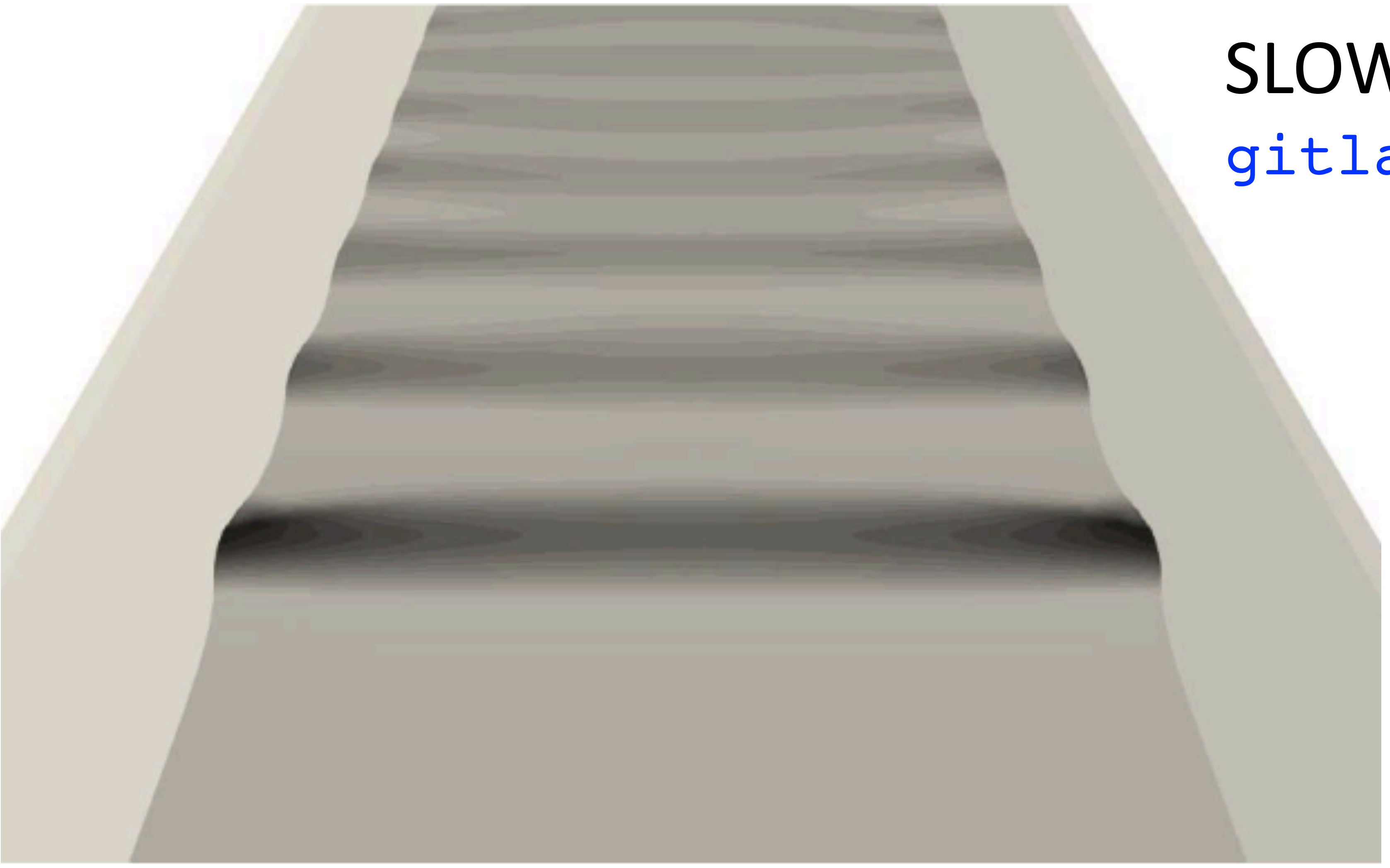
■ SGN banks + Treske banks

● SGN axis x Treske axis

2D simulations (trapezoidal)

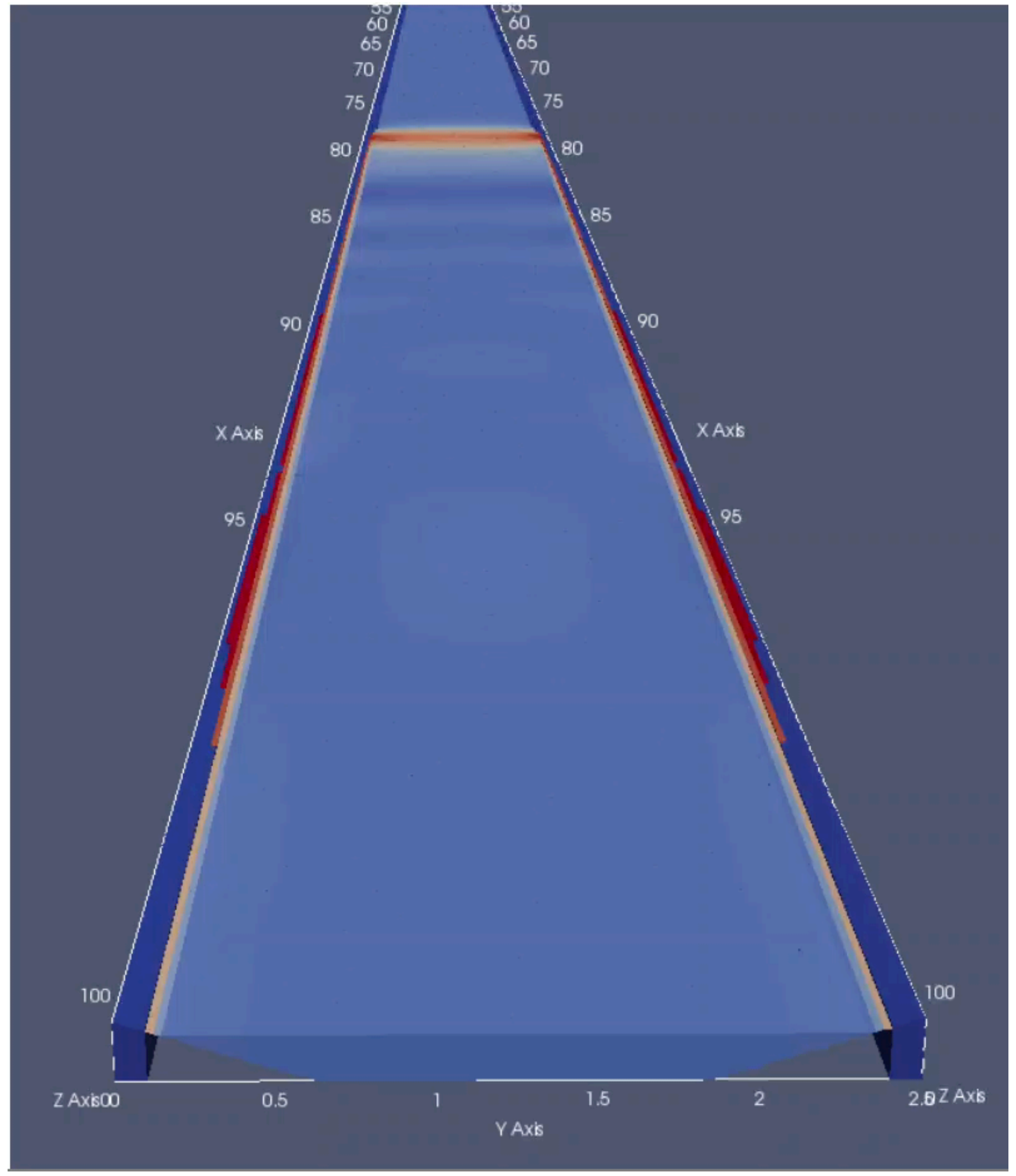
Dispersive-like waves

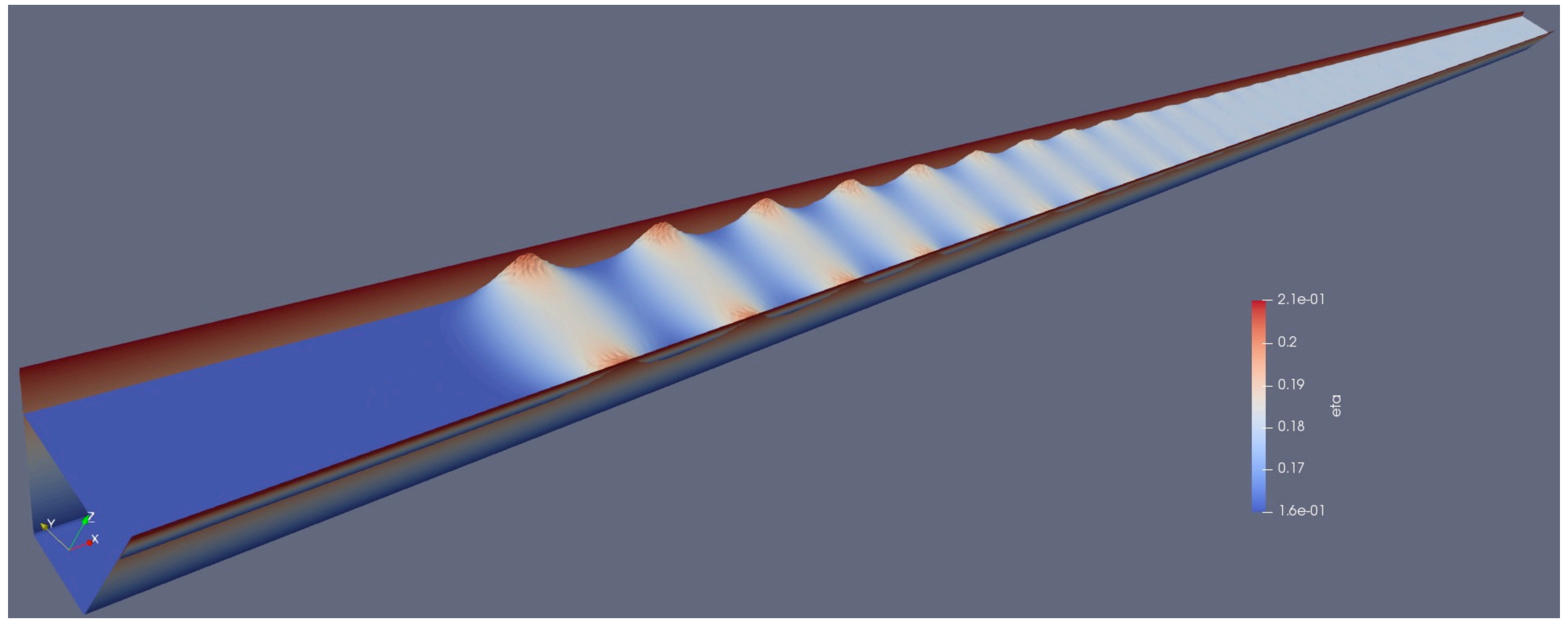
Shallow water simulations with different codes for Fr = 1.05



$Fr = 1.05$

SLOWS, developed by inria
`gitlab.inria.fr/slows-public-group/slows_public`



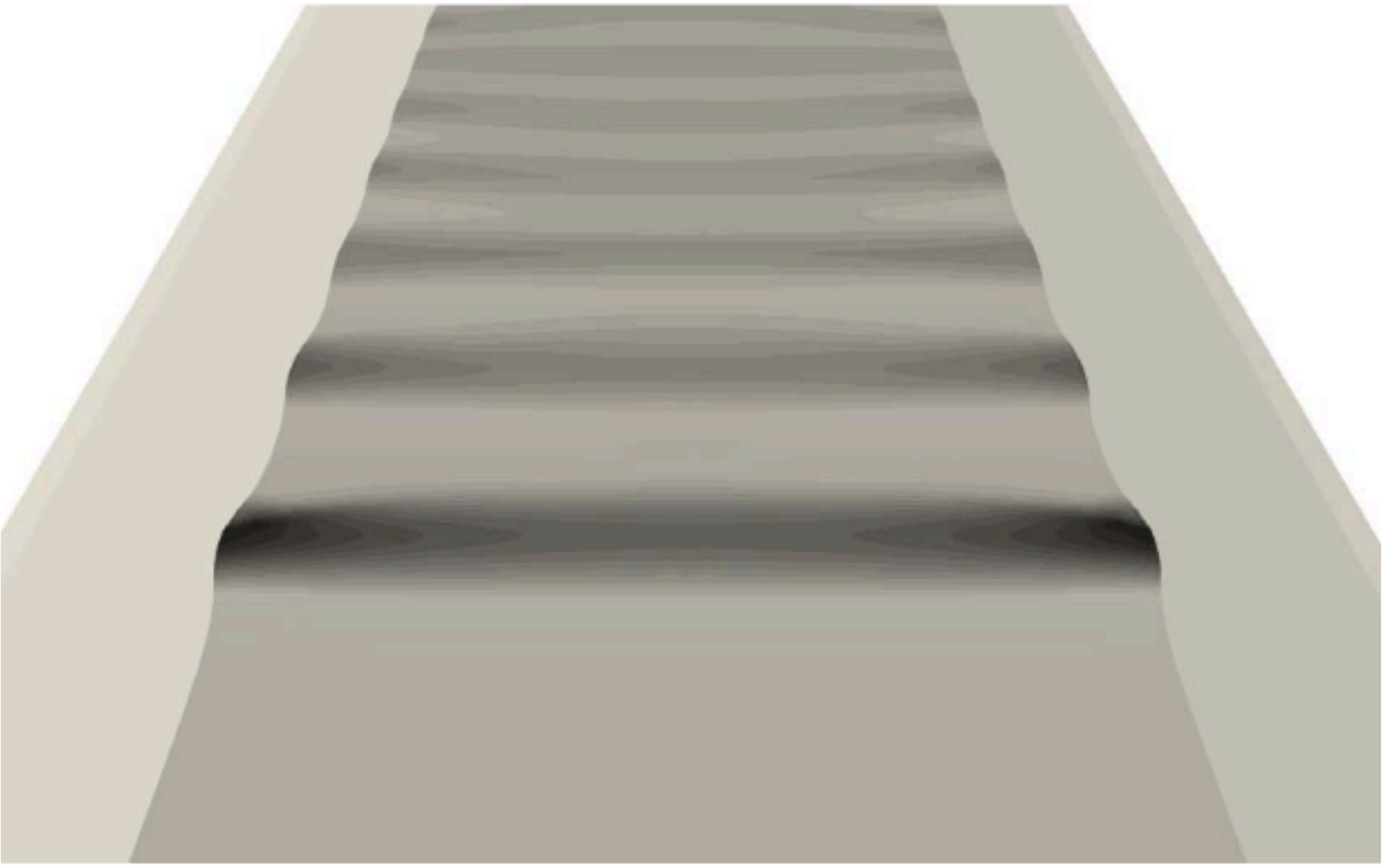


UHAINA, by the French Geophysical survey
www.brgm.fr

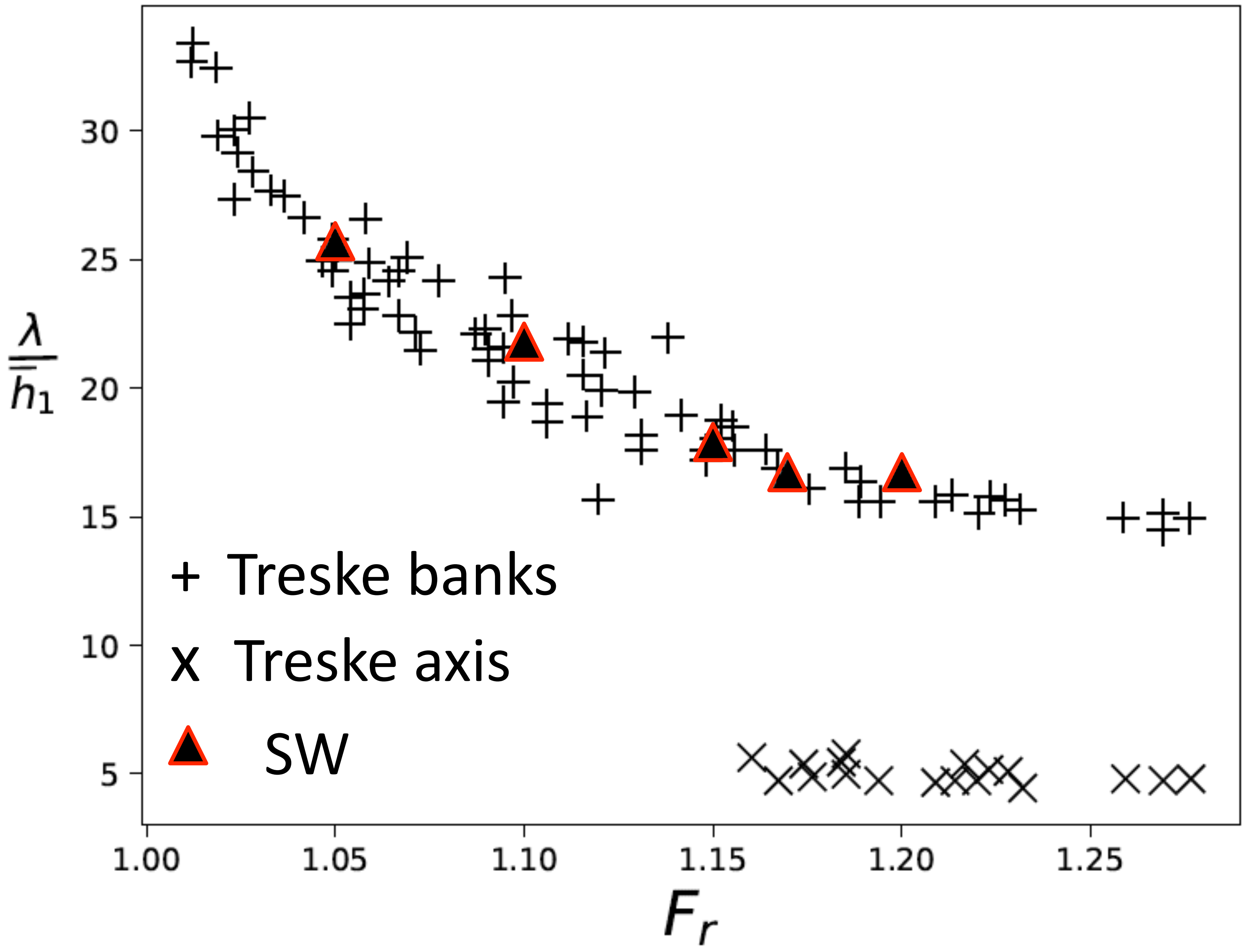
Eole-SW, developed by PRINCIPIA
www.principia-group.com

2D simulations (trapezoidal)

Dispersive-like waves



$Fr = 1.05$



Dispersive waves described by the hyperbolic shallow water eq.s !

With a discontinuous initial state !

A geometrical fully nonlinear dispersive model for (weakly) dispersive-like waves in channels

Result from **Chassagne et al**, JFM 2019 :

under appropriate scaling assumptions a 1D transverse averaged

wave equation from the 2D linearized SW equations with prismatic section

The above model predicts within quite some quantitative accuracy the wavelengths for the low Froude waves (cf later).

We describe here a fully nonlinear variant (**joint work with S. Gavryliuk**)

GGN model | Dispersive like

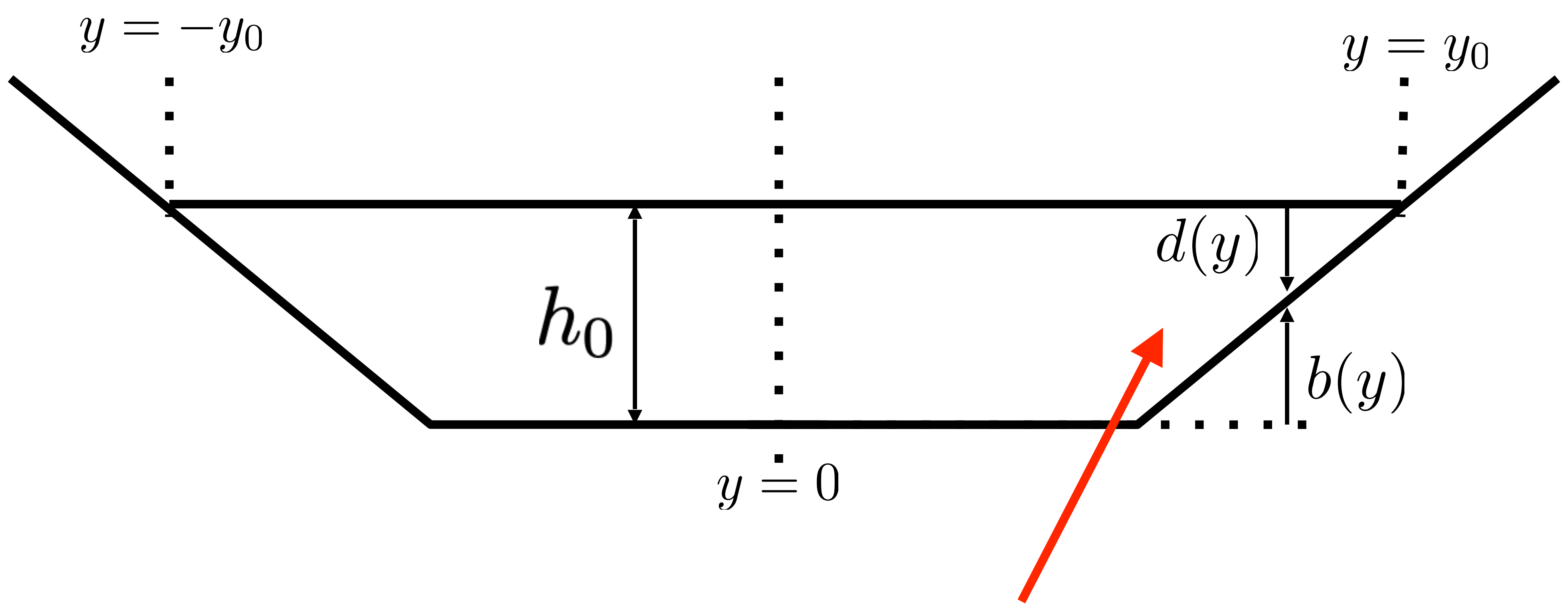
$b = b(y)$

prismatic channels

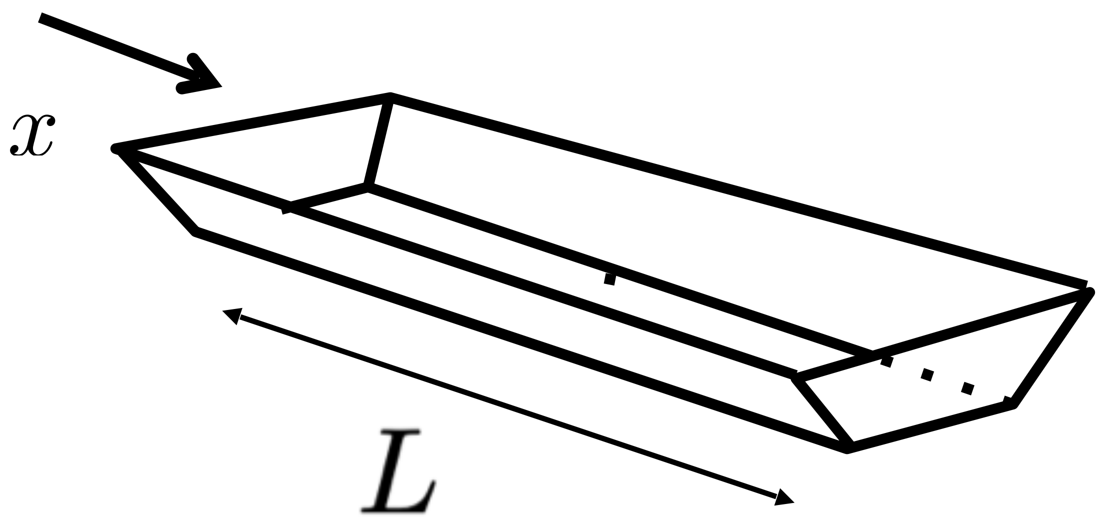
$$\partial_t h + \partial_x(hu) + \partial_y(hv) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + \partial_y(huv) = 0$$

$$\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2 + gh^2/2) = -gh\partial_y b(y)$$



