Equivariant Euler characteristics and sheaf resolvents

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Abstract: For certain tame abelian covers of arithmetic surfaces X/Y we obtain striking formulas, involving a quadratic form derived from intersection numbers, for the equivariant Euler characteristics of both the canonical sheaf $\omega_{X/Y}$ and also its square root $\omega_{X/Y}^{1/2}$. These formulas allow us to carry out explicit calculations; in particular, we are able to exhibit examples where these two Euler characteristics and that of the structure sheaf of X are all different and non-trivial. Our results are obtained by using resolvent techniques together with the local Riemann-Roch approach developed in [CPT].

Introduction

Let N/K be a finite Galois extension of number fields with Galois group G. A number of interesting arithmetic modules may be associated to such an extension: the ring of algebraic integers $\mathcal{O}_N$ of N; the codifferent $\mathcal{D}_{N/K}^{-1}$ of N/K; and, when the ramification subgroups of N/K are of odd order, the square root of the codifferent $\mathcal{D}_{N/K}^{-1/2}$. Their structure as Galois modules has been studied extensively. When N/K is at most tamely ramified, they are all three locally free $\mathbf{Z}[G]$-modules. It was proved in [T1] and [T2] that, for any tame Galois extension N/K, the classes of $\mathcal{O}_N$ and $\mathcal{D}_{N/K}^{-1}$ in $\mathrm{Cl}(\mathbf{Z}[G])$, the locally free class group of $\mathbf{Z}[G]$-modules, are equal; that is to say, $\mathcal{O}_N$ is a selfdual $\mathbf{Z}[G]$-module. If in addition we suppose that G is of odd order, then in fact $\mathcal{O}_N$, $\mathcal{D}_{N/K}^{-1}$ and $\mathcal{D}_{N/K}^{-1/2}$ are all three free over $\mathbf{Z}[G]$. This result for the ring of integers is a consequence of the Fröhlich conjecture, which was proved in [T3]. The result for the square root of the codifferent was obtained in [ET]. In such a situation where one of the modules $\mathcal{O}_N$, $\mathcal{D}_{N/K}^{-1}$ and $\mathcal{D}_{N/K}^{-1/2}$

are $\mathbf{Z}[G]$-free, we shall say that the module in question has a normal integral basis (abbreviated NIB).

Over the past ten, or so, years a number of articles have been devoted to the study of the analogues of such Galois module problems in higher dimensions. The study of such questions for arithmetic surfaces is the central topic of this paper.

Throughout this article G denotes a finite group of *odd* order . We consider a G-cover $\pi : X \to Y$ of schemes which are projective and flat over $Spec(\mathbf{Z})$. For a G-equivariant locally free sheaf $\mathcal{F}$ on X, the coherent cohomology groups $H^i(X, \mathcal{F})$ are finitely generated $\mathbf{Z}[G]$-modules which are of considerable interest. Their study leads us to consider the hypercohomology complex $\mathbf{R\Gamma}(X, \mathcal{F})$ of $\mathbf{Z}[G]$-modules. When the action of G is tame, i.e. when for any point x of X the order e_x of the inertia group I_x of x is coprime to the residue field characteristic of x, this complex is perfect; that is to say it is isomorphic, in the derived category of complexes of $\mathbf{Z}[G]$-modules, to a bounded complex of projective $\mathbf{Z}[G]$-modules. The fact that this complex is perfect, under such hypotheses, was first shown in [CE] (see also [CEPT3] and [P], Section 2). This then leads to the following definition:

The sheaf $\mathcal{F}$ is said to have a normal integral basis when the complex $\mathbf{R\Gamma}(X, \mathcal{F})$, in the derived category of complexes of $\mathbf{Z}[G]$-modules, is isomorphic to a bounded complex of free $\mathbf{Z}[G]$-modules .

Following Chinburg in [C1], we can associate to $\mathcal{F}$ an equivariant Euler characteristic $\chi^P(\mathcal{F})$ in $\mathrm{Cl}(\mathbf{Z}[G])$, which measures the obstruction to the existence of a NIB for $\mathcal{F}$. More precisely, the sheaf $\mathcal{F}$ has a NIB if and only if $\chi^P(\mathcal{F})$ is trivial. We observe that in the situation described above where we take $X = \mathrm{Spec}(\mathcal{O}_N)$, it follows from the above mentioned theorems that the coherent sheaves associated to $\mathcal{O}_N$, $\mathcal{D}_{N/K}^{-1}$ and $\mathcal{D}_{N/K}^{-1/2}$ all have a NIB, since their Euler characteristics are precisely the classes of these modules in $\mathrm{Cl}(\mathbf{Z}[G])$.

We now introduce the analogous sheaves of modules in the geometric setting. We start by recalling that the different divisor of X/Y is the divisor on X given by

$$D_{X/Y} = \sum_x (e_x - 1)x$$

with x running over the set of codimension 1 points of X. We shall be particularly interested in the following G-equivariant sheaves: the structural sheaf $\mathcal{O}_X$; the canonical sheaf $\omega_{X/Y} = \mathcal{O}_X(D_{X/Y})$ of the cover $X \to Y$; and its square root $\omega_{X/Y}^{1/2} = \mathcal{O}_X(\frac{1}{2}D_{X/Y})$.

The main object of our study is the existence, or otherwise, of a NIB for these sheaves by computing and comparing their equivariant Euler characteristics. In the case when G is abelian and X is a G-torsor over Y, the morphism $\pi : X \to Y$ is etale and hence the sheaves $\omega_{X/Y}$ and $\omega_{X/Y}^{1/2}$ both coincide with the structural sheaf $\mathcal{O}_X$, and furthermore they have a NIB by a theorem of Pappas in [P]. We shall see presently that when the cover $\pi : X \to Y$ is not etale then our sheaves do not necessarily have NIB. A new approach to such questions has been recently developed in [CPT]. Our results depend strongly on the use of this paper, and illustrate how their new techniques permit the efficient calculation of such Euler characteristics.

We now describe the contents of the paper. In Section 1 we describe the situation that we wish to study; here we introduce our notation and we state our main results. In Section 2 we present some general results on equivariant duality; the content of this section derives from some working notes of T. Chinburg and G. Pappas, and we are most grateful for their permission to use their work here. In Section 3 we introduce the notions of a sheaf resolvent and a divisor resolvent; these two concepts play a central role throughout this paper. The main theorems are proved in Sections 4 and 5. We conclude with the detailed study of some examples in Section 6; we are extremely grateful to Arnaud Jehanne for his help with the computations in 6.c.

1 Notation and main results

Let G be a finite abelian group of odd order n, and let R denote either a Dedekind domain or a complete discrete valuation ring; in all cases we denote the field of fractions of R by K. In the case where R is a valuation ring, we shall assume that R contains the n-th roots of unity and that its residue field k is perfect and of characteristic prime to n. We consider a regular flat projective scheme $Y \to S = \mathrm{Spec}(R)$. The fibres are of constant dimension and we denote this fibral dimension by d. Let $\pi : X \to Y$ be a G-cover which is generically a G-torsor on Y with X is regular. Note that it follows from the assumptions that $\pi : X \to Y$ is flat (see Remark 3.1.a in [CPT]). When $R = \mathbf{Z}$ we suppose that the ramification locus of this cover is supported on a finite set of rational primes Σ which is disjoint with the set of prime divisors of the order of G.

In order to state our results we now suppose that $R = \mathbf{Z}$ and $d = 1$. We consider a G-equivariant, coherent and invertible sheaf $\mathcal{F}$ on X. For any $p \in \Sigma$ we denote by $\mathbf{Z}'_p$ the subring of $\mathbf{Q}_p^c$ obtained by adjoining the n-th roots of unity to $\mathbf{Z}_p$. By forming the base changes of $\pi : X \to Y$ by $\mathrm{Spec}(\mathbf{Z}_p) \to \mathrm{Spec}(\mathbf{Z})$ and by $\mathrm{Spec}(\mathbf{Z}'_p) \to \mathrm{Spec}(\mathbf{Z})$, we obtain G-covers $\pi_p : X_p \to Y_p$ and $\pi'_p : X'_p \to Y'_p$. We denote by $\mathcal{F}'_p$ the X'_p- sheaf obtained from $\mathcal{F}$ by pullback.

For an equivariant sheaf $\mathcal{F}'_p$ as above, for any $\mathbf{Q}_p^c$-character φ of G and for a codimension one point y of Y'_p, in Section 2 we will define a rational number $v_y(\mathcal{F}'_{p,\varphi})$, which depends on the ramification of y in the cover $X'_p \to Y'_p$. We shall then use these rational numbers to define the *local resolvent divisor of* $\mathcal{F}$ *at* p by setting

$$r_p(\mathcal{F}, \varphi) = \sum_y v_y(\mathcal{F}'_{p,\varphi})y, \tag{1.1}$$

where y runs over the set of codimension one points of Y'_p which are contained in the special fiber $Y'^{(s)}_p$ of $Y'_p \to \mathrm{Spec}(\mathbf{Z}'_p)$. We may consider $r_p(\mathcal{F}, \varphi)$ as a vector with rational coordinates. Presently we will see that $nv_y(\mathcal{F}'_{p,\varphi})$ is an integer for any such y, so that $n.r_p(\mathcal{F}, \varphi)$ is a divisor of Y'_p. These resolvent divisors play a similar role to that of Lagrange resolvents in the algebraic number field setting. For codimension one points y and z of $Y'^{(s)}_p$, we denote their intersection number by $y \cdot z$. Recall that this integer is the degree of the line bundle $\mathcal{O}_{Y'_p}(y)$ restricted to z. Let $\omega_{Y'_p}$ be the canonical sheaf of $Y'_p \to \mathbf{Z}'_p$ and denote its first Chern class by $c_1(\omega_{Y'_p})$. Then we define the integer $c_1(\omega_{Y'_p}) \cdot y$ as the degree of the 0-cycle $c_1(\omega_{Y'_p}) \cap y$ of Y'_p (see Chapter 2 in [Fu]). Finally we set:

$$r_p(\mathcal{F}, \varphi)^2 = \sum_{y,z} v_y(\mathcal{F}'_{p,\varphi})v_z(\mathcal{F}'_{p,\varphi})y \cdot z \ , \tag{1.2}$$

$$c_1(\omega_{Y'_p}) \cdot r_p(\mathcal{F}, \varphi) = \sum_y v_y(\mathcal{F}'_{p,\varphi})c_1(\omega_{Y'_p}) \cdot y, \tag{1.3}$$

and

$$T_p(\mathcal{F}, \varphi) = r_p(\mathcal{F}, \varphi)^2 + c_1(\omega_{Y'_p})r_p(\mathcal{F}, \varphi). \tag{1.4}$$

We observe that $r_p(\mathcal{F}, \varphi)^2$ may be thought of as the quadratic form, defined by the intersection matrix, evaluated on a local resolvent divisor; while $c_1(\omega_{Y'_p}) \cdot r_p(\mathcal{F}, \varphi)$ may be thought of as a linear form, evaluated on the same local resolvent divisor.

Suppose now that there exists a locally free $\mathcal{O}_Y$-sheaf $\omega_{Y/S}^{1/2}$ with the property that $\omega_{Y/S}^{1/2} \otimes \omega_{Y/S}^{1/2} = \omega_{Y/S}$; we shall refer to $\omega_{Y/S}^{1/2}$ as a square root of $\omega_{Y/S}^{1/2}$.

We then define a twist $\tilde{\mathcal{F}}$ of $\mathcal{F}$ by setting

$$\tilde{\mathcal{F}} = \mathcal{F} \otimes \pi^*(\omega_{Y/S}^{1/2}) \ . \tag{1.5}$$

It follows from the adjunction formula that $\tilde{\omega}_{X/Y}^{1/2}$ has the property that $\tilde{\omega}_{X/Y}^{1/2} \otimes \tilde{\omega}_{X/Y}^{1/2} = \omega_{X/S}$ and we therefore denote this sheaf by $\omega_{X/S}^{1/2}$.

