General boundary conditions for the Boussinesq-Abbott model with varying topography

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SORBONNE UNIVERSITÉ

Institut de Mathématiques de Bordeaux

Objectives

Project supported by IMPT and ANR Bourgeons

Supervision : David Lannes and Philippe Bonneton

Long term goal: study extreme waves in littoral area

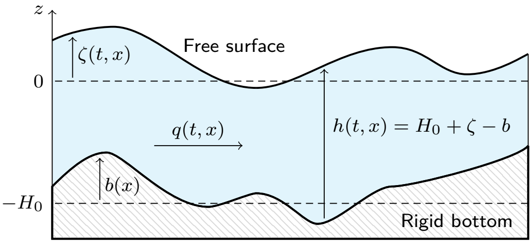
Outline

Boussinesq-Abbott model with varying bottom:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 + h_b \mathcal{T}) \partial_t q + \partial_x f(\zeta, q) = -gh\partial_x b \end{cases} \quad \text{(BA)}$$

with $h_b = H_0 - b$ (depth at rest) and

$$f(\zeta, q) = \frac{q^2}{h} + \frac{gh^2}{2}, \qquad h_b \mathcal{T}(\cdot) = -\frac{1}{3}\partial_x(h_b^2 \partial_x \cdot) \ + \ \text{lower order terms}$$



The PDEs

Initial-boundary value problem:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 + h_b \mathcal{T})\partial_t q + \partial_x f(\zeta, q) = -gh\partial_x b \end{cases} \quad \text{in } (0, \ell) \tag{BA}$$

completed with

$$(\zeta, q)_{|t=0} = (\zeta_0, q_0), \qquad \zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t)$$

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How to recover $q_{|x=0,\ell}(t)$? (missing data)

- Hyperbolic case ($h_b \mathcal{T} \equiv 0$): Riemann invariants
- Dispersive case: need to invert $(1 + h_b \mathcal{T}) \rightarrow$ requires knowledge on $\partial_t q_{|x=0,\ell}$

Some references

[WKS99] Wei, Kirby, Sinha. "Generation of waves in Boussinesq models using a source function method" (1999)

[DMS09] Dougalis, Mitsotakis, Saut. "On initial-boundary value problems for a Boussinesq system of BBM-BBM type in a plane domain" (2009)

[LW20] Lannes, Weynans. "Generating boundary conditions for a Boussinesq system" (2020)

[BL22] Beck, Lannes. "Freely floating objects on a fluid governed by the Boussinesq equations" (2022)

[LR24] Lannes, R. "General boundary conditions for a Boussinesq model with varying bathymetry" (2024)

Reformulation of the model (flat bottom)

Flat bottom case ($b \equiv \text{Cst}$): discharge eq. simplifies to

$$(1 - \kappa^2 \partial_{xx}^2)\partial_t q + \partial_x f(\zeta, q) = 0 \qquad \text{in } (0, \ell)$$

Fix $0 \leq t \leq T$, then $y(x) = \partial_t q(t, x)$ satisfies an ODE of the form

$$\begin{cases} y - \kappa^2 y'' = \phi(x) \\ y(0) = \dot{q}_{|_0}, \quad y(\ell) = \dot{q}_{|_\ell} \end{cases}$$

Equivalently: $y = y_h + y_b$ with $\begin{cases} y_h - \kappa^2 y_h'' = 0 \\ y_h(0) = \dot{q}_{|_0}, \quad y_h(\ell) = \dot{q}_{|_\ell} \end{cases}$ and $\begin{cases} y_b - \kappa^2 y_b'' = \phi(x) \\ y_b(0) = y_b(\ell) = 0 \end{cases}$

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Define R^0 as the inverse of $(1-\kappa^2\partial_{xx}^2)$ with **homogeneous Dirichlet** conditions at $x = 0, \ell$

$$\Rightarrow \quad \partial_t q = \underbrace{-R^0\partial_x f}_{y_b} + \underbrace{\mathfrak{s}_{(0)}(x)\dot{q}_{|_{x=0}} + \mathfrak{s}_{(\ell)}(x)\dot{q}_{|_{x=\ell}}}_{y_h}$$

$$\text{where} \quad \begin{cases} (1-\kappa^2\partial_{xx}^2)\mathfrak{s}_{(0)} = 0 \\ \mathfrak{s}_{(0)}(0) = 1, \quad \mathfrak{s}_{(0)}(\ell) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (1-\kappa^2\partial_{xx}^2)\mathfrak{s}_{(\ell)} = 0 \\ \mathfrak{s}_{(\ell)}(0) = 0, \quad \mathfrak{s}_{(\ell)}(\ell) = 1 \end{cases} . \qquad (1)$$

Reformulation of the model (flat bottom)

Note R^1 the inverse of $(1 - \kappa^2 \partial^2_{xx})$ with **homogeneous Neumann** conditions at $x = 0, \ell$

$$\boxed{\Rightarrow R^0 \partial_x = \partial_x R^1}$$

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Proposition 1 (D. Lannes, R.)