$b = b(y)$



GGN model

Dispersive like

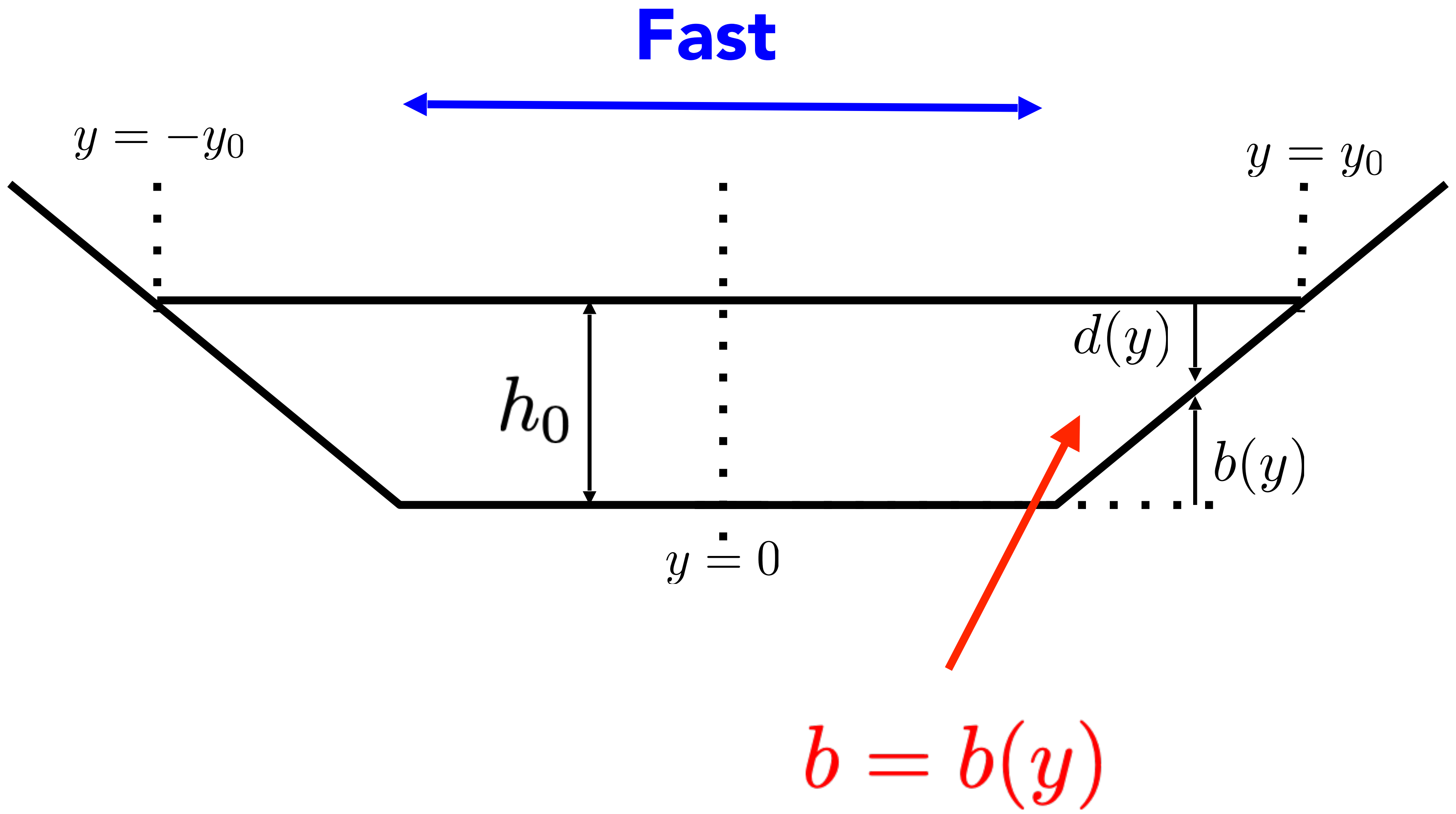
Smallness ansatz (all is along y here !!!)

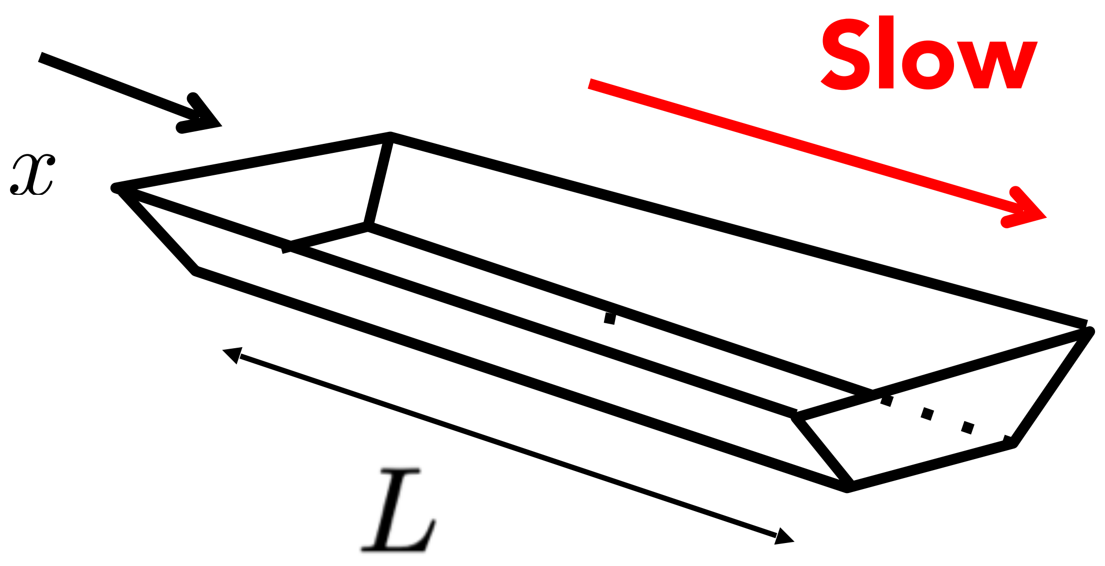
$$\boxed{\tau_y \ll \tau_x}$$

$$L = \tau_x \sqrt{g h_0}$$

$$\ell = \tau_y \sqrt{g h_0} \ll L$$

$$\longrightarrow \quad \boxed{\varepsilon = \ell / L \ll 1}$$





GGN model | Dispersive like

Smallness ansatz (all is along y here !!!)

$$b^* = bh_0 \ , \quad \zeta^* = \zeta h_0 \ , \quad d^* = h_0 d$$

$$x^* = xL \ \ , \quad y^* = y\ell \boxed{= \varepsilon y L} \ , \quad t^* = t\frac{L}{\sqrt{gh_0}}$$

$$u^* = \sqrt{gh_0} \ , \quad v^* \boxed{= \varepsilon\sqrt{gh_0}}$$

$$\boxed{\varepsilon = \ell/L \ll 1}$$

Ansatz on boundary conditions and geometry, and averaging

It is assumed that

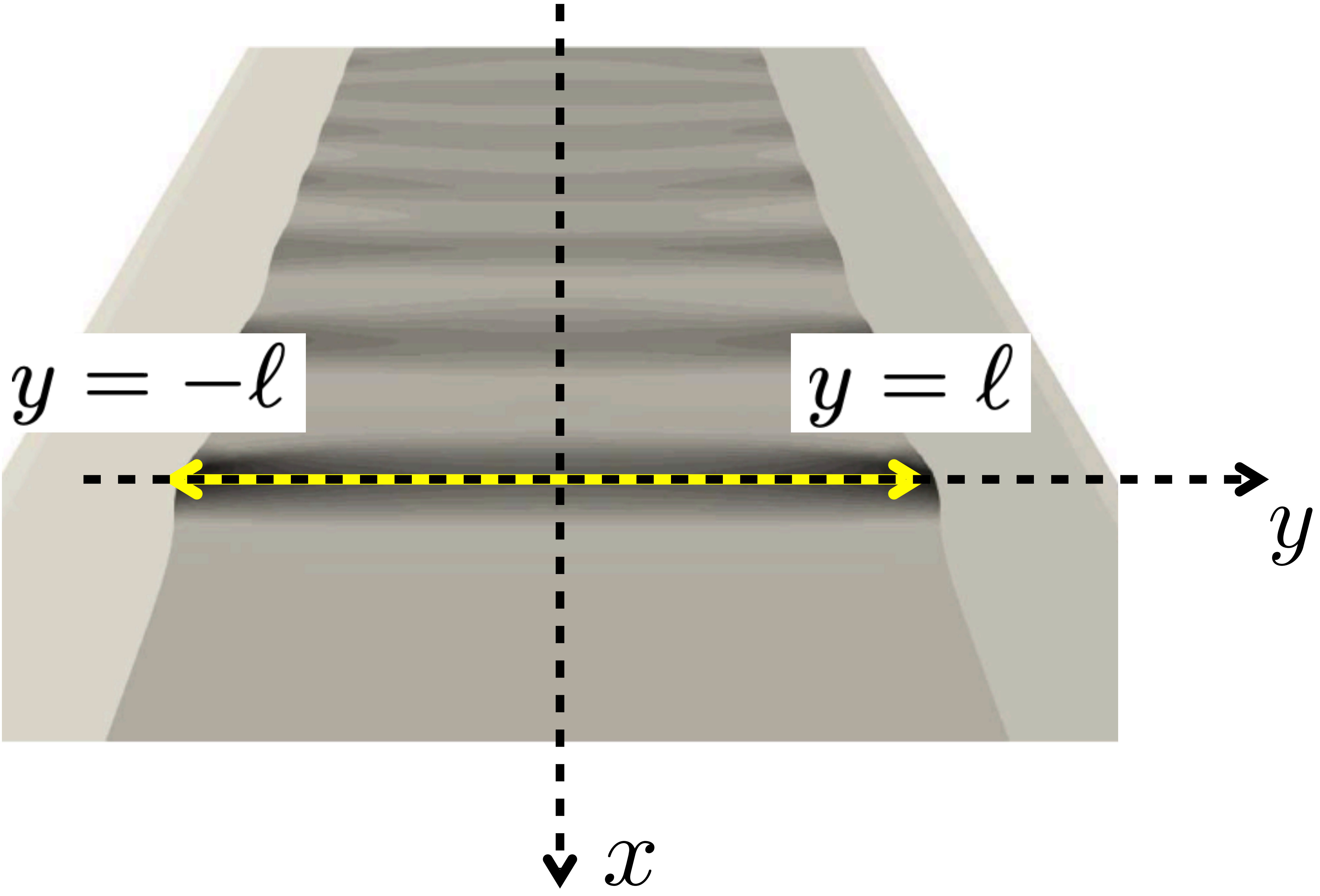
$$hv(t, x, y = \ell) = hv(t, x, y = -\ell)$$

valid for

- straight walls (v =0)

- periodicity

- banks (h=0)



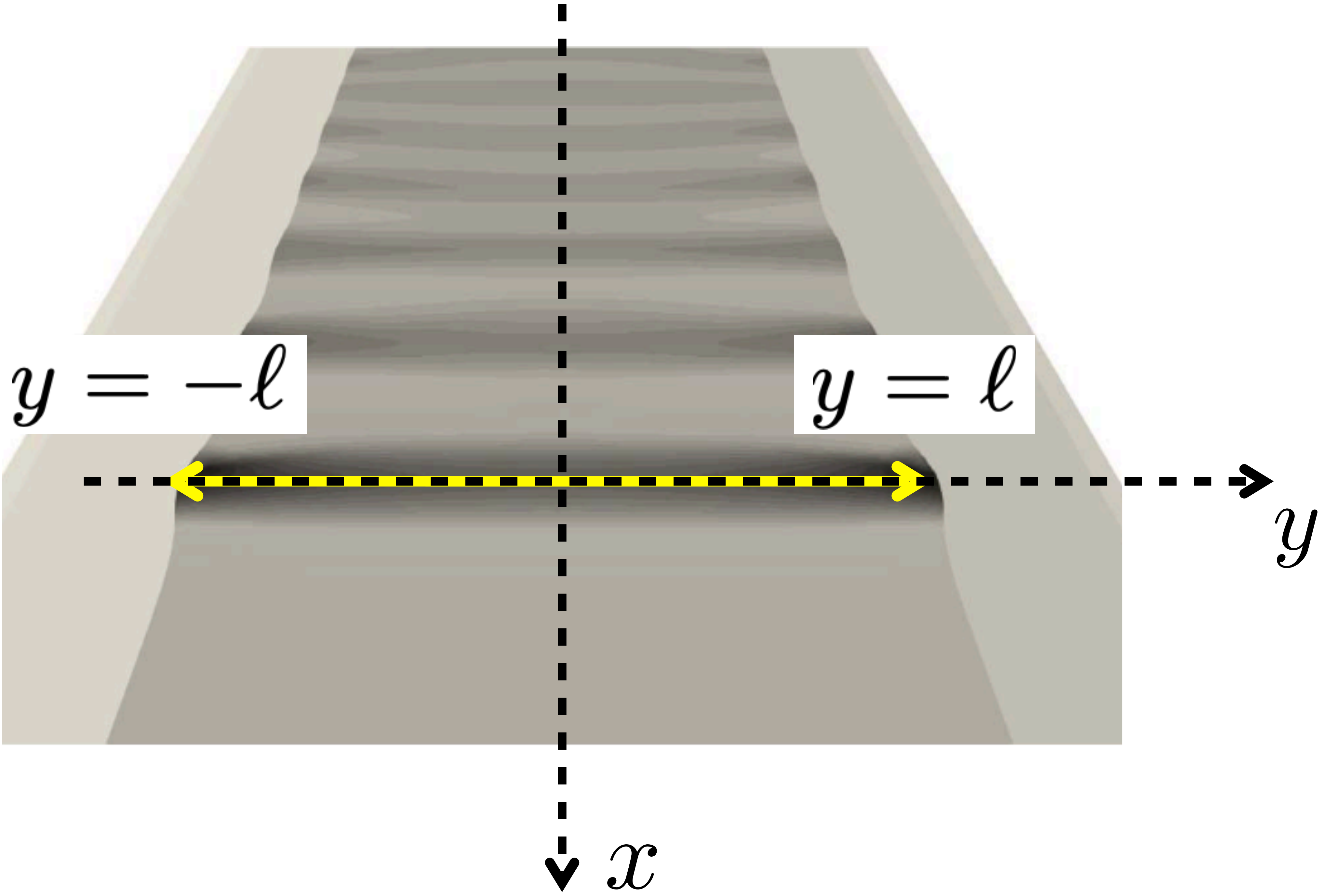
Ansatz on boundary conditions and geometry, and averaging

Transverse averaging

$$\overline{(\cdot)} = \frac{1}{2\ell}\int_{-\ell}^{\ell} (\cdot)(t,x,y)dy$$

Favre transverse averaging

$$\langle\cdot\rangle = \frac{1}{2\ell\bar{h}}\int_{-\ell}^{\ell} h(t,x,y)(\cdot)(t,x,y)dy$$



Ansatz on boundary conditions and geometry, and averaging

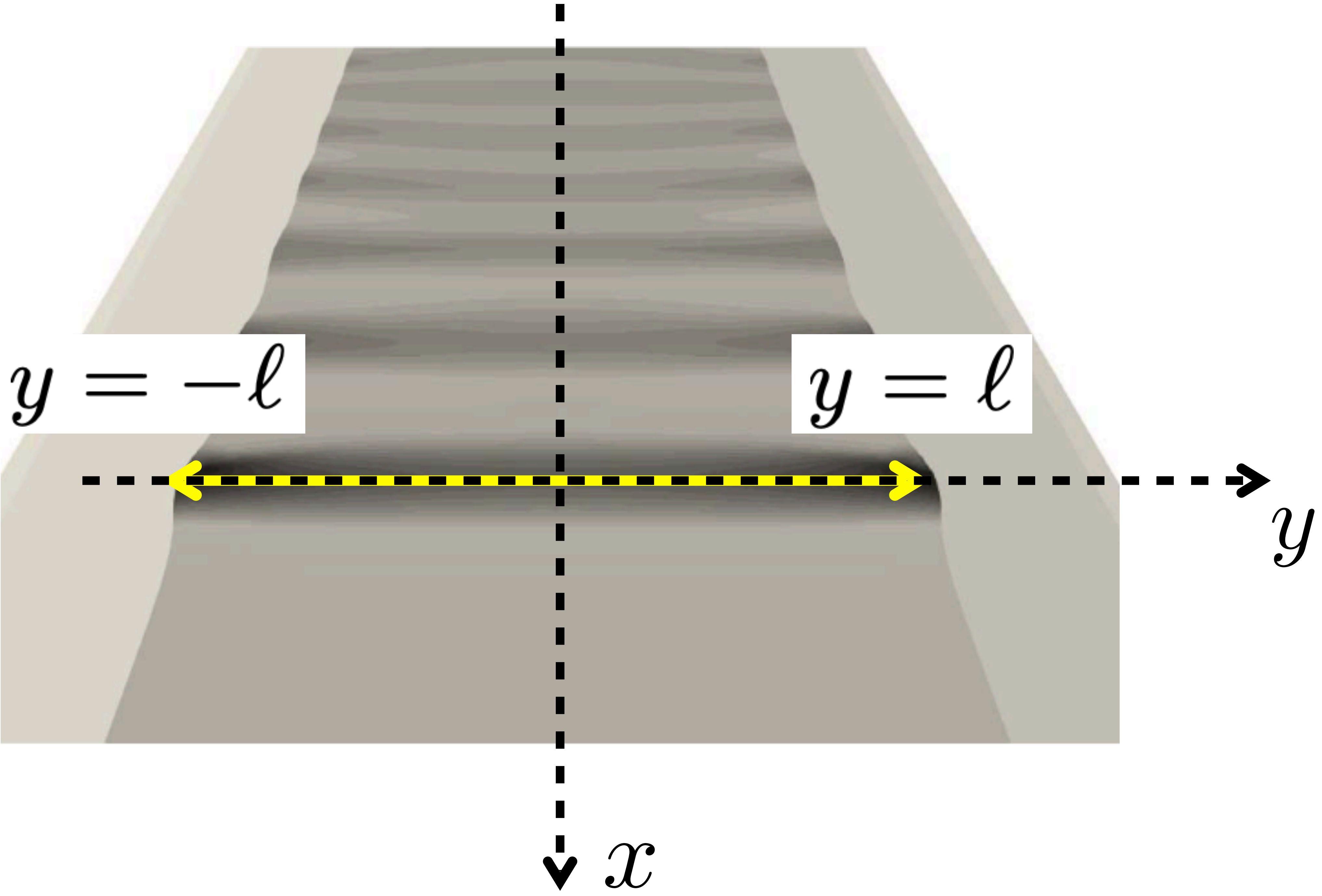
In general even for prismatic channels

$$\ell = \ell(x,t)$$

following e.g. **Peregrine** JFM 1968 , **Teng and Wu** JFM 1992

we assume $\ell =$ const

- exact for straight walls and
- exact for periodic $b(y)$
- for banks we accept a small geometrical approximation (cf. asymptotic analysis later)



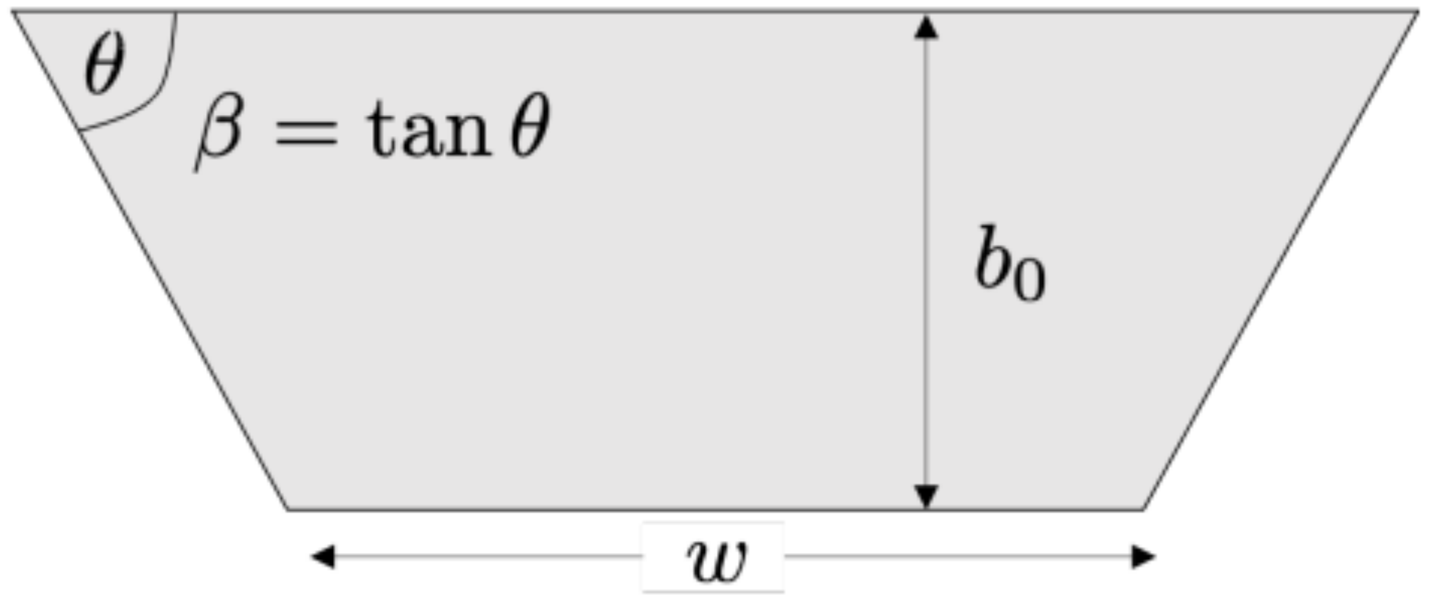
Ansatz on boundary conditions and geometry, and averaging

Additional assumption: wide channel wrt depth (rivers, human made channels)

$$\frac{b-\bar{b}}{\bar{h}} = \mathcal{O}(\varepsilon^{\gamma}),\ \gamma > 0$$

For trapezoidal sections equivalent to

$$\frac{b_0}{w\ \tan\theta} = \mathcal{O}(\epsilon^{\gamma})$$



Dimensionless eq.s and exact averages

$$h_t + (hu)_x + (hv)_y = 0,$$
$$u_t + uu_x + vu_y + (h+b)_x = 0,$$
$$\varepsilon^2(v_t + uv_x + vv_y) + (h+b)_y = 0,$$

lead to

$$\overline{h}_t + (\overline{h}\langle u\rangle)_x = 0,$$
$$(\overline{h}\langle u\rangle)_t + \boxed{(\overline{hu^2} + \frac{1}{2}\overline{h^2})_x} = 0,$$
$$h + b = \overline{h} + \overline{b} - \varepsilon^2\left\{\int_{-1}^{y}\frac{Dv}{Dt}ds - \overline{\int_{-1}^{y}\frac{Dv}{Dt}ds}\right\}$$