Once and for all we fix a sufficiently large finite Galois extension E of $\mathbf{Q}$. We denote the group of finite ideles of E by $J_f(E)$; we let R_G denote the additive group of the virtual E-characters of G and we let t denote the group homomorphism

$$t : \mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_f(E)) \rightarrow \mathrm{Cl}(\mathbf{Z}[G]) \tag{1.6}$$

of Fröhlich's so-called " Hom-description ", (see Chapter 2 in [F1]). For each finite prime l of $\mathbf{Z}$ we fix an embedding $j_l : \mathbf{Q}^c \rightarrow \mathbf{Q}_l^c$ and we denote the closure of $j_l(E)$ in $\mathbf{Q}_l^c$ by E_l. The group $J_l(E) = (E \otimes \mathbf{Q}_l)^\times$ may be identified with the Galois submodule of $J_f(E)$ consisting of the finite ideles of E which are equal to 1 outside l; we shall therefore view $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_l(E))$ as a subgroup of $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_f(E))$. We therefore obtain, via j_l, a homomorphism $j_l : E \otimes \mathbf{Q}_l \rightarrow E_l$; we also obtain an isomorphism $\varphi \mapsto (\varphi)^{j_l}$ between the $\mathbf{Q}^c$-characters of G and the $\mathbf{Q}_l^c$-characters of G and therefore an isomorphism between R_G and $R_{G,l}$, the additive groups of their virtual characters. It is well known, (see for instance Lemma II.2.1 of [F1]), that the homomorphism

$$j_l^* : \mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_l(E)) \rightarrow \mathrm{Hom}_{G_{\mathbf{Q}_l}}(R_{G,l}, E_l^*)$$

defined by $j_l^*(f)(\varphi) = f(\varphi^{j_l^{-1}})^{j_l}$ is an isomorphism. We shall often abbreviate the notation $j_l^*(f)$ to f^*. Our main results are the following:

Theorem 1.7 *With the above notation and under the above assumptions on X, Y, G, and for both $\mathcal{F} = \omega_{X/Y}$ or $\omega_{X/Y}^{1/2}$ we have the following equality in the class group $Cl(\mathbf{Z}[G])$:*

$$2\chi^P(\mathcal{O}_X) - 2\chi^P(\mathcal{F}) = \prod_{p \ \in \Sigma} t(g_p)$$

where g_p^* is the element of $Hom_{G_{\mathbf{Q}_p}}(R_{G,p}, E_p^*)$ defined on irreducible $\mathbf{Q}_p^c$-characters φ of G by

$$g_p^*(\varphi) = p^{T_p(\mathcal{F},\varphi) - T_p(\mathcal{O}_X,\varphi)} \ .$$

Theorem 1.8 *With the above notation and under the above assumptions on X, Y, G, suppose there is an $\mathcal{O}_Y$-sheaf $\omega_{Y/S}^{1/2}$ as above; then for both $\mathcal{F} = \omega_{X/Y}$ or $\omega_{X/Y}^{1/2}$ we have the following equality in the class group Cl($\mathbf{Z}[G]$)*

$$2\chi^P(\tilde{\mathcal{O}}_X) - 2\chi^P(\tilde{\mathcal{F}}) = \prod_{p\ \in\Sigma} t(\tilde{g}_p)$$

where $\tilde{g}_p^*$ is the element of $Hom_{G_{\mathbf{Q}_p}}(R_{G,p}, E_p^*)$ defined on irreducible $\mathbf{Q}_p^c$-characters φ of G by

$$\tilde{g}_p^*(\varphi) = p^{r_p(\mathcal{F},\varphi)^2 - r_p(\mathcal{O}_X,\varphi)^2} \quad .$$

Remarks.

1. The factor 2 in our formulas derives from the need to consider the determinant of cohomology of bundles of even rank in [CPT]; note that this factor can be removed in certain situations. (See Section 3 in [CPT] for further details.)

2. Since the sheaves $\mathcal{F}$, considered in the above theorems, are isomorphic to the structural sheaf $\mathcal{O}_X$ on the general fiber, it should be possible to apply the techniques developed in Section 8 of [CEPT1] to compute the difference between their Euler characteristics. This would lead, under certain assumptions, to a generalisation of our theorems to non-abelian G. The advantage of our method is that it yields an attractive formula, involving quadratic elements associated to resolvent divisors. One can hope that the two methods will provide quite different formulas, which it will be interesting to compare. This is something that we plan to do in the near future.

We denote by $\mathcal{M}$ the maximal order of $\mathbf{Q}[G]$ and we let $D(\mathbf{Z}[G])$ be the kernel of the group homomorphism $\mathrm{Cl}(\mathbf{Z}[G]) \to \mathrm{Cl}(\mathcal{M})$ induced by extension of scalars. Using the representative for the class $2\chi(\mathcal{O}_X) - 2\chi(\omega_{X/Y})$ in $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_f(E))$ given in Theorem 1.7 above, in Section 5 we will show:

Theorem 1.9 *Let G be a finite abelian l-group.*
i) The order of the class $2\chi^P(\mathcal{O}_X) - 2\chi^P(\omega_{X/Y})$ is a power of l.
ii) If l is a regular prime number, then $2\chi^P(\mathcal{O}_X) - 2\chi^P(\omega_{X/Y}) \in D(\mathbf{Z}[G])$.
iii) If G is of order l, then $2\chi^P(\mathcal{O}_X) = 2\chi^P(\omega_{X/Y})$.

We observe that this theorem provides a generalisation for arithmetic surfaces (up to a factor of 2) of Theorem 3.1 of [T2].

Remarks.
1. It follows from Theorem 4.3.2 in [CPET1] that the results of this last theorem remain true if we replace $\chi^P(\omega_{X/Y})$ by $\chi^P(\Omega^1_{X/\mathbf{Z}})$.
2. The class group $\mathrm{Cl}(\mathbf{Z}[G])$ carries a natural duality involution $x \mapsto x^D$ (see Section 2). The anti-selfduality of $\chi^P(\omega^{1/2}_{X/\mathbf{Z}})$ in dimension 2 will be shown in Section 2 to follow from a general result proved for arbitrary dimension (see Corollary 2.6).

Let $p \equiv 1 \bmod 24$ be a prime number and let $X_1(p)$ be the model over $\mathrm{Spec}(\mathbf{Z})$ of the modular curve associated to the congruence subgroup $\Gamma_1(p)$ as described in Section 4 in [CPT]. The group $\Gamma = (\mathbf{Z}/p\mathbf{Z})^*/\{\pm 1\}$ acts faithfully on $X_1(p)$. For a prime divisor l of $p-1$, with $l > 3$, we let H be the subgroup of Γ of index l and $G = \Gamma/H$. We consider the projective schemes $X = X_1(p)/H$ and $Y = X_1(p)/\Gamma$. Then $\pi : X \to Y$ is a G-cover to which we can apply our general results. Let $\mathcal{P}$ be the prime ideal of $\mathbf{Q}(\zeta_l)$ defined by the chosen embedding $j_p : \mathbf{Q}^c \to \mathbf{Q}_p^c$ and we let $\bar{\mathcal{P}}$ denote its image under complex conjugation. For $1 \leq a < l$ we denote by σ_a the $\mathbf{Q}$-automorphism of $\mathbf{Q}(\zeta_l)$ induced by $\zeta_l \mapsto \zeta_l^a$. Since G is of prime order, by a theorem of Rim, any non-trivial abelian character of G induces an isomorphism between $\mathrm{Cl}(\mathbf{Z}[G])$ and $\mathrm{Cl}(\mathbf{Z}[\zeta_l])$. For $i \in \{1, 2\}$ we define the elements s_i in the group ring $\mathbf{Z}[G]$ by

$$s_i = \sum_{1 \leq a < l/2} a^i \sigma_a^{-1}.$$

Theorem 1.10 *For a suitable choice of non-trivial abelian character φ of G, we have equalities in $Cl(\mathbf{Z}[\zeta_l])$:*
i) $\varphi(2\chi^P(\omega^{1/2}_{X/Y}) - 2\chi^P(\mathcal{O}_X)) = [\mathcal{P}\overline{\mathcal{P}}]^{\frac{p-1}{12l}s_1}$.
ii) $\varphi(2l\chi^P(\omega^{1/2}_{X/Y})) = [\mathcal{P}\overline{\mathcal{P}}]^{\frac{p-1}{12l}s_2}$ *and* $\varphi(2l\chi^P(\mathcal{O}_X)) = [\mathcal{P}\overline{\mathcal{P}}]^{\frac{p-1}{12l}(s_2 - ls_1)}$.

Using this we immediately deduce:

Corollary 1.11 *Let h_l^+ be the class number of the maximal real subfield of $\mathbf{Q}(\zeta_l)$ and assume that $h_l^+ = 1$; then we have:*

$$2\chi^P(\mathcal{O}_X) = 2\chi^P(\omega^{1/2}_{X/Y}), \quad 2l\chi^P(\mathcal{O}_X) = 2l\chi^P(\omega^{1/2}_{X/Y}) = 0 \ .$$

If in addition l is a regular prime number, then

$$2\chi(\mathcal{O}_X) = 2\chi(\omega_{X/Y}) = 2\chi(\omega^{1/2}_{X/Y}) = 0.$$

Remark. 1. The Lefschetz-Riemann-Roch theorems of [CEPT2] and the techniques developed in [P] provide efficient tools for determining the prime-to-l part of the Euler characteristic of the sheaves that we have considered here.

This corollary can be used to provide families of covers $X \to Y$ for which our sheaves have Euler characteristic of order two . In Section 6 we will see that Theorem 1.10 can also be used to construct families of examples where such Euler characteristics have order greater than 2. For instance for $p = 182857$ and $l = 401$ the cover $X \to Y$ provides us with a tame cover such that: $\chi^P(\mathcal{O}_X)$, $\chi^P(\omega_{X/Y})$ and $\chi^P(\omega_{X/Y}^{1/2})$ all have order greater than 2; where $2\chi^P(\mathcal{O}_X) = 2\chi^P(\omega_{X/Y})$; but where $\chi^P(\mathcal{O}_X) \neq \chi^P(\omega_{X/Y}^{1/2})$.

2 An equivariant Duality Theorem.

In this section we do not impose any restriction of the dimension of Y and we no longer suppose G to be abelian. However, we note that if G is abelian, then there is a simpler proof of the results of this section by following [RD] and working over $\mathrm{Spec}(\mathbf{Z}[G])$ as the base.

The action of complex conjugation on the characters of G induces an involutary automorphism on $\mathrm{Cl}(\mathbf{Z}[G])$. Thus if $[M]$ is an element of $\mathrm{Cl}(\mathbf{Z}[G])$ represented by $f \in \mathrm{Hom}_{G_\mathbf{Q}}(R_G, J(E))$, we define $\overline{[M]}$ as the element of $\mathrm{Cl}(\mathbf{Z}[G])$ represented by $\overline{f}$ where for $\chi \in R_G$

$$\overline{f}(\chi) = f(\overline{\chi})$$

and one checks that this automorphism maps $\ker(t)$ in 1.6 into itself. We denote by $\mathrm{Cl}(\mathbf{Z}[G])^+$ (resp. $\mathrm{Cl}(\mathbf{Z}[G])^-$) the subgroup of elements of $x \in \mathrm{Cl}(\mathbf{Z}[G])$ with the property that $\bar{x} = x$ (resp. $\bar{x} = -x$).

Clearly the group automorphism $f \mapsto f^{-1}$ on $\mathrm{Hom}_{G_\mathbf{Q}}(R_G, J(E))$ induces an involution on $\mathrm{Cl}(\mathbf{Z}[G])$. By composing these two involutions we obtain a further involution which we denote by $[M] \mapsto [M]^D$. In Proposition 3 of Appendix A, IX in [F1], Fröhlich gives an interpretation of this latter involution by proving that for any locally free $\mathbf{Z}[G]$-module M one has the equality:

$$[M]^D = [M^D] \tag{2.1}$$

where $M^D = \mathrm{Hom}_\mathbf{Z}(M, \mathbf{Z})$ is the $\mathbf{Z}$-linear dual of M. Such involutions were used to provide some of the first restrictions on the Galois module structure of algebraic rings of integers and their ideals. In the previous section we associated to certain G-equivariant sheaves $\mathcal{F}$ on X an equivariant Euler characteristic $\chi^P(\mathcal{F})$ in $\mathrm{Cl}(\mathbf{Z}[G])$; it is then natural to consider the image of this class under the above involution. The initial aim of this section is to give an interpretation of the class $\chi^P(\mathcal{F})^D$ in terms of a duality functor for Y-sheaves, which extends Fröhlich's above result. We provide just such a description in Corollary 2.5, and we note that the relative dimension d of X over $\mathrm{Spec}(\mathbf{Z})$ appears explicitly in this result. The result that we give is derived from an equivariant generalization of the duality theorem for projective morphisms given in Theorem 11.1 in III of [RD]. This "equivariant" version was first given in some unpublished notes of Ted Chinburg and George Pappas.

