Lifting Artin-Schreier covers with maximal wild monodromy

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Abstract

Let k be an algebraically closed field of characteristic $p > 0$. We consider the problem of lifting p-cyclic covers of $\mathbb{P}^1_k$ as p-cyclic covers of the projective line over some DVR under the condition that the wild monodromy is maximal. We answer positively the question for covers birational to $w^p - w = tR(t)$ for some additive polynomial $R(t)$.

1 Introduction

Let (R, v) be a complete discrete valuation ring of mixed characteristic $(0, p)$ with fraction field K containing a primitive p-th root of unity ζ_p and algebraically closed residue field k. The stable reduction theorem states that given a smooth, projective, geometrically connected curve C/K of genus $g(C) \geq 2$, there exists a unique minimal Galois extension M/K called *the monodromy extension of* C/K such that $C_M := C \times M$ has stable reduction over M. The group $G = \mathrm{Gal}(M/K)$ is the *monodromy group of* C/K.

Let us consider the case where $\phi : C \to \mathbb{P}^1_K$ is a p-cyclic cover. Let $\mathcal{C}$ be the stable model of C_M/M and $\mathrm{Aut}_k(\mathcal{C}_k)^\#$ be the subgroup of $\mathrm{Aut}_k(\mathcal{C}_k)$ of elements acting trivially on the reduction in $\mathcal{C}_k$ of the ramification locus of $\phi \times \mathrm{Id}_M : C_M \to \mathbb{P}^1_M$ (see [Liu02] 10.1.3 for the definition of the reduction map of C_M). One derives from the stable reduction theorem the following injection

$$\mathrm{Gal}(M/K) \hookrightarrow \mathrm{Aut}_k(\mathcal{C}_k)^\#. \tag{1}$$

When the p-Sylow subgroups of these groups are isomorphic, one says that the *wild monodromy is maximal*. We are interested in realization of smooth covers as above such that the p-adic valuation of $|\,\mathrm{Aut}_k(\mathcal{C}_k)^\#|$ is large compared to the genus of $\mathcal{C}_k$ and having maximal wild monodromy. Moreover, we will study the ramification filtration and the Swan conductor of their monodromy extension.

Recall that a big action is a pair (X, G) where X/k is a smooth, projective, geometrically connected curve of genus $g(X) \geq 2$ and G is a finite p-group of k-automorphisms of X/k such that $|G| > \frac{2p}{p-1} g(X)$. According to [LM05] Theorem 1.1 II f), if (X, G) is a big action, then one has that $|G| \leq \frac{4p}{(p-1)^2} g(X)^2$ with equality if and only if X/k is birationally given by $w^p - w = tR(t)$ where $R(t) \in k[t]$ is an additive polynomial. In this case, G is an extra-special p-group and equals the p-Sylow subgroup $G_{\infty,1}(X)$ of the subgroup of $\mathrm{Aut}_k(X)$ leaving $t = \infty$ fixed.

This motivates the following question, with the above notations, given a big action (C, G) such that $|G| = \frac{4p}{(p-1)^2} g(X)^2$, is it possible to find a field K and a a p-cyclic cover C/K of $\mathbb{P}^1_K$ such that $\mathcal{C}_k \simeq X$, that $G \simeq \mathrm{Aut}(\mathcal{C}_k)^\#_1$ is a p-Sylow subgroup of $\mathrm{Aut}(\mathcal{C}_k)^\#$ and the curve C/K has maximal wild monodromy ?

Let $n \in \mathbb{N}^\times$, $q = p^n$, $\lambda = \zeta_p - 1$ and $K = \mathbb{Q}_p^{\mathrm{ur}}(\lambda^{1/(1+q)})$. For any additive polynomial $R(t) \in k[t]$ of degree q, let X/k be curve defined by $w^p - w = tR(t)$. In section 3, we prove the following

Theorem 1.1. *There exists a p-cyclic cover C/K of $\mathbb{P}^1_K$ such that $\mathcal{C}_k \simeq X$, one has $G_{\infty,1}(X) \simeq \mathrm{Aut}(\mathcal{C}_k)^\#_1$ and the curve C/K has maximal wild monodromy M/K. The extension M/K is the decomposition field of an explicitly given polynomial and the group $\mathrm{Gal}(M/K) \simeq \mathrm{Aut}_k(\mathcal{C}_k)^\#_1$ is an extra-special p-group of order pq^2.*

The group $G_{\infty,1}(\mathcal{C}_k) = \mathrm{Aut}_k(\mathcal{C}_k)^\#_1$ is endowed with the ramification filtration $(G_{\infty,i}(\mathcal{C}_k))_{i\geq 0}$ which is easily seen to be :

$$G_{\infty,0}(\mathcal{C}_k) = G_{\infty,1}(\mathcal{C}_k) \supsetneq \mathrm{Z}(G_{\infty,0}(\mathcal{C}_k)) = G_{\infty,2}(\mathcal{C}_k) = \cdots = G_{\infty,1+q}(\mathcal{C}_k) \supsetneq \{1\}.$$

Moreover, $G := \mathrm{Gal}(M/K)$ being the Galois group of a finite extension of K, it is endowed with the ramification filtration $(G_i)_{i\geq 0}$. Since $G \simeq G_{\infty,1}(\mathcal{C}_k)$ it is natural to ask for the behaviour of $(G_i)_{i\geq 0}$ under (1), that is to compare $(G_i)_{i\geq 0}$ and $(G_{\infty,i}(\mathcal{C}_k))_{i\geq 0}$. In the general case, the arithmetic is quite tedious due to the expression of the lifting of X/k. Actually we could not obtain a numerical example for the easiest case when $p = 3$. Nonetheless, when $p = 2$, one computes the conductor exponent $f(\mathrm{Jac}(C)/K)$ of $\mathrm{Jac}(C)/K$ and its Swan conductor $\mathrm{sw}(\mathrm{Jac}(C)/K)$:

Theorem 1.2. *Under the hypotheses of Theorem 1.1, if $p = 2$ the lower ramification filtration of G is :*

$$G = G_0 = G_1 \supsetneq \mathrm{Z}(G) = G_2 = \cdots = G_{1+q} \supsetneq \{1\}.$$

Then, $f(\mathrm{Jac}(C)/K) = 2q+1$ *and* $\mathrm{sw}(\mathrm{Jac}(C)/\mathbb{Q}_2^{\mathrm{ur}}) = 1$.

Remarks :

1. In Theorem 1.1, one actually obtains a family of liftings C/K of X/k with the announced properties. It is worth noting that there are finitely many additive polynomials $R_0(t) \in k[t]$ such that $w^p - w = tR(t)$ is k-isomorphic to $w^p - w = tR_0(t)$ (see [LM05] 8.2), so we have to solve the problem in a somehow generic way. In [CM11], we obtain the analogous of Theorem 1.1 and Theorem 1.2 for $p \geq 2$ in the easier case $R(t) = t^q$.

2. For $p = 3$, the easiest non-trivial case is such that $[M : K] = 243$, that is why we could not even do computations using Magma to guess the behaviour of the ramification filtration of the monodromy extension for $p > 2$. Nonetheless, one shows that if $p \geq 3$, the lower ramification filtration of G is

$$G = G_0 = G_1 \supsetneq G_2 = \cdots = G_u = \mathrm{Z}(G) \supsetneq \{1\},$$

   where $u \in 1 + q\mathbb{N}$.

3. The value $\mathrm{sw}(\mathrm{Jac}(C)/\mathbb{Q}_2^{\mathrm{ur}}) = 1$ is the smallest one among abelian varieties over $\mathbb{Q}_2^{ur}$ with non tame monodromy extension. That is, in some sense, a counter part of [BK05] and [LRS93] where an upper bound for the conductor exponent is given and it is shown that this bound is actually achieved.