Velocity profiles 1

Assuming no vorticity

$$\begin{cases} \dfrac{D}{Dt}\left(\dfrac{\omega}{h}\right) = 0 \\ \omega_{t=0} = 0 \end{cases}$$

➡

$$\partial_y u - \varepsilon^2 \partial_x v = 0$$

Integrating in y we get:

$$u = \langle u \rangle + \varepsilon^2 \left\{ \int_{-1}^{y} v_x ds - \langle \int_{-1}^{y} v_x ds \rangle \right\}$$

Leading to

$$\overline{hu^2} = \overline{h} \langle u \rangle^2 + \mathcal{O}(\varepsilon^4)$$

Velocity profiles 2

$$h + b = \overline{h} + \overline{b} - \varepsilon^2 \left\{ \int_{-1}^{y} \frac{Dv}{Dt} ds - \overline{\int_{-1}^{y} \frac{Dv}{Dt} ds} \right\}$$ and $$\overline{h}_t + (\overline{h}\langle u \rangle)_x = 0$$

Combine the above and assume no net transverse mass flux $\int_{-1}^{1} hv\, dy = 0$

➡ $$v = (\overline{S} - S)\langle u \rangle_x + \mathcal{O}(\varepsilon^2), \quad S = \int_{-1}^{y} (\overline{b} - b(y'))dy'$$

Velocity profiles 2

$$h + b = \overline{h} + \overline{b} - \varepsilon^2 \left\{ \int_{-1}^{y} \frac{Dv}{Dt} ds - \overline{\int_{-1}^{y} \frac{Dv}{Dt} ds} \right\}$$ and $$\overline{h}_t + (\overline{h}\langle u \rangle)_x = 0$$

Combine the above and assume no net transverse mass flux $\int_{-1}^{1} hv\,dy = 0$

➡ $$v = (\overline{S} - S)\langle u \rangle_x + \mathcal{O}(\varepsilon^2), \quad S = \int_{-1}^{y} (\overline{b} - b(y'))dy'$$

REMARK For symmetric channels $\overline{S} = 0$ which allows to show $v(-\ell) = v(\ell) = \mathcal{O}(\varepsilon^2)$

In this case geometrical (on ℓ) and BCs errors are bounded by $\mathcal{O}(\varepsilon^2)$

Dispersive-like behaviour

We can now compute

$$\overline{h^2} = \overline{h}^2 + 2\varepsilon^2 \overline{\frac{d\sigma}{dy}\mathsf{M}} + \text{const} + \mathcal{O}(\varepsilon^4)$$

where $\sigma(y) = \overline{S} - S(y)$ and (setting $\tau = 1/\overline{h}$ and $\dot{\tau} = \partial_t \tau + \langle u \rangle \partial_x \tau$)

$$\mathsf{M} = \int_{-1}^{y} \left(\frac{\sigma(y')}{1 - \tau \dfrac{d\sigma(y')}{dy'}} \ddot{\tau} + \frac{\sigma(y') \dfrac{d\sigma(y')}{dy'}}{\left(1 - \tau \dfrac{d\sigma(y')}{dy}\right)^2} \dot{\tau}^2 \right) dy' + \frac{\sigma^2(y)}{2\left(1 - \tau \dfrac{d\sigma(y')}{dy}\right)^2} \dot{\tau}^2$$

Lagrangian structure

For symmetric channels

$$\overline{\frac{d\sigma}{dy}\mathsf{M}} = -\frac{\delta \mathcal{L}}{\delta \tau} := -\left(\partial_\tau \mathcal{L} - \frac{D}{Dt}(\partial_{\dot{\tau}}\mathcal{L})\right)$$

with

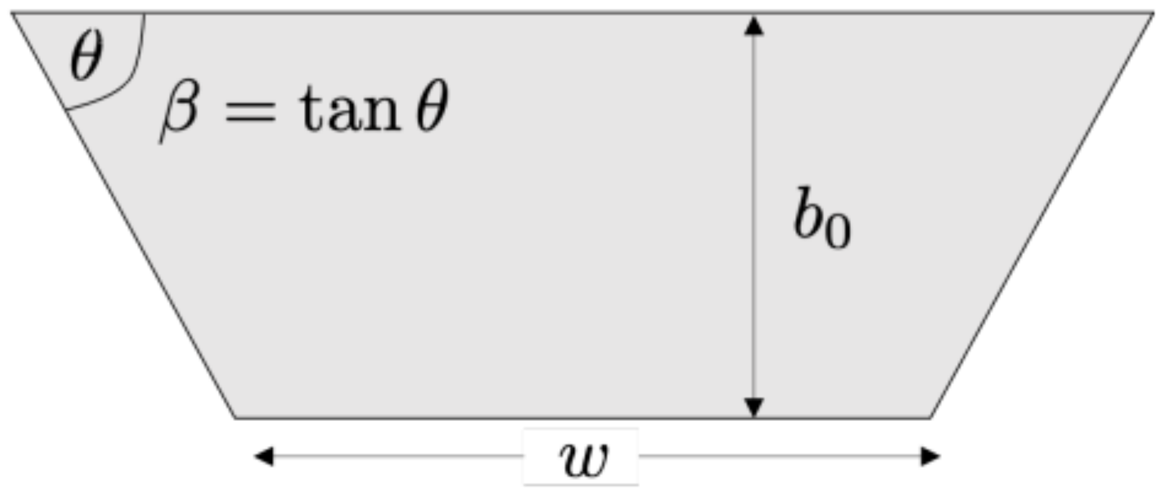
$$\mathcal{L} = \overline{\frac{d\sigma}{dy}\mathsf{N}}\,, \quad \mathsf{N} = \frac{\dot{\tau}^2}{2}\int_{-1}^{y} \frac{\sigma(y')}{1-\tau\dfrac{d\sigma(y')}{dy'}}dy'$$

and with the abuse of notation

$$\dot{f} = \frac{Df}{Dt} = \partial_t f + \langle u \rangle \partial_x f$$

Lagrangian structure: simplified model

$$\frac{b-\bar{b}}{\bar{h}} = \mathcal{O}(\varepsilon^{\gamma}),\ \gamma > 0$$



For symmetric wide/shallow channels

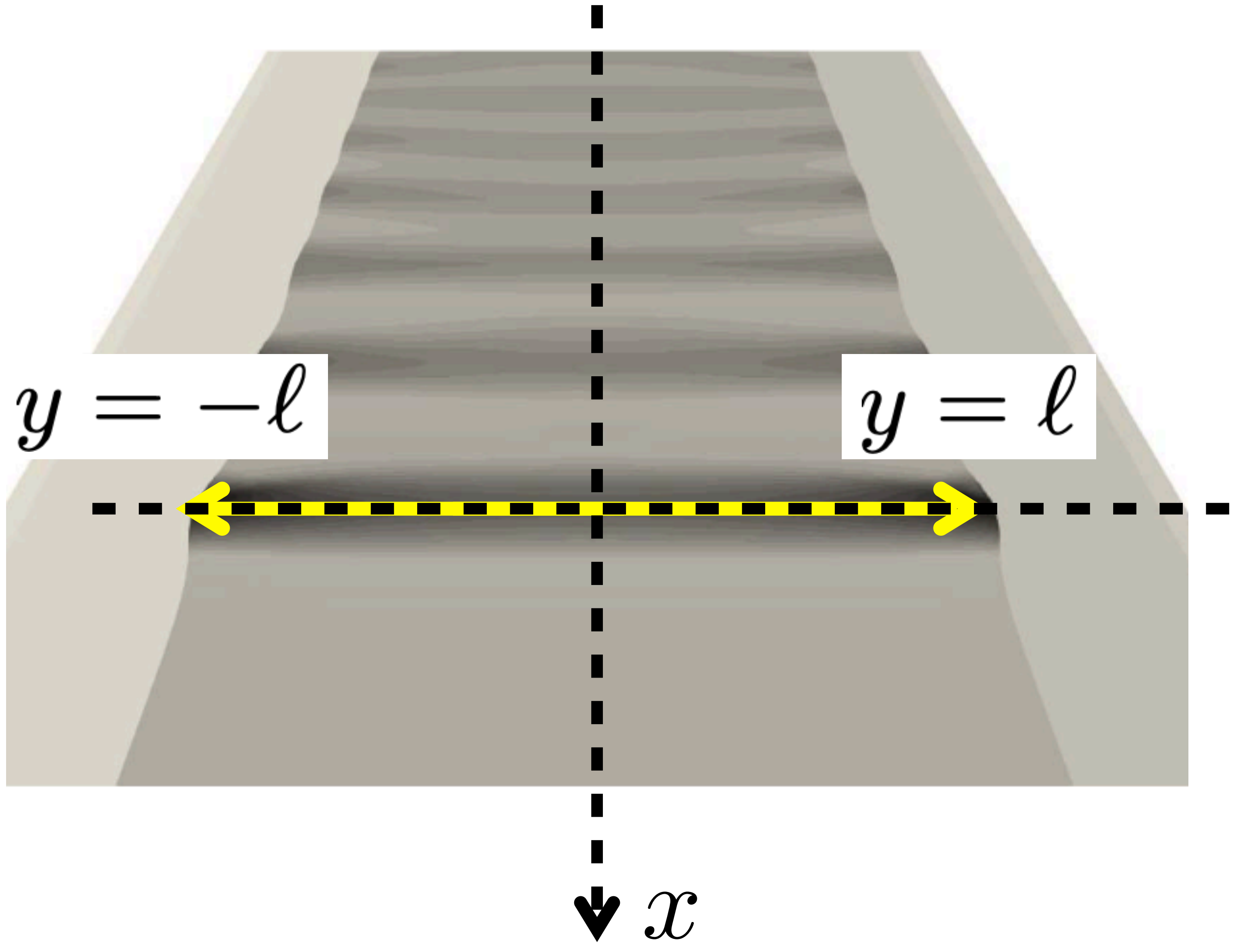
$$\mathcal{L} = -\overline{S^2}\frac{\dot{\tau}^2}{2} + \mathcal{O}(\varepsilon^{\gamma})$$

with $S = \int_{-1}^{y} (\bar{b} - b(y'))dy'$ and now

$$\overline{h^2} = -\frac{\delta L}{\delta \tau} + \mathcal{O}(\varepsilon^{2+2\gamma})\ , \quad L = \frac{1}{\tau} + \mathcal{L}$$

(simplified) Geometrical Green-Naghdi equations

Neglecting small terms we obtain the system of equations for tansverse averaged depth and velocity (averages removed for simplicity)



$$S(y) = \int_{-\ell}^{y} (\bar{b} - b(s))ds$$

$$h_t + (hu)_x = 0$$
$$(hu)_t + (hu^2 + gh^2/2 + p)_x = 0$$
$$E(h,u)_t + F(h,u)_x = 0$$

where

$$p = -\overline{S^2}\ddot{\tau}\ , \quad E = g\frac{h^2}{2} + h\frac{u^2}{2} + \overline{S^2}h\frac{\dot{\tau}^2}{2}\ , \quad F = huE + u(g\frac{h^2}{2} + p)$$

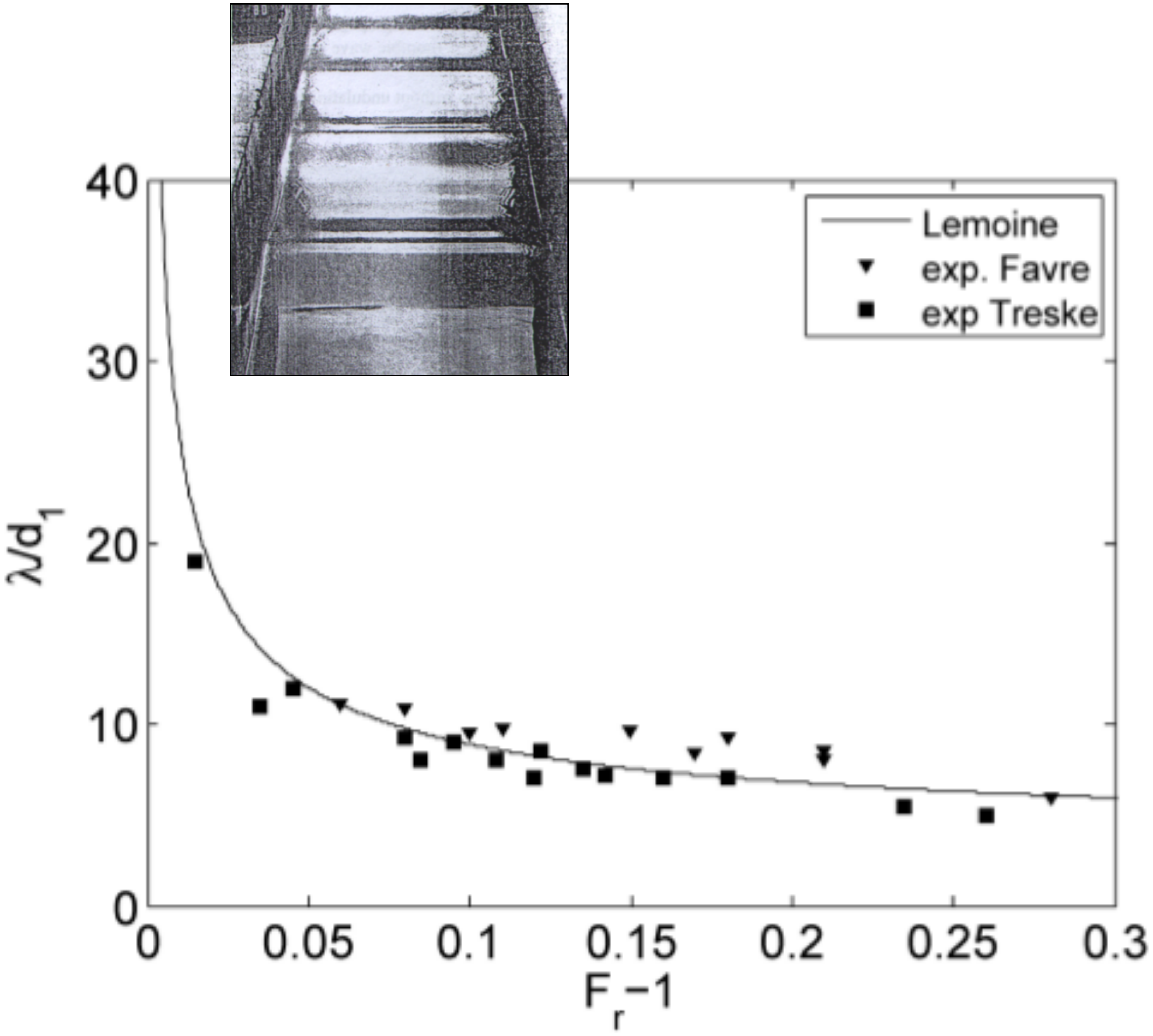
Model properties

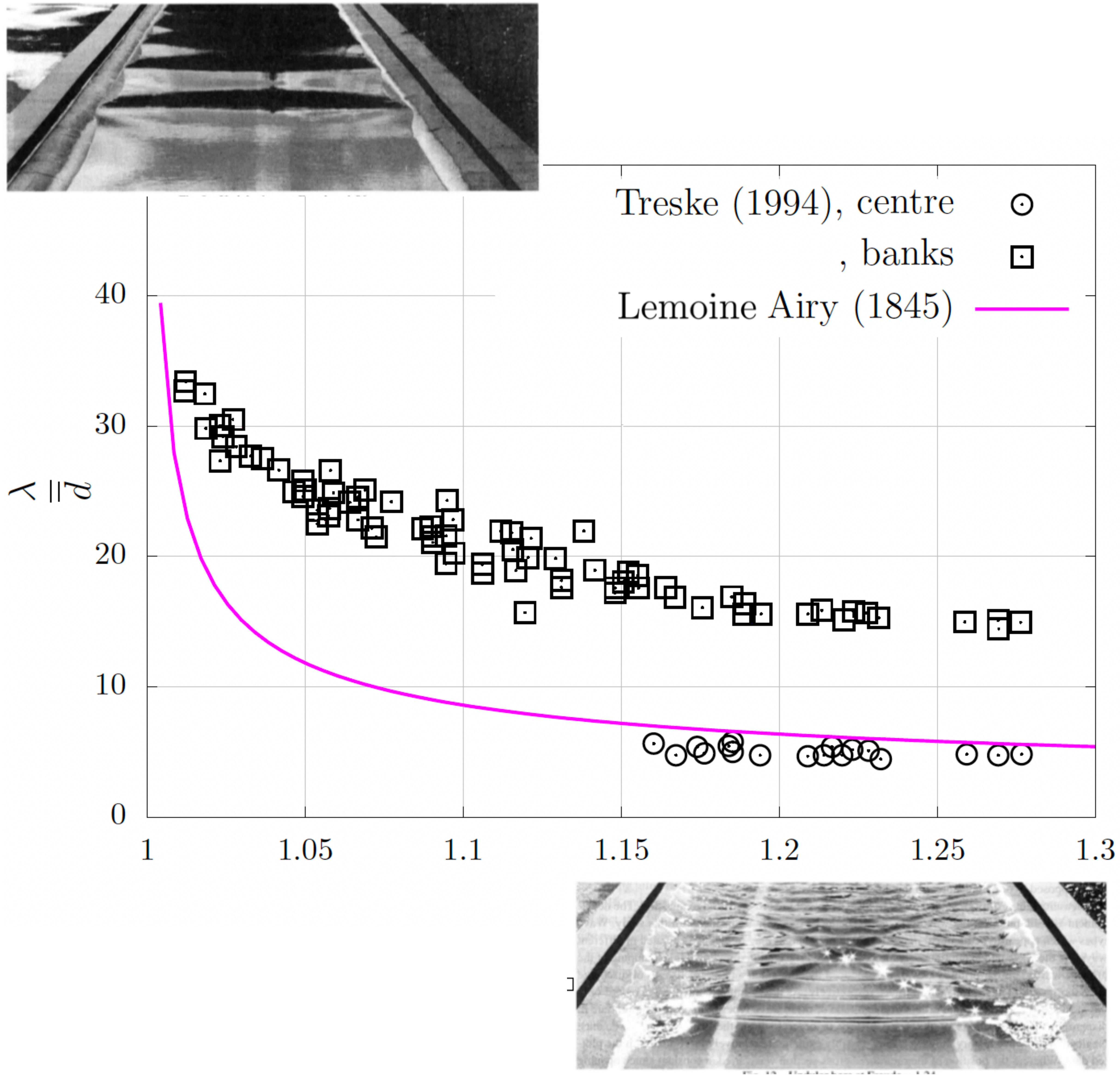
- Galilean invariance
- Variational (Lagrangian) formulation and the energy conservation law
- Exhibits several families of travelling wave solutions (solitons, periodic, composite)
- Physically relevant dispersion relation (cf. next)
- Consistent with all relevant BCs, and hypotheses (for practical interest: banks and walls)

More in S. Gavrilyuk and M. Ricchiuto, A geometrical Green-Naghdi type system for dispersive-like waves in prismatic channels, https://arxiv.org/abs/2408.08625, in revision on Journal of Fluid Mechanics

GGN model | Dispersive properties

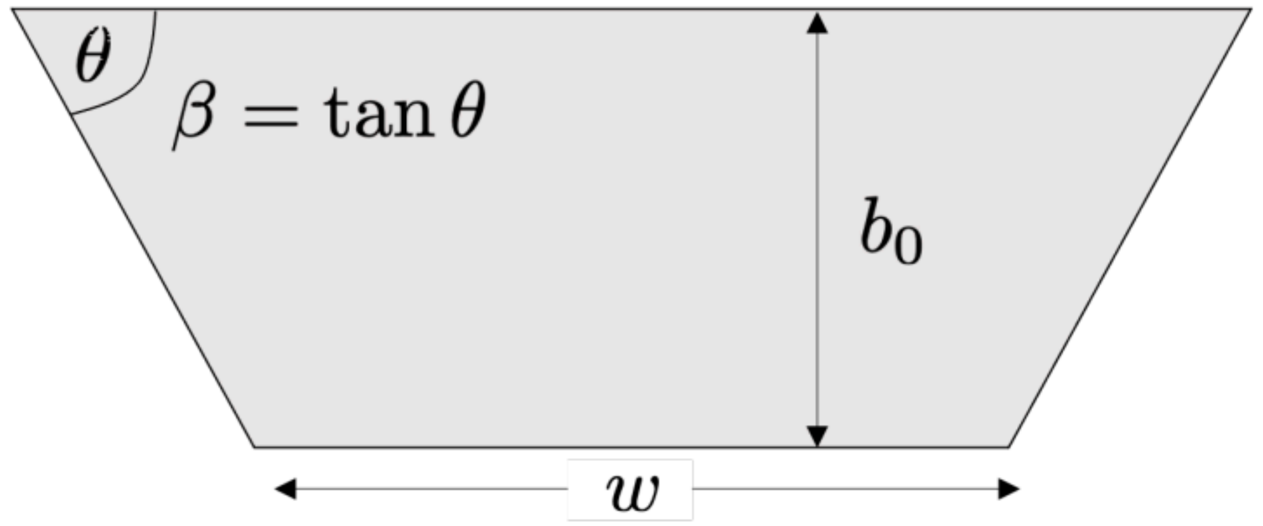
Rankine Hugoniot (non-dispersive) —> $C_b = U_2 + C_\lambda$ <— Phase celerity