We start by introducing a small amount of further notation. Our main references are Section 2 of [C2] and Section 2 of [P]. As previously, G is a finite group and Y is a projective flat scheme over $\mathbf{Z}$. Let $K(Y,G)$ (resp. $K(\mathbf{Z},G)$) be the homotopy category of $\mathcal{O}_Y[G]$-modules (resp. $\mathbf{Z}[G]$-modules) and let $K^+(Y,G)$ (resp. $K^+(\mathbf{Z},G)$ be the subcategory of complexes in $K(Y,G)$ (resp. $K(\mathbf{Z},G)$) which are bounded below and which have coherent (resp. finitely generated) cohomology. We let $D(Y,G)$, $D^+(Y,G)$, $D(\mathbf{Z},G)$ and $D^+(\mathbf{Z},G)$ be their respective derived categories . We define $K(Y)$, $K^+(Y)$, $D(Y)$ and $D^+(Y)$ (resp. $K(\mathbf{Z})$, $K^+(\mathbf{Z})$, $D(\mathbf{Z})$ and $D^+(\mathbf{Z})$) similarly by considering complexes of $\mathcal{O}_Y$-modules (resp. $\mathbf{Z}$-modules). The global section functor Γ has a right derived functor

$$R\Gamma^+ : D^+(Y,G) \to D^+(\mathbf{Z},G)$$

(see Section 2 in II of [RD]).

In Theorem 1.1 of [C1], Chinburg proved:

Theorem 2.2 *If $F^\bullet$ is a bounded complex in $K^+(Y,G)$ with the property that each stalk of each term of $F^\bullet$ is a cohomologically trivial G-module, then $R\Gamma^+(F^\bullet)$ is isomorphic in $D^+(\mathbf{Z},G)$ to a finite complex of finitely generated projective $\mathbf{Z}[G]$-modules.*

This fact led him to associate to any such complex an Euler characteristic $\chi^P(R\Gamma^+(F^\bullet))$ in the class group $\mathrm{Cl}(\mathbf{Z}[G])$. When $\pi : X \to Y$ is a tame G-cover, then, under the hypotheses of Section 1, one can prove that if $\mathcal{F}$ is any G-equivariant coherent sheaf on X, then the complex of $D^+(Y,G)$, which is $\pi_*(\mathcal{F})$ in degree 0 and 0 elsewhere, satisfies the conditions required by Chinburg's theorem. In this case the class $\chi^P(\mathcal{F})$, referred to in the Introduction, coincides with the class $\chi^P(R\Gamma^+(\pi_*(\mathcal{F})))$ defined above.

For any $F^\bullet$ in $K(Y,G)$ and any $T^\bullet$ in $K(Y)$ we have the complex $\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(F^\bullet, T^\bullet)$ in $K(Y,G)$ (see Section 3 in II of[RD]). We thereby obtain a bifunctor:

$$\mathbf{Hom}^\bullet_{\mathcal{O}_Y} : K(Y,G)^0 \times K(Y) \to K(Y,G)$$

and a derived bifunctor

$$R\mathbf{Hom}^\bullet_{\mathcal{O}_Y} : D(Y,G)^0 \times D^+(Y) \to D^+(Y,G)$$

where, as usual, the superscript 0 denotes the opposite category. The bifunctors $\mathbf{Hom}^\bullet_{\mathbf{Z}}$ and $R\mathbf{Hom}^\bullet_{\mathbf{Z}}$ are defined in the same way. For any $F^\bullet$ in $D(Y,G)$ and $T^\bullet$ in $D^+(Y)$ we may then consider the Euler characteristic $\chi^P(R\Gamma^+(R\mathbf{Hom}^\bullet_{\mathcal{O}_Y})(F^\bullet, T^\bullet))$ in $\mathrm{Cl}(\mathbf{Z}[G])$. The following theorem is a special case of the above mentioned equivariant duality theorem:

Theorem 2.3 *(Chinburg, Pappas.) Let $F^\bullet$ be a bounded complex in $K^+(Y,G)$ as in Theorem 2.2. Let $h : Y \to S = Spec(\mathbf{Z})$ be the structural morphism. Then one has the equality of Euler characteristics in $Cl(\mathbf{Z}[G])$*

$$\chi^P(R\Gamma^+(R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(F^\bullet, h^!(\mathcal{O}_S)))) = \chi^P(R\mathbf{Hom}^\bullet_{\mathbf{Z}}(R\Gamma^+(F^\bullet), \mathbf{Z})) \ .$$

Remark The construction of $h^!$ is given in III of [RD] .

To give a very brief idea of the proof of the theorem we remark that the equality of these two Euler characteristics is a consequence of the existence of an isomorphism $\Gamma(\theta_h)$ in $D^+(\mathbf{Z})$:

$$\Gamma(\theta_h) : R\Gamma^+(R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(F^\bullet, h^!(\mathcal{O}_S))) \to R\mathbf{Hom}^\bullet_{\mathbf{Z}}(R\Gamma^+(F^\bullet), \mathbf{Z})$$

and this latter isomorphism is obtained by applying the global section functor to the duality isomorphism θ_h of Section 11 in III of [RD].

Corollary 2.4 *Since Y is flat over S, the fibres all have constant dimension which we denote by d. We then have the equality:*

$$(-1)^d \chi^P(R\Gamma^+(R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(F^\bullet, \omega_{Y/S}))) = \chi^P(R\mathbf{Hom}^\bullet_{\mathbf{Z}}(R\Gamma^+(F^\bullet), \mathbf{Z})) \ .$$

PROOF. It follows from [RD] that $h^!(\mathcal{O}_S)$ is the complex $\omega_{Y/S}[d]$ whose only non-zero term is $\omega_{Y/S}$ in degree $-d$. Moreover we have the equality:

$$\chi^P(R\Gamma^+(R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(F^\bullet, \omega_{Y/S}[d]))) = (-1)^d \chi^P(R\Gamma^+(R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(F^\bullet, \omega_{Y/S}))) \ .$$

The corollary therefore follows immediately from the theorem.

Corollary 2.5 *With the hypotheses and notation of the previous corollary, we have the following equality in Cl($\mathbf{Z}[G]$):*

$$\chi^P(\mathcal{F})^D = (-1)^d \chi^P(R\Gamma^+(R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(\pi_*(\mathcal{F}), \omega_{Y/S}))) \ .$$

PROOF. We consider the complex $\pi_*(\mathcal{F})$ whose only non-zero term is $\pi_*(\mathcal{F})$ in degree 0. Since the action (X, G) is tame we can apply Theorem 2.2 to deduce that the complex $R\Gamma^+(\pi_*(\mathcal{F}))$ is perfect. Let $M^\bullet$ be a bounded complex of finitely generated projective $\mathbf{Z}[G]$-modules which is isomorphic to $R\Gamma^+(\pi_*(\mathcal{F}))$ in $D^+(\mathbf{Z}, G)$. Then $R\mathbf{Hom}^\bullet_{\mathbf{Z}}(M^\bullet, \mathbf{Z}) = \mathbf{Hom}(M^\bullet, \mathbf{Z})$ is the class in $D^+(\mathbf{Z}, G)$ of the complex whose $(-j)^{th}$ term is $\mathrm{Hom}_{\mathbf{Z}}(M^j, \mathbf{Z}) = (M^j)^D$. Therefore it follows from Fröhlich's duality result that we have the equality in Cl($\mathbf{Z}[G]$):

$$\chi^P(R\mathbf{Hom}^\bullet_{\mathbf{Z}}(R\Gamma^+(\pi_*(\mathcal{F})), \mathbf{Z})) = \chi^P(\mathcal{F})^D \ .$$

The result now follows from the previous corollary.

We now wish use the above result to derive some properties for the Euler characteristics of the G-equivariant sheaves on X that we have considered.

Corollary 2.6 *Suppose that the hypotheses of the previous corollaries are satisfied. Then*

i)

$$\chi^P(\mathcal{O}_X)^D = (-1)^d \chi^P(\omega_{X/S}).$$

ii) Moreover, if the ramification indices of $X \to Y$ are odd and if there exists an $\mathcal{O}_Y$-line bundle $\omega^{1/2}_{Y/S}$, with the property that $\omega^{1/2}_{Y/S} \otimes \omega^{1/2}_{Y/S} = \omega_{Y/S}$, then

$$\chi^P(\omega^{1/2}_{X/S})^D = (-1)^d \chi^P(\omega^{1/2}_{X/S}) \ .$$

In particular, when $d = 1$ the class $\chi^P(\mathcal{O}_X) - \chi^P(\omega_{X/S})$ belongs to Cl($\mathbf{Z}[G]$)$^-$ and the class $\chi^P(\omega^{1/2}_{X/S})$ belongs to Cl($\mathbf{Z}[G]$)$^+$.

Proof. From the duality formula of Corollary 2.5 we see that, in order to prove this corollary, we need to evaluate $R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(\pi_*(\mathcal{F}), \omega_{Y/S})$ for $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{F} = \omega_{X/S}^{1/2}$. Since the action (X, G) is tame, in both cases $\pi_*(\mathcal{F})$ is a locally free $\mathcal{O}_Y[G]$-module. Therefore $R\mathbf{Hom}^\bullet_{\mathcal{O}_Y}(\pi_*(\mathcal{F}), \omega_{Y/S}) = \mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\mathcal{F}), \omega_{Y/S})$, and so now we consider $\mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\mathcal{F}), \omega_{Y/S})$. First we observe that

$$\mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\mathcal{F}), \omega_{Y/S}) \simeq \mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\mathcal{F}), \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \omega_{Y/S}.$$

From Propositions 4.25. and 4.32 in VI of [L] we deduce that when $\mathcal{F} = \mathcal{O}_X$ we have

$$\mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\mathcal{O}_X), \mathcal{O}_Y) \simeq \pi_*(\omega_{X/Y}) \ .$$

Therefore we have isomorphisms of $\mathcal{O}_Y[G]$-modules

$$\mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\mathcal{O}_X), \omega_{Y/S}) \simeq \pi_*(\omega_{X/Y}) \otimes_{\mathcal{O}_Y} \omega_{Y/S} \simeq \pi_*(\omega_{X/S})$$

with the latter isomorphism following from the adjunction and the projection formulas. This then shows i).

Now let $\mathcal{F} = \omega_{X/S}^{1/2}$. From the projection formula we deduce that $\pi_*(\omega_{X/S}^{1/2}) = \pi_*(\omega_{X/Y}^{1/2}) \otimes_{\mathcal{O}_Y} \omega_{Y/S}^{1/2}$. Therefore

$$\mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\omega_{X/S}^{1/2}), \mathcal{O}_Y) \simeq \mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\omega_{X/Y}^{1/2}), \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathbf{Hom}_{\mathcal{O}_Y}(\omega_{Y/S}^{1/2}, \mathcal{O}_Y) \ .$$

By use of the trace pairing we know that $\pi_*(\omega_{X/Y}^{1/2})$ is a self-dual $\mathcal{O}_Y[G]$-module; also, since $\omega_{Y/S}^{1/2}$ is invertible, we know that $\mathbf{Hom}_{\mathcal{O}_Y}(\omega_{Y/S}^{1/2}, \mathcal{O}_Y) = \omega_{Y/S}^{-1/2}$. In conclusion we deduce from the above that

$$\mathbf{Hom}_{\mathcal{O}_Y}(\pi_*(\omega_{X/S}^{1/2}), \omega_{Y/S}) \simeq \pi_*(\omega_{X/Y}^{1/2}) \otimes_{\mathcal{O}_Y} \omega_{Y/S}^{-1/2} \otimes_{\mathcal{O}_Y} \omega_{Y/S} \simeq \pi_*(\omega_{X/S}^{1/2}) \ .$$

This then completes the proof of the corollary.