2 Background

Notations. Let (R, v) be a complete discrete valuation ring (DVR) of mixed characteristic $(0, p)$ with fraction field K and algebraically closed residue field k. We denote by π_K a uniformizer of R and assume that K contains a primitive p-th root of unity ζ_p. Let $\lambda := \zeta_p - 1$. If L/K is an algebraic extension, we will denote by π_L(resp. v_L, resp. L°) a uniformizer for L (resp. the prolongation of v to L such that $v_L(\pi_L) = 1$, resp. the ring of integers of L). If there is no possible confusion we note v for the prolongation of v to an algebraic closure K^{alg} of K.

1. *Stable reduction of curves.* The first result is due to Deligne and Mumford (see for example [Liu02] for a presentation following Artin and Winters).

Theorem 2.1 (Stable reduction theorem)**.** *Let C/K be a smooth, projective, geometrically connected curve over K of genus $g(C) \geq 2$. There exists a unique finite Galois extension M/K minimal for the inclusion relation such that C_M/M has stable reduction. The stable model $\mathcal{C}$ of C_M/M over M° is unique up to isomorphism. One has a canonical injective morphism :*

$$\mathrm{Gal}(M/K) \overset{i}{\hookrightarrow} \mathrm{Aut}_k(\mathcal{C}_k). \tag{2}$$

Remarks :

1. Let's explain the action of $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ on $\mathcal{C}_k/k$. The group $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ acts on $C_M := C \times M$ on the right. By unicity of the stable model, this action extends to $\mathcal{C}$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sigma} & \mathcal{C} \\ \downarrow & & \downarrow \\ M^\circ & \xrightarrow{\sigma} & M^\circ \end{array}$$

   Since $k = k^{\mathrm{alg}}$ one gets $\sigma \times k = \mathrm{Id}_k$, whence the announced action. The last assertion of the theorem characterizes the elements of $\mathrm{Gal}(K^{\mathrm{alg}}/M)$ as the elements of $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ that trivially act on $\mathcal{C}_k/k$.

2. If $p > 2g(C)+1$, then C/K has stable reduction over a tamely ramified extension of K. We will study examples of covers with $p \leq 2g(C) + 1$.

3. Our results will cover the elliptic case. Let E/K be an elliptic curve with additive reduction. If its modular invariant is integral, then there exists a smallest extension M of K over which E/K has good reduction. Else E/K obtains split multiplicative reduction over a unique quadratic extension of K (see [Kra90]).

Definition 2.1. *The extension M/K is the monodromy extension of C/K. We call* $\mathrm{Gal}(M/K)$ *the monodromy group of C/K. It has a unique p-Sylow subgroup* $\mathrm{Gal}(M/K)_1$ *called the wild monodromy group. The extension $M/M^{\mathrm{Gal}(M/K)_1}$ is the wild monodromy extension.*

From now on we consider smooth, projective, geometrically integral curves C/K of genus $g(C) \geq 2$ birationally given by $Y^p = f(X) := \prod_{i=0}^{t}(X - x_i)^{n_i}$ with $(p, \sum_{i=0}^{t} n_i) = 1$, $(p, n_i) = 1$ and $\forall\ 0 \leq i \leq t, x_i \in R^\times$. Moreover, we assume that $\forall i \neq j,\ v(x_i - x_j) = 0$, that is to say, the branch locus

$B = \{x_0, \ldots, x_t, \infty\}$ of the cover has *equidistant geometry*. We denote by *Ram* the ramification locus of the cover.

Remark : We only ask p-cyclic covers to satisfy Raynaud's theorem 1' [Ray90] condition, that is the branch locus is K-rational with equidistant geometry. This has consequences on the image of (2).

Proposition 2.1. *Let $\mathcal{T} = \mathrm{Proj}(M^\circ[X_0, X_1])$ with $X = X_0/X_1$. The normalization $\mathcal{Y}$ of $\mathcal{T}$ in $K(C_M)$ admits a blowing-up $\tilde{\mathcal{Y}}$ which is a semi-stable model of C_M/M. The dual graph of $\tilde{\mathcal{Y}}_k/k$ is a tree and the points in Ram specialize in a unique irreducible component $D_0 \simeq \mathbb{P}^1_k$ of $\tilde{\mathcal{Y}}_k/k$. There exists a contraction morphism $h : \tilde{\mathcal{Y}} \to \mathcal{C}$, where $\mathcal{C}$ is the stable model of C_M/M and*

$$\mathrm{Gal}(M/K) \hookrightarrow \mathrm{Aut}_k(\mathcal{C}_k)^\#, \tag{3}$$

where $\mathrm{Aut}_k(\mathcal{C}_k)^\#$ is the subgroup of $\mathrm{Aut}_k(\mathcal{C}_k)$ of elements inducing the identity on $h(D_0)$.

Proof. see [CM11]. □

Remark : The component D_0 is the so called *original component*.

Definition 2.2. *If* (3) *is surjective, we say that C has* maximal monodromy*. If $v_p(|\,\mathrm{Gal}(M/K)|) = v_p(|\,\mathrm{Aut}_k(\mathcal{C}_k)^\#|)$, we say that C has* maximal wild monodromy*.*

Definition 2.3. *The valuation on $K(X)$ corresponding to the discrete valuation ring $R[X]_{(\pi_K)}$ is called the* Gauss valuation *v_X with respect to X. We then have*

$$v_X\left(\sum_{i=0}^{m} a_i X^i\right) = \min\{v(a_i),\ 0 \leq i \leq m\}.$$

Note that a change of variables $T = \frac{X-y}{\rho}$ for $y, \rho \in R$ induces a Gauss valuation v_T. These valuations are exactly those that come from the local rings at generic points of components in the semi-stables models of $\mathbb{P}^1_K$.

2. *Extra-special p-groups.* The Galois groups and automorphism groups that we will have to consider are p-groups with peculiar group theoretic properties (see for example [Hup67] Kapitel III §13 or [Suz86] for an account on extra-special p-groups). We will denote by $\mathrm{Z}(G)$ (resp. $\mathrm{D}(G)$, $\Phi(G)$) the center (resp. the derived subgroup, the Frattini subgroup) of G. If G is a p-group, one has $\Phi(G) = \mathrm{D}(G)G^p$.

Definition 2.4. *An extra-special p-group is a non abelian p-group G such that* $\mathrm{D}(G) = \mathrm{Z}(G) = \Phi(G)$ *has order p.*

Proposition 2.2. *Let G be an extra-special p-group.*

1. *Then* $|G| = p^{2n+1}$ *for some* $n \in \mathbb{N}^\times$*.*
2. *One has the exact sequence*

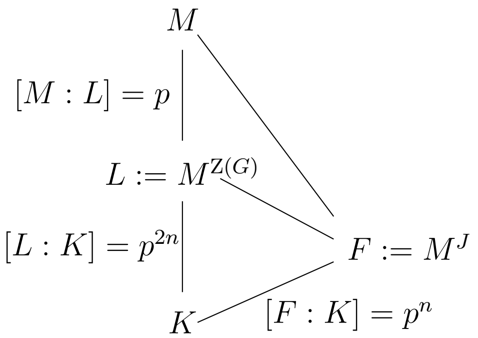
$$0 \to \mathrm{Z}(G) \to G \to (\mathbb{Z}/p\mathbb{Z})^{2n} \to 0.$$

3. *The group G has an abelian subgroup J such that* $\mathrm{Z}(G) \subseteq J$ *and* $|J/\mathrm{Z}(G)| = p^n$*.*

3. *Galois extensions of complete DVRs.* Let L/K be a finite Galois extension with group G. Then G is endowed with a *lower ramification filtration* $(G_i)_{i\geq -1}$ where G_i is the *i-th lower ramification group* defined by $G_i := \{\sigma \in G \mid v_L(\sigma(\pi_L) - \pi_L) \geq i+1\}$. The integers i such that $G_i \neq G_{i+1}$ are called *lower breaks*. For $\sigma \in G - \{1\}$, let $i_G(\sigma) := v_L(\sigma(\pi_L) - \pi_L)$. The group G is also endowed with a *higher ramification filtration* $(G^i)_{i\geq -1}$ which can be computed from the G_i's by means of the *Herbrand's function* $\varphi_{L/K}$. The real numbers t such that $\forall \epsilon > 0,\ G^{t+\epsilon} \neq G^t$ are called *higher breaks*.