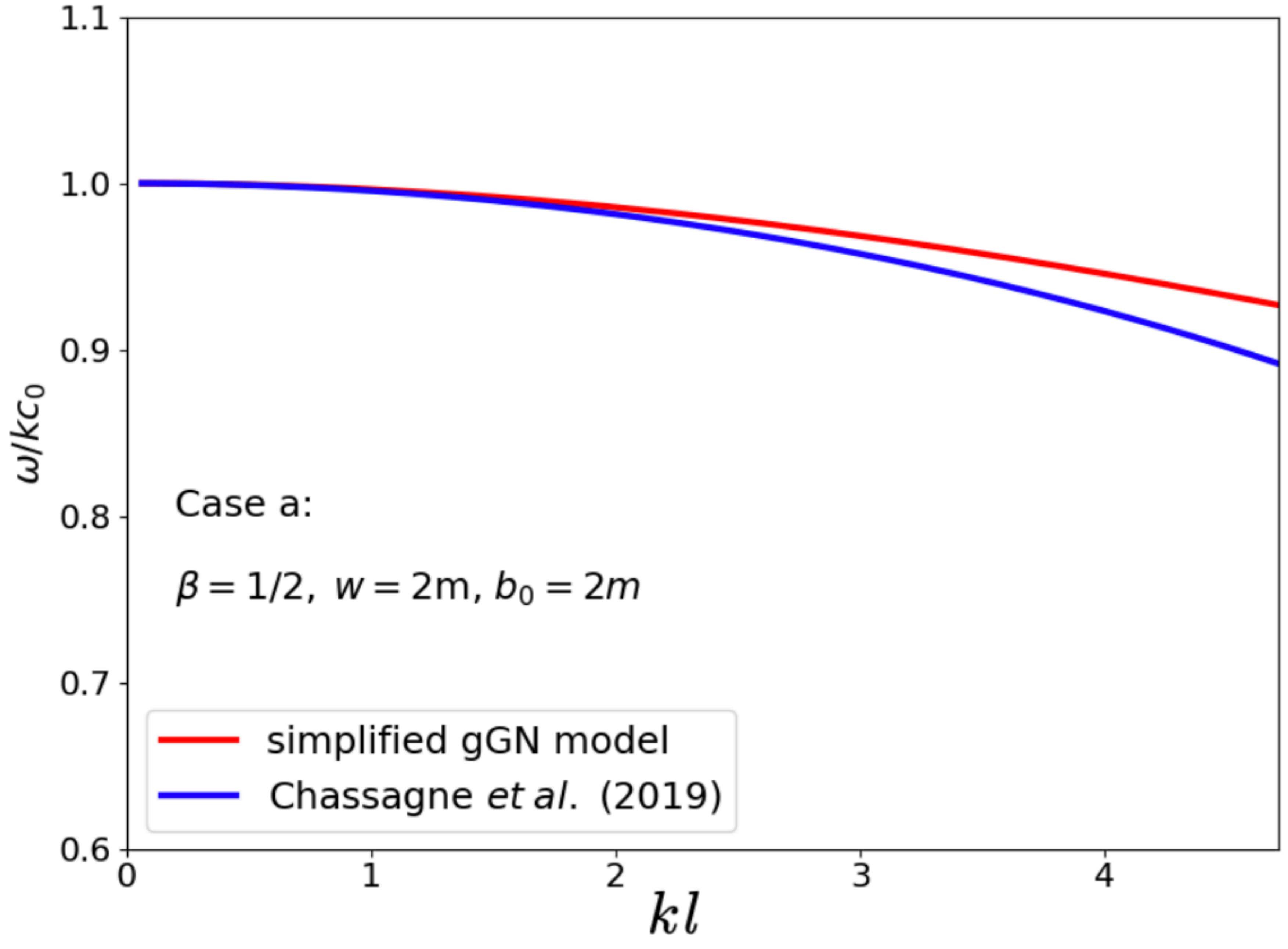


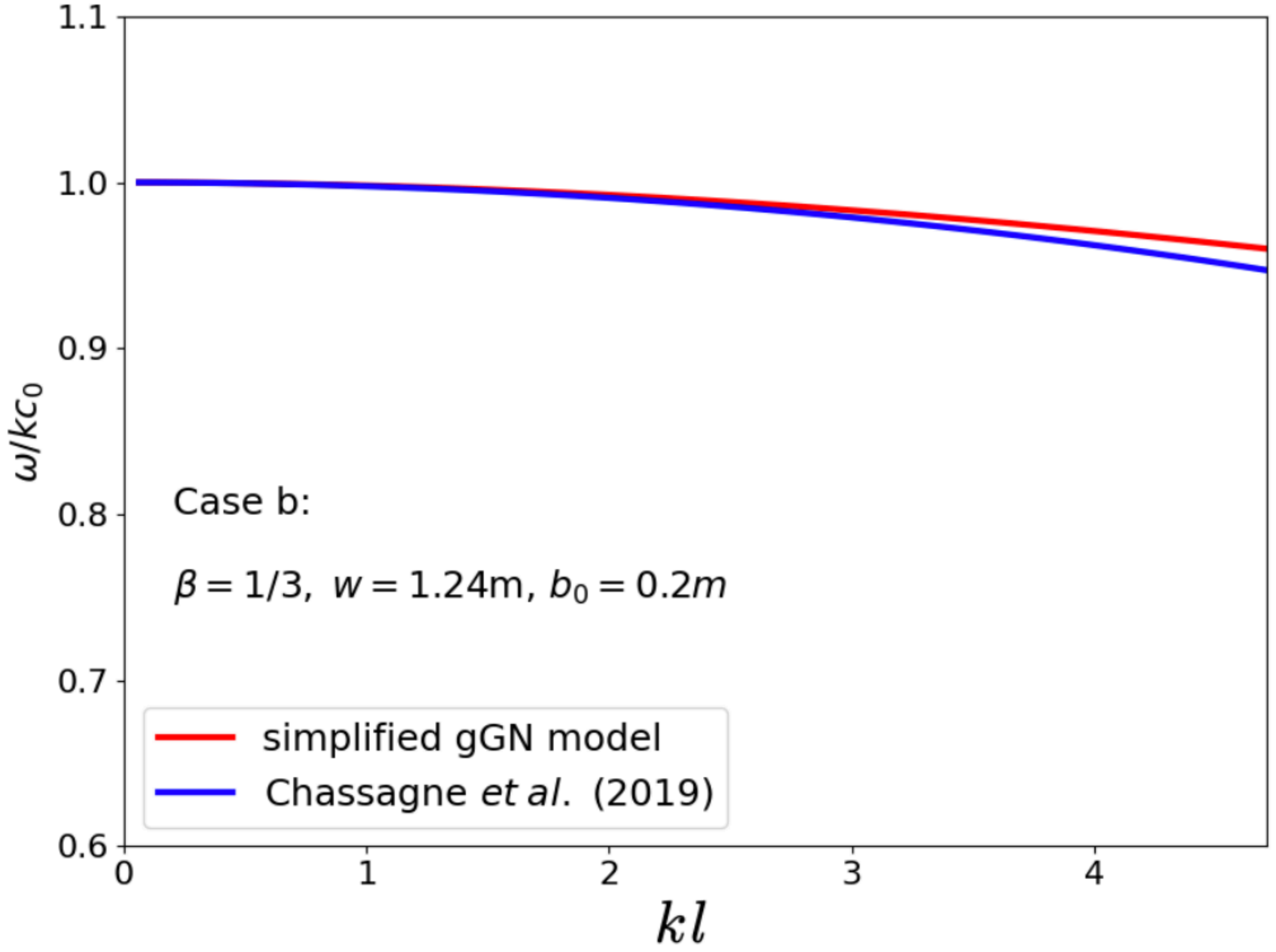


GGN model

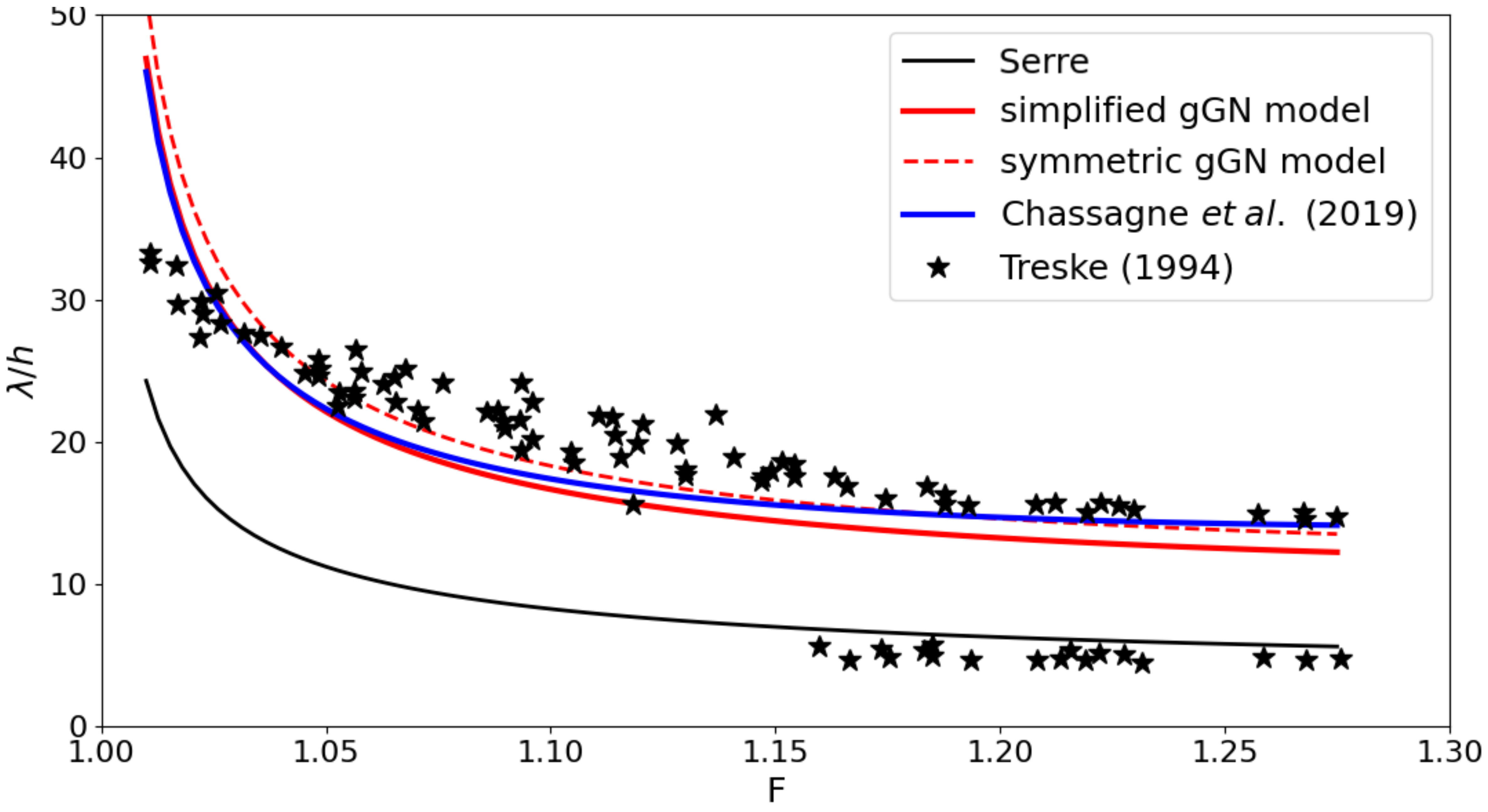
Dispersive properties







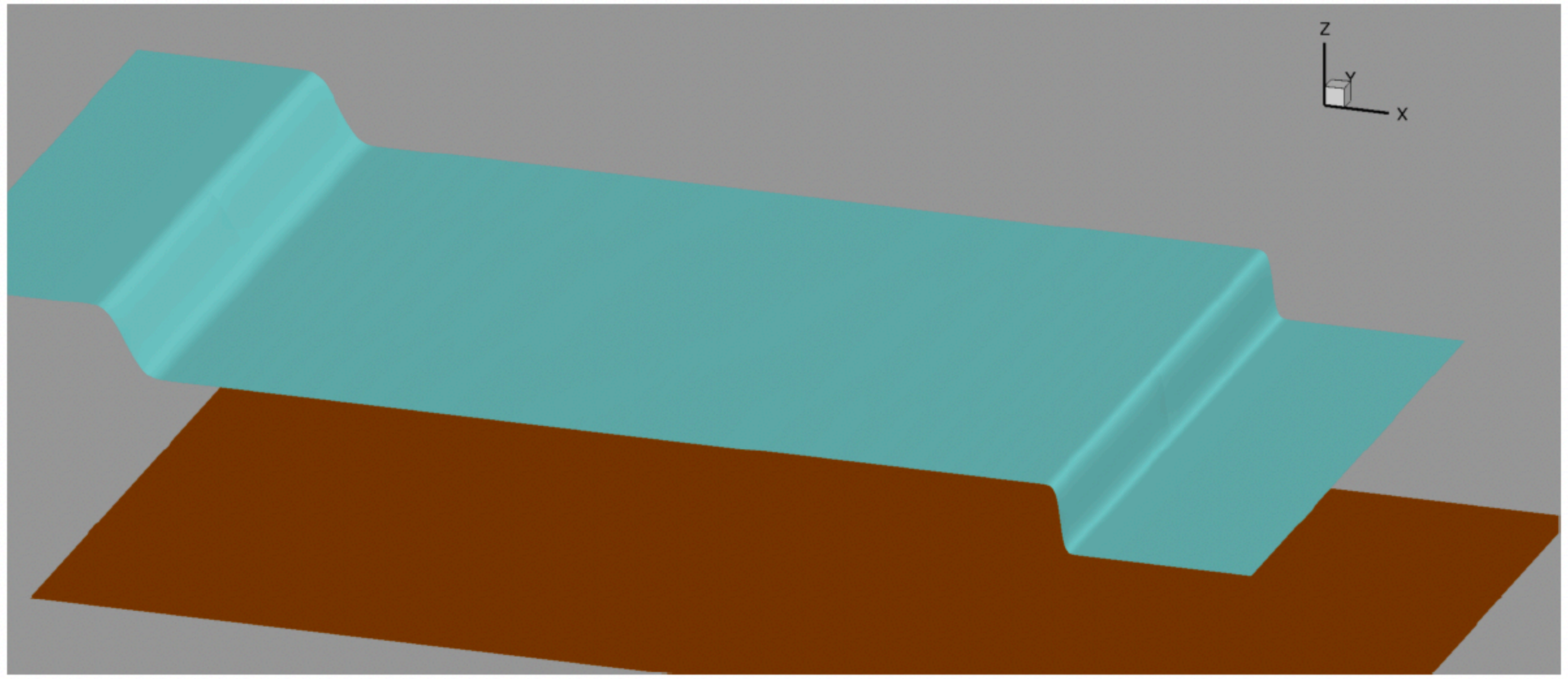
Lemoine analogy

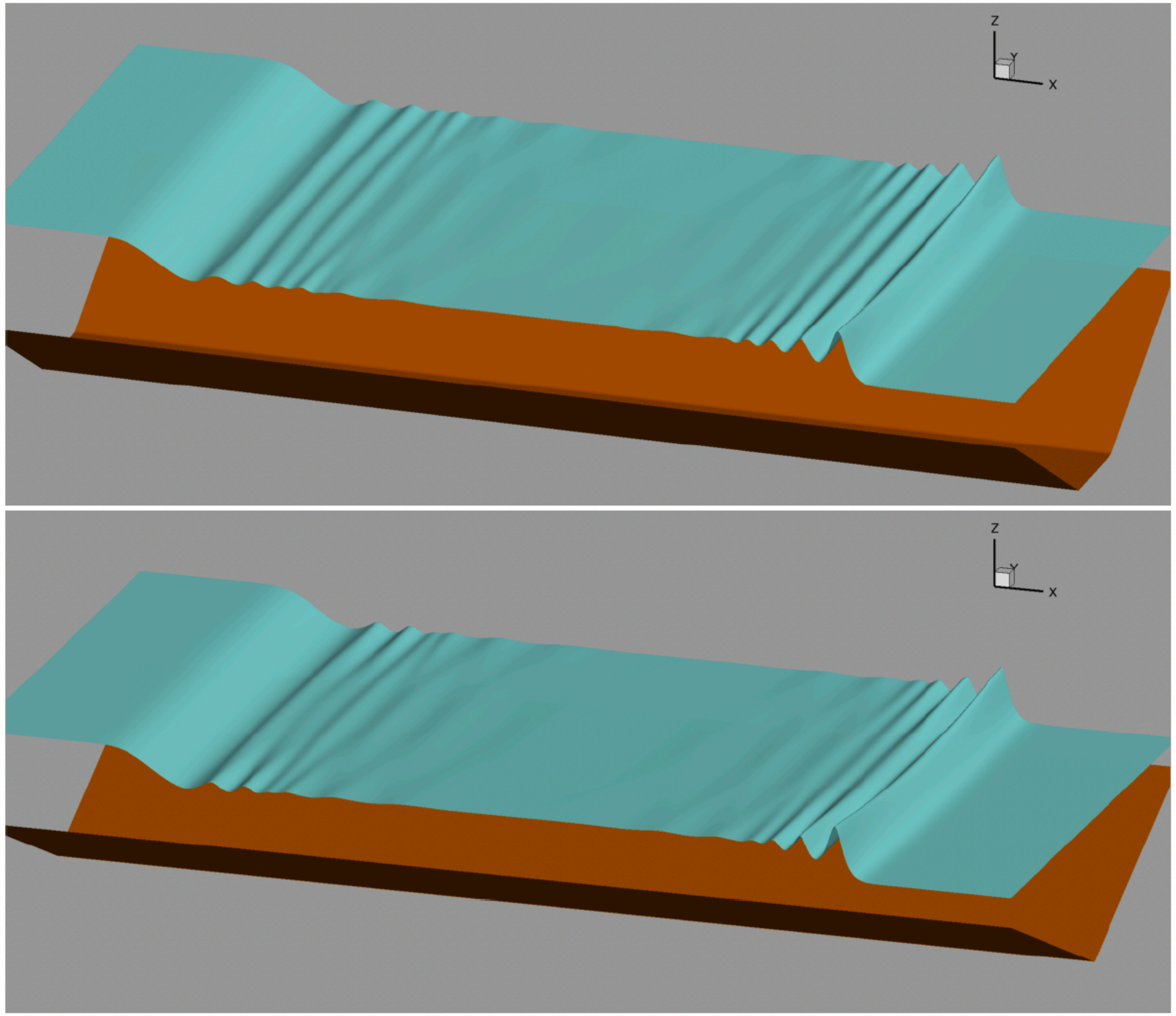


Numerical test: Riemann problem

$$\frac{h_2}{h_1} = 1.125\,, \quad \chi = 0 \text{ (no bathymetry)}$$

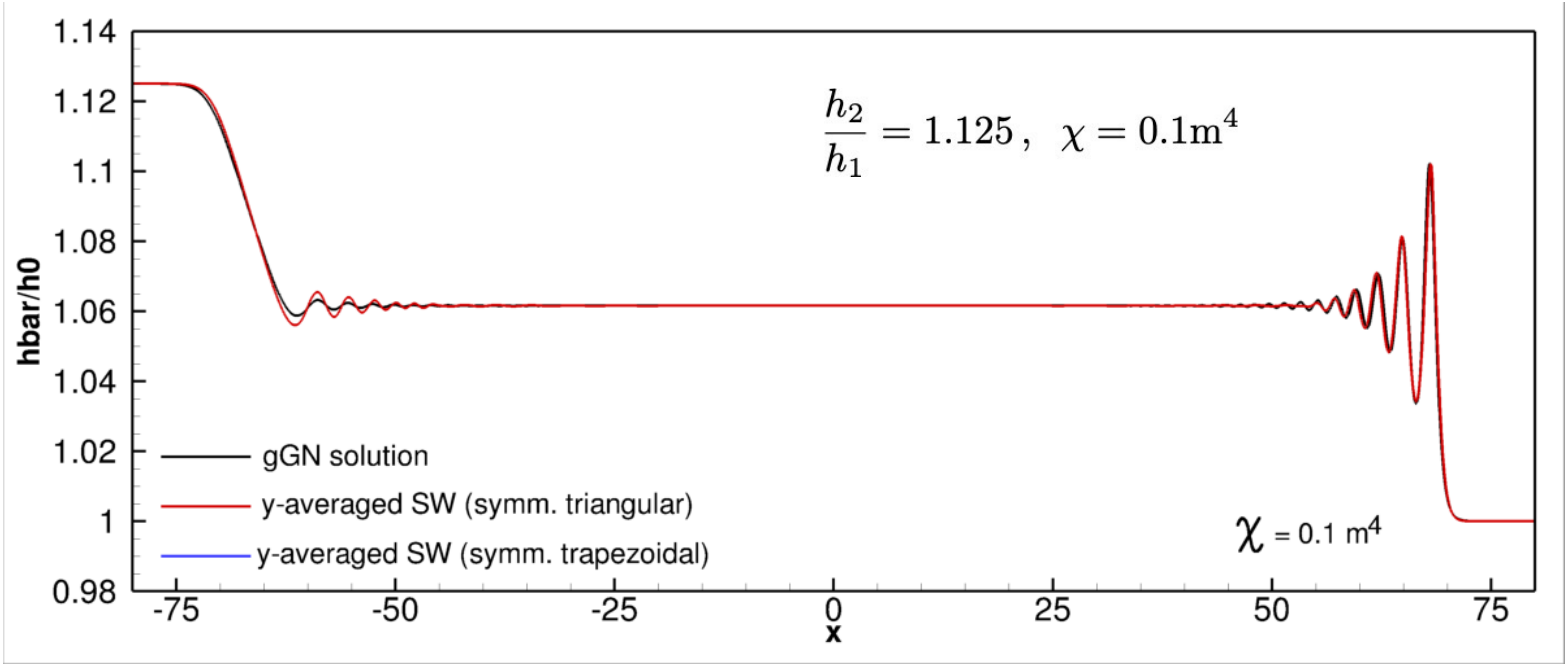
$$\frac{h_2}{h_1} = 1.125\,, \quad \chi = 0.6\text{m}^4$$





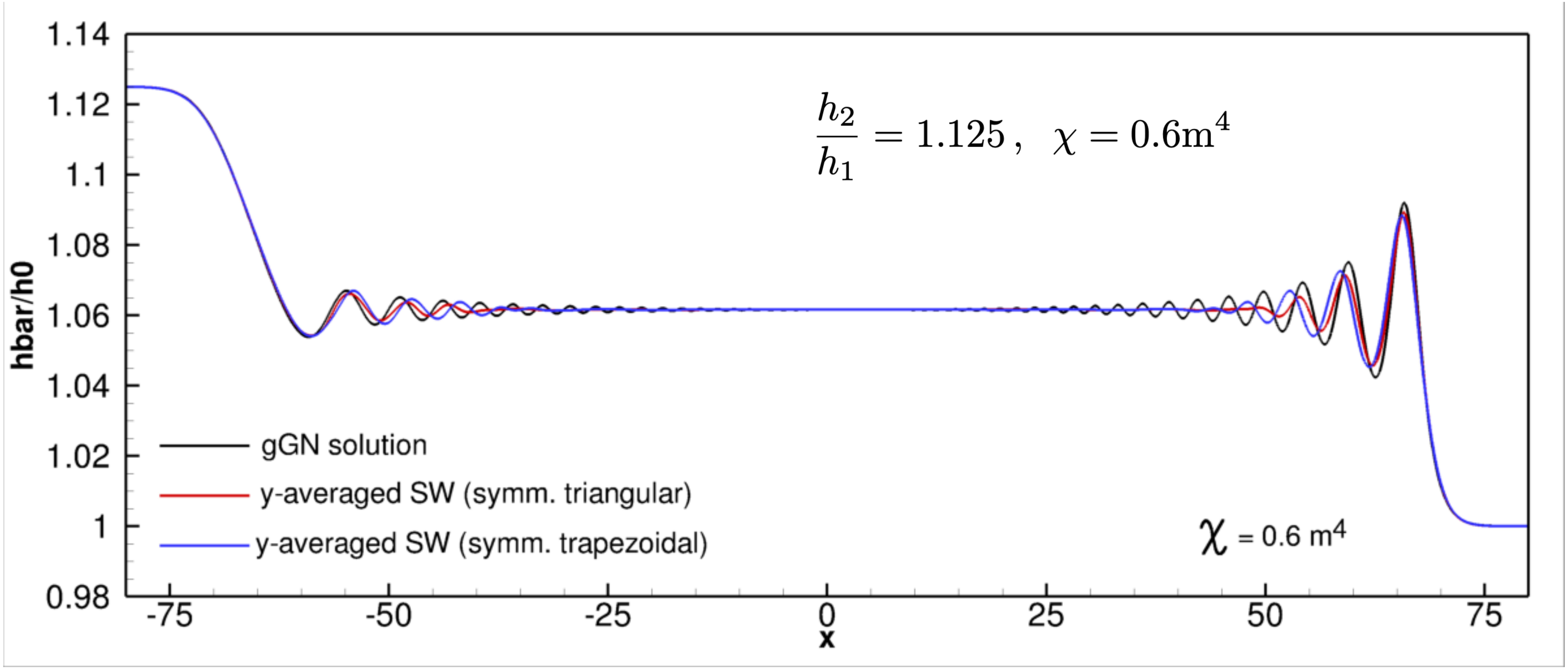
2D shallow water simulations (mesh converged)

Numerical test: Riemann problem



GGN model

Numerical test: Riemann problem



$$\frac{h_2}{h_1} = 1.125\,, \quad \chi = 0.6\mathrm{m}^4$$

GGN model	**Favre experiment**

Fig. 8. Undular bore at Froude ~ 1.04.