3 Sheaf and divisor resolvents

In this section we assume that G is a finite group of exponent n and that R is a complete discrete valuation ring whose residue class field k is of characteristic p, which is coprime to n, and that R contains the nth roots of unity. We assume that $Y \to S = \mathrm{Spec}(R)$ is of absolute dimension $d+1$. We associate to G the constant group scheme over S, which we again denote by G_S; we let $G_S^D = \mathrm{Spec}(R[G])$ be the Cartier dual of G_S, and G_Y^D denotes the fiber product $G_S^D \times_S Y$. For any scheme Z, we denote by $\mathcal{K}_Z$ the sheaf of stalks of meromorphic functions on Z. This is the sheaf associated to the presheaf defined on affine open subschemes $\mathcal{U}$ of Z by $Frac(\mathcal{O}_Z(\mathcal{U}))$, where we let $Frac(A)$ denote the ring of fractions of the ring A.

Let $\mathcal{F}$ be a coherent invertible subsheaf of $\mathcal{K}_X$ whose support is contained in the branch locus of the cover $\pi : X \to Y$. We assume that $\mathcal{F} = \mathcal{O}_X(D)$ where $D = \sum_x d_x x$ is a divisor of X with $gD = D$ for any $g \in G$ so that d_x depends only on the G-orbit of x and furthermore $d_x = 0$ whenever the inertia subgroup I_x of x is trivial. We consider the sheaf $\pi_*(\mathcal{F})$: this is a G-equivariant, locally free $\mathcal{O}_Y$-module which may be viewed as a G_Y^D-module (see for instance 5.17 in II of [H]). We suppose that this is an invertible G_Y^D-module. For future reference we note that, since the cover is tame, the structure sheaf $\mathcal{O}_X$, the canonical sheaf $\omega_{X/Y}$ and, if the ramification of X/Y is odd, the sheaf $\omega_{X/Y}^{1/2}$ have this property. Any character $\varphi : G \to \Gamma(S, \mathcal{O}_S^*)$ of G provides us with an S-point of G_S^D and hence, by base change, with a Y-point $\varphi : Y \to G_Y^D$ of G_Y^D.

Definition 3.1. We define the resolvent sheaf associated to $\mathcal{F}$ and φ to be the invertible Y-sheaf:

$$\mathcal{F}_\varphi = \varphi^*((\pi_*)(\mathcal{F})) \ .$$

$\mathcal{F}_\varphi$ is easily seen to be the subsheaf of $\pi_*(\mathcal{F})$ consisting of those local sections on which G acts by the character φ.

n-fold multiplication induces a homomorphism of Y-sheaves

$$\mu : \pi_*(\mathcal{K}_X)^{\otimes n} \to \mathcal{K}_X$$

which, by restriction, identifies $\mathcal{F}_\varphi^{\otimes n}$ with an invertible subsheaf of $\mathcal{K}_Y$ that may be defined by a divisor. The principal aim of this section is to give an explicit expression for this divisor.

Let x be a codimension one point of X, let I_x be the inertia group of x and e_x be the order of I_x. The action of I_x on the cotangent space at x defines a faithful character ψ_x with values in k^*. Since n is coprime to the characteristic of k and since R is a complete discrete valuation ring, we may view ψ_x as taking values in R^*. Hence for any character φ of G there exists a unique integer $n(\varphi, x)$, $0 \le n(\varphi, x) < e_x$, such that the restriction to I_x of φ is equal to $\psi_x^{n(\varphi,x)}$. We set $\pi(x) = y$. Since the points of x above y

are all conjugate under the action of G, both the group I_x and the integer $n(\varphi,x)$ depend only on y. Moreover, as noted previously, since the divisor $D=\sum_x d_x x$ is G-invariant, the integer d_x also depends only on y. We shall therefore denote these objects $I_y, n(\varphi,y)$ and d_y. For any rational number a we denote its integral part by $[a]$ and its fractional part by $\{a\}=a-[a]$. We set

$$f_{\mathcal{F}}(\varphi,y)=\frac{d_y}{e_y}-\{\frac{n(\varphi,y)+d_y}{e_y}\}\ . \tag{3.1}$$

Proposition 3.2 *For any abelian character φ of G the map μ identifies $\mathcal{F}_\varphi^{\otimes n}$ with $\mathcal{O}_Y(F_{\mathcal{F}}(\varphi))$ with*

$$F_{\mathcal{F}}(\varphi)=\sum_y n f_{\mathcal{F}}(\varphi,y)y$$

where y runs over the set of codimension one points of Y which are contained in the special fiber of $Y\to S$.

Proof. We may assume, for the purposes of the proof, that the decomposition subgroup of x is equal to the inertia subgroup. We consider a codimension one point y on the special fiber of Y and we fix a point x on X such that $\pi(x)=y$. For the sake of simplicity we will write e for e_y, d for d_y, I for I_y etc.. Since R is a complete discrete valuation ring with residue characteristic p coprime to n and which contains the nth roots of unity, we are in a tame Kummer situation and so we can choose a uniformising parameter ω_x of $\mathcal{O}_{X,x}$ with the property that $\omega_y=\omega_x^e$ is a uniformiser of $\mathcal{O}_{Y,y}$. From the very definition of $\mathcal{F}$ we know that $\mathcal{F}_x=\omega_x^{-d}\mathcal{O}_{X,x}$. We define q and r by the equality $-d=qe+r$ subject to the restriction that $0\le r<e$. Then $\omega_x^{-d}=\omega_y^q\omega_x^r$. Again, since we are in a tame Kummer situation, we know that $\alpha=\frac{1}{e}(1+\omega_x+...\omega_x^{e-1})$ is a free basis of $\mathcal{O}_{X,x}$ as an $\mathcal{O}_{Y,y}[I]$-module. It then follows easily that $\omega_y^q\omega_x^r\alpha$ is a free basis of $\mathcal{F}_x$ over $\mathcal{O}_{Y,y}[I]$. This implies that the stalk of $\mathcal{F}_\varphi^n$ at y is given by:

$$\omega_y^{-n.f_{\mathcal{F}}(\varphi)}\mathcal{O}_{Y,y}=\mathcal{F}_{\varphi,y}^n=(\omega_y^q\omega_x^r\alpha\mid\psi_x^{n(\varphi)})^n\mathcal{O}_{Y,y}$$

where for $a\in\mathcal{F}_x$ and a character θ of I, $(a\mid\theta)$ denotes the Lagrange resolvent

$$(a\mid\theta)=\sum_{g\in I}a^g\theta(g^{-1}).$$

By the standard properties of Lagrange resolvents and the definition of the cotangent character ψ_x we have the following equalities:

$$(\omega_y^q\omega_x^r\alpha\mid\psi_x^{n(\varphi)})=\omega_y^q(\omega_x^r\alpha\mid\psi_x^{n(\varphi)})=\omega_y^q\omega_x^r(\alpha\mid\psi_x^{n(\varphi)-r})\ .$$

By an easy computation we obtain that

$$(\alpha \mid \psi_x^{n(\varphi)-r}) = \sum_{g\in I_y} \alpha^{g^{-1}} \psi_x^{n(\varphi)-r}(g) =$$

$$\frac{1}{e}\sum_{g\in I_y}\sum_{0\le k<e} \omega_x^k \psi_x^{-k-r+n(\varphi)}(g) = \frac{1}{e}\sum_{0\le k<e}\omega_x^k \sum_{g\in I_y}\psi_x^{-k-r+n(\varphi)}(g) \ .$$

We conclude that $(\alpha \mid \psi_x^{n(\varphi)-r}) = \omega_x^{n(\varphi)-r}$ if $r \le n(\varphi)$ and $e + n(\varphi) - r$ otherwise. Piecing this together we obtain that $-n.f_{\mathcal{F}}(\varphi) = \frac{n}{e}(n(\varphi) + eq)$ if $r \le n(\varphi)$ and $\frac{n}{e}(n(\varphi) + e(q + 1))$ otherwise. The proposition now follows at once from this equality.

As we have seen in the proof of the above proposition, the rational numbers $f_{\mathcal{F}}(\varphi, y)$ are defined by the equalities

$$f_{\mathcal{F}}(\varphi, y) = -\frac{1}{n} v_y(\mathcal{F}^n_{\varphi,y}) \ . \tag{3.3}$$

From now on we will denote this number by $-v_y(\mathcal{F}_\varphi)$.

Definition 3.2. The resolvent divisor of $\mathcal{F}$ at φ is defined to be

$$r(\mathcal{F}, \varphi) = \sum_y v_y(\mathcal{F}_{\varphi,y})y \tag{3.4}$$

where y runs over the set of codimension one points of Y which are contained in the special fiber of $Y \to S$.

We note that $n.r(\mathcal{F}, \varphi)$ is a divisor on Y with the property that

$$\mathcal{F}_\varphi^{\otimes n} = \mathcal{O}_Y(-n.r(\mathcal{F}, \varphi)) \ .$$

We will sometimes abuse notation and write $\mathcal{F}_\varphi = \mathcal{O}_Y(-r(\mathcal{F}, \varphi))$.

Examples. The following three examples are central to our study.

1. If $\mathcal{F}$ is the structural sheaf of X, then $D = 0$ and from the above proposition it follows that

$$v_y(\mathcal{O}_{X,\varphi}) = \frac{n(\varphi, y)}{e_y} \quad \text{and} \quad r(\mathcal{O}_X, \varphi) = \sum_y \frac{n(\varphi, y)}{e_y} y \ . \tag{3.5}$$

Note that this is Lemma 3.5 of [CPT].

2. We let $\mathcal{S}(\varphi)$ denote the set of codimension one points y of Y contained in the special fiber of $Y \to S$, with the property that $n(\varphi, y) > 0$. We set

$$f(\varphi) = \sum_{y \in \mathcal{S}(\varphi)} y. \tag{3.6}$$

If we take $\mathcal{F} = \omega_{X/Y}$, then $d_y = e_y - 1$ and we get

$$r(\omega_{X/Y}, \varphi) = r(\mathcal{O}_X, \varphi) - f(\varphi)$$

by using Proposition 3.2 and the fact that

$$\{\frac{n(\varphi, y) + e_y - 1}{e_y}\} = \frac{e_y - 1}{e_y} + \frac{n(\varphi, y)}{e_y} - 1.$$

3. We let $\mathcal{S}'(\varphi)$ denote the set of codimension one points y of Y contained in the special fiber of $Y \to S$, with the property that that $n(\varphi, y) > e_y/2$. We set

$$f'(\varphi) = \sum_{y \in \mathcal{S}'(\varphi)} y \ . \tag{3.7}$$

If we take $\mathcal{F} = \omega_{X/Y}^{1/2}$, then $d_y = (e_y - 1)/2$ and we get

$$r(\omega_{X/Y}^{1/2}, \varphi) = r(\mathcal{O}_X, \varphi) - f'(\varphi)$$

by using Proposition 3.2 and the fact that

$$\{\frac{n(\varphi, y) + e_(y - 1)/2}{e_y}\} = \frac{e_y - 1}{2e_y} + \frac{n(\varphi, y)}{e_y} \quad \text{if} \quad n(\varphi, y) < e_y/2,$$

$$\{\frac{n(\varphi, y) + e_(y - 1)/2}{e_y}\} = \frac{e_y - 1}{2e_y} + \frac{n(\varphi, y)}{e_y} - 1 \quad \text{if} \quad n(\varphi, y) > e_y/2.$$

From Proposition 3.2 it follows that

$$r(\omega_{X/Y}^{1/2}, \varphi) = r(\mathcal{O}_X, \varphi) - f'(\varphi).$$

Corollary 3.8 *Let φ be an abelian character of G and let $\bar{\varphi}$ be its complex conjugate. Then:*

i)

$$r(\mathcal{O}_X, \varphi) + r(\mathcal{O}_X, \bar{\varphi}) = f(\varphi)$$

ii)

$$r(\omega_{X/Y}^{1/2}, \varphi) + r(\omega_{X/Y}^{1/2}, \bar{\varphi}) = 0$$

iii)

$$r(\omega_{X/Y}^{1/2}, \varphi) = r(\mathcal{O}_X, \varphi^2) - r(\mathcal{O}_X, \varphi) \ .$$

Proof. Part (i) follows from Example 1 above and part (ii) comes from Example 3. To prove part (iii) we note that if $n(\varphi, y) < e_y/2$ then $n(\varphi^2, y) = 2n(\varphi, y)$ and if $n(\varphi, y) > e_y/2$ then $n(\varphi^2, y) = 2n(\varphi, y) - e_y$, and so

$$r(\mathcal{O}_X, \varphi^2) - r(\mathcal{O}_X, \varphi) = (\sum_y \frac{2n(\varphi, y)}{e_y}) - f'(\varphi)$$

$$= 2r(\mathcal{O}_X, \varphi) - f'(\varphi) = r(\mathcal{O}_X, \varphi) + r(\omega_{X/Y}^{1/2}, \varphi).$$

Remark. We denote the sheaf $\mathcal{O}_Y(-f(\varphi))$ by $F(\varphi)$. From the above corollary we deduce the following equalities of invertible $\mathcal{O}_Y$-sheaves:

$$\mathcal{O}^n_{X,\varphi} \otimes \mathcal{O}^n_{X,\bar{\varphi}} = F(\varphi)^n, \quad (\omega^{1/2}_{X/Y,\varphi})^n \otimes (\omega^{1/2}_{X/Y,\bar{\varphi}})^n = \mathcal{O}_Y$$

and we observe that the first equality provides us with a geometric analogue of Theorem 18 in [F2].