Lemma 2.1. *Let M/K be a Galois extension such that* $\mathrm{Gal}(M/K)$ *is an extra-special p-group of order* p^{2n+1}*. Assume that* $\mathrm{Gal}(M^{\mathrm{Z}(G)}/K)_2 = \{1\}$*, then the break t of* $M/M^{\mathrm{Z}(G)}$ *is such that* $t \in 1 + p^n\mathbb{N}$*.*

Proof. According to Proposition 2.2 *3.*, there exists an abelian subgroup J with $\mathrm{Z}(G) \subseteq J \subseteq G$ and $|J/\mathrm{Z}(G)| = p^n$. Thus, one has the following diagram



Let t be the lower break of M/L, then t is a lower break of M/F and $\varphi_{M/F}(t) = \varphi_{L/F}(\varphi_{M/L}(t))$ is a higher break of M/F. Since $\varphi_{M/L}(t) = t$, one has $\varphi_{M/F}(t) = \varphi_{L/F}(t)$. Since $\mathrm{Gal}(L/K)_2 = \{1\}$, one has $\mathrm{Gal}(L/F)_2 = \{1\}$ and $\varphi_{L/F}(t) = 1 + \frac{t-1}{p^n}$. The Hasse-Arf Theorem applied to the abelian extension M/F implies that $1 + \frac{t-1}{p^n} \in \mathbb{N} - \{0\}$, thus $t \in 1 + p^n\mathbb{N}$.

□

4. *Torsion points on abelian varieties.* Let A/K be an abelian variety over K with potential good reduction and $\ell \neq p$ be a prime number. We denote by $A[\ell]$ the ℓ-torsion group of $A(K^{\mathrm{alg}})$ and by $T_\ell(A) = \varprojlim A[\ell^n]$ (resp. $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}_\ell$) the Tate module (resp. ℓ-adic Tate module) of A.

The following result may be found in [Gur03] (paragraph 3). We recall it for the convenience of the reader.

Lemma 2.2. *Let $k = k^{\mathrm{alg}}$ be a field with* $\mathrm{char}\, k = p \geq 0$ *and C/k be a projective, smooth, integral curve. Let $\ell \neq p$ be a prime number and H be a finite subgroup of* $\mathrm{Aut}_k(C)$ *such that* $(|H|, \ell) = 1$. *Then*

$$2g(C/H) = \dim_{\mathbb{F}_\ell} \mathrm{Jac}(C)[\ell]^H.$$

If $\ell \geq 3$, then $L = K(A[\ell])$ is the minimal extension over which A/K has good reduction. It is a Galois extension with group G (see [ST68]). We denote by r_G (resp. 1_G) the character of the regular (resp. unit) representation of G. We denote by I the inertia group of K^{alg}/K. For further explanations about conductor exponents see [Ser67], [Ogg67] and [ST68].

Definition 2.5. *1. Let*

$$a_G(\sigma) := -i_G(\sigma), \quad \sigma \neq 1,$$
$$a_G(1) := \sum_{\sigma \neq 1} i_G(\sigma),$$

and $\mathrm{sw}_G := a_G - r_G + 1_G$. *Then, a_G is the character of a $\mathbb{Q}_\ell[G]$-module and there exists a projective $\mathbb{Z}_\ell[G]$-module* Sw_G *such that* $\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ *has character* sw_G.

2. We still denote by $T_\ell(A)$ (resp. $A[\ell]$) the $\mathbb{Z}_\ell[G]$-module (resp. $\mathbb{F}_\ell[G]$-module) afforded by $G \to \mathrm{Aut}(T_\ell(A))$ (resp. $G \to \mathrm{Aut}(A[\ell])$). Let

$$\mathrm{sw}(A/K) := \dim_{\mathbb{F}_\ell} \mathrm{Hom}_G(\mathrm{Sw}_G, A[\ell]),$$
$$\epsilon(A/K) := \mathrm{codim}_{\mathbb{Q}_\ell} V_\ell(A)^I.$$

The integer $f(A/K) := \epsilon(A/K) + \mathrm{sw}(A/K)$ is the so called conductor exponent of A/K and $\mathrm{sw}(A/K)$ is the Swan conductor of A/K.

Proposition 2.3. *Let $\ell \neq p$, $\ell \geq 3$ be a prime number.*

1. *The integers* $\mathrm{sw}(A/K)$ *and* $\epsilon(A/K)$ *are independent of* ℓ*.*
2. *One has*
$$\mathrm{sw}(A/K) = \sum_{i \geq 1} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\ell} A[\ell]/A[\ell]^{G_i}.$$
Moreover, for ℓ large enough, $\epsilon(A/K) = \dim_{\mathbb{F}_\ell} A[\ell]/A[\ell]^{G_0}$.

Remark : It follows from the definition that $\mathrm{sw}(A/K) = 0$ if and only if $G_1 = \{1\}$. The Swan conductor is a measure of the wild ramification.

5. *Automorphisms of Artin-Schreier covers.* See [LM05] for further results on this topic. Let $R(t) \in k[t]$ be a monic additive polynomial and A_R/k be the smooth, projective, geometrically irreducible curve birationally given by $w^p - w = tR(t)$. There is a so called Artin-Schreier morphism $\pi : A_R \to \mathbb{P}^1_k$. The automorphism t_a of $\mathbb{P}^1_k$ given by $t \mapsto t + a$ with $a \in k$ has a prolongation $\tilde{t_a}$ to A_R if there is a commutative diagram

$$\begin{array}{ccc} A_R & \xrightarrow{\tilde{t_a}} & A_R \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^1_k & \xrightarrow{t_a} & \mathbb{P}^1_k \end{array}$$

Proposition 2.4. *Let $n \geq 1$, $q := p^n$ and $R(t) := \sum_{k=0}^{n-1} \bar{u}_k t^{p^k} + t^q \in k[t]$. The automorphism of $\mathbb{P}^1_k$ given by $t \mapsto t + a$ with $a \in k$ has a prolongation to A_R/k if and only if one has*

$$a^{q^2} + (2\bar{u}_0 a)^q + \sum_{k=1}^{n-1} (\bar{u}_k^q a^{qp^k} + (\bar{u}_k a)^{q/p^k}) + a = 0.$$

3 Main theorem

We start by fixing notations that will be used throughout this section.

Notations. We denote by $\mathfrak{m}$ the maximal ideal of $(K^{\mathrm{alg}})^\circ$. Let $n \in \mathbb{N}^\times$, $q := p^n$, $a_n := (-1)^q(-p)^{p+p^2+\cdots+q}$ and $\forall\, 0 \leq i \leq n-1$, $d_i := p^{n-i+1} + \cdots + q$. We denote by $\mathbb{Q}_p^{\mathrm{ur}}$ the maximal unramified extension of $\mathbb{Q}_p$ and we put

$K := \mathbb{Q}_p^{\mathrm{ur}}(\lambda^{1/(1+q)})$. Let $\underline{\rho} := (\rho_0, \dots, \rho_{n-1})$ where $\forall\ 0 \leq k \leq n-1$, $\rho_k \in K$, $\rho_k = u_k \lambda^{p(q-p^k)/(1+q)}$ and $v(u_k) = 0$ or $u_k = 0$. For $c \in R$, let

$$f_{c,\rho}(X) := 1 + \sum_{k=0}^{n-1} \rho_k X^{1+p^k} + cX^q + X^{1+q},$$

$$\text{and } s_{1,\rho}(X) := 2\rho_0 X + \sum_{k=1}^{n-1} \rho_k X^{p^k} + X^q.$$

One defines the *modified monodromy polynomial* $L_{c,\rho}(X)$ by

$$s_{1,\rho}(X)^q - a_n f_{c,\rho}(X)^{q-1}(c+X) - (-1)^q \sum_{k=1}^{n-1} (\rho_k X)^{q/p^k} (-p)^{d_k} f_{c,\rho}(X)^{q(p^k-1)/p^k}.$$

Let $C_{c,\rho}/K$ and A_u/k be the smooth projective integral curves birationally given respectively by $Y^p = f_{c,\rho}(X)$ and $w^p - w = \sum_{k=0}^{n-1} \bar{u}_k t^{1+p^k} + t^{1+q}$.