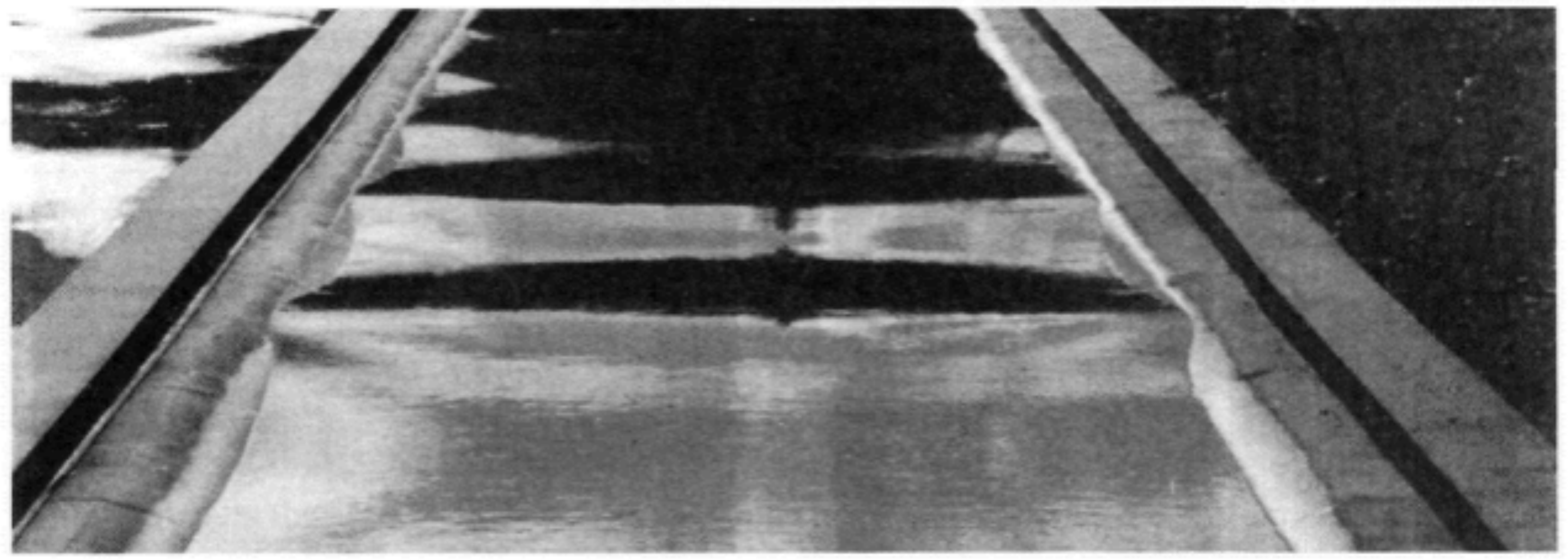


Fig. 9. Undular bore at Froude ~ 1.06.

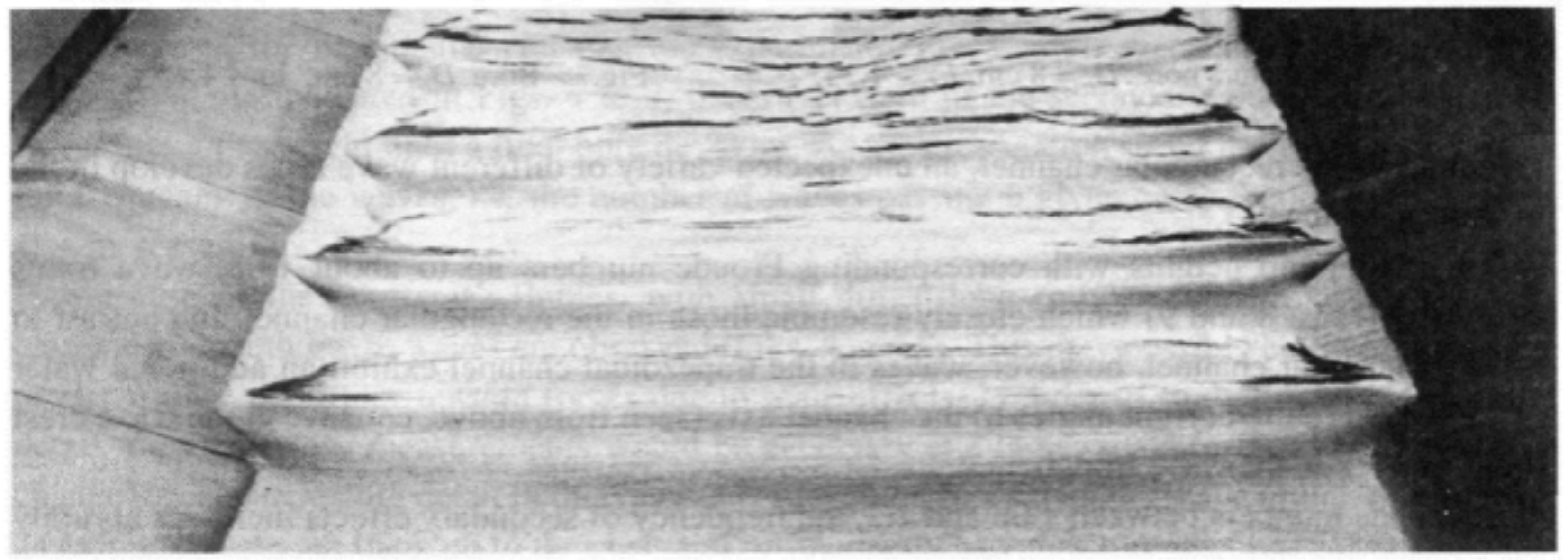


Fig. 10. Undular bore at Froude ~ 1.10.

Fr

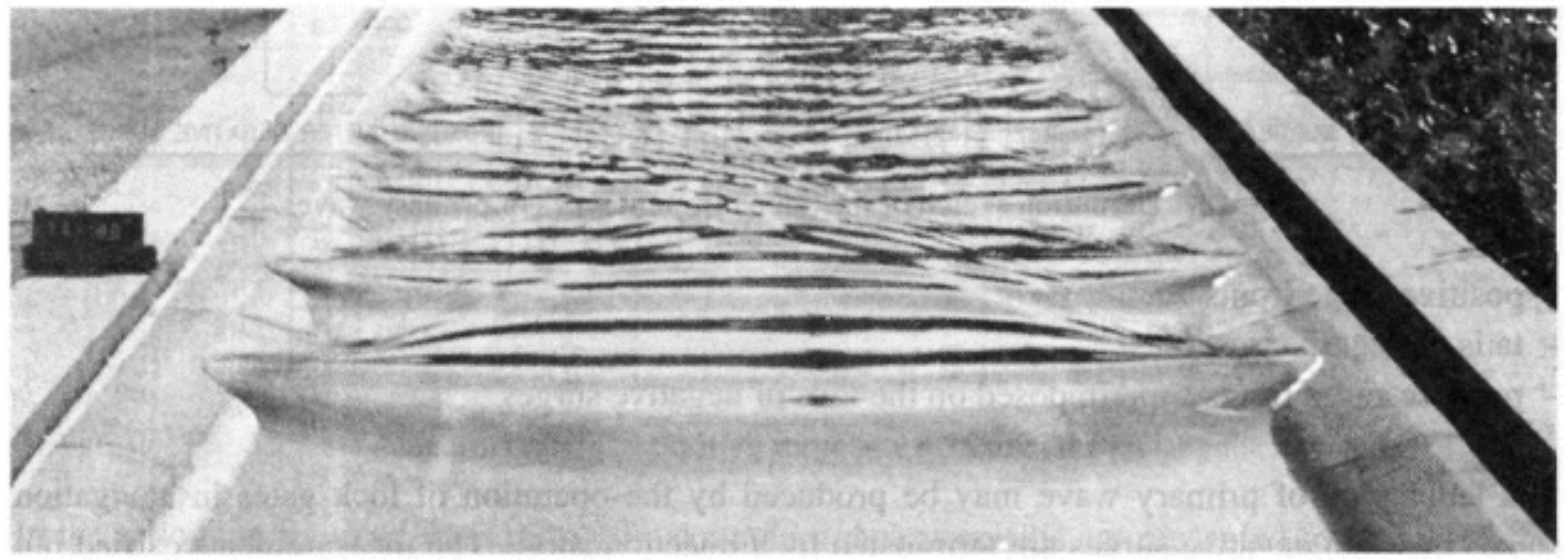


Fig. 11. Undular bore at Froude ~ 1.12.

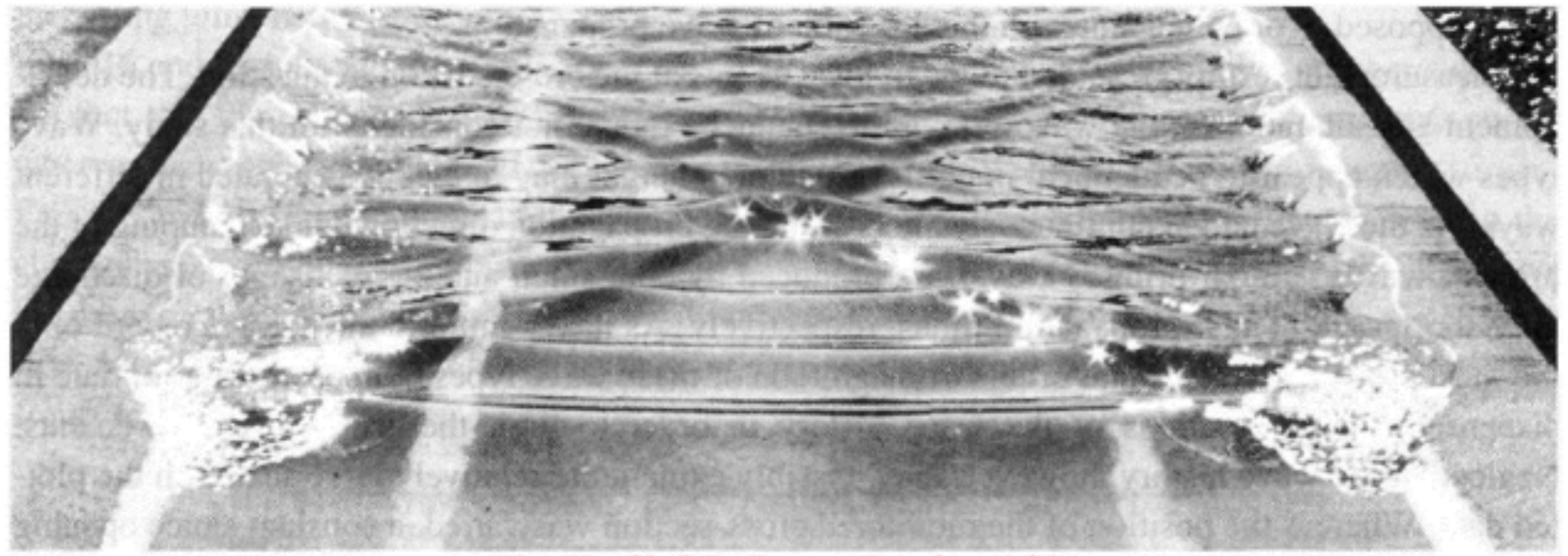


Fig. 12. Undular bore at Froude ~ 1.24.

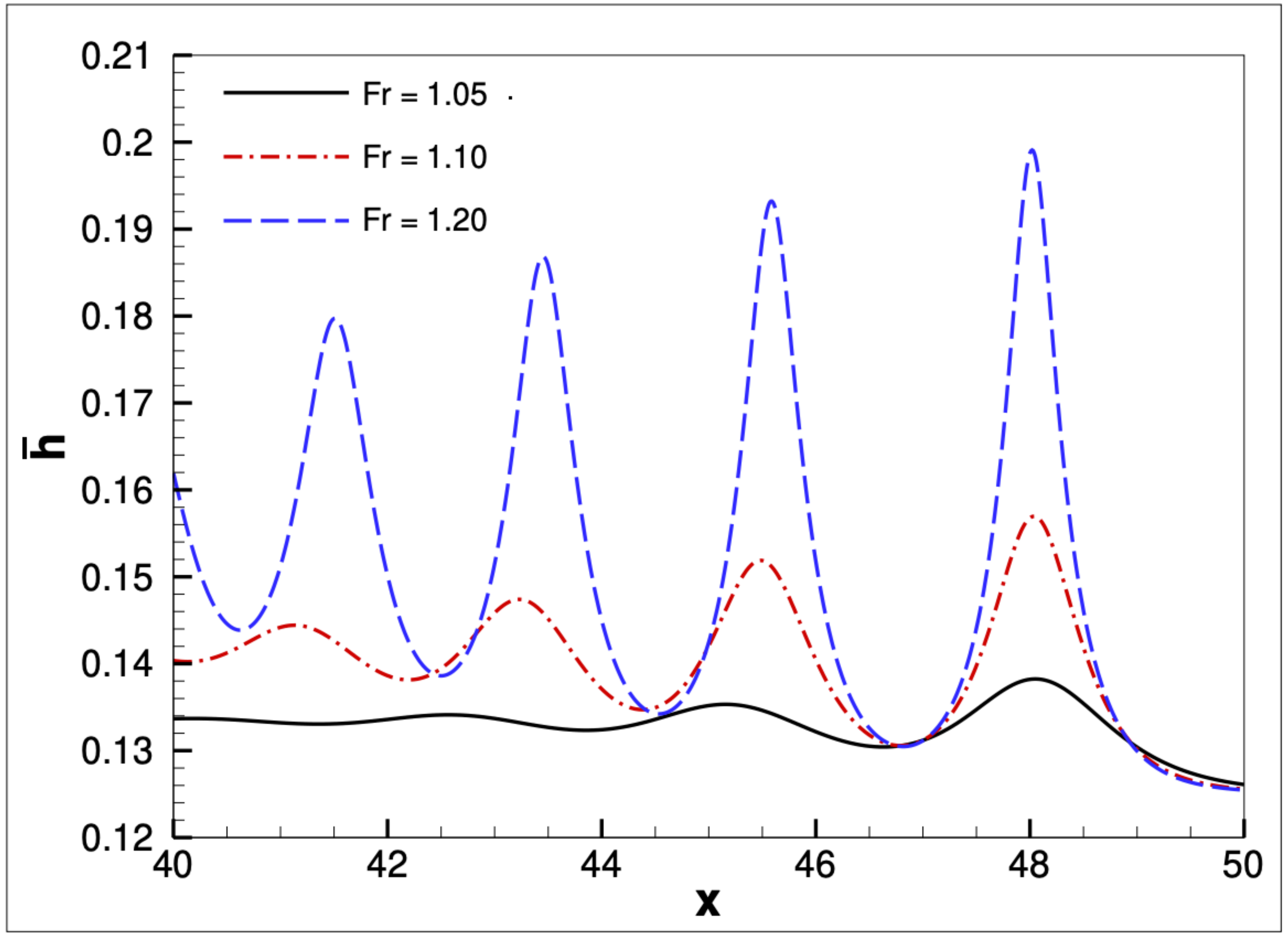
Fig. 13. Bore at Froude ~ 1.35.

GGN model | **Favre experiment**

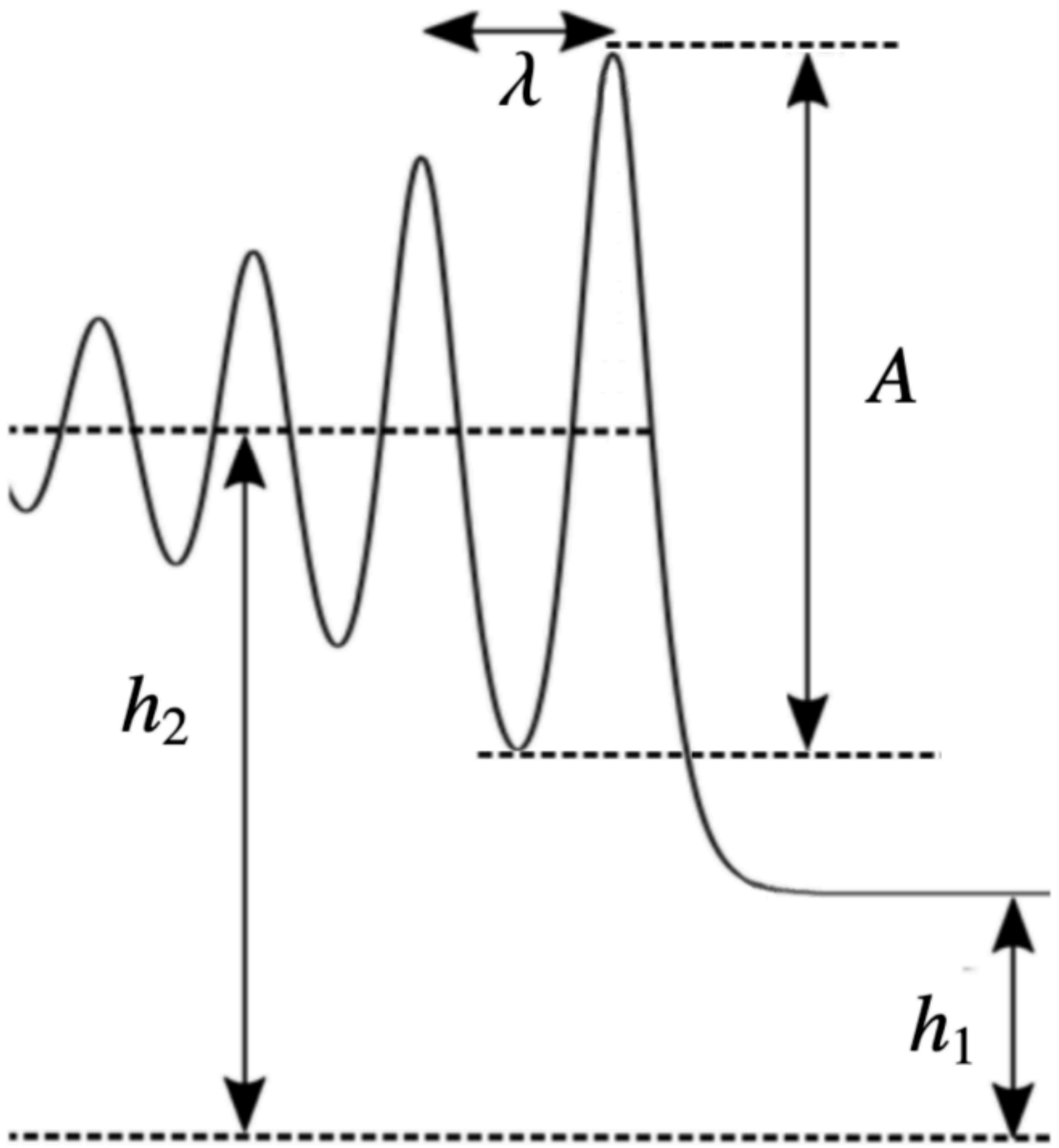
$$\bar{h}(x) = \bar{h}_1 + \frac{\bar{h}_2 - \bar{h}_1}{2}(1 - \tanh(x/\alpha))$$

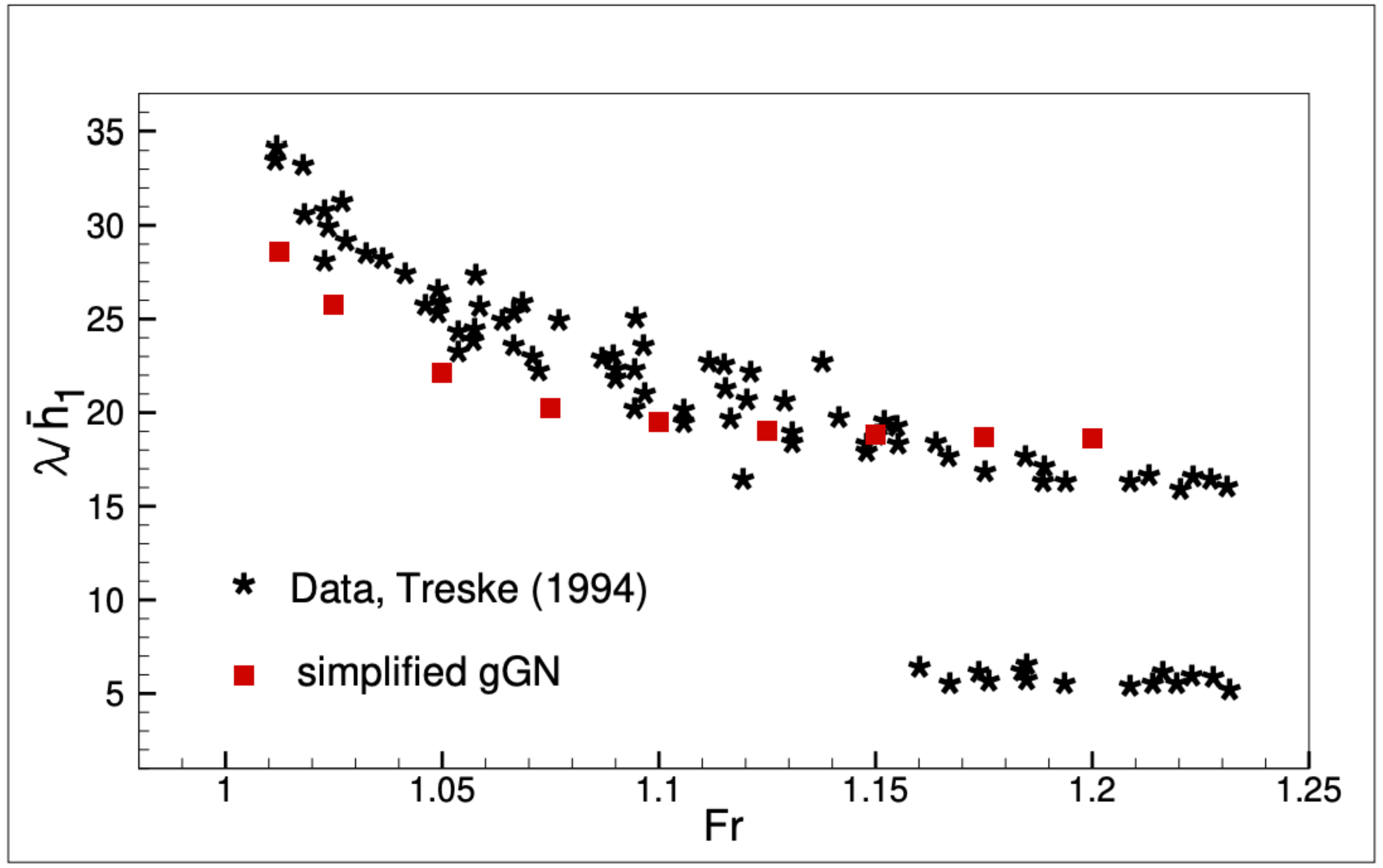
$$u(x) = \frac{u_2}{2}(1 - \tanh(x/\alpha))$$