4 Euler characteristics and representatives.

4.a The Riemann-Roch Theorem

Let R be a Dedekind domain. Henceforth we suppose that $Y \to S = \mathrm{Spec}(R)$ is of absolute dimension 2 and we consider a G-cover of Y satisfying the assumptions of Section 1.

To any locally free coherent G-sheaf $\mathcal{F}$ on X we can associate the following two invariants: the first is the equivariant Euler characteristic $\chi^P(\mathcal{F}) \in \mathrm{Cl}(R[G])$, which was introduced in Section 1 (see [C1] for further details). We now briefly recall the construction of the second invariant that we wish to study. Recall that the sheaf $\pi_*(\mathcal{F})$ may be viewed as a locally free coherent G^D_Y-sheaf. Let $\tilde{h} : G^D_Y \to G^D_S$ be the base change of $h : Y \to S$ by $G^D_S \to S$. Then the total derived image $R\tilde{h}(\pi_*(\mathcal{F}))$ in the derived category of complexes of G^D_S-sheaves is represented by a perfect complex. Therefore, following Knudsen and Mumford, we can define the invertible $R[G]$-module $\det(R\tilde{h}(\pi_*(\mathcal{F}))$ and consider its class in $\mathrm{Pic}(G^D_S) = \mathrm{Pic}(R[G])$. Since G is abelian this latter group may be identified with the class group $\mathrm{Cl}(R[G])$ and we have the following equality in the class group $\mathrm{Cl}(R[G])$, (see 2.c in [P] and also Section 3 of [CPT]):

$$\chi^P(\mathcal{F}) = [\det(R\tilde{h}(\pi_*(\mathcal{F}))] \ .$$

In fact we shall be interested in twice this class and so, with the notation of [CPT], we set

$$\delta(\mathcal{F}) = \det(R\tilde{h}(\pi_*(\mathcal{F}))^{\otimes 2}.$$

For $i, 1 \leq i \leq 2$, we let $\mathcal{F}_i$ denote a coherent, invertible, G-equivariant sheaf on X, defined by a divisor D_i. We assume that $D_1 \leq D_2$ and that both these divisors are supported on the branch locus of $\pi : X \to Y$. Our aim is to describe $\psi(\mathcal{F}_1, \mathcal{F}_2) = 2(\chi^P(\mathcal{F}_2) - \chi^P(\mathcal{F}_1))$ in $\mathrm{Cl}(R[G])$. It follows from the above that $\psi(\mathcal{F}_1, \mathcal{F}_2) = \delta(\mathcal{F}_2) \otimes \delta(\mathcal{F}_1)^{-1}$. Since $D_1 \leq D_2$ we have a natural injection of sheaves $f : \pi_*(\mathcal{F}_1) \to \pi_*(\mathcal{F}_2)$ and so we have the equality

$$\psi(\mathcal{F}_1, \mathcal{F}_2) = [\delta(\pi_*(\mathcal{F}_1) \to \pi_*(\mathcal{F}_2))]$$

where $\pi_*(\mathcal{F}_1) \xrightarrow{f} \pi_*(\mathcal{F}_2)$ denotes the perfect complex of G_Y^D-sheaves with terms in positions -1 and 0. We shall denote by $\delta(f)$ the locally free $R[G]$-module defined by the "δ-image" of this complex; that is to say, the determinant of the push down, via f, of this complex in $\mathrm{Cl}(R[G])$. With this notation, the previous equality may be re-written as

$$\psi(\mathcal{F}_1, \mathcal{F}_2) = [\delta(f)] \ .$$

Observe that $\delta(f)$ actually defines a submodule of $K[G]$. Since G is abelian, the determination of this module reduces to the computation of $\varphi(\delta(f))$ for each abelian character φ of G. By pullback, any such character induces a morphism of sheaf resolvents

$$\varphi^*(f) : \mathcal{F}_{1,\varphi} \to \mathcal{F}_{2,\varphi}$$

and, by the functorial properties of the determinant of the cohomology, it follows that we have $\varphi(\delta(f)) = \delta(\varphi^*(f))$.

The module $\delta(f)$ may of course be described by its local components, and so we now consider the local situation where R is a complete discrete valuation ring. We introduce a small amount of further notation. Let k denote the residue class field of R. For any scheme $f : T \to \mathrm{Spec}(k)$ we let $G_0(T)$ denote the Grothendieck group of coherent sheaves on T. We identify $G_0(\mathrm{Spec}(k))$ with $\mathbf{Z}$. Assuming f to be proper, we define the Euler characteristic of a coherent sheaf $\mathcal{F}$ on T by the equality:

$$\mathcal{X}(T, \mathcal{F}) = f_*([\mathcal{F}])$$

where $[\mathcal{F}]$ is the class of $\mathcal{F}$ in $G_0(T)$.
We let $h_s : Y_s \to \mathrm{Spec}(k)$ denote the special fiber of Y. The complex

$$c_\varphi : \mathcal{F}_{1,\varphi} \stackrel{\varphi^*(f)}{\to} \mathcal{F}_{2,\varphi}$$

is then a complex of locally free $\mathcal{O}_Y$-modules which is exact off Y_s. Thus the cokernel of $\varphi^*(f)$, which we denote by $\mathcal{T}_\varphi(\mathcal{F}_1, \mathcal{F}_2)$, is a coherent sheaf of $\mathcal{O}_Y$-modules which is supported on Y_s. It therefore defines a class in the Grothendieck group $G_0(Y_s)$ of coherent Y_s-modules. Let λ be a uniformizing parameter of R. Then it is easily shown that

$$\varphi(\delta(f)) = R\lambda^{-2\mathcal{X}(\mathcal{T}_\varphi(\mathcal{F}_1,\mathcal{F}_2))}$$

where we view the Euler characteristic $\mathcal{X}(\mathcal{T}_\varphi(\mathcal{F}_1, \mathcal{F}_2)) \in \mathbf{Z}$ as above. The remainder of this section will be devoted to the computation of the Euler characteristic $\mathcal{X}(\mathcal{T}_\varphi(\mathcal{F}_1, \mathcal{F}_2))$ for particular choices of $\mathcal{F}_1$ and $\mathcal{F}_2$.

For any integer $m \geq 0$ we let $A_m(T)$ denote the group of m-cycles on T modulo rational equivalence and we set

$$A_*(T) = \oplus_{0 \leq m} A_m(T) \ .$$

We write $A_*(T)_{\mathbf{Q}} = A_*(T) \otimes \mathbf{Q}$ and, as usual, we identify $A_0(\mathrm{Spec}(k))$ with $\mathbf{Z}$. The general Riemann-Roch theorem provides us with homomorphisms $\tau_T : G_0(T) \to A_*(T)_{\mathbf{Q}}$ which are covariant for proper morphisms. For any coherent sheaf $\mathcal{F}$ on T, the Riemann-Roch theorem states that

$$\mathcal{X}(T, \mathcal{F}) = f_*(\tau_T([\mathcal{F}])) \ .$$

In this equality the right-hand side is the push forward of the zero component of $\tau_T([\mathcal{F}])$ to $A_0(Spec(k))_{\mathbf{Q}}$ identified with $\mathbf{Q}$.

For any invertible $\mathcal{O}_Y[G]$-sheaf $\mathcal{F}$ and any abelian $\mathbf{Q}_p^c$-character φ of G, as previously, we consider the resolvent divisor $r(\mathcal{F}, \varphi) = \sum_y v_y(\mathcal{F}_\varphi) y$, (see Definition 3.2 in Section 3), and we define $T(\mathcal{F}, \varphi)$ to be the integer

$$T(\mathcal{F}, \varphi) = r(\mathcal{F}, \varphi)^2 + c_1(\omega_{Y/S}) \cdot r(\mathcal{F}, \varphi) \ .$$

Recall that for two codimension one points y, z on the special fiber Y_s of Y, we denote their intersection number by $y \cdot z$ and $c_1(\omega_{Y/S}) \cdot y$ is the degree of the 0-cycle $c_1(\omega_{Y/S}) \cap y$ on Y (see Chapter 2 of[Fu]). In the case where we have a "square root" $\omega_{Y/S}^{1/2}$ of $\omega_{Y/S}$, in 1.5 we have defined the twist $\tilde{\mathcal{F}}$ of $\mathcal{F}$ by the rule

$$\tilde{\mathcal{F}} = \mathcal{F} \otimes \pi^*(\omega_{Y/S}^{1/2}).$$

Proposition 4.1 *Let $\mathcal{F}$ denote either $\omega_{X/Y}$ or $\omega_{X/Y}^{1/2}$. Then for any abelian $\mathbf{Q}_p^c$-character φ of G one has the following equalities:*
i)

$$2\chi(\mathcal{T}_\varphi(\mathcal{O}_X, \mathcal{F})) = T(\mathcal{F}, \varphi) - T(\mathcal{O}_X, \varphi) \ .$$

ii) If there exists a "square root" of $\omega_{Y/S}$, then

$$2\chi(\mathcal{T}_\varphi(\tilde{\mathcal{O}}_X, \tilde{\mathcal{F}})) = r(\mathcal{F}, \varphi)^2 - r(\mathcal{O}_X, \varphi)^2 \ .$$

Proof. Since the proofs of these equalities are similar, we shall only give the proof of *ii*) when $\mathcal{F} = \omega_{X/Y}$ and will leave the proof of the remaining cases to the reader. Let φ be an abelian character of G and let c_φ be the complex of Y-line bundles

$$\mathcal{O}_{X,\varphi} \otimes_{\mathcal{O}_Y} \omega_{Y/S}^{1/2} \stackrel{\varphi^*(\tilde{f})}{\rightarrow} \omega_{X/Y,\varphi} \otimes_{\mathcal{O}_Y} \omega_{Y/S}^{1/2}$$

at degrees -1 and 0. Since c_φ is exact outside Y_s and provides a locally free resolution of $\mathcal{T}_\varphi(\tilde{\mathcal{O}}_X, \tilde{\omega}_{X/Y})$, (henceforth abbreviated to $\tilde{\mathcal{T}}_\varphi$), it follows from the Riemann-Roch Theorem (see Theorems 18.2 and 18.3 in [Fu]) that

$$\chi(\tilde{\mathcal{T}}_\varphi) = (h_s)_*((ch_{Y_s}^Y(c_\varphi) \cap Td(h))_0)$$

where $(h_s)_*$ is the push forward $A_0(Y_s)_{\mathbf{Q}} \rightarrow A_0(\mathrm{Spec}(k))_{\mathbf{Q}} = \mathbf{Q}$, $ch_{Y_s}^Y(c_\varphi)$ is the localized Chern character which lives in the bivariant group $A(Y_s \rightarrow Y)_{\mathbf{Q}}$, and $Td(h)$ is the Todd class of h, which is an element of $A_*(Y)_{\mathbf{Q}}$ that we will describe presently. We first observe that the complex c_φ may be written as $\omega_{Y/S}^{1/2} \otimes d_\varphi$, where d_φ is the complex with terms in degrees -1 and 0.