Theorem 3.1. *The curve $C_{c,\rho}/K$ has potential good reduction isomorphic to A_u/k.*

1. *If $v(c) \geq v(\lambda^{p/(1+q)})$, then the monodromy extension of $C_{c,\rho}/K$ is trivial.*
2. *If $v(c) < v(\lambda^{p/(1+q)})$, let y be a root of $L_{c,\rho}(X)$ in K^{alg}. Then $C_{c,\rho}$ has good reduction over $K(y, f_{c,\rho}(y)^{1/p})$. If $L_{c,\rho}(X)$ is irreducible over K, then $C_{c,\rho}/K$ has maximal wild monodromy. The monodromy extension of $C_{c,\rho}/K$ is $M = K(y, f_{c,\rho}(y)^{1/p})$ and $G = \mathrm{Gal}(M/K)$ is an extra-special p-group of order pq^2.*
3. *Assume that $c = 1$. The polynomial $L_{1,\rho}(X)$ is irreducible over K. The lower ramification filtration of G is*

$$G = G_0 = G_1 \supsetneq G_2 = \cdots = G_u = \mathrm{Z}(G) \supsetneq \{1\},$$

with $u \in 1 + q\mathbb{N}$. Moreover, if $p = 2$, then $u = 1 + q$, one has $f(\mathrm{Jac}(C_{1,\rho})/K) = 2q + 1$ and $\mathrm{sw}(\mathrm{Jac}(C_{1,\rho})/\mathbb{Q}_2^{\mathrm{ur}}) = 1$.

Proof. 1. Assume that $v(c) \geq v(\lambda^{p/(1+q)})$. Set $\lambda^{p/(1+q)}T = X$ and $\lambda W + 1 = Y$. Then, the equation defining $C_{c,\rho}/K$ becomes

$$(\lambda W + 1)^p = 1 + \sum_{k=0}^{n-1} \rho_k \lambda^{p(1+p^k)/(1+q)} T^{1+p^k} + c\lambda^{pq/(1+q)} T^q + \lambda^p T^{1+q}.$$

After simplification by λ^p and reduction modulo π_K this equation gives :

$$w^p - w = \sum_{k=0}^{n-1} \bar{u}_k t^{1+p^k} + at^q + t^{1+q},\ a \in k. \tag{4}$$

By Hurwitz formula the genus of the curve defined by (4) is seen to be that of $C_{c,\rho}/K$. Applying [Liu02] 10.3.44, there is a component in the stable reduction birationally given by (4). The stable reduction being a tree, the curve $C_{c,\rho}/K$ has good reduction over K.

2. The proof is divided into eight steps. Let y be a root of $L_{c,\rho}(X)$.

Step I : *One has* $v(y) = v(a_n c)/q^2$.
By expanding $L_{c,\rho}(X)$, one shows that its Newton polygon has a single slope $v(a_n c)/q^2$. The polynomial $L_{c,\rho}(X)$ has degree q^2 and its leading (resp. constant) coefficient has valuation 0 (resp. $v(a_n c)$). One examines monomials from $a_n f_{c,\rho}^{q-1}(X)(c+X)$. Since $v(c) < v(\lambda^{p/(1+q)})$, one checks that

$$\forall 1 \le i \le q^2 - 1,\ \frac{v(a_n)}{q^2 - i} \ge \frac{v(a_n c)}{q^2}.$$

Then one examines monomials from $(\rho_i X)^{q/p^i} p^{d_i} f_{c,\rho}(X)^{q(p^i-1)/p^i}$. They have degree at least q/p^i, thus one checks that

$$\forall\ 1 \le i \le n-1,\ \frac{q/p^i v(\rho_i) + d_i v(p)}{q^2 - q/p^i} \ge \frac{v(a_n c)}{q^2}.$$

The monomial X^{q^2} in $s_{1,\rho}(X)^q$ corresponds to the point $(0,0)$ in the Newton polygon of $L_{c,\rho}(X)$, the other monomials of $s_{1,\rho}(X)^q$ produce a slope greater than $v(\rho_i)/(q-p^i)$ and one checks that

$$\forall\ 0 \le i \le n-1,\ \frac{v(\rho_i)}{q - p^i} \ge \frac{v(a_n c)}{q^2}.$$

Note that **Step I** implies that $v(f_{c,\rho}(y)) = 0$, we will use this remark throughout this proof.

Step II : *Define* S *and* T *by* $\lambda^{p/(1+q)}T = (X - y) = S$. *Then* $f_{c,\rho}(S+y)$ *is congruent modulo* $\lambda^p\mathfrak{m}[T]$ *to*

$$f_{c,\rho}(y) + s_{1,\rho}(y)S + \sum_{k=0}^{n-1} \rho_k S^{1+p^k} + \sum_{k=1}^{n-1} \rho_k y S^{p^k} + (c+y)S^q + S^{1+q}.$$

Using the following formula for $A \in K^{\mathrm{alg}}$ with $v(A) > 0$ and $B \in (K^{\mathrm{alg}})^\circ[T]$

$$k \geq 1,\ (A+B)^{p^k} \equiv (A^{p^{k-1}} + B^{p^{k-1}})^p \bmod\ p^2\mathfrak{m}[T],$$

one computes mod $\lambda^p\mathfrak{m}[T]$

$$f_{c,\rho}(y+S) = 1 + \sum_{k=0}^{n-1} \rho_k (y+S)^{1+p^k} + (y+S)^{1+q} + c(y+S)^q$$

$$\equiv 1 + \rho_0(y+S)^2 + \sum_{k=1}^{n-1} \rho_k (y+S)(y^{p^{k-1}} + S^{p^{k-1}})^p + (y+S+c)(y^{q/p} + S^{q/p})^p.$$

Using **Step I**, one checks that for all $1 \leq k \leq n-1$

$$\rho_k (y^{p^{k-1}} + S^{p^{k-1}})^p \equiv \rho_k (y^{p^k} + S^{p^k}) \bmod\ \lambda^p\mathfrak{m}[T],$$

and $(y^{q/p} + S^{q/p})^p \equiv y^q + S^q \bmod\ \lambda^p\mathfrak{m}[T]$. It follows that

$$f_{c,\rho}(y+S) \equiv 1 + \rho_0(y+S)^2 + \sum_{k=1}^{n-1} \rho_k (y+S)(y^{p^k} + S^{p^k}) + (y+c+S)(y^q + S^q).$$

One easily concludes from this last expression.

Step III : *Let $R_1 := K[y]^\circ$. For all $0 \leq i \leq n$, one defines $A_i(S) \in R_1[S]$ and $B_i \in R_1$ by induction :*

$$B_n := -s_{1,\rho}(y),\ \ \forall\ 1 \leq i \leq n-1,\ B_i := \frac{f_{q,c}(y) B_{i+1}^p}{(-p f_{c,\rho}(y))^p} - y\rho_{n-i},$$

$$\text{and } B_0 := \frac{f_{c,\rho}(y) B_1^p}{(-p f_{c,\rho}(y))^p},$$

$$A_0(S) :=\ 0 \text{ and } \forall\ 0 \leq i \leq n-1\ \ SA_{i+1}(S) := SA_i(S) - \frac{B_{i+1} S^{q/p^{i+1}}}{p f_{q,c}(y)^{(p-1)/p}}.$$

Then for all $0 \leq i \leq n-1$, $v(B_{i+1}) = (1 + \cdots + p^i)v(p)/p^i + v(c)/p^{i+1}$ and modulo $\lambda^{\frac{pq^2}{q+1}}\mathfrak{m}$ one has

$$B_n^q \equiv \frac{a_n}{(-1)^q} f_{c,\rho}(y)^{q-1} B_0 + \sum_{k=1}^{n-1} (\rho_k y)^{q/p^k} (-p)^{d_k} f_{c,\rho}(y)^{q(p^k-1)/p^k}. \tag{5}$$

We prove the claim about $v(B_{i+1})$ by induction on i. Using **Step I**, one checks that $\forall\ 0 \leq k \leq n-1,\ v(\rho_k y^{p^k}) > v(y^q)$, so $v(B_n) = v(y^q)$. Assume that we have shown the claim for i, then one checks that $v((B_{i+1}/p)^p) < v(y\rho_{n-i})$ and one deduces $v(B_i)$ from the definition of B_i. According to the expression of $v(B_i)$, one has $\forall\, 0 \leq i \leq n, A_i(S) \in R_1[S]$.