Let (ζ, q) *be such that the* **compatibility conditions** *hold:*

$$\begin{cases} \zeta_{|t=0}(0) = g_0(0) \\ -\partial_x q_{|t=0}(0) = \dot{g}_0(0) \end{cases}, \qquad \begin{cases} \zeta_{|t=0}(\ell) = g_\ell(0) \\ -\partial_x q_{|t=0}(\ell) = \dot{g}_\ell(0) \end{cases}.$$

Then the two assertions are equivalent:

1. *The pair* (ζ, q) *satisfies the IBVP* (BA) *with* $\zeta(\cdot, 0) = g_0$ *and* $\zeta(\cdot, \ell) = g_\ell$

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2. *The pair* (ζ, q) *satisfies the IVP*

$$\begin{cases}\partial_t\zeta + \partial_x q = 0\\ \partial_t q + \partial_x(R^1 f) = \mathfrak{s}_{(0)}\dot{q}_{|_0} + \mathfrak{s}_{(\ell)}\dot{q}_{|_\ell}\end{cases} \quad \text{in } (0,\ell), \tag{2}$$

with the trace equations

$$\begin{pmatrix}\mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0)\\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell)\end{pmatrix}\begin{pmatrix}\dot{q}_{|_0}\\ \dot{q}_{|_\ell}\end{pmatrix} = \frac{1}{\kappa^2}\begin{pmatrix}(R^1-\mathrm{id})_{|_0} f\\ (R^1-\mathrm{id})_{|_\ell} f\end{pmatrix} - \begin{pmatrix}\ddot{g}_0\\ \ddot{g}_\ell\end{pmatrix} \tag{3}$$

Reformulation of the model (flat bottom)

Proof: $\boxed{\Rightarrow}$

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x (R^1 f) = \mathfrak{s}_{(0)} \dot{q}_{|0} + \mathfrak{s}_{(\ell)} \dot{q}_{|\ell} \end{cases}$$

Reformulation of the model (flat bottom)

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Proof: $\boxed{\Rightarrow}$

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_x\Big(\partial_t q + \partial_x (R^1 f) = \mathfrak{s}_{(0)} \dot{q}_{|0} + \mathfrak{s}_{(\ell)} \dot{q}_{|\ell}\Big) \end{cases} \quad \Rightarrow \quad -\partial_{tt}\zeta + \underbrace{\partial^2_{xx}(R^1 f)}_{\frac{1}{\kappa^2}(R^1 - \mathrm{id})f} = \mathfrak{s}'_{(0)} \dot{q}_{|0} + \mathfrak{s}'_{(\ell)} \dot{q}_{|\ell}$$

By construction

$$(1 - \kappa^2 \partial^2_{xx}) R^1 = \mathrm{id} \;\Rightarrow\; \partial^2_{xx} R^1 = \kappa^{-2}(R^1 - \mathrm{id})$$

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Taking the trace at $x = 0, \ell$ and using that $\zeta_{|x=0,\ell} = g_{0,\ell}$:

$$\begin{cases} -\ddot{g}_0 + \dfrac{1}{\kappa^2}(R^1 - \mathrm{id})_{|x=0} f(\zeta, q) = \mathfrak{s}'_{(0)}(0) \dot{q}_{|0} + \mathfrak{s}'_{(\ell)}(0) \dot{q}_{|\ell} \\ -\ddot{g}_\ell + \dfrac{1}{\kappa^2}(R^1 - \mathrm{id})_{|x=\ell} f(\zeta, q) = \mathfrak{s}'_{(0)}(\ell) \dot{q}_{|0} + \mathfrak{s}'_{(\ell)}(\ell) \dot{q}_{|\ell} \end{cases}$$