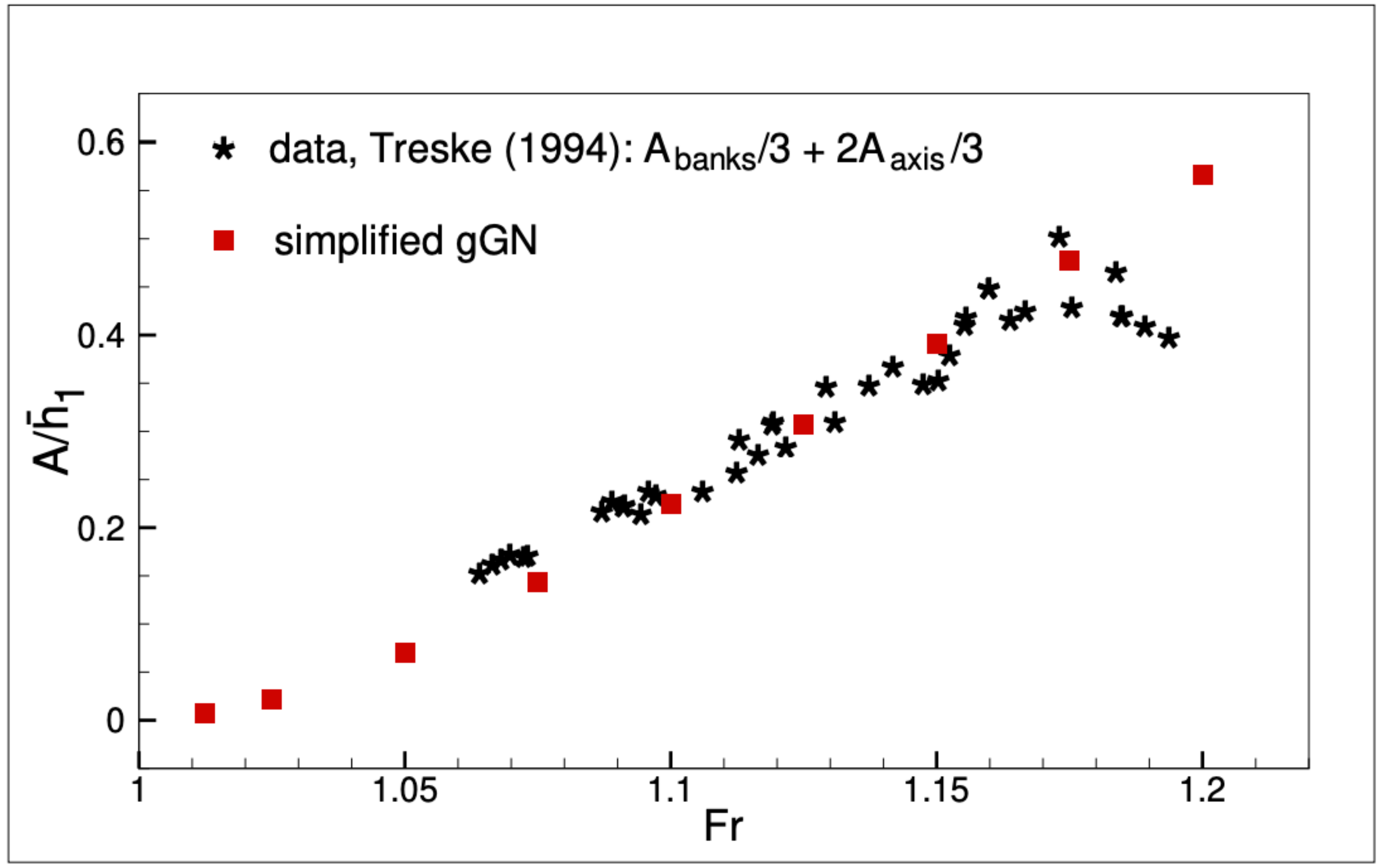
- $\bar{h}_1$, $\bar{h}_2$ from jump conditions of non dispersive limit
- $S(y)$ evaluated using the post-bore section height
- Froude numbers from 1.0125 to 1.20



GGN model	Favre experiment







Energy stability

vs

Energy conservation

Dilemma

- Hyperbolics : thermodynamics and viscous regularisations agree and allow to provide physically correct notion of dissipative solutions

- Work by M.Lukácová-Medvidová and co-workers exploits this for rigorous convergence of several different schemes for complex systems of PDEs

- Dispersive models: notion of admissible solutions of initially discontinuous data purely geometrical (some sort of generalized Lax condition), no notion of dissipation.

  In fact energy is always conserved (see **El** J. Nonlin. Sci. 2005, **Hoefer** J. Nonlin. Sci. 2014,

  or **Arnold, Camassa, Ding** St.Appl.Math 2024)

- Energy dissipation for a scheme is a natural stability property, but does not agree with any thermodynamics. It is arguable that one should aim for this property (instead of conservation ..)

What does practice tell us ????

Example 1: Serre-Green-Naghdi equations

In **Ranocha & Ricchiuto** arxiv.org/abs/2408.02665 Num.Meth.PDEs (submitted) framework to obtain mass, momentum, energy conservative schemes for (several variants of) the SGN equations using a combination of

- split form of the equation (combination of conservative/non-conservative)
- summation by parts discrete operators
- arbitrary order approximations (FD, Fourier, FE)

$$h_t + (hu)_x = 0,$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2 + \tilde{p}\right)_x = 0,$$

$$\tilde{p} = -\frac{1}{3}\left(h^3(\dot{u})_x - 2h^3u_x^2\right),$$

$$\underbrace{\left(\frac{1}{2}gh^2 + \frac{1}{2}hu^2 + \frac{1}{6}h(\dot{h})^2\right)_t}_{=E} + \underbrace{\left(gh^2u + \frac{1}{2}hu^3 + \frac{1}{6}h(\dot{h})^2u + \tilde{p}u\right)_x}_{=F} = 0.$$

Numerics vs physics | Example 2: Boussinesq model by Wincler and Liu

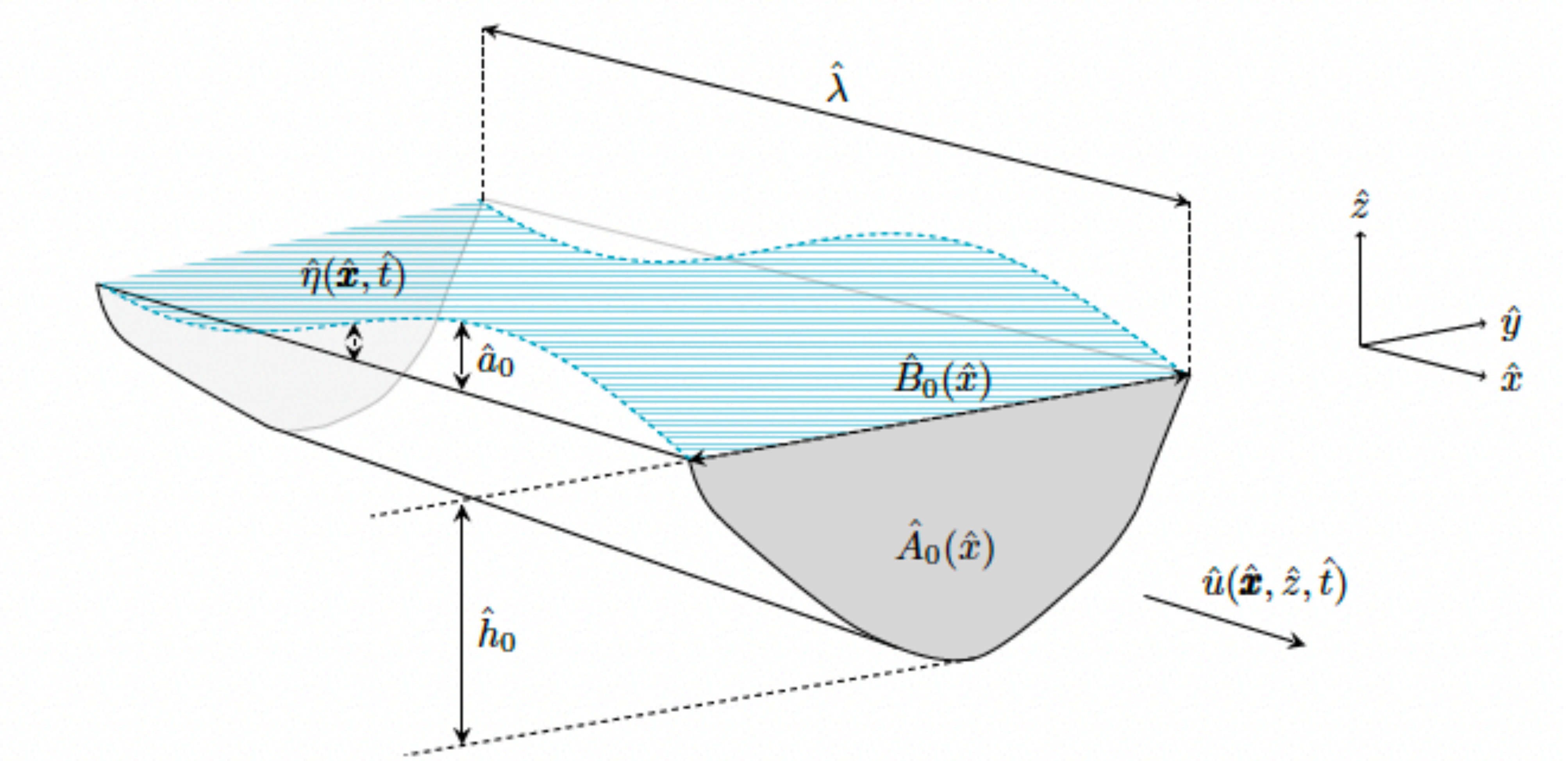
In **Jouy et al** Appl.Math.Mod 2024 non-dissipative approximation for the Boussinesq equations by **Winckler and Liu** J. Fluid.Mech. 2015 (WL model) modelling dispersive waves in channels of arbitrary section, using a combination of

- reformulation as section averaged hyperbolic SWE with a point source satisfying an elliptic PDE
- entropy conservative FV for the section averaged SWEs
  based on a generalization of Tadmor's shuffle conservation condition
- FE treatment of the elliptic operator

$$\partial_t A + \partial_x(Au) = 0$$

$$\partial_t(Au) + \partial_x(Au^2 + gK) = A\phi$$

$$\phi - \partial_x(\gamma^* \partial_x \phi) = \partial_x(\gamma^* \partial_x \theta)$$

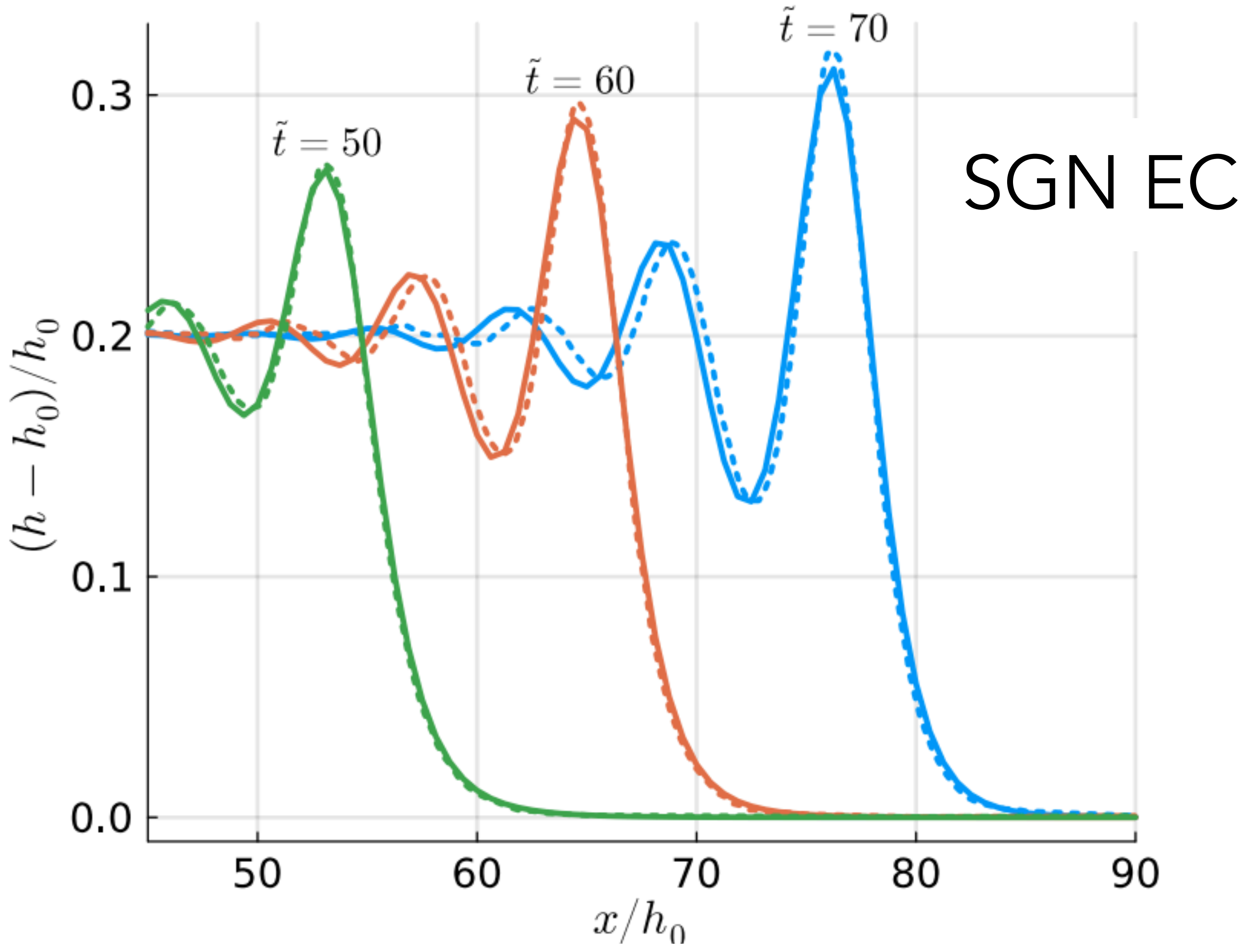


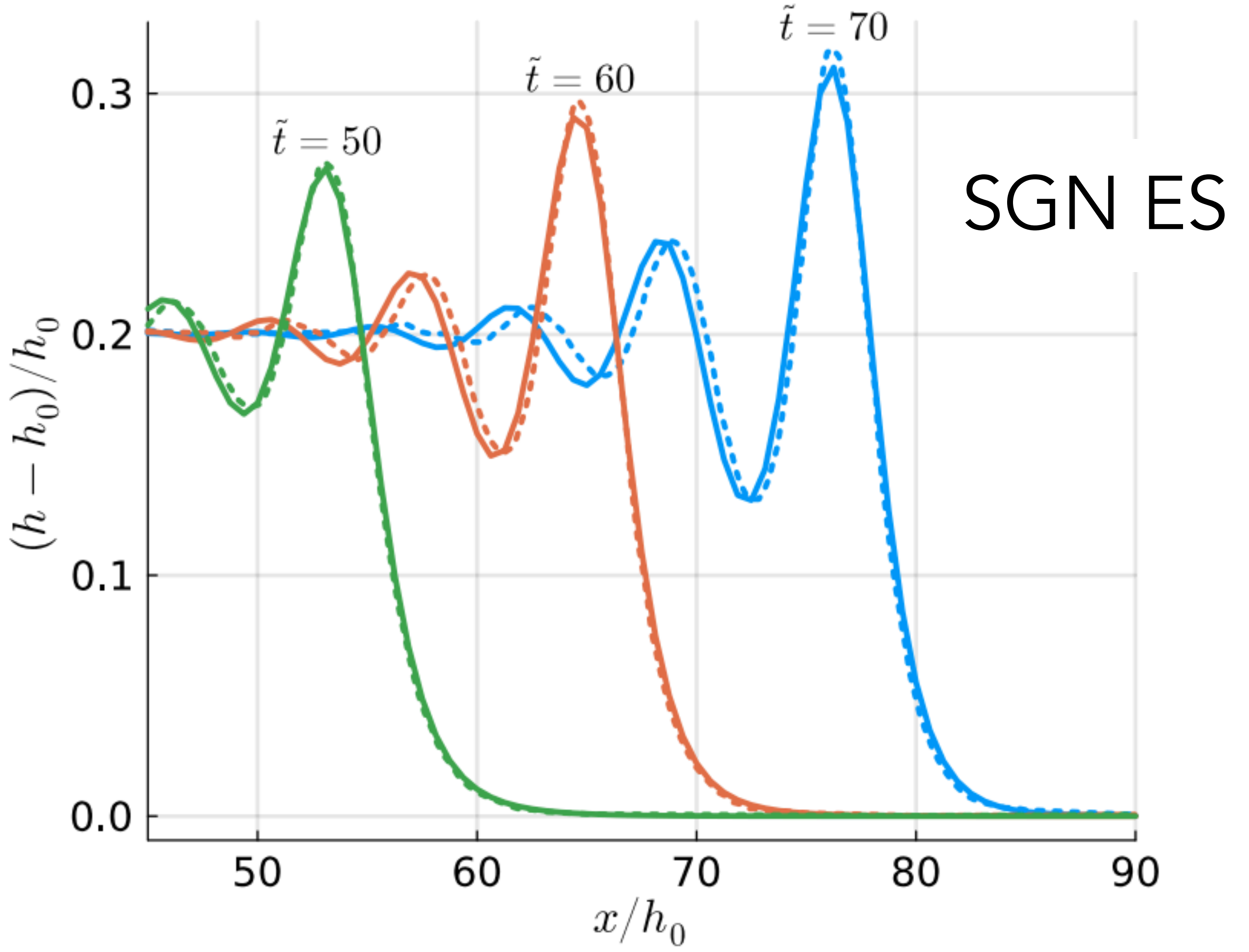
$$A = \int_{-\ell}^{\ell} h(y)dy\ , \quad K = \int A(h)dh + K_0$$