Then we prove the second part of **Step III**. From the definition of the B_i's one obtains that for all $1 \leq i \leq n-1$

$$B_{n-i+1}^{q/p^{i-1}} = (-p)^{q/p^{i-1}} f_{c,\rho}(y)^{q(p-1)/p^i}(y\rho_i + B_{n-i}(y))^{q/p^i}. \tag{6}$$

Using **Step I** and $v(B_{n-1})$ one checks that $\forall\, 1 \leq k \leq q/p - 1$

$$p^q \binom{q/p}{k} (y\rho_1)^k B_{n-1}^{q/p-k} \equiv 0 \bmod \lambda^{pq^2/(1+q)}\mathfrak{m},$$

so $p^q(y\rho_1 + B_{n-1})^{q/p} \equiv p^q((y\rho_1)^{q/p} + B_{n-1}^{q/p}) \bmod \lambda^{pq^2/(1+q)}\mathfrak{m}$. Thus, applying equation (6) with $i = 1$, one gets

$$\begin{aligned} B_n^q &= (-p)^q f_{c,\rho}(y)^{q(p-1)/p}(y\rho_1 + B_{n-1})^{q/p} \\ &\equiv (-p)^q f_{c,\rho}(y)^{q(p-1)/p}((y\rho_1)^{q/p} + B_{n-1}^{q/p}) \bmod \lambda^{pq^2/(1+q)}\mathfrak{m}. \end{aligned}$$

One checks using **Step I** and $v(B_{n-i})$ that $\forall\, 1 \leq i \leq n-1$ and $1 \leq k \leq q/p^i - 1$

$$p^{q+\cdots+q/p^{i-1}} \binom{q/p^i}{k} B_{n-i}^{q/p^i-k}(y\rho_i)^k \equiv 0 \bmod \lambda^{pq^2/(1+q)}\mathfrak{m},$$

then by induction on i, using equation (6), one shows that modulo $\lambda^{pq^2/(1+q)}\mathfrak{m}$

$$B_n^q \equiv (-p)^{p+\cdots+q} f_{c,\rho}(y)^{q-1}B_0 + \sum_{k=1}^{n-1}(\rho_k y)^{q/p^k}(-p)^{d_k} f_{c,\rho}(y)^{q(p^k-1)/p^k}.$$

Step IV : *One has modulo* $\lambda^p\mathfrak{m}[T]$

$$f_{c,\rho}(S+y) \equiv f_{c,\rho}(y) + s_{1,\rho}(y)S + \sum_{k=0}^{n-1}\rho_k S^{1+p^k} + \sum_{k=1}^{n-1} y\rho_k S^{p^k} + B_0 S^q + S^{1+q}.$$

Since $L_{c,\rho}(y) = 0$, one has

$$s_{1,\rho}(y)^q = a_n f_{c,\rho}(y)^{q-1}(c+y) + (-1)^q \sum_{k=1}^{n-1}(\rho_k y)^{q/p^k}(-p)^{d_k} f_{c,\rho}(y)^{q(p^k-1)/p^k}. \tag{7}$$

Using $B_n := -s_{1,\rho}(y)$, equations (5) and (7) one gets

$$a_n f_{c,\rho}(y)^{q-1}(c + y - B_0) \equiv 0 \bmod \lambda^{pq^2/(q+1)}\mathfrak{m}.$$

which is equivalent to $S^q(y+c-B_0) \equiv 0 \bmod \lambda^p\mathfrak{m}[T]$. Then, **Step IV** follows from **Step II**.

Step V : *One has*

$$f_{c,\rho}(S+y) \equiv (f_{c,\rho}(y)^{1/p} + SA_n(S))^p + \sum_{k=0}^{n-1} \rho_k S^{1+p^k} + S^{1+q} \bmod \lambda^p\mathfrak{m}[T].$$

Let $R := \sum_{k=0}^{n-1} \rho_k S^{1+p^k} + S^{1+q} + s_{1,\rho}(y)S$. Since $B_n = -s_{1,\rho}(y)$ one has

$$\begin{aligned}
&(f_{c,\rho}(y)^{1/p} + SA_n(S))^p + \sum_{k=0}^{n-1} \rho_k S^{1+p^k} + S^{1+q} \\
&= (f_{c,\rho}(y)^{1/p} + SA_n(S))^p + B_nS + R \\
&= \Big(f_{c,\rho}(y)^{1/p} + SA_{n-1}(S) - \frac{B_nS}{pf_{q,c}(y)^{(p-1)/p}}\Big)^p + B_nS + R \\
&= (f_{c,\rho}(y)^{1/p} + SA_{n-1}(S))^p + \Big(\frac{-B_nS}{pf_{q,c}(y)^{(p-1)/p}}\Big)^p + B_nS + R + \Sigma, \qquad (8)
\end{aligned}$$

where

$$\Sigma = \sum_{k=1}^{p-1} \binom{p}{k} (f_{c,\rho}(y)^{1/p} + SA_{n-1}(S))^{p-k} \Big(\frac{-B_nS}{pf_{q,c}(y)^{(p-1)/p}}\Big)^k. \qquad (9)$$

Using the expression of $v(B_n)$ computed in **Step III**, one checks that the terms with $k \geq 2$ in (9) are zero modulo $\lambda^p\mathfrak{m}[T]$. It implies the following relations

$$\begin{aligned}
\Sigma + B_nS &\equiv B_nS\left[1 - \frac{(f_{c,\rho}(y)^{1/p} + SA_{n-1}(S))^{p-1}}{f_{c,\rho}(y)^{(p-1)/p}}\right] \\
&\equiv \frac{B_nS}{f_{c,\rho}(y)^{(p-1)/p}}\Big[f_{c,\rho}(y)^{(p-1)/p} - (f_{c,\rho}(y)^{1/p} + SA_{n-1}(S))^{p-1}\Big] \\
&\equiv \frac{B_nS}{f_{c,\rho}(y)^{(p-1)/p}}\Big[-\sum_{k=1}^{p-1}\binom{p-1}{k} f_{c,\rho}(y)^{(p-1-k)/p}(SA_{n-1}(S)^k\Big] \\
&\equiv 0 \bmod \lambda^p\mathfrak{m}[T], \text{ since for } k \geq 1,\ B_nS^{k+1} \equiv 0 \bmod \lambda^p\mathfrak{m}[T].
\end{aligned}$$

According to the definition of B_{n-1} (see **Step III**) one obtains

$$(8) \equiv (f_{c,\rho}(y)^{1/p} + SA_{n-1}(S))^p + R + B_{n-1}S^p + y\rho_1 S^p \bmod \lambda^p\mathfrak{m}[T]. \qquad (10)$$

Using the same process, one shows by induction on i that (8) is congruent to

$$(f_{c,\rho}(y)^{1/p}+SA_{i+1}(S))^p+B_{i+1}S^{p^{n-i-1}}+\sum_{k=1}^{n-i-1} y\rho_k S^{p^k}+R \mod \lambda^p\mathfrak{m}[T]. \quad (11)$$