$$\mathcal{O}_{X,\varphi} \stackrel{\varphi^*(f)}{\rightarrow} \omega_{X/Y,\varphi} \ .$$

It therefore follows from Proposition 18.1 (c) in [Fu] that $ch_{Y_s}^Y(c_\varphi) = ch(\omega_Y^{1/2}) \cap ch_{Y_s}^Y(d_\varphi)$. We now consider the complex $d_\varphi^{\otimes n}$. For the sake of simplicity we set $F(\varphi) = -n.r(\mathcal{O}_X, \varphi)$ and $R(\varphi) = n.f(\varphi)$. It follows from Proposition 3.2 that $d_\varphi^{\otimes n}$ is the complex $\mathcal{O}_Y(F(\varphi)) \rightarrow \mathcal{O}_Y(F(\varphi) + R(\varphi))$. We observe that once again $d_\varphi^{\otimes n}$ can be written as the tensor product $\mathcal{O}_Y(F(\varphi)) \otimes b_\varphi$ where b_φ is the new complex $\mathcal{O}_Y \rightarrow \mathcal{O}_Y(R(\varphi))$. Since $R(\varphi)$ is an effective divisor and since we assume that X and Y are relative curves, it follows from Propositi on 3.10 (b) and (e) in [CPT] that

$$ch_{Y_s}^Y(b_\varphi) = [R(\varphi)] + \frac{[R(\varphi)]^2}{2}$$

and therefore

$$ch^Y_{Y_s}(d_\varphi^{\otimes n}) = [R(\varphi)] + c_1(\mathcal{O}_Y(F(\varphi))) \cap [R(\varphi)] + \frac{[R(\varphi)]^2}{2}.$$

Since for any integer q we know that $ch^{Y,q}_{Y_s}(d_\varphi^{\otimes n}) = n^q ch^{Y,q}_{Y_s}(d_\varphi)$, it follows that

$$ch^Y_{Y_s}(d_\varphi) = \frac{[R(\varphi)]}{n} + \frac{c_1(\mathcal{O}_Y(F(\varphi))) \cap [R(\varphi)]}{n^2} + \frac{[R(\varphi)]^2}{2n^2} \ .$$

Since $c_\varphi = \omega^{1/2}_{Y/S} \otimes d_\varphi$, we finally obtain that

$$ch^Y_{Y_s}(c_\varphi) = \frac{[R(\varphi)]}{n} + \frac{[R(\varphi)] \cap c_1(\omega^{1/2}_{Y/S})}{n} + \frac{c_1(\mathcal{O}_Y(F(\varphi))) \cap [R(\varphi)]}{n^2} + \frac{[R(\varphi)]^2}{2n^2} \ .$$

From the very definition of $Td(h)$ we deduce that

$$Td_1(h) = -\frac{c_1(\omega_{Y/S})}{2} = -c_1(\omega^{1/2}_{Y/S})$$

(see (3.e) in [CPT]). Therefore, piecing the above together, we obtain that

$$2\chi(\tilde{\mathcal{T}}_\varphi) = 2(ch^Y_{Y_s}(c_\varphi) \cap Td(h))_0 = 2\frac{c_1(\mathcal{O}_Y(F(\varphi))) \cap [R(\varphi)]}{n^2} + \frac{[R(\varphi)]^2}{n^2} \ .$$

Since we have the equalities

$$\frac{[R(\varphi)]^2}{n^2} = f(\varphi)^2 \ \text{ and } \ \frac{c_1(\mathcal{O}_Y(F(\varphi))) \cap [R(\varphi)]}{n^2} = -r(\mathcal{O}_X, \varphi) \cdot f(\varphi) \ ,$$

it suffices to use the equalities of Example 2 in Section 3 in order to deduce the required formula from above .

4.b Proof of Theorems 1.7 and 1.8

Our hypotheses and our notations are those of Theorems 1.7 and 1.8. Since the proofs are similar we will give the proof of Theorem 1.7 and leave the proof of Theorem 1.8 to the reader. Our aim is to find a representative for the class $2(\chi^P(\mathcal{F}) - \chi^P(\mathcal{O}_X))$; that is to say an element g in the group $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_f(E))$ with the property that

$$2(\chi^P(\mathcal{F}) - \chi^P(\mathcal{O}_X)) = t(g) \ .$$

For any element p of Σ we check that the map

$$u_p : \varphi \mapsto p^{T_p(\mathcal{O}_X,\varphi)-T_p(\mathcal{F},\varphi)}$$

belongs to $\mathrm{Hom}_{G_{\mathbf{Q}_p}}(R_{G,p}, E_p^*)$. Therefore we can define g_p to be the unique element of $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_p(E))$ such that $g_p^* = u_p$. We set $g = \prod_{p\in\Sigma} g_p$. Our aim now is to show that g is a representative for our class.

Since we have a group isomorphism

$$\mathbf{Q}_p[G]^* \simeq \mathrm{Hom}_{G_{\mathbf{Q}_p}}(R_{G,p}, E_p^*)$$

given by $x \mapsto (\varphi \mapsto \varphi(x))$, we may define a_p to be the unique element of $\mathbf{Q}_p[G]^*$ corresponding to u_p under the above isomorphism. By taking $a_p = 1$ for $p \notin \Sigma$ and a_p as above for $p \in \Sigma$, we obtain a finite idele $(a_p)_p$ of $\mathbf{Q}[G]$ and thus an element in $\mathrm{Pic}(\mathbf{Z}[G])$ by considering the class of the fractional ideal $\cap_p \mathbf{Z}_p[G]a_p \cap \mathbf{Q}[G]$. This class corresponds to $t(g)$ under the identification between $\mathrm{Cl}(\mathbf{Z}[G])$ and $\mathrm{Pic}(\mathbf{Z}[G])$. Under this identification, as seen previously, $2(\chi^P(\mathcal{F}) - \chi^P(\mathcal{O}_X))$ identifies with $\psi(\mathcal{O}_X, \mathcal{F})$. Therefore, in order to prove the theorem, it suffices to prove that for each finite place p we have the equality $\psi(\mathcal{O}_X, \mathcal{F})\mathbf{Z}_p = \mathbf{Z}_p[G]a_p$. When $p \notin \Sigma$ this follows from the definition of $\psi(\mathcal{O}_X, \mathcal{F})$. For $p \in \Sigma$, since p is coprime to n, it suffices to show that, for any abelian $\mathbf{Q}_p^c$-character φ of G, we have $\varphi(\psi(\mathcal{O}_X, \mathcal{F})\mathbf{Z}'_p) = \varphi(a_p)\mathbf{Z}'_p$, where, as previously, $\mathbf{Z}'_p$ is obtained by adjoining the n-th roots of unity to $\mathbf{Z}_p$. Since the functor δ commutes with base change, this last equality is equivalent to the equality:

$$\psi(\mathcal{O}_{X'_p}, \mathcal{F}'_p) = p^{T_p(\mathcal{O}_X,\varphi)-T_p(\mathcal{F},\varphi)}\mathbf{Z}'_p = p^{T(\mathcal{O}_{X'_p},\varphi)-T(\mathcal{F}'_p,\varphi)}\mathbf{Z}'_p \ .$$

This now follows from Proposition 4.1.(i) because p is a uniformizing parameter of $\mathbf{Z}'_p$.

5 The class $\chi(\mathcal{O}_X) - \chi(\omega_{X/Y})$

The aim of this section is to prove Theorem 1.9. We now assume that G is an abelian group of order l^N where l is a prime number, $l \geq 3$ and $l \notin \Sigma$. For any $p \in \Sigma$ we let $\mathbf{Z}'_p$ be the discrete valuation ring obtained by adjoining to $\mathbf{Z}_p$ a primitive l^N-th root of unity and we let $\mathbf{Q}_p(\zeta_{l^N})$ denote its field of fractions. Let $\pi_p : X'_p = X \otimes_{\mathbf{Z}} \mathbf{Z}'_p \to Y'_p = Y \otimes_{\mathbf{Z}} \mathbf{Z}'_p$ denote the G-cover obtained from

$\pi : X \to Y$ by base change. There is then an action of $V_p = \mathrm{Gal}(K_p/\mathbf{Q}_p)$ on $\mathbf{Z}'_p$, and hence on X'_p and Y'_p, and therefore on the codimension one points of X'_p and Y'_p. Since the actions of G and V_p on X'_p commute, it follows that for any $\omega \in V_p$ and any codimension one point x of X, one has $I_{x^\omega} = I_x$ and $\psi_{x^\omega} = \psi_x^\omega$ (recall that ψ_x is the character of I_x defined by the action of I_x on the cotangent space in x). This implies that for any abelian character $\varphi : G \to \mathbf{Z}_p'^{\times}$, $\omega \in V_p$ we will have $n(\varphi^\omega, x^\omega) = n(\varphi, x)$. As previously, when $x \mapsto y$ we shall write I_y and $n(\varphi, y)$ for I_x and $n(\varphi, x)$.

Let E be the number field obtained by adjoining to $\mathbf{Q}$ the l^N-th roots of unity. In Theorem 1.7 we proved that

$$2(\chi(\mathcal{O}_X) - \chi(\omega_{X/Y})) = \prod_{p \in \Sigma} t(g_p)$$

where g_p is the element of $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_p(E))$ defined by

$$g_p^*(\varphi) = p^{T_p(\omega_{X/Y},\varphi) - T_p(\mathcal{O}_X,\varphi)}.$$

Let p be an element of Σ and let $S(\varphi)$ denote the set of codimension one points y of Y'_p contained in the special fiber of $Y'_p \to Spec(\mathbf{Z}'_p)$ such that $n(\varphi, y) > 0$. We know from Example 2 of Section 3 that, for any abelian character φ of G

$$r_p(\omega_{X/Y}, \varphi) = r_p(\mathcal{O}_X, \varphi) - f(\varphi) \quad .$$

Moreover, in 3.e. of [CPT] we find that

$$c_1(\omega_{Y_p}) \cdot y = -y^2 - 2\chi(y, \mathcal{O}_y).$$

Using 1.4 and Example 2 in Section3, we can use the above to deduce that

$$T_p(\omega_{X/Y}, \varphi) - T_p(\mathcal{O}_X, \varphi) = f(\varphi)^2 + \sum_{y \in S(\varphi)} (y^2 + 2\chi(y, \mathcal{O}_y)) - 2f(\varphi) \cdot r_p(\mathcal{O}_X, \varphi)$$

where, as before, $y \cdot z$ (resp. y^2) denotes the intersection (resp. self-intersection) number of y and z (resp. y). We set:

$$a(\varphi) = f(\varphi)^2 + \sum_{y \in S(\varphi)} (y^2 + 2\chi(y, \mathcal{O}_y)) \ .$$

Since $a(\varphi) - 2f(\varphi)r(\mathcal{O}_X, \varphi)$ is an Euler characteristic, and since by definition $a(\varphi)$ is an integer, it follows that $2f(\varphi) \cdot r_p(\mathcal{O}_X, \varphi)$ must also be an integer.

We observe that for any element $\omega \in \mathrm{Gal}(\mathbf{Q}_p^c/\mathbf{Q}_p)$ there exists r_ω, coprime to l, such that $\varphi^\omega = \varphi^{r_\omega}$. We therefore deduce that $S(\varphi) = S(\varphi^\omega)$ and hence $a(\varphi) = a(\varphi^\omega)$. This implies that g_p may be written in $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J_p(E))$ as a product $u_p h_p$ where for any abelian character φ we define $u_p^*(\varphi) = p^{\ a(\varphi)}$ and $h_p^*(\varphi) = p^{-2f(\varphi)\cdot r(\mathcal{O}_{X_p},\varphi)}$.

The theorem will be a consequence of the following two lemmas.