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$$\begin{cases} -\ddot{g}_0 + \dfrac{1}{\kappa^2}(R^1 - \mathrm{id})_{|x=0} f(\zeta, q) = \mathfrak{s}'_{(0)}(0)\dot{q}_{|0} + \mathfrak{s}'_{(\ell)}(0)\dot{q}_{|\ell} \\ -\ddot{g}_\ell + \dfrac{1}{\kappa^2}(R^1 - \mathrm{id})_{|x=\ell} f(\zeta, q) = \mathfrak{s}'_{(0)}(\ell)\dot{q}_{|0} + \mathfrak{s}'_{(\ell)}(\ell)\dot{q}_{|\ell} \end{cases} \quad \& \quad \begin{cases} \zeta_{|t=0}(x = 0, \ell) = g_{0,\ell}(0) \\ -\partial_x q_{|t=0}(x = 0, \ell) = \dot{g}_{0,\ell}(0) \end{cases}$$

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Following previous steps we also have

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Compatibility conditions $\Rightarrow$ solution given by $(\zeta_{|0}, \zeta_{|\ell}) = (g_0, g_\ell)$

More general boundary conditions

Possibility to enforce general boundary conditions

$$\xi_0^+(\zeta_{|_0}, q_{|_0})(t) = g_0(t), \qquad \xi_\ell^-(\zeta_{|_\ell}, q_{|_\ell})(t) = g_\ell(t). \tag{4}$$

For instance, $\xi^\pm$ given by q or Saint-Venant Riemann invariants

$$\mathcal{R}_\pm(U) = \frac{q}{h} \pm 2\sqrt{gh}$$

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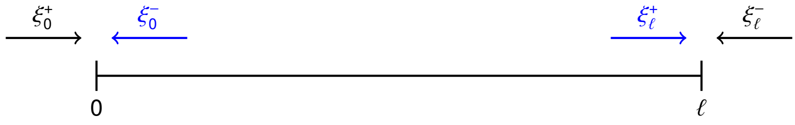
Adapt trace ODE in terms of missing data (outgoing information ξ_0^- and ξ_ℓ^+)

$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q(\xi_0^+, \xi_0^-) \\ q(\xi_\ell^+, \xi_\ell^-) \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \mathrm{id})_{|_0} f \\ (R^1 - \mathrm{id})_{|_\ell} f \end{pmatrix} - \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \zeta(\xi_0^+, \xi_0^-) \\ \zeta(\xi_\ell^+, \xi_\ell^-) \end{pmatrix}$$



Case of a varying topography ($b \not\equiv$ Cst)

$$\boxed{(1 + h_b\mathcal{T})\partial_t q + \partial_x f(\zeta, q) = -gh\partial_x b}$$

where $h_b\mathcal{T}(\cdot) = -\frac{1}{3}\partial_x(h_b^2\partial_x\cdot)\ +$ lower order terms

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Note R_b^0 the inverse of $(1 + h_b\mathcal{T})$ with **homogeneous Dirichlet cond.** and

$$\begin{cases} (1 + h_b\mathcal{T})\mathfrak{s}_{(b,0)} = 0 \\ \mathfrak{s}_{(b,0)}(0) = 1, \quad \mathfrak{s}_{(b,0)}(\ell) = 0 \end{cases} \qquad \begin{cases} (1 + h_b\mathcal{T})\mathfrak{s}_{(b,\ell)} = 0 \\ \mathfrak{s}_{(b,\ell)}(0) = 0, \quad \mathfrak{s}_{(b,\ell)}(\ell) = 1 \end{cases}$$

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$$\Rightarrow \boxed{\partial_t q = -R_b^0\partial_x f - gR_b^0(h\partial_x b) + \mathfrak{s}_{(b,0)}\dot{q}_{|_0} + \mathfrak{s}_{(b,\ell)}\dot{q}_{|_\ell}}$$

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$$\nexists R_b^1 = \left(\sum_{i=0}^{2} c_i \partial_x^i \right)^{-1} \quad \text{such that} \quad R_b^0 \partial_x = \partial_x R_b^1$$

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However we can construct R_b^1 verifying for some $(\alpha, \beta)(x)$

$$R_b^0 (\partial_x + \alpha) = (\partial_x + \beta) R_b^1$$

$$\Rightarrow \ \partial_t q = -\partial_x R_b^1 f \underbrace{- \beta R_b^1 f + R_b^0 (\alpha f - g h \partial_x b)}_{\mathfrak{B}(\zeta, q)} + \mathfrak{s}_{(b,0)} \dot{q}_{|_0} + \mathfrak{s}_{(b,\ell)} \dot{q}_{|_\ell}$$

Case of a varying topography ($b \not\equiv \mathrm{Cst}$)

Proposition 2 (D. Lannes, R.)