Thus, one applies equation (11) with $i=0$

$$(8) \equiv (f_{c,\rho}(y)^{1/p}+SA_1(S))^p+B_1S^{q/p}+\sum_{k=1}^{n-1} y\rho_k S^{p^k}+R \mod \lambda^p\mathfrak{m}[T].$$

One defines Σ' by $(f_{c,\rho}(y)^{1/p}+SA_1(S))^p = f_{c,\rho}(y)+(SA_1(S))^p+\Sigma'$. From $pf_{c,\rho}(y)^{(p-1)/p}SA_1(S) = -B_1S^{q/p}$ (see the definition of $SA_1(S)$) one gets

$$\Sigma'+B_1S^{q/p} = \sum_{k=2}^{p-1}\binom{p}{k} f_{c,\rho}(y)^{(p-k)/p}(SA_1(S))^k,$$

so using the expression of $v(B_1)$ computed in **Step III**, one checks that $\Sigma'+B_1S^{q/p}\equiv 0 \mod \lambda^p\mathfrak{m}[T]$. From the definition of $SA_1(S)$ and B_0 one has $(SA_1(S))^p = B_0S^q$, thus

$$(8) \equiv f_{c,\rho}(y)+B_0S^q+\sum_{k=1}^{n-1} y\rho_k S^{p^k}+R \mod \lambda^p\mathfrak{m}[T].$$

Then, **Step V** follows from **Step IV** and this last relation.

Step VI : *The curve $C_{c,\rho}/K$ has good reduction over $K(y, f_{c,\rho}(y)^{1/p})$.* According to **Step V**, the change of variables in $K(y, f_{c,\rho}(y)^{1/p})$

$$X=\lambda^{p/(1+q)}T+y=S+y \quad\text{and}\quad Y=\lambda W+f_{c,\rho}(y)^{1/p}+SA_n(S),$$

induces in reduction $w^p-w=\sum_{k=0}^{n-1}\bar{u}_k t^{1+p^k}+t^{1+q}$ with genus $g(C_{c,\rho})$. So [Liu02] 10.3.44 implies that this change of variables gives the stable model. Note that the ρ_k's were chosen to obtain this equation for the special fiber of the stable model.

Step VII : *For any distinct roots y_i, y_j of $L_{c,\rho}(X)$, $v(y_i-y_j)=v(\lambda^{p/(1+q)})$.* The changes of variables $\lambda^{p/(1+q)}T = X-y_i$ and $\lambda^{p/(1+q)}T = X-y_j$ induce equivalent Gauss valuations of $K(C_{c,\rho})$, else applying [Liu02] 10.3.44 would contradict the uniqueness of the stable model. Thus $v(y_i-y_j)\geq v(\lambda^{p/(1+q)})$.

One checks that $v(f'_{c,\rho}(y))>0$, $\forall\, 0\leq k\leq n-1$ $v(\rho_k^{q/p^k}p^{d_k}q/p^k)>v(a_n)$, $v(s'_{1,\rho}(y))>0$, $v(s_{1,\rho}(y))=v(y^q)$ and $v(qs_{1,\rho}(y)^{q-1}s'_{1,\rho}(y))>v(a_n)$, so

$$v(L'_{c,\rho}(y)) = v(a_n) = (q^2-1)v(\lambda^{p/(1+q)}).$$

Taking into account that $L'_{c,\rho}(y_i) = \prod_{j \neq i}(y_i - y_j)$ and $\deg L_{c,\rho}(X) = q^2$, one obtains $v(y_i - y_j) = v(\lambda^{p/(1+q)})$.

Step VIII : *If $L_{c,\rho}(X)$ is irreducible over K, then $K(y, f_{c,\rho}(y)^{1/p})$ is the monodromy extension M of $C_{c,\rho}/K$ and $G := \mathrm{Gal}(M/K)$ is an extra-special p-group of order pq^2.*
Let $(y_i)_{i=1,\dots,q^2}$ be the roots of $L_{c,\rho}(X)$, $L := K(y_1, \dots, y_{q^2})$ and M/K be the monodromy extension of $C_{c,\rho}/K$. Any $\tau \in \mathrm{Gal}(L/K) - \{1\}$ is such that $\tau(y_i) = y_j$ for some $i \neq j$. Thus, the change of variables

$$X = \lambda^{p/(1+q)}T + y_i \text{ and } Y = \lambda W + f_{c,\rho}(y)^{1/p} + SA_n(S),$$

induces the stable model and τ acts on it by :

$$\tau(T) = \frac{X - y_j}{\lambda^{p/(1+q)}}, \quad \text{hence} \quad T - \tau(T) = \frac{y_j - y_i}{\lambda^{p/(1+q)}}.$$

According to **Step VII**, τ acts non-trivially on the stable reduction. It follows that $L \subseteq M$. Indeed if $\mathrm{Gal}(K^{\mathrm{alg}}/M) \not\subseteq \mathrm{Gal}(K^{\mathrm{alg}}/L)$ it would exist $\sigma \in \mathrm{Gal}(K^{\mathrm{alg}}/M)$ inducing $\bar{\sigma} \neq \mathrm{Id} \in \mathrm{Gal}(L/K)$, which would contradict the characterization of $\mathrm{Gal}(K^{\mathrm{alg}}/M)$ (see remark after Theorem 2.1) .

According to [LM05], the p-Sylow subgroup $\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}$ of $\mathrm{Aut}_k(\mathcal{C}_k)^{\#}$ is an extra-special p-group of order pq^2. Moreover, one has :

$$0 \to \mathrm{Z}(\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}) \to \mathrm{Aut}_k(\mathcal{C}_k)_1^{\#} \to (\mathbb{Z}/p\mathbb{Z})^{2n} \to 0,$$

where $(\mathbb{Z}/p\mathbb{Z})^{2n}$ is identified with the group of translations $t \mapsto t+a$ extending to elements of $\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}$. Therefore we have morphisms

$$\mathrm{Gal}(M/K) \overset{i}{\hookrightarrow} \mathrm{Aut}_k(\mathcal{C}_k)_1^{\#} \overset{\varphi}{\to} \mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}/\mathrm{Z}(\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}).$$

The composition is seen to be surjective since the image contains the q^2 translations $t \mapsto t + \overline{(y_i - y_1)/\lambda^{p/(1+q)}}$. Consequently, $i(\mathrm{Gal}(M/K))$ is a subgroup of $\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}$ of index at most p. So it contains $\Phi(\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}) = \mathrm{Z}(\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#}) = \mathrm{Ker}\,\varphi$. It implies that i is an isomorphism and $[M : K] = pq^2$. By **Step VI**, one has $M \subseteq K(y, f_{q,c}(y)^{1/p})$, hence $M = K(y, f_{q,c}(y)^{1/p})$.

We show that $K(y_1)/K$ is Galois and that $\mathrm{Gal}(M/K(y_1)) = \mathrm{Z}(G)$. Indeed, $M/K(y_1)$ is p-cyclic and generated by σ defined by :

$$\sigma(y_1) = y_1 \text{ and } \sigma(f_{c,\rho}(y_1)^{1/p}) = \zeta_p f_{c,\rho}(y_1)^{1/p}.$$

According to **Step VI**, σ acts on the stable model by :

$$\sigma(S) = S, \quad \sigma(Y) = Y = \lambda\sigma(W) + \zeta_p f_{c,\rho}(y_1)^{1/p} + SA_n(S).$$

Hence

$$\sigma(W) = W - f_{c,\rho}(y_1)^{1/p}.$$

It follows that, in reduction, σ induces a morphism that generates $\mathrm{Z}(\mathrm{Aut}_k(\mathcal{C}_k)_1^{\#})$. It implies that $K(y_1)/K$ is Galois, $\mathrm{Gal}(M/K(y_1)) = \mathrm{Z}(G)$ and $\mathrm{Gal}(K(y_1)/K) \simeq (\mathbb{Z}/p\mathbb{Z})^{2n}$.