Lemma 5.1 *$t(u_p)$ belongs to $D(\mathbf{Z}[G])$.*

PROOF. The key observation is that for any abelian character φ of G and any $\omega \in G_{\mathbf{Q}}$ from the above we know that

$$a(\varphi_p) = a((\varphi^\omega)_p)\ .$$

It therefore follows that the character map v_p, whose value on an abelian character φ is given by $v_p(\varphi) = p^{\ a(\varphi_p)}$, belongs to $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, \mathbf{Q}^\times)$. Let w_p be the element of $\mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, J(E))$ defined on abelian characters of G by the rule:

$$w_p(\varphi)_v = \begin{cases} 1 \text{ if } \ v \text{ divides } \ p \\ p^{-a(\varphi_p)} \text{ if } v \text{ if not} \end{cases}$$

and so by definition $w_p \in \mathrm{Hom}_{G_{\mathbf{Q}}}(R_G, U(E))$. Since $(v_p w_p)^* = u_p^*$ we deduce that $v_p w_p = u_p$. Therefore

$$t(u_p) = t(v_p)t(w_p) = t(w_p)\ .$$

Since $t(w_p) \in D(\mathbf{Z}[G])$ the lemma is proved.

If F is a number field and x is a finite idele of F, then the content of x is a fractional ideal of F, which we denote by $c(x)$.

Lemma 5.2 *i) There exists an integer $m \geq 0$ such that $l^m t(h_p)$ belongs to $D(\mathbf{Z}[G])$.*

ii) If G is of order l, then $t(h_p)$ belongs to $D(\mathbf{Z}[G])$.

PROOF. In order to prove *i*), it suffices to show that for any non-trivial abelian character φ of G there exists an integer $s \geq 0$ such that $l^s c(h_p(\varphi))$ is a principal ideal. For such a character φ, let F be the field $\mathbf{Q}(\varphi)$ and let $\mathcal{P}$ be the prime ideal of F defined by the restriction of j_p to F. It follows from the definition that $v_{\mathcal{P}}(c(h_p(\varphi))) = -2f(\varphi_p)r_p(\mathcal{O}_X, \varphi_p)$. The Galois group H_p of the extension $\mathbf{Q}_p(\varphi_p)/\mathbf{Q}_p$ can be identified via j_p with the decomposition group of $\mathcal{P}$. For any integer a, coprime to l, we denote by σ_a the automorphism of E defined by the property that on a primitive l^N-th root of unity ζ we have $\sigma_a(\zeta) = \zeta^a$. We then let $\{a_i, 1 \leq i \leq q\}$ be a set of integers such that the $\{\sigma_{a_i}\}$ are a set of representatives of $\mathrm{Gal}(F/\mathbf{Q})/H_p$. We have the equalities

$$v_{\mathcal{P}\sigma_a^{-1}}(c(h_p(\varphi))) = v_{\mathcal{P}}((c(h_p(\varphi))^{\sigma_a}) = v_{\mathcal{P}}(c(h_p(\varphi^a)) = -2f(\varphi_p^a) \cdot r_p(\mathcal{O}_X, \varphi_p^a) \ .$$

Since it is clear that $f(\varphi_p^a) = f(\varphi_p)$, we deduce from Example 1 in Section 3 that

$$f(\varphi_p^a) \cdot r_p(\mathcal{O}_X, \varphi_p^a) = \sum_{y \in S(\varphi_p)} (f(\varphi_p) \cdot y) \frac{n(\varphi_p^a, y)}{e_y} \ .$$

Therefore we obtain

$$c(h_p(\varphi)) = \mathcal{P}^{\alpha(\varphi)} \text{ with } \alpha(\varphi) = -2 \sum_{y \in S(\varphi_p)} (f(\varphi_p) \cdot y) \sum_{1 \leq i \leq q} \frac{n(\varphi_p^{a_i}, y)}{e_y} \sigma_{a_i}^{-1} \ .$$

The group $V_p = Gal(\mathbf{Q}_p(\zeta_{l^N})/\mathbf{Q}_p)$ acts on $S(\varphi_p)$. We let $\{y_j, 1 \leq j \leq r\}$ be a set of orbit representatives of $S(\varphi_p)$ under the action of V_p. For any j, $1 \leq j \leq r$, let $V_{p,j}$ denote the isotropy group of the codimension one point y_j. Since $V_{p,j}$ is a subgroup of $\mathbf{Q}_p(\zeta_{l^N})/\mathbf{Q}_p(\psi_{y_j})$, we note that it must be an l-group. For any $u \in V_p$ and any $y \in S(\varphi_p))$, we have

$$f(\varphi_p) \cdot y^u = \sum_{z \in S(\varphi_p)} z \cdot y^u = \sum_{z \in S(\varphi_p)} z^u \cdot y^u = f(\varphi_p) \cdot y$$

because we know that $z^u \cdot y^u = z \cdot y$. Therefore we deduce that $\alpha(\varphi)$ may be written as:

$$\alpha(\varphi) = -2 \sum_{1 \leq j \leq r} (f(\varphi_p) \cdot y_j) A_j(\varphi_p)$$

with

$$A_j(\varphi_p) = \sum_{1 \leq i \leq q} \sum_{u \in V_p/V_{p,j}} \frac{n(\varphi_p^{a_i}, y_j^u)}{e_{y_j^u}} \sigma_{a_i}^{-1} \ .$$

Let l^{m_j} denote the order of $V_{p,j}$. Using the equality $n(\varphi_p^{a_i}, y_j^u) = n(\varphi_p^{a_i u^{-1}}, y_j)$ and the fact that $e_y = e_{y^u}$, from the above we obtain that

$$l^{m_j} A_j(\varphi_p) = \sum_{1 \leq i \leq q} \sum_{u \in V_p} \frac{n(\varphi_p^{a_i}, y_j^u)}{e_{y_j^u}} \sigma_{a_i}^{-1} = \sum_{1 \leq i \leq q} \sum_{v \in V_p} \frac{n(\varphi_p^{a_i v}, y_j)}{e_{y_j}} \sigma_{a_i}^{-1}$$

and since for $v \in Gal(\mathbf{Q}_p(\zeta_{l^N}) : \mathbf{Q}_p(\varphi_p)))$ we have $n(\varphi^v) = n(\varphi)$ we get

$$l^{m_j} A_j(\varphi_p) = [\mathbf{Q}_p(\zeta_{l^N}) : \mathbf{Q}_p(\varphi_p)](\sum_{1 \le i \le q} \sum_{v \in H_p} \frac{n(\varphi_p^{a_i v}, y_j)}{e_{y_j}} \sigma_{a_i}^{-1} .$$

Since the integer $[\mathbf{Q}_p(\zeta_{l^N}) : \mathbf{Q}_p(\varphi_p)]$ is a power of l we deduce that for each j there exists $n_j \in \mathbf{Z}$ such that

$$A_j(\varphi_p) = l^{n_j} \sum_{1 \le i \le q} \sum_{v \in H_p} \frac{n(\varphi_p^{a_i v}, y_j)}{e_{y_j}} \sigma_{a_i}^{-1} .$$

Let l^s denote the order of φ. Since for any $v \in H_p$, we know that $\mathcal{P}^{\sigma_v} = \mathcal{P}$, we can deduce from the previous equality and the definition of $\alpha(\varphi)$ that there exists a positive integer m and for any $j, 1 \le j \le q$, an integer a_j such that

$$c(h_p(\varphi))^{l^m} = \mathcal{P}^{l^m \alpha(\varphi)}$$

with

$$l^m \alpha(\varphi) = -2 \sum_{1 \le j \le r} a_j b_j(\varphi) \quad \text{and} \quad b_j(\varphi) = \sum_{1 \le a \le l^s, (a,l)=1} \frac{n(\varphi_p^a, y_j)}{e_{y_j}} \sigma_a^{-1} .$$

We now wish to study the sums $b_j(\varphi)$. Let y be one of the $\{y_j\}$ and let

$$b(\varphi) = \sum_{1 \le a \le l^s, (a,l)=1} \frac{n(\varphi_p^a, y)}{e_y} \sigma_a^{-1} .$$

We write $e_y = l^n, \varphi_p \mid I_y = \psi_y^{ul^m}$ with $(u, l) = 1$ and $0 < ul^m < l^n$. It follows that the order of $\varphi_p \mid I_y$ is equal to l^{n-m}; if we put $t = (n - m)$ and recall that we denote the order of φ_p by l^s, then we have $t \le s$. One checks easily that

$$\frac{n(\varphi_p^a, e_y)}{e_y} = \{\frac{au}{l^t}\}.$$

We have the equality of sets of integers:

$$\{c + kl^t, 1 \le c < l^t, (c, l) = 1, 0 \le k < l^{s-t}\} = \{1 \le a < l^s, (a, l) = 1\} .$$

This implies that $b(\varphi)$ may be rewritten:

$$b(\varphi) = \sum_{1 \le c < l^t, (c,l)=1} \{\frac{uc}{l^t}\} \sum_{0 \le k < l^{s-t}} \sigma_{c+kl^t}^{-1} .$$

If we let M_y denote the field $\mathbf{Q}(\zeta_{l^t})$, then it follows that

$$b(\varphi) = \sigma_u \theta(M_y) Tr_{\mathbf{Q}(\varphi)/M_y} \tag{5.3}$$

where $\theta(M_y)$ is the Stickelberger element of the field M_y. We then deduce from Stickelberger's Theorem that for any j there exists an integer N_j such that $l^{N_j} b_j(\varphi)$ annihilates $\mathcal{P}$ and therefore that there exists a power of l which annihilates $c(h_p(\varphi))$, as required for the first part of the lemma.

In the special case where G has order l we note that in the above $m_j = n_j = 0$ for any j and that $\mathbf{Q}_p(\zeta_{l^N}) = \mathbf{Q}_p(\varphi_p)$. Therefore for any j there exists r_j such that $A_j(\varphi_p) = \sigma_{r_j}\theta$, where θ is the Stickelberger element of $\mathbf{Q}(\zeta_l)/\mathbf{Q}$. We therefore conclude that $c(h_p(\varphi))$ is principal. The lemma is now proved.

We now can complete the proof of Theorem 1.9. It follows from Lemmas 5.1 and 5.2 that for any $p \in \Sigma$ there exists an integer d such that $t(g_p)^{l^d}$ belongs to $D(\mathbf{Z}[G])$. Moreover one can choose d equal to 0 when G is of order l. This proves Theorem 1.9. *i*) and *iii*), since the group $D(\mathbf{Z}[G])$ is itself an l-group which is trivial when G is of order l. Let $\mathcal{M}$ denote the unique maximal order of the group algebra $\mathbf{Q}[\mathbf{G}]$. When l is regular, the order of the class group $\mathrm{Cl}(\mathcal{M})$ is coprime with l. Therefore, in this case, for any $p \in \Sigma$, the class $t(g_p)$ is itself an element of $D(\mathbf{Z}[G])$. This proves Theorem 1.9 (*ii*).

6 Some non-trivial Euler characteristics

This section contains the proof of Theorem 1.10. Our aim here is to use the theory of modular curves to provide some detailed examples of the above general results.

6.a A modular example

We let p be a prime number with $p \equiv 1 \bmod 24$ and $X_1 = X_1(p)$ be the model over Spec $(\mathbf{Z})$ of the modular curve associated to the congruence subgroup $\Gamma_1(p)$ as considered in Section 4 of [CPT]. The group $\Gamma/\{\pm 1\}$ acts faithfully on X_1. Let l be a prime divisor of $(p-1)$ with $l > 3$ and let H be the subgroup of Γ of index l. Since $p \equiv 1 \bmod 24$ we know that 6 divides the order of H. We consider the quotients $Y = X_1/\Gamma$ and $X = X_1/H$ of X_1. The scheme X is projective and flat over Spec $(\mathbf{Z})$ and is acted on tamely by the cyclic group $G = \Gamma/H$ of order l. The morphism $\pi : X \to Y$ is a cover which fulfils all the hypotheses required in this paper (see Theorems 4.2 and 4.3 in [CPT]). It follows from the work of various authors that $X[1/p] \to Y[1/p]$ is a G-torsor which implies that the set Σ defined previously is reduced to $\{p\}$. Moreover the special fiber over p is reduced with normal crossings and has two irreducible components D_0 and D_∞ both isomorphic to $\mathbf{P}^1_{\mathbf{F}_p}$. If we let y_0 (resp. y_∞) be the generic point of D_0 (resp. D_∞), then the intersection number $y_0.y_\infty$ is equal to $(p-1)/12$. Therefore we deduce that $y_0^2 = y_\infty^2 = (1-p)/12$ (see for instance Proposition 1.21 in Chapter 9 of [L]). Finally, it follows once again from Theorem 4.3 in [CPT] that π is totally ramified over y_0 and non-ramified over y_∞. If x_0 denotes the codimension one point of X above y_0, then we denote by ψ_0 the character of $G = I_{x_0}$ defining the action of I_{x_0} on the cotangent space at x_0.