$$\theta = \partial_x(\zeta + u^2/2g)$$

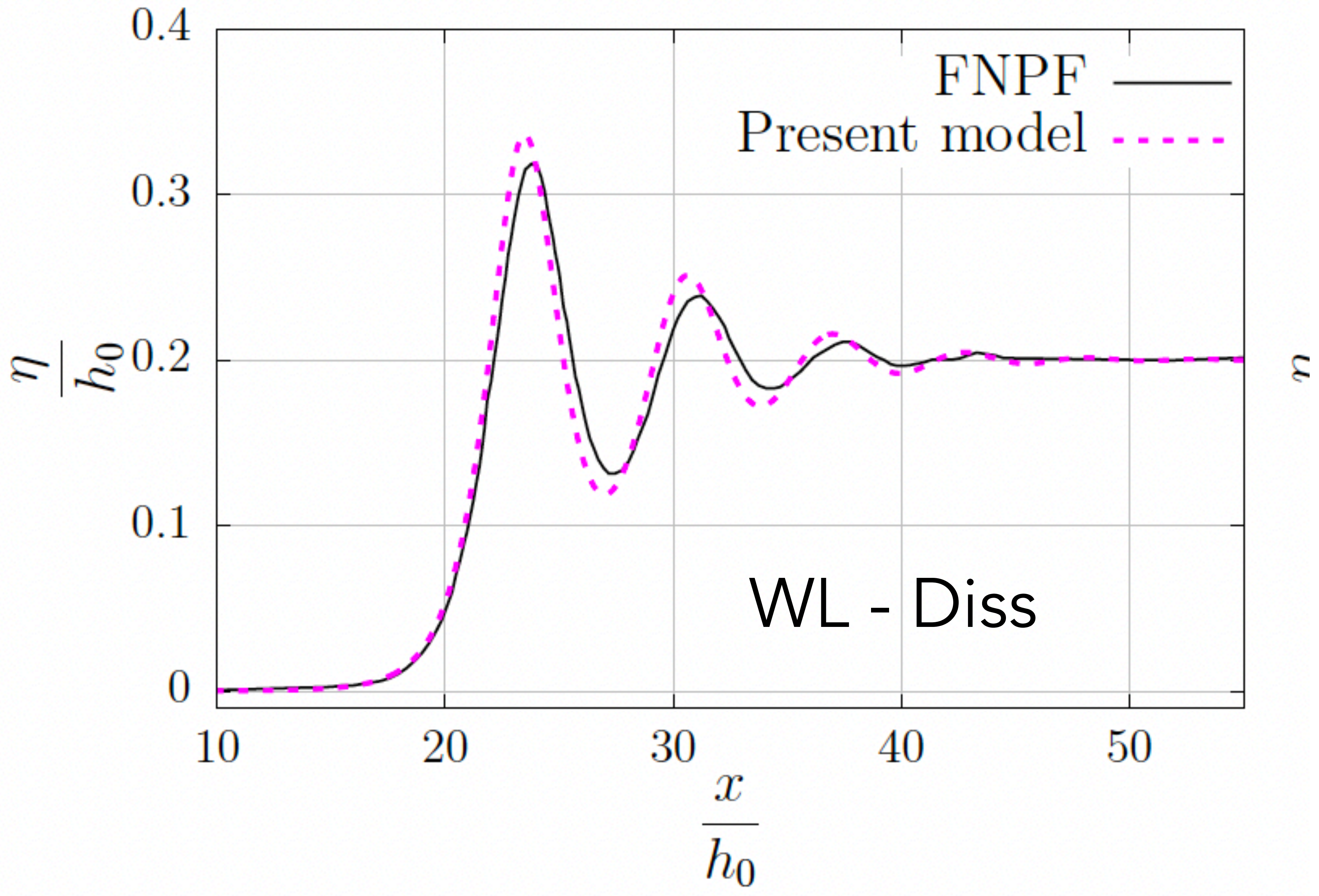
Numerics vs physics

Example: short term, no friction





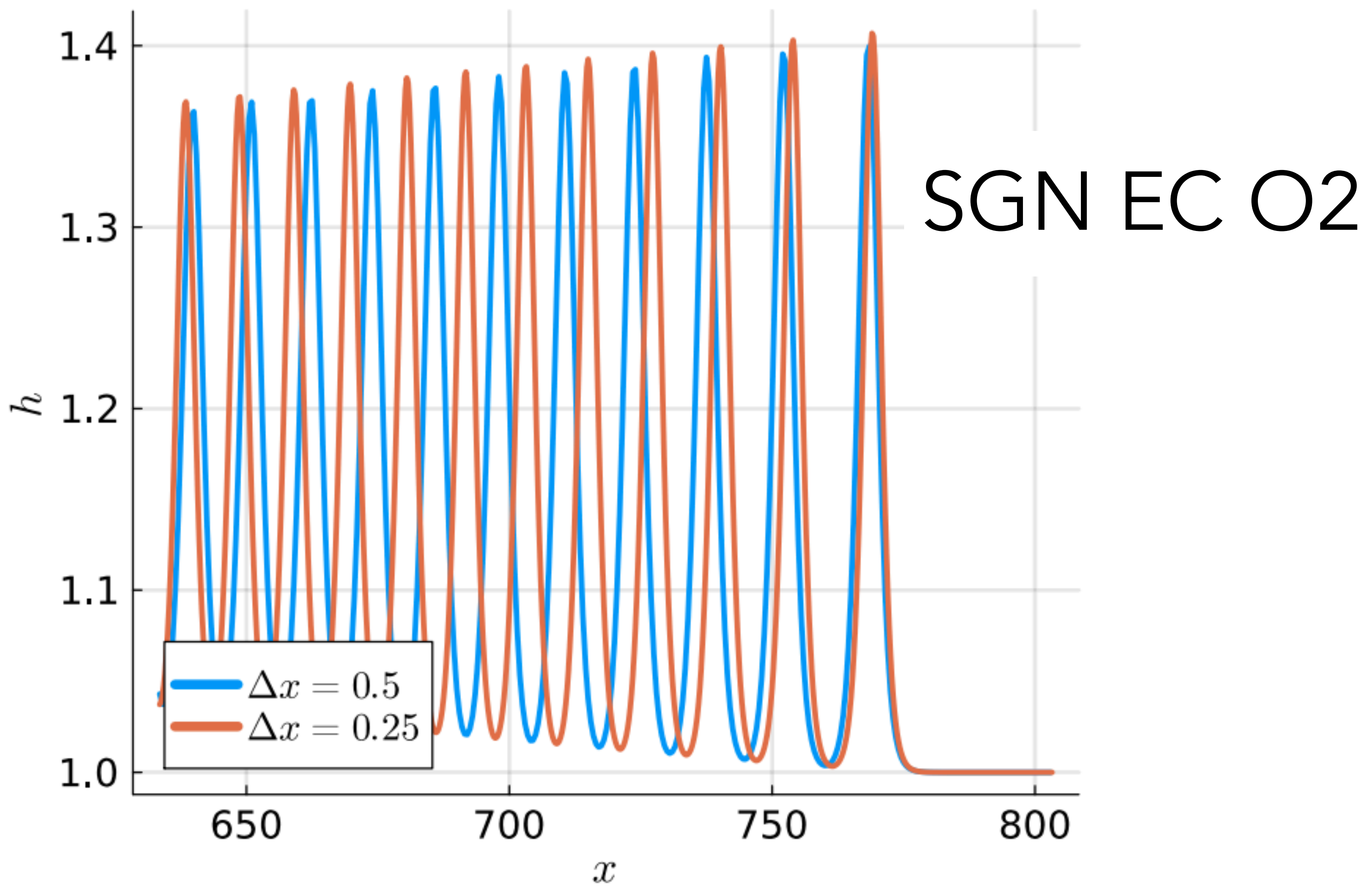
Jouy et al, Appl.Math.Mod. 2024

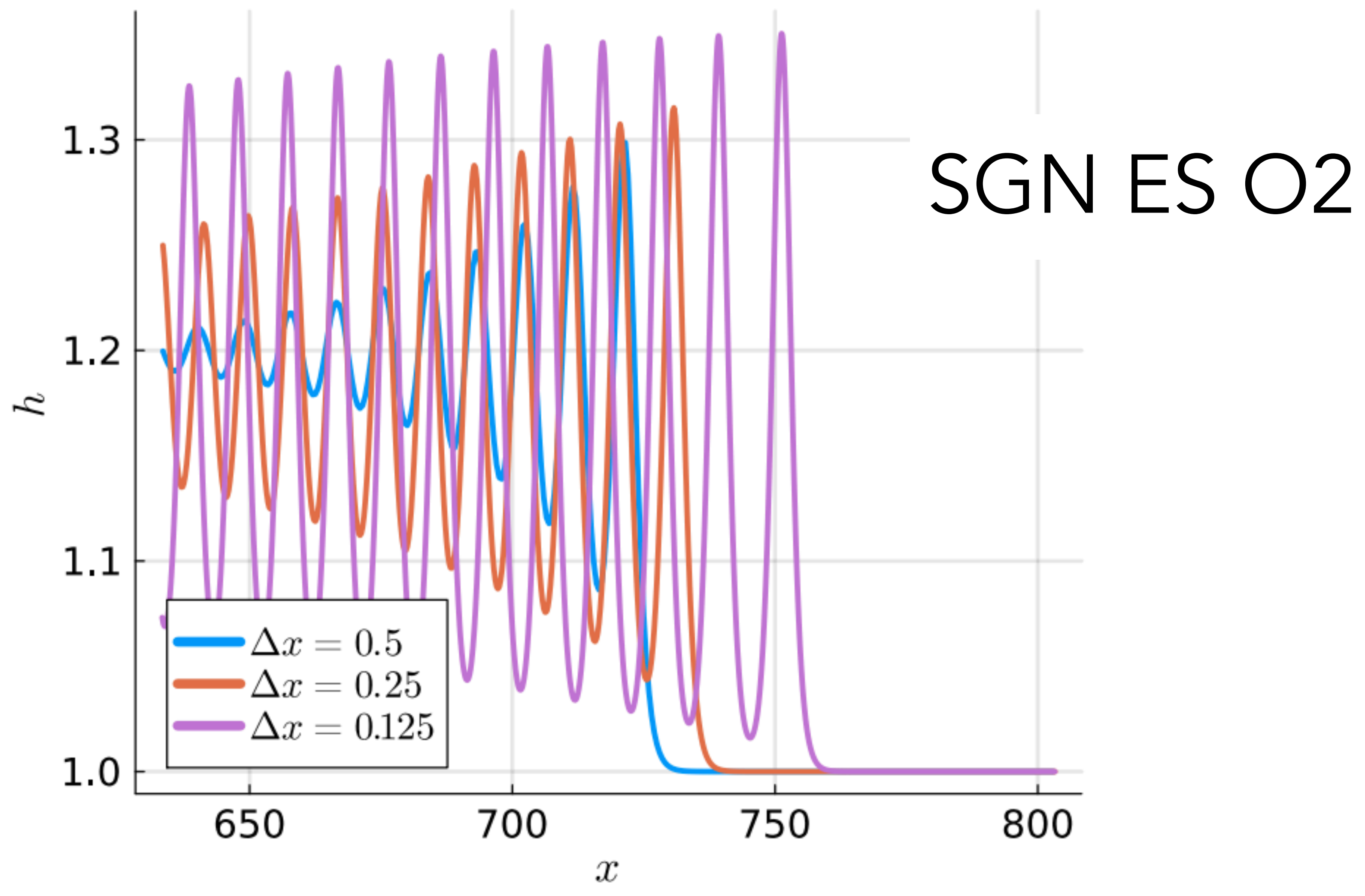


For short term propagation there is no visible impact of numerical dissipation for fixed/comparable order/mesh size

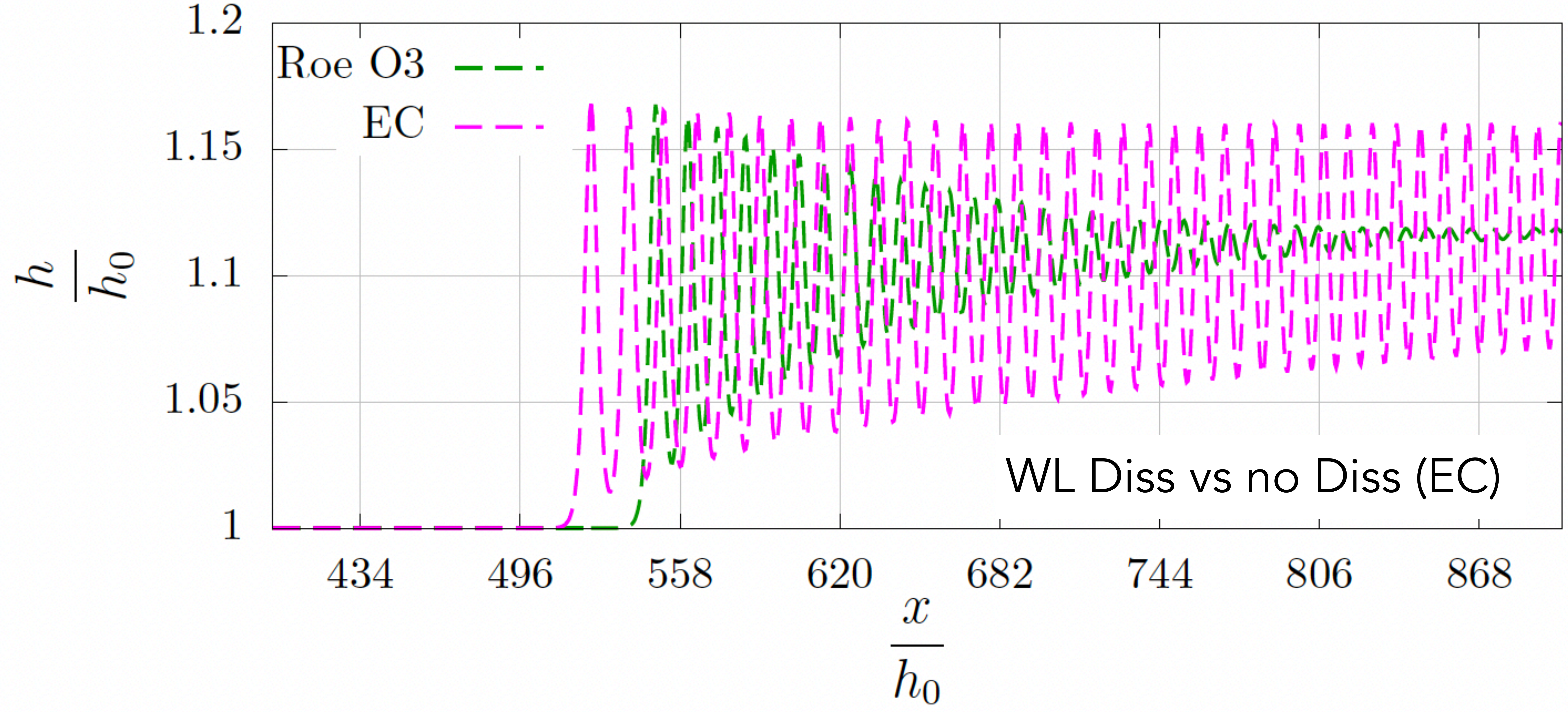
Numerics vs physics

Example: long term, no friction





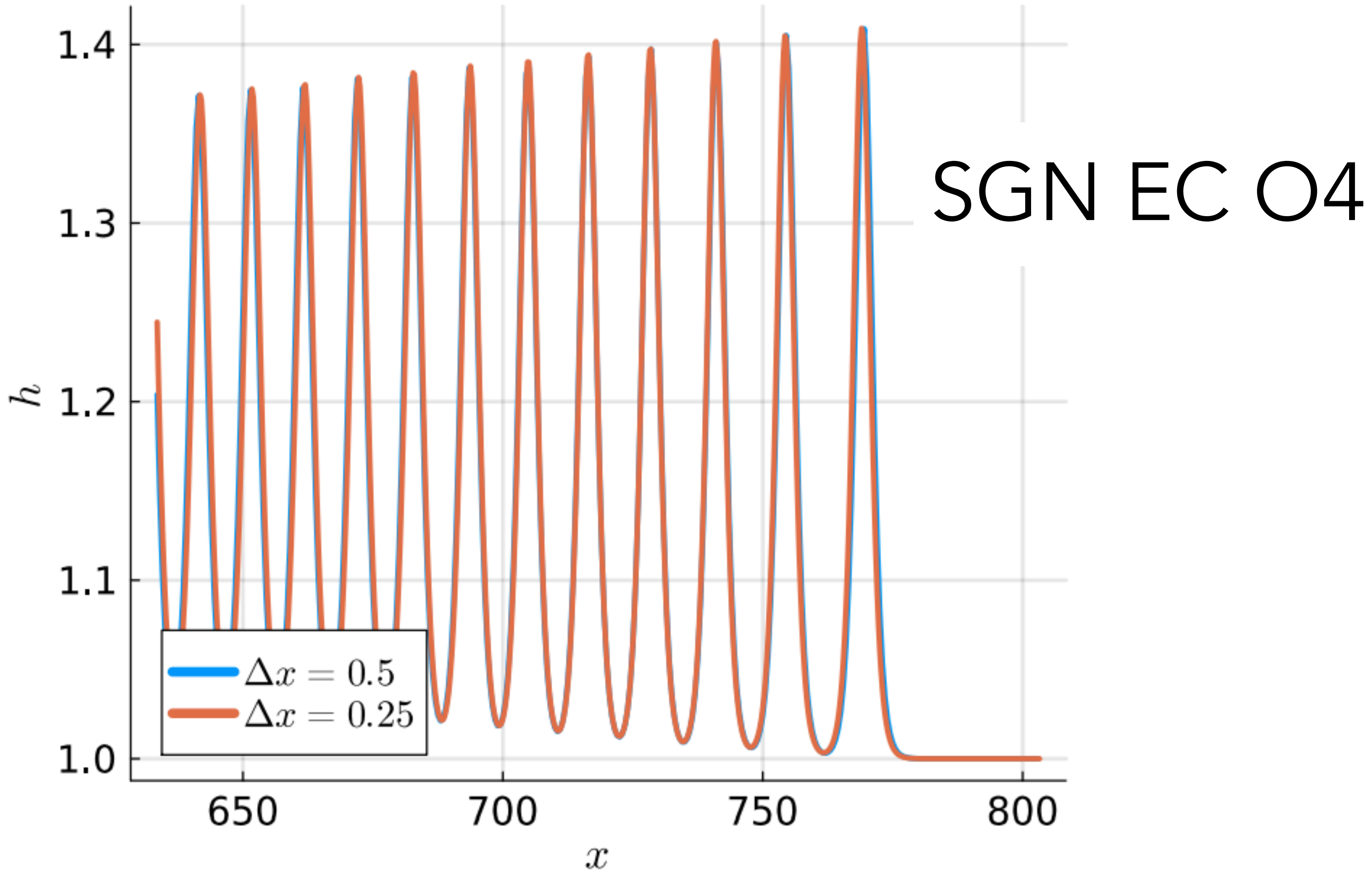
Jouy, 2024

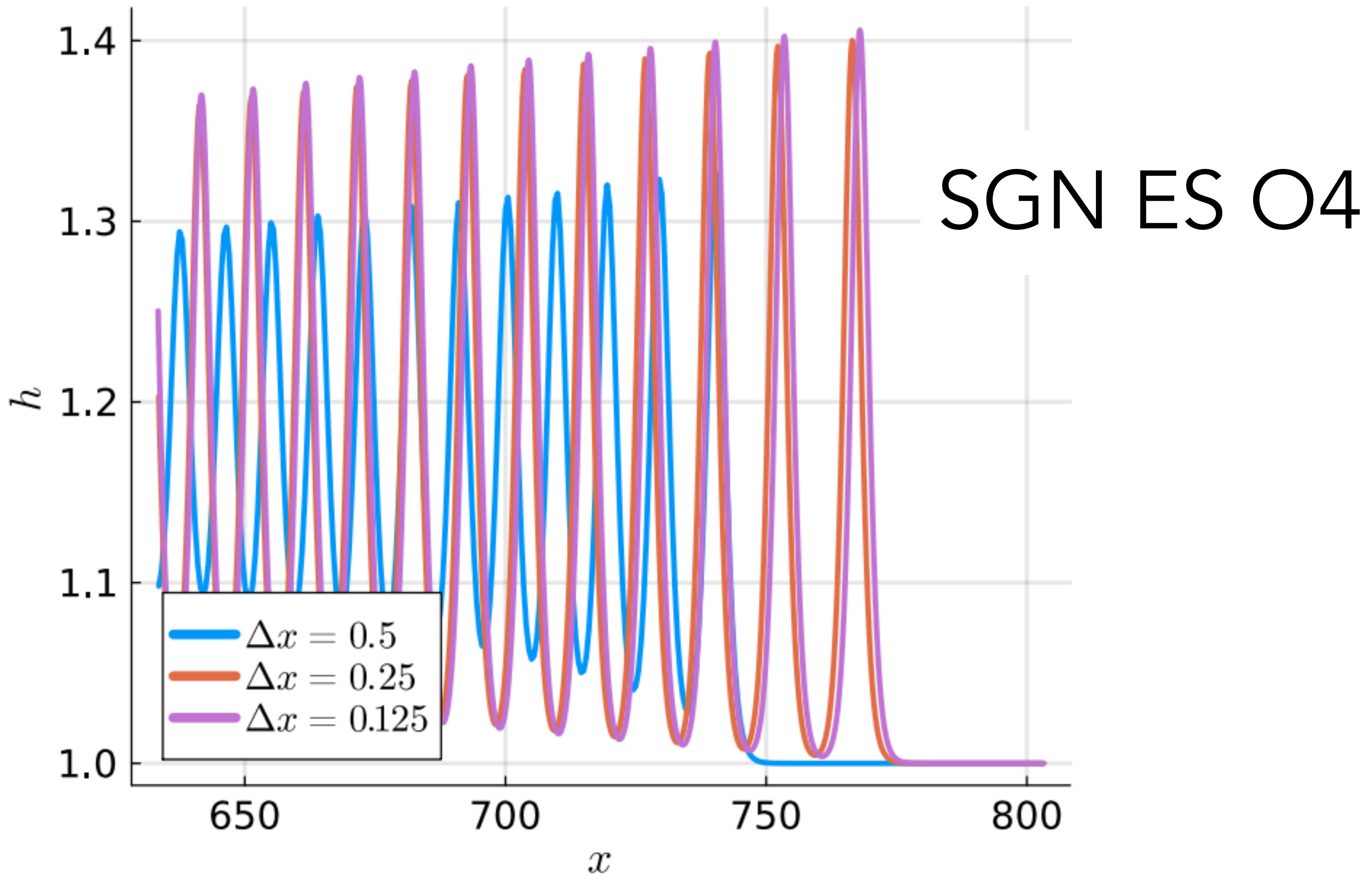


For long term propagation there is a strong impact of numerical dissipation on amplitudes and phase !

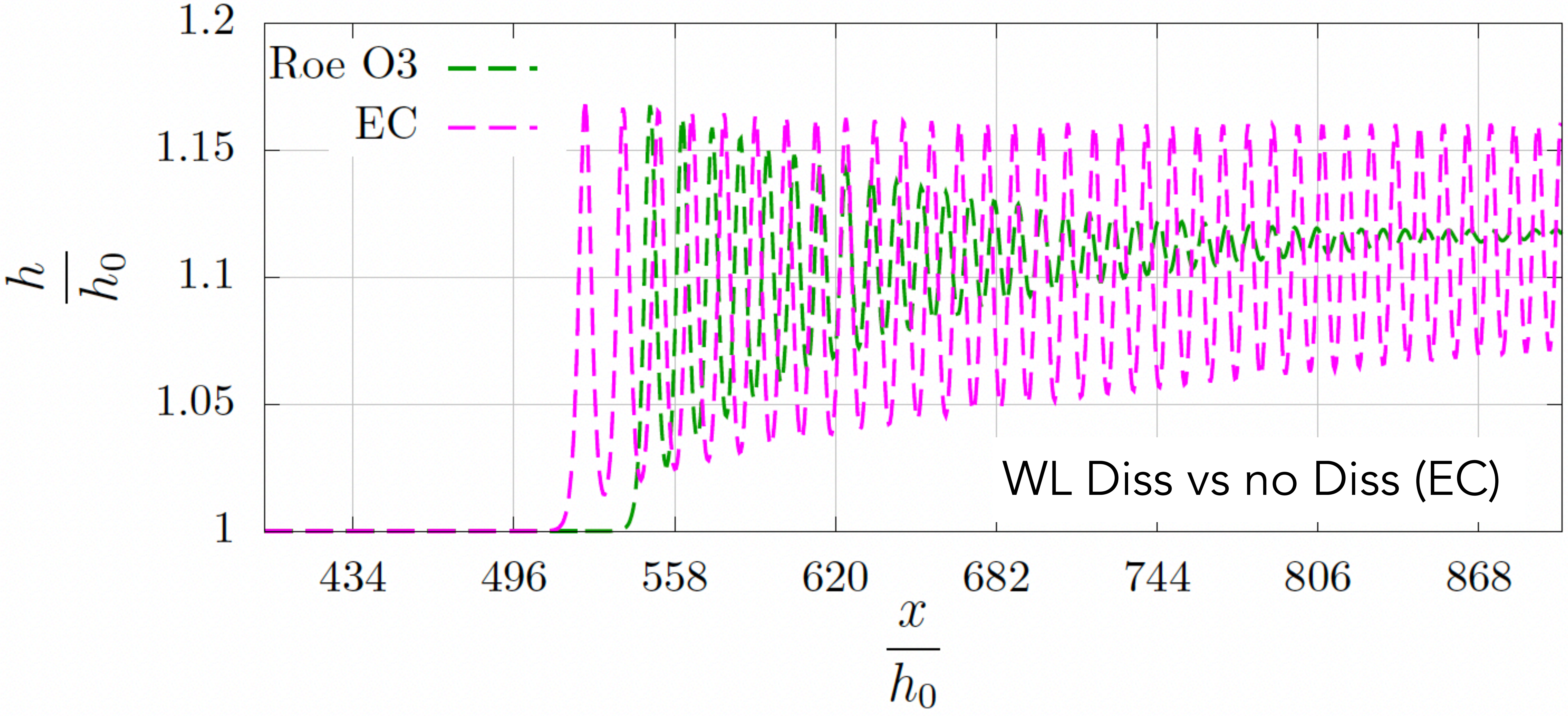
Numerics vs physics

Example: long term, no friction





Jouy, 2024

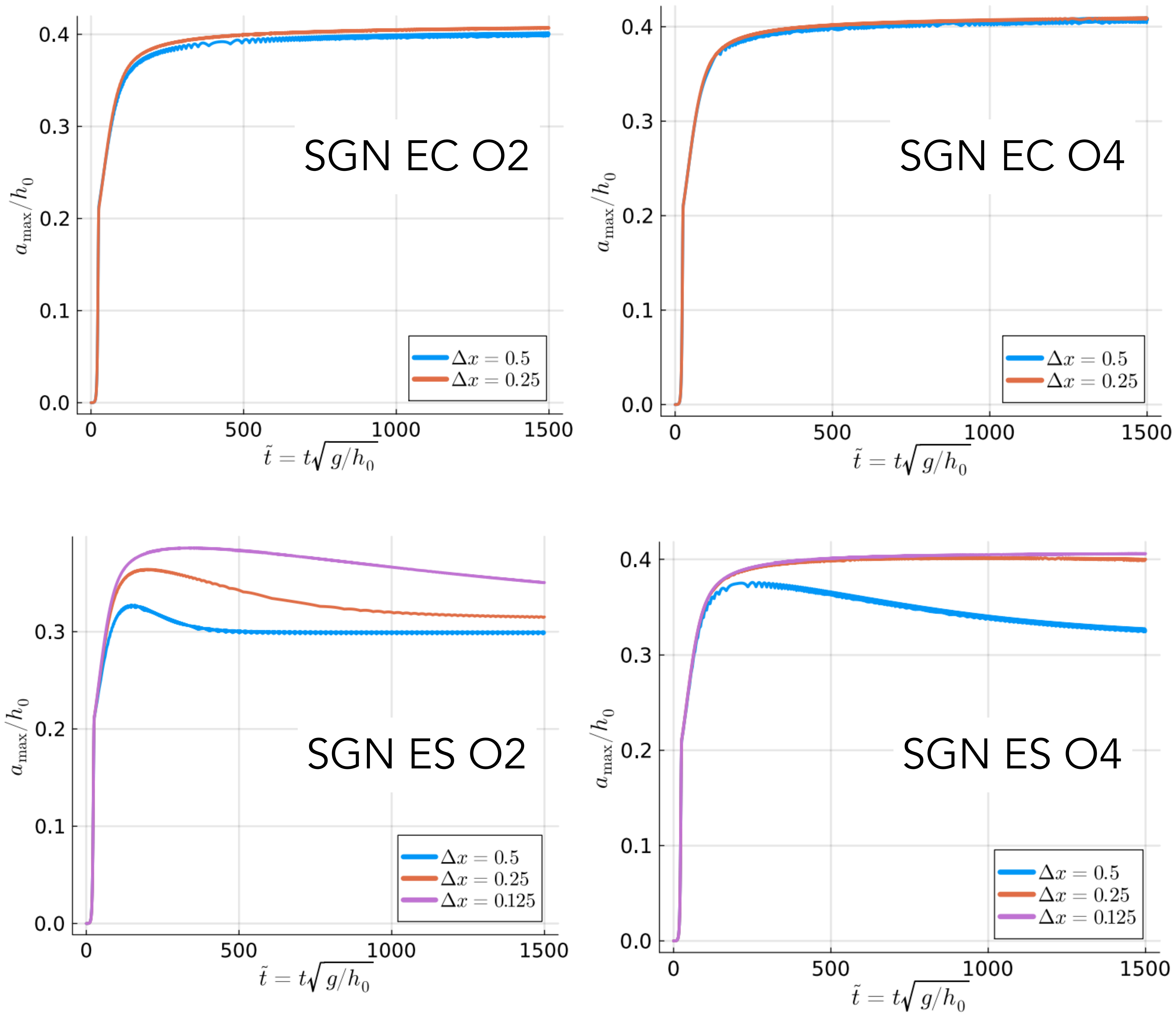


For long term propagation there is a strong impact of numerical dissipation on amplitudes and phase !