3. Let $L_\rho(X) := L_{1,\rho}(X)$, $f_\rho(X) := f_{1,\rho}(X)$, $s_\rho(y) := s_{1,\rho}(y)$, y be a root of $L_\rho(X)$ and $b_n := (-1)(-p)^{1+p+\cdots+p^{n-1}}$. Note that $b_n^p = a_n$, $L = K(y)$ and we do not assume $p = 2$ until **Step E**.

Step A : *The polynomial $L_\rho(X)$ is irreducible over K.*
Let $\tilde{s} := s_\rho(y) - y^q$, $\sigma := \sum_{k=1}^{q} \binom{q}{k} \tilde{s}^k y^{q(q-k)}$ and $R_1 := \sum_{k=1}^{p-1} \binom{p}{k} y^{kq^2/p}(-b_n)^{p-k}$. Since $L_\rho(y) = 0$ one has

$$y^{q^2} + \sigma = s_\rho(y)^q = a_n f_\rho(y)(1+y) + \sum_{k=1}^{n-1} (\rho_k y)^{q/p^k}(-p)^{d_k}(-1)^q f_\rho(y)^{q(p^k-1)/p^k}.$$

It implies that $(y^{q^2/p} - b_n)^p$ equals

$$a_n\left[f_\rho(y)(1+y) + (-1)^p\right] + \sum_{k=1}^{n-1} (\rho_k y)^{q/p^k}(-p)^{d_k}(-1)^q f_\rho(y)^{q(p^k-1)/p^k} + R_1 - \sigma.$$

We are going to remove monomials with valuation greater than $v(a_n y)$ in the above expression by taking p-th roots. Note that if $\forall i \geq 1$, $\rho_i = 0$, then one could skip most of **Step A** (see equation (14)). Assume that $\rho_i \neq 0$ for some $i \geq 1$, let $j := \max\{1 \leq i \leq n-1, \rho_i \neq 0\}$ and $l := \min\{1 \leq i \leq n-1, \rho_i \neq 0\}$. The following relations are straight forward computations using **Step I** :

$$v(f_\rho(y)(1+y) + (-1)^p) = v(y), \quad v(\tilde{s}) = v(\rho_j y^{p^j}), \quad v(\sigma) = q v(\tilde{s}), \tag{12}$$
$$v\left(\sum_{k=1}^{n-1} (\rho_k y)^{q/p^k}(-p)^{d_k}(-1)^q f_\rho(y)^{q(p^k-1)/p^k}\right) = v((\rho_l y)^{p^{n-l}} p^{d_l}).$$

Then one checks that

$$v(R_1) > v(a_n y) > v((\rho_l y)^{p^{n-l}} p^{d_l}) > v(\sigma). \tag{13}$$

It implies that $v((y^{q^2/p} - b_n)^p) = q v(\tilde{s})$, so one considers $(y^{q^2/p} - b_n + \tilde{s}^{q/p})^p$. By expanding this last expression, using (12), (13) and taking into account

$$v(\sum_{k=1}^{q-1} \binom{q}{k} \tilde{s}^k y^{q(q-k)}) > v(a_n y), \quad v(\sum_{k=1}^{p} \binom{p}{k} (y^{q^2/p} - b_n)^k \tilde{s}^{(p-k)q/p}) > v(a_n y),$$

one obtains that $pv(y^{q^2/p} - b_n + \tilde{s}^{q/p}) = v((\rho_l y)^{p^{n-l}} p^{d_l})$, leading us to consider

$$(y^{q^2/p} - b_n + \tilde{s}^{q/p} + (\rho_l y)^{q/p^{l+1}} (-p)^{d_l/p} f_\rho(y)^{q(p^l-1)/p^{l+1}})^p.$$

By expanding this expression and using (12) and (13) one easily checks that it has valuation $v((\rho_{l_1} y)^{p^{n-l_1}} p^{d_{l_1}})$ where $l_1 := \min\{l+1 \leq i \leq n-1, \rho_i \neq 0\}$. By induction one shows that

$$t := y^{q^2/p} - b_n + \tilde{s}^{q/p} + \sum_{k=1}^{n-1} (\rho_k y)^{q/p^{k+1}} (-p)^{d_k/p} f_\rho(y)^{q(p^k-1)/p^{k+1}}, \qquad (14)$$

satisfies $pv(t) = v(a_n y)$. Then $v_L(p^{q^2} t^{-(p-1)(q+1)}) = v_L(p)/q^2 = [L : \mathbb{Q}_p^{\mathrm{ur}}]/q^2$, so q^2 divides $[L : K]$. It implies that $L_\rho(X)$ is irreducible over K.

Step B : *Reduction step.*
The last non-trivial group G_{i_0} of the lower ramification filtration $(G_i)_{i \geq 0}$ of $G := \mathrm{Gal}(M/K)$ is a subgroup of $\mathrm{Z}(G)$ ([Ser79] IV §2 Corollary 2 of Proposition 9) and as $\mathrm{Z}(G) \simeq \mathbb{Z}/p\mathbb{Z}$, it follows that $G_{i_0} = \mathrm{Z}(G)$.

According to **Step VIII** the group $H := \mathrm{Gal}(M/L)$ is $\mathrm{Z}(G)$. Consequently, the filtration $(G_i)_{i \geq 0}$ can be deduced from that of M/L and L/K (see [Ser79] IV §2 Proposition 2 and Corollary of Proposition 3).

Step C : *Let $\sigma \in \mathrm{Gal}(L/K) - \{1\}$, then $v(\sigma(t) - t) = q^2 v(\pi_K)$.*
Let $y' := \sigma(y)$, one deduces the following easy lemma from **Step VII**.

Lemma 3.1. *For any $n \geq 0$, $v(y^n - y'^n) \geq nv(y)$.*

Recall the definition $\tilde{s} := 2\rho_0 y + \sum_{k=1}^{n-1} \rho_k y^{p^k}$. First one shows that modulo $(y - y')^{q^2/p}\mathfrak{m}$ one has

$$\sigma(\tilde{s})^{q/p} - \tilde{s}^{q/p} \equiv (2\rho_0)^{q/p}(y'^{q/p} - y^{q/p}) + \sum_{k=1}^{n-1} \rho_k^{q/p}(y'^{qp^k/p} - y^{qp^k/p}). \qquad (15)$$

Indeed, let $(m_i)_{i=0,\dots,n-1} \in \mathbb{N}^n$ such that $m_0 + m_1 + \cdots + m_{n-1} = q/p$ and $t := m_0 + m_1 p + \cdots + m_{n-1} p^{n-1}$, then using lemma 3.1 one checks that

$$v(p\rho_0^{m_0}\rho_1^{m_1} \dots \rho_{n-1}^{m_{n-1}}(y^t - y'^t)) > \frac{q^2}{p} v(y - y').$$

This inequality implies (15).

Let $1 \leq k \leq n-1$ and write $f_\rho(y)^{(p^k-1)q/p^{k+1}} = 1 + \sum_{i \in I_k} \alpha_{i,k} y^i$, for some set I_k. Then

$$\begin{aligned} &y'^{q/p^{k+1}} f_\rho(y')^{(p^k-1)q/p^{k+1}} - y^{q/p^{k+1}} f_\rho(y)^{(p^k-1)q/p^{k+1}} \\ &= y'^{q/p^{k+1}} - y^{q/p^{k+1}} + \sum_{i\in I_k} \alpha_{i,k}(y'^i - y^i). \end{aligned}$$

Let $i \in I_k$. Consider the case when $v(\alpha_{i,k}) \geq v(\rho_h)$ for some $0 \leq h \leq n-1$, then using **Step VII**, one checks that $\forall\ 1 \leq k \leq n-1$, $v(\alpha_{i,k}) \geq v(\rho_h) > qv(y'-y)/p^{k+1}$. If this case does not occur, then according to the expression of $f_\rho(y)$ one has $i \geq q/p^{k+1} + q$ and using lemma 3.1 one checks that $v(y'^i - y^i) > qv(y'-y)/p^{k+1}$. In any case $v(\alpha_{i,k}(y'^i - y^i)) > qv(y'-y)/p^{k+1}$ and one checks that

$$v(p^{d_k/p}\rho_k^{q/p^{k+1}}\alpha_{i,k}(y'^i - y^i)) > q^2 v(y'-y)/p. \tag{16}$$