Let (ζ, q) *be such that the* **compatibility conditions** *hold:*

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$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x (R_b^1 f) = \mathfrak{B}(\zeta, q) + \mathfrak{s}_{(b,0)} \dot{q}_{|0} + \mathfrak{s}_{(b,\ell)} \dot{q}_{|\ell} \end{cases} \quad \text{in } (0, \ell), \tag{5}$$

with the trace equations

$$\begin{pmatrix} \mathfrak{s}'_{(b,0)}(0) & \mathfrak{s}'_{(b,\ell)}(0) \\ \mathfrak{s}'_{(b,0)}(\ell) & \mathfrak{s}'_{(b,\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_{|0} \\ \dot{q}_{|\ell} \end{pmatrix} + V_{boundary}(\zeta_{|0,\ell}, q_{|0,\ell}) = V_{interior}[\zeta, q] - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix} \tag{6}$$

where $V_{boundary}$, $V_{interior}$ *are known.*

Numerical experiment

- Standard finite differences for nonlocal terms and trace equations
- Finite volumes for interior equations (Lax-Friedrichs or MacCormack)

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Asymptotic stability

Question: starting from a wrong initial condition, can we recover the reference solution by enforcing appropriate boundary conditions?

Numerical experiment

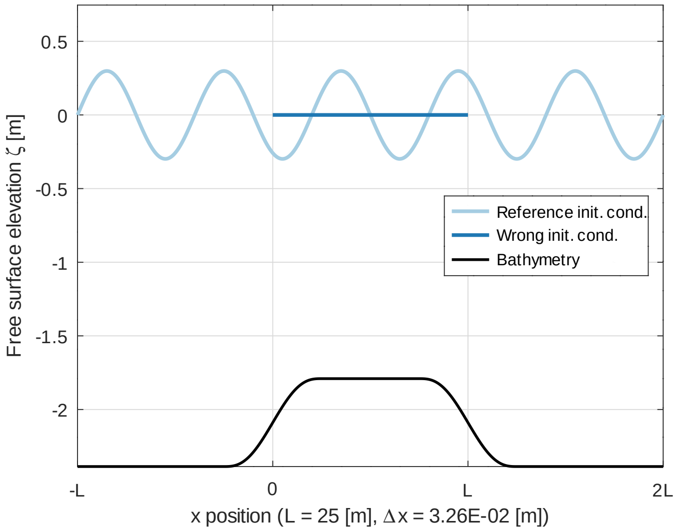

Figure: Sine over bump

Numerical experiment

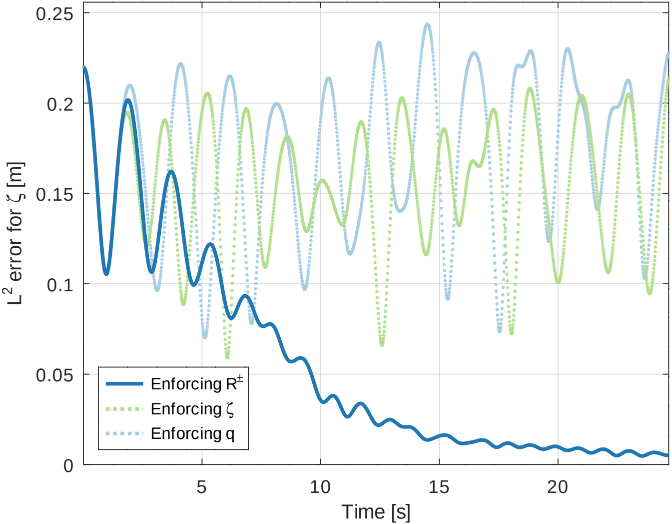

Figure: L^2 error

Conclusion

- Boussinesq-Abbott model is accurate, but boundary conditions are challenging
- Reformulation strategy allows to recover missing data
- Extension to varying bathymetries & general boundary conditions
- Numerical validation and experiments (asymptotic stability)

Perspectives: high order DGFEM code, improved dispersion relation

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Thank you for your attention!