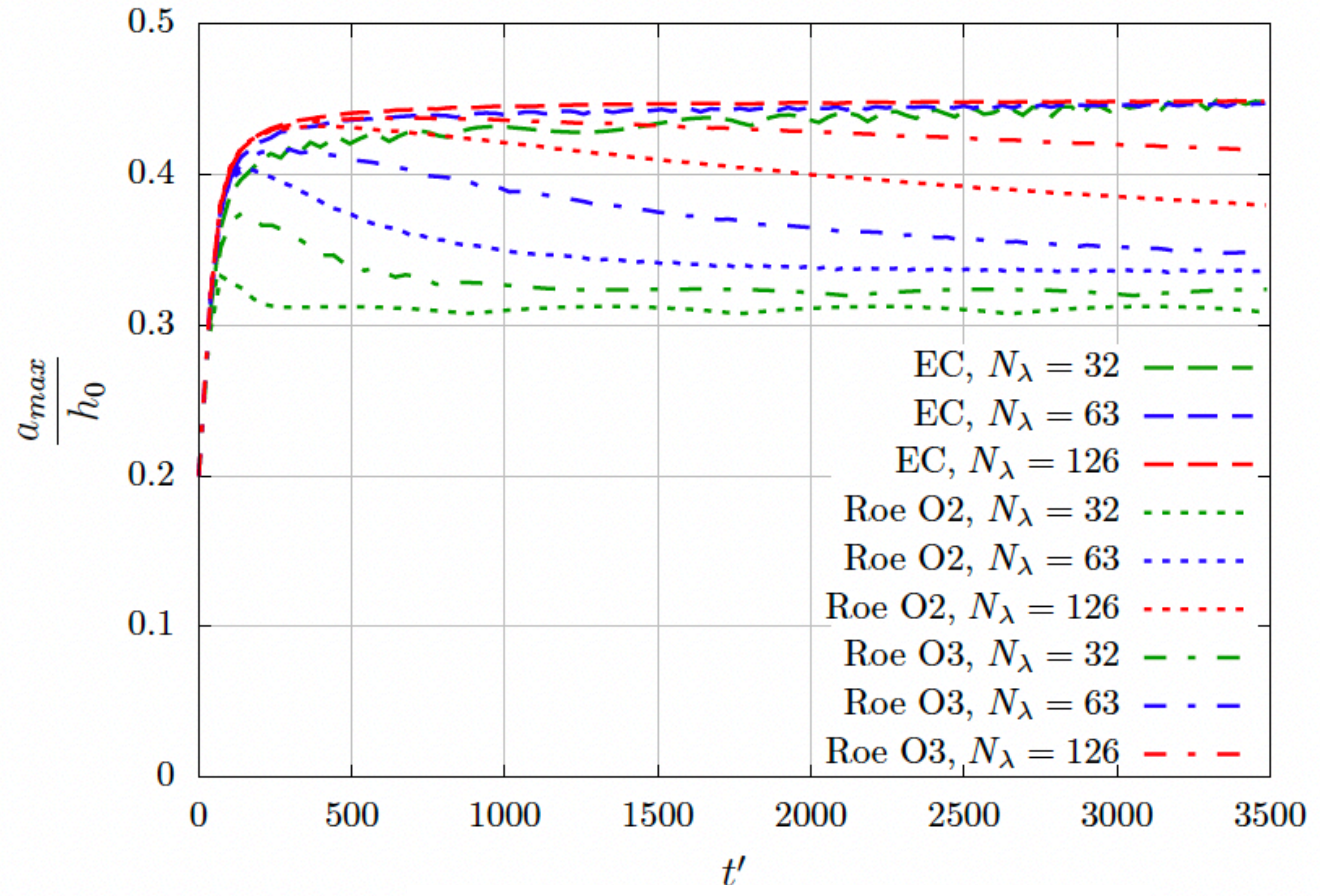
also for high orders …

Numerics vs physics

Example: long term, no friction



Jouy et al, Appl.Math.Mod. 2024

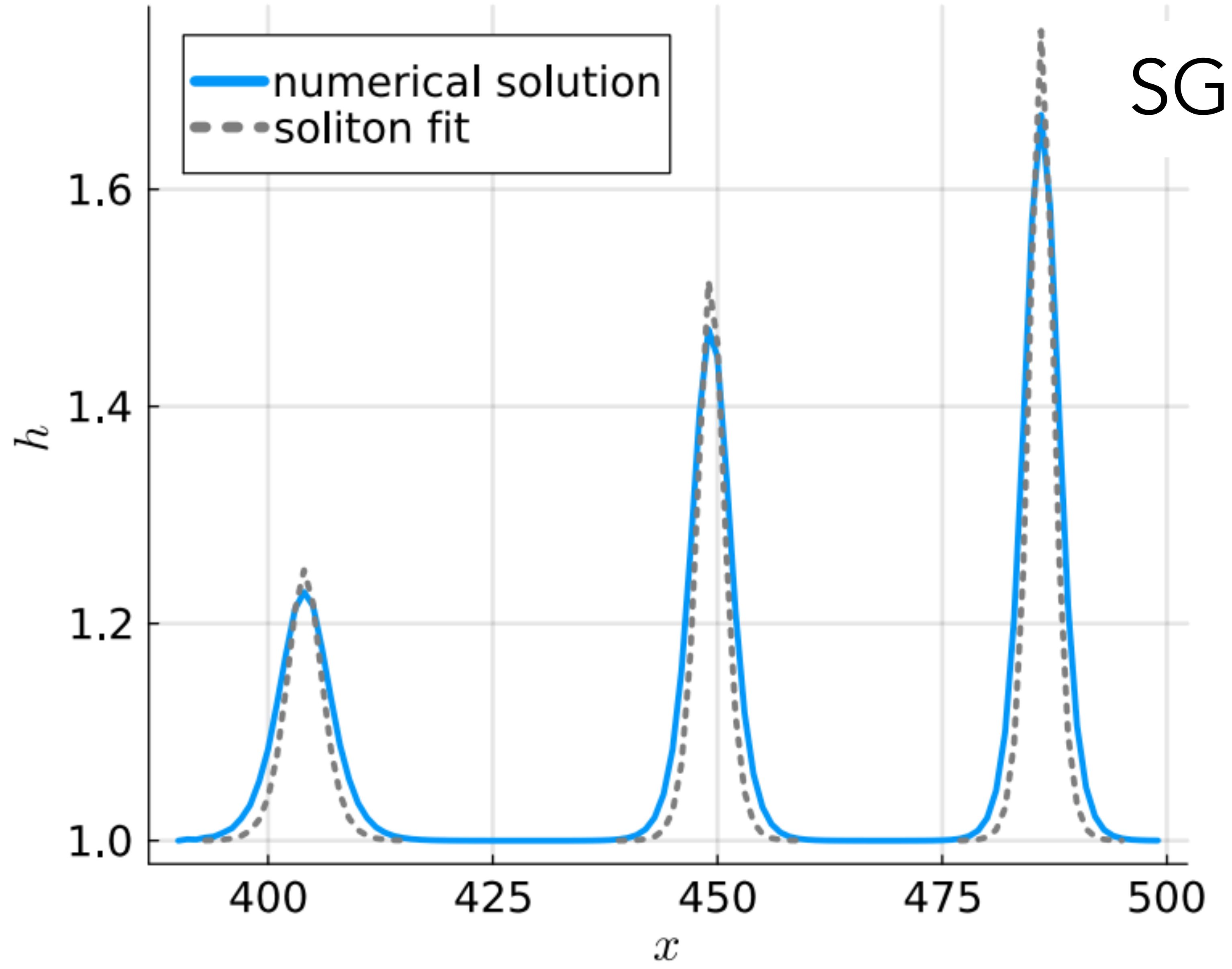


WL Diss vs no Diss (EC)

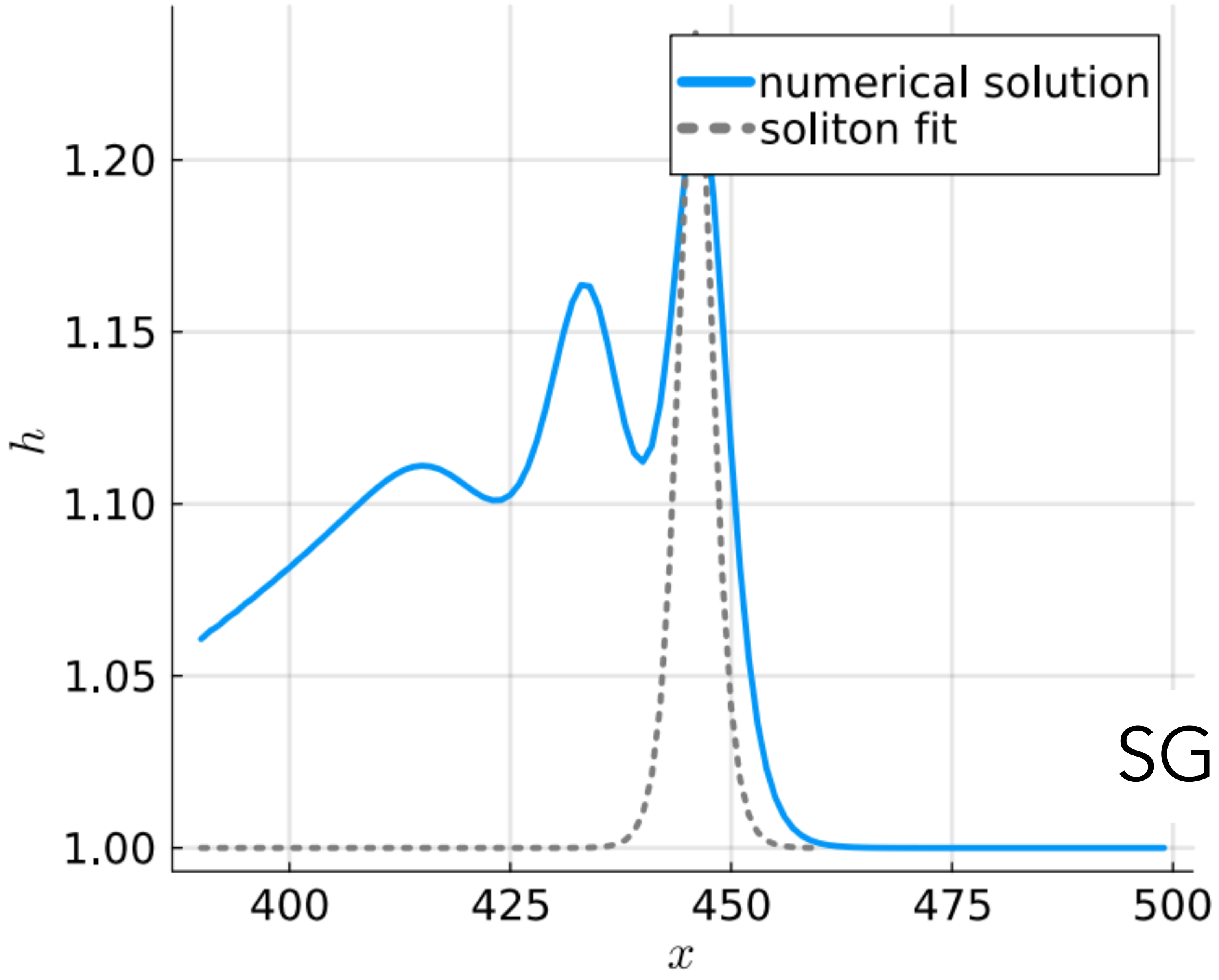
For very long term numerical dissipation stabilizes undular bores with
MESH DEPENDENT amplitude and
MESH DEPENDENT phase …

Numerics vs physics

Example: very long term, no friction

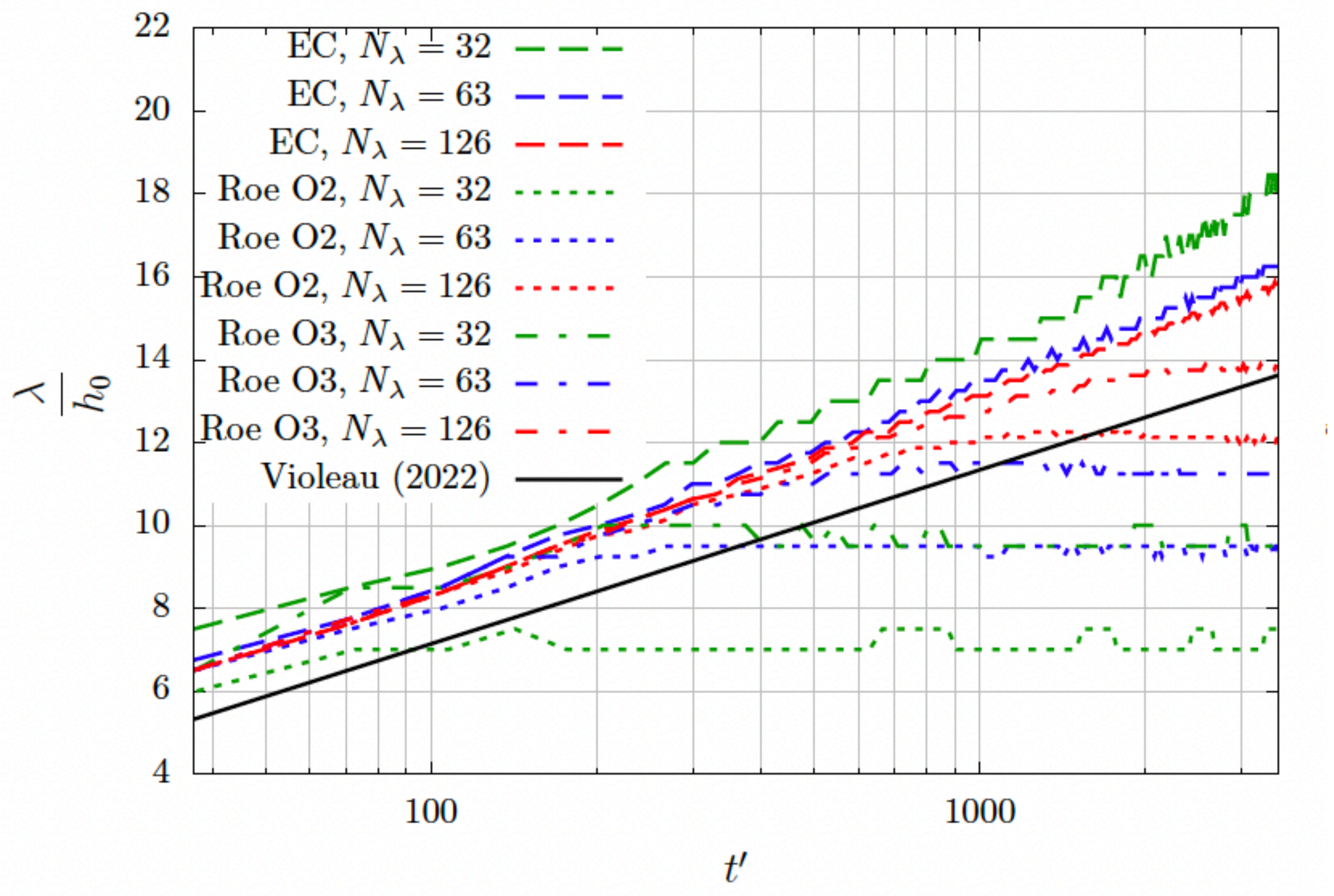


SGN EC O2



SGN ES O2

Jouy et al, Appl.Math.Mod. 2024

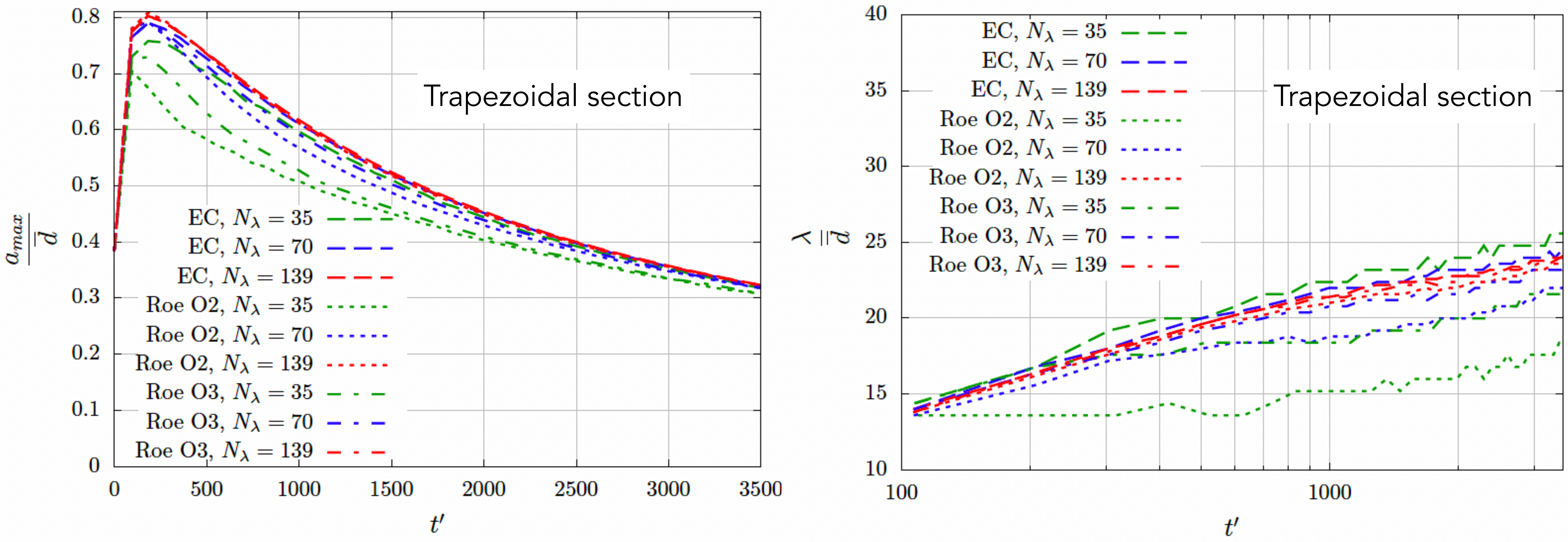


WL Diss vs no Diss (EC)

For very long term numerical dissipation stabilizes undular bores with
MESH DEPENDENT amplitude and
MESH DEPENDENT phase …

Numerics vs physics

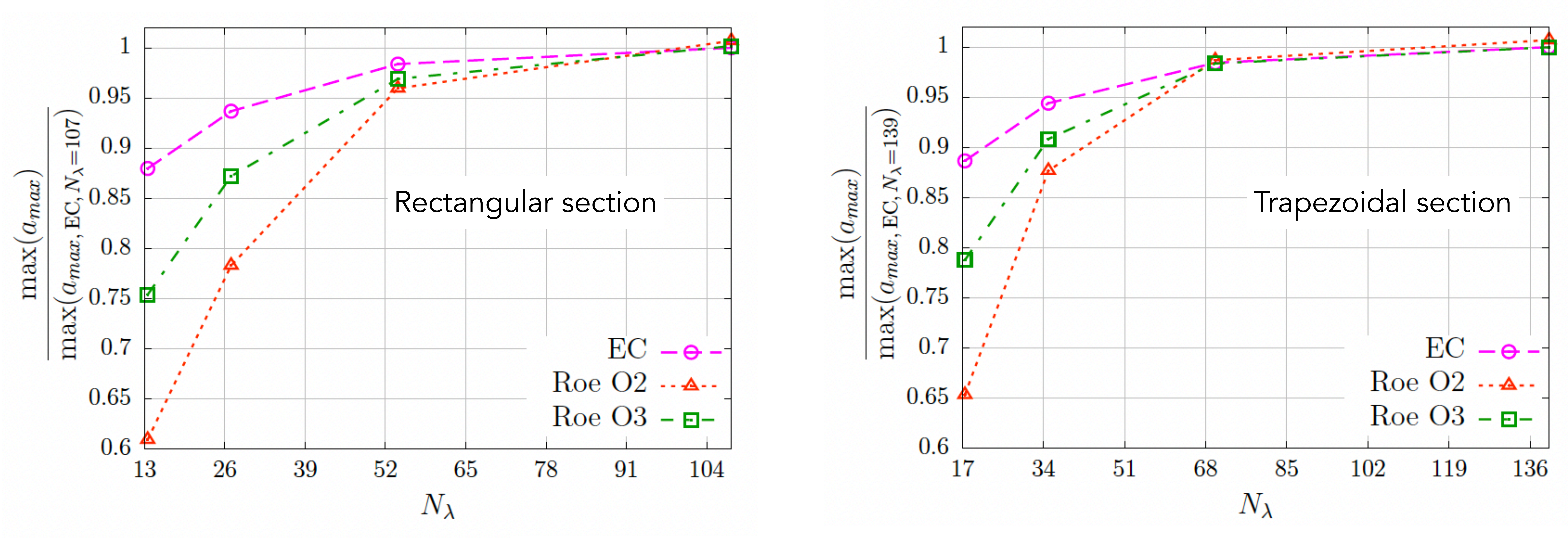
Example: long term, with friction



WL Diss vs no Diss (EC), with friction

Jouy et al, Appl.Math.Mod. 2024

Example: long term, with friction



WL Diss vs no Diss (EC), with friction

EC costs "roughly" as much as Roe-O2

Jouy, 2024

Some take away messages

Part I: the study of Favre waves has revealed importance of “transverse" dispersion

- “Transverse" dispersion is related to the geometry of the bathymetry
- “Transverse" dispersion is hydrostatic, and is well approximated by the SW equations
- **Model for vertical AND horizontal dispersive processes in 1D ? What is the transition mechanism ?**

Part II: long time simulations revealed (not surprisingly) the impact of numerical dissipation

- Short times reveal no real impact (unless extremely coarse meshes are used)
- Long times: numerical dissipation stabilises mesh depended waves (also in presence of real dissipation)
- Long times: energy conservative/non-dissipative approaches allow to work on coarse meshes
- **What notion of stability ? How to concile the two without increasing the computational cost ?**

References

Content of the presentation

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