Before describing our result we need to introduce a small amount of further notation. We let $\mathcal{P}$ be the prime ideal of the cyclotomic field $\mathbf{Q}(\zeta_l)$ defined by the chosen embedding $j_p : \mathbf{Q}^{\mathbf{c}} \to \mathbf{Q}_p^c$. For any abelian character θ of G we denote by θ_p the $\mathbf{Q}_p^c$-character of G obtained by composing θ with j_p. We denote by φ the non-trivial character of G such that $\varphi_p = \psi_0$. The character φ induces a group homomorphism $\lambda \mapsto \varphi(\lambda)$ from the group of ideles of $\mathbf{Q}[G]$ to the group of ideles of $\mathbf{Q}(\zeta_l)$. We denote by $c(\varphi(\lambda))$ the content of this idele. For any number field L and any fractional ideal I of L we denote by $[I]$ the class of I in the class group $\mathrm{Cl}(O_L)$ of O_L. Let $x \in \mathrm{Cl}(\mathbf{Z}[G])$ have representative λ in the group of ideles of $\mathbf{Q}[G]$. Since G is a group of prime order l, the map $x \mapsto [c(\varphi(\lambda))]$, induces a group isomorphism from $\mathrm{Cl}(\mathbf{Z}[G])$ into $\mathrm{Cl}(\mathbf{Z}[\zeta_l])$ that we also denote by φ.

For $\mathcal{F}$ equal to either $\mathcal{O}_X$ or $\omega_{X/Y}^{1/2}$ and for any abelian character θ of G we consider the rational number $T_p(\mathcal{F}, \theta)$:

$$T_p(\mathcal{F}, \theta) = r_p(\mathcal{F}, \theta)^2 + c_1(\omega_{Y_p}) \cdot r_p(\mathcal{F}, \theta) \ . \tag{6.1}$$

We know that l divides $y_0 \cdot y_\infty$; it follows from Proposition 4.1 that $T_p(\mathcal{O}_X, \theta) - T_p(\omega_{X/Y}^{1/2}, \theta)$ is an integer and from 3.13 in [CPT] we know that the same is true for $lT_p(\mathcal{O}_X, \theta)$. Let us first consider the class

$$V_{X/Y} = 2\chi^P(\omega_{X/Y}^{1/2}) - 2\chi^P(\mathcal{O}_X) \ .$$

By Theorem 1.7 we conclude that

$$\varphi(V_{X/Y}) = [\mathcal{P}]^{\sum_{1\leq a<l}(T_p(\mathcal{O}_X,\psi_0^a)-T_p(\omega_{X/Y}^{1/2},\psi_0^a))\sigma_a^{-1}} .$$

The description of $\varphi(l\chi(\mathcal{O}_X)$ follows from Theorem 3.13 in [CPT]. Since $lT_p(\mathcal{O}_X,\theta)$ is an integer, for any abelian character θ we obtain that

$$\varphi(2l\chi(\mathcal{O}_X)) = [\mathcal{P}]^{-l\sum_{1\leq a<l}T_p(\mathcal{O}_X,\psi_0^a)\sigma_a^{-1}}.$$

We deduce from the previous equalities that

$$\varphi(2l\chi(\omega_{X/Y})^{1/2}) = [\mathcal{P}]^{-l\sum_{1\leq a<l}T_p(\omega_{X/Y}^{1/2},\psi_0^a)\sigma_a^{-1}} .$$

6.b Proof of Theorem 1.10

From 3.5 we know that $r_p(\mathcal{O}_X,\psi_0^a) = \frac{a}{l}y_0$. On the other hand from (3.15) in Section 3 in [CPT] we know that

$$c_1(\omega_{Y_p})\cdot y_0 = -y_0^2 - 2\chi(y_0,\mathcal{O}_{y_0}) = -y_0^2 - 2. \tag{6.2}$$

Piecing this together with 6.1 we see that we have shown that

$$-lT_p(\mathcal{O}_X \mid \psi_0^a) = \frac{p-1}{12l}a^2 + (2-\frac{p-1}{12})a$$

and so

$$\varphi(2l\chi(\mathcal{O}_X)) = [\mathcal{P}]^{\frac{p-1}{12l}\sum_{1\leq a<l}a^2\sigma_a^{-1}+(2-\frac{p-1}{12})l\theta}$$

where θ is the Stickelberger element of $\mathbf{Q}(\zeta_l)$. Since we know from Stickelberger's theorem that $l\theta$ annihilates the class group of $\mathbf{Q}(\zeta_l)$, we deduce that

$$\varphi(2l\chi(\mathcal{O}_X)) = [\mathcal{P}]^{\frac{p-1}{12l}\sum_{1\leq a<l}a^2\sigma_a^{-1}}.$$

We apply the results of Examples in Section 3 and consider separately the cases when $a < l/2$ and when $a > l/2$.

If $a < l/2$, then we obtain that

$$r_p(\omega_{X/Y}^{1/2},\psi_0^a) = r_p(\mathcal{O}_X,\psi_0^a)$$

and so

$$T_p(\mathcal{O}_X,\psi_0^a) - T_p(\omega_{X/Y}^{1/2},\psi_0^a) = 0.$$

If now $a > l/2$, then

$$r_p(\omega_{X/Y}^{1/2}, \psi_0^a) = r_p(\mathcal{O}_X, \psi_0^a) - y_0$$

and so

$$T_p(\mathcal{O}_X, \psi_0^a) - T_p(\omega_{X/Y}^{1/2}, \psi_0^a) = \frac{p-1}{6}(1 - \frac{a}{l}) - 2.$$

Therefore we have proved that

$$\varphi(V_{X/Y}) = \overline{\mathcal{P}}^{\frac{p-1}{6l}s_1 - 2s_0}$$

where for any integer $i \in \{0, 1, 2\}$ we have defined s_i in $\mathbf{Z}[G]$ by

$$s_i = \sum_{1 \leq a < l/2} a^i \sigma_a^{-1} \ .$$

In order to conclude we need some elementary relationships in $\mathbf{Z}[G]$ that we prove in the next lemma.

Lemma 6.3 *One has the following equalities:*
i) $s_0 = (2\sigma_{-1} - \sigma_{(l-1)/2}^{-1})\theta$.
ii) $(1 - \sigma_{-1})s_1 = (\sigma_{(l+1)/2}^{-1} - 1)(l\theta)$.
iii) $\sum_{1 \leq a < l} a^2 \sigma_a^{-1} = (1 + \sigma_{-1})s_2 + l\sigma_{-1}(ls_0 - 2s_1)$.

Proof. The proof of *iii)* is immediate. We deduce *ii)* from *i)* via the relationship:

$$l\theta = (1 - \sigma_{-1})s_1 + l\sigma_{-1}s_0 \ .$$

Therefore it suffices to prove *i)*. We first observe that

$$2\sigma_2^{-1}(l\theta) = \sum_{1 \leq a < l} (2a\sigma_{2a}^{-1})$$

can be decomposed as the sum

$$2\sigma_2^{-1}(l\theta) = \sum_{1 \leq a < l/2} 2a\sigma_{2a}^{-1} + \sum_{l/2 < a < l} (2a - l)\sigma_{2a-l}^{-1} + l \sum_{l/2 < a < l} \sigma_{2a-l}^{-1} \ .$$

We now observe that the sum of the first two terms is precisely $l\theta$. Moreover, using the equality $\sigma_{2a-l}^{-1} = \sigma_{\frac{l-1}{2}}\sigma_{l-a}^{-1}$, we express the last term of the sum as $\sigma_{\frac{l-1}{2}}s_0$. Therefore we have proved that

$$2\sigma_2^{-1}\theta = \theta + \sigma_{\frac{l-1}{2}}s_0$$

and i) follows.

From the lemma and Stickelberger's theorem we deduce that s_0 annihilates the class group of $\mathbf{Z}[\zeta_l]$. Therefore we obtain that

$$\varphi(V_{X/Y}) = [\mathcal{P}]^{\frac{p-1}{6l}s_1} \text{ and } \varphi(2l(\chi(\mathcal{O}_X))) = [\mathcal{P}\overline{\mathcal{P}}]^{\frac{p-1}{12l}s_2}[\overline{\mathcal{P}}]^{-\frac{p-1}{6}s_1} .$$

It follows also from part (ii) of the lemma that $[\mathcal{P}]^{s_1} = [\overline{\mathcal{P}}]^{s_1}$ and thus that $[\mathcal{P}]^{2s_1} = [\mathcal{P}\overline{\mathcal{P}}]^{s_1}$. The formulas of Theorem 1.10 follow immediately.

6.c Non existence of NIB

We conclude this section by giving examples where our invariants are non-trivial. Our strategy is to evaluate the norm of these classes in the class group of the quadratic subfield k of $\mathbf{Q}(\zeta_l)$. We will denote the norm from $\mathbf{Q}(\zeta_l)$ to k by N. Since $\mathbf{Q}(\zeta_l)/k$ contains no unramified subextension F/k with $F \neq k$, N induces a surjective group homomorphism from $\mathrm{Cl}(\mathbf{Z}[\zeta_l])$ onto $\mathrm{Cl}(\mathcal{O}_k)$ (see for instance Theorem 10.1 of [W]). When $l \equiv 3 \bmod 4$ the field k is quadratic imaginary. It therefore follows from Theorem 1.10.i that all the classes belong to the kernel of N. This implies a certain restriction on their orders.

We next consider the case where $l \equiv 1 \bmod 4$, so that $k = \mathbf{Q}(\sqrt{l})$. The Galois group of $\mathbf{Q}(\zeta_l)$ over k consists of the set $\{\sigma_a, 1 \leq a < l\}$ such that a is a square mod l. We let A (resp. B) denote the set of $a, 1 \leq a < l/2$, such that a is (resp. is not) a square mod l. Since -1 is a square, we see immediately that Card(A) = Card(B). For $i \in \{1, 2\}$ we set

$$t_i = \sum_{a \in A} a^i - \sum_{b \in B} b^i.$$

By taking the norm of both sides of the equalities in Theorem 1.10, and writing $\beta = N(\mathcal{P})$, we obtain that

$$N(\varphi(V_{X/Y})) = [\beta]^{\frac{p-1}{6l}t_1} , \quad N(\varphi(2l\chi(\omega_{X/Y}^{1/2}))) = [\beta]^{\frac{p-1}{6l}t_2}$$

and

$$N(\varphi(2l\chi(\mathcal{O}_X))) = [\beta]^{\frac{p-1}{6l}(t_2 - lt_1)}.$$

To conclude we consider the case $l = 401$. The class number of $\mathbf{Q}(\sqrt{l})$ is 5. The integers t_1 and t_2 are independent of p. By computation we obtain that $t_1 \equiv 4 \bmod 5$, $t_2 \equiv 3 \bmod 5$ and $t_2 - lt_1 \equiv 4 \bmod 5$. Therefore for any $p \equiv 1 \bmod 24l$ with $p \not\equiv 1 \bmod 5$ with the property that β is not principal in k, we obtain three non trivial classes. The smallest prime satisfying these properties is $p = 182857$. This example therefore provides us with a tame cover of surfaces $\pi : X \to Y$ where $2\chi(\mathcal{O}_X)$, $2\chi(\omega_{X/Y})$ and $2\chi(\omega_{X/Y}^{1/2})$ are all non trivial, where $2\chi(\mathcal{O}_X) = 2\chi(\omega_{X/Y})$ but where $2\chi(\mathcal{O}_{X/Y}) \neq 2\chi(\omega_{X/Y}^{1/2})$.

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