Taking into account (14), (15) and (16), one gets mod $(y'-y)^{q^2/p}\mathfrak{m}$

$$\begin{aligned} \sigma(t) - t \equiv\ & y'^{q^2/p} - y^{q^2/p} + (2\rho_0)^{q/p}(y'^{q/p} - y^{q/p}) \qquad (17) \\ &+ \sum_{k=1}^{n-1} \rho_k^{q/p}(y'^{qp^k/p} - y^{qp^k/p}) + \sum_{k=1}^{n-1}(-p)^{d_k/p}\rho_k^{q/p^{k+1}}(y'^{q/p^{k+1}} - y^{q/p^{k+1}}). \end{aligned}$$

Using lemma 3.1, it is now straight forward to check the following relations mod $(y'-y)^{q^2/p}\mathfrak{m}$.

$$\begin{aligned} y'^{q^2/p} - y^{q^2/p} &\equiv (y'-y)^{q^2/p}, \\ \rho_k^{q/p}(y'^{qp^k/p} - y^{qp^k/p}) &\equiv \rho_k^{q/p}(y'-y)^{qp^k/p}, \\ (-p)^{d_k/p}\rho_k^{q/p^{k+1}}(y'^{q/p^{k+1}} - y^{q/p^{k+1}}) &\equiv (-p)^{d_k/p}\rho_k^{q/p^{k+1}}(y'-y)^{q/p^{k+1}}. \end{aligned}$$

Using **Step VII**, one sees that each of these three elements has valuation $q^2 v(y'-y)/p$, thus one gets

$$\begin{aligned} (\sigma(t)-t)^p \equiv & (y'-y)^{q^2} + (2\rho_0)^q(y'-y)^q + \sum_{k=1}^{n-1}\rho_k^q(y'-y)^{qp^k} \qquad (18) \\ &+ \sum_{k=1}^{n-1}(-p)^{d_k}\rho_k^{q/p^k}(y'-y)^{q/p^k} \mod (y'-y)^{q^2}\mathfrak{m}. \end{aligned}$$

Now recall **Step VII**, the definitions of the ρ_k's and of λ, then for some $v \in R^\times$ and $\Sigma \in R$

$$\rho_k = u_k\lambda^{p(q-p^k)/(1+q)}, \quad y'-y = v\lambda^{p/(1+q)} \text{ and } -p = \lambda^{p-1} + p\lambda\Sigma.$$

Since $q^2 v(y'-y) = \frac{pq^2}{1+q} v(\lambda)$, equation (18) becomes

$$(\sigma(t)-t)^p \equiv \lambda^{\frac{q^2 p}{1+q}} \left[v^{q^2} + (2u_0 v)^q + \sum_{k=1}^{n-1} (u_k^q v^{qp^k} + (u_k v)^{q/p^k}) \right] \bmod \lambda^{\frac{q^2 p}{1+q}} \mathfrak{m}.$$

From the action of σ on the stable reduction (see **Step VIII**), one has that the automorphism of $\mathbb{P}^1_k$ given by $t \mapsto t + \bar{v}$ has a prolongation to A_u/k, so Proposition 2.4 implies that

$$\bar{v}^{q^2} + (2\bar{u}_0 \bar{v})^q + \sum_{k=1}^{n-1} (\bar{u}_k^q \bar{v}^{qp^k} + (\bar{u}_k \bar{v})^{q/p^k}) + \bar{v} = 0. \tag{19}$$

Assume that $\bar{v}^{q^2} + (2\bar{u}_0\bar{v})^q + \sum_{k=1}^{n-1} (\bar{u}_k^q \bar{v}^{qp^k} + (\bar{u}_k \bar{v})^{q/p^k}) = 0$, then from (19) one has $\bar{v} = 0$, which contradicts $v \in R^\times$. It implies that $v(\sigma(t)-t) = q^2 v(\lambda)/(1+q) = q^2 v(y-y')/p = q^2 v(\pi_K)$.

Step D : *The ramification filtration of L/K is :*

$$(G/H)_0 = (G/H)_1 \supsetneq (G/H)_2 = \{1\}.$$

Since $K/\mathbb{Q}_p^{\mathrm{ur}}$ is tamely ramified of degree $(p-1)(q+1)$, one has $K = \mathbb{Q}_p^{\mathrm{ur}}(\pi_K)$ with $\pi_K^{(p-1)(q+1)} = p$ for some uniformizer π_K of K. In particular $z := \pi_K^{q^2}/t$, is a uniformizer of L. Let $\sigma \in \mathrm{Gal}(L/K) - \{1\}$, then

$$\sigma(z) - z = \frac{t - \sigma(t)}{\sigma(t)t} \pi_K^{q^2} = \frac{t-\sigma(t)}{\pi_K^{q^2}} \frac{\pi_K^{q^2}}{t} \frac{\pi_K^{q^2}}{\sigma(t)}.$$

Using **Step C** one obtains $v(\sigma(z)-z) = 2v(z)$, i.e. $(G/H)_2 = \{1\}$.

Step E : *From now on, we assume $p = 2$. Let $s := (q+1)(2q^2-1)$. There exist $u, h \in L$ and $r \in \pi_L^s \mathfrak{m}$ such that $v_L(2y^{q/2}h) = s$ and*

$$f_\rho(y) u^2 = 1 + \rho_{n-1} y^{1+q/2} + 2y^{q/2} h + r.$$

To prove the first statement we note that, from the definition of $f_\rho(y)$, one has $f_\rho(y) = 1 + T$ with $v(T) = qv(y)$ and $L_\rho(y) = 0$, thus

$$\left(\frac{s_\rho^{q/2}(y)}{b_n}\right)^2 = f_\rho(y)^{q-1}(1+y) + \sum_{k=1}^{n-1} \frac{(\rho_k y)^{q/2^k}}{2^{2+\cdots+2^{n-k}}} f_\rho(y)^{q(2^k-1)/2^k},$$

and $f_\rho(y)^{q-1}(1+y) = 1 + y + \sum_{k=1}^{q-1} \binom{q-1}{k} T^k (1+y).$

Then, we put $\tilde{\Sigma} := \sum_{k=1}^{q-1} \binom{q-1}{k} T^k(1+y)$ and

$$h := \frac{s_\rho^{q/2}(y)}{b_n} + \sum_{k=1}^{n-1} \frac{(\rho_k y)^{q/2^{k+1}}}{2^{1+\cdots+2^{n-k-1}}} f_\rho(y)^{q(2^k-1)/2^{k+1}} - 1.$$

Then one computes

$$\begin{aligned} h^2 &= \Big[\frac{s_\rho^{q/2}(y)}{b_n} + \sum_{k=1}^{n-1} \frac{(\rho_k y)^{q/2^{k+1}}}{2^{1+\cdots+2^{n-k-1}}} f_\rho(y)^{q(2^k-1)/2^{k+1}}\Big]^2 + 1 - 2(h+1) \\ &= (\frac{s_\rho^{q/2}(y)}{b_n})^2 + \sum_{k=1}^{n-1} \frac{(\rho_k y)^{q/2^k}}{2^{2+\cdots+2^{n-k}}} f_\rho(y)^{q(2^k-1)/2^k} + \Sigma_1 + 1 - 2(h+1) \\ &= 2 + y + 2\sum_{k=1}^{n-1} \frac{(\rho_k y)^{q/2^k}}{2^{2+\cdots+2^{n-k}}} f_\rho(y)^{q(2^k-1)/2^k} + \Sigma_1 + \tilde{\Sigma} - 2(h+1). \end{aligned}$$

In **Step III**, we proved that $v(B_n) = qv(y) = 2v(b_n)/q$ where $B_n = -s_\rho(y)$, so $v(\frac{s_\rho^{q/2}(y)}{b_n}) = 0$ and one checks using **Step I** that

$$v(2) > v(y), \text{ and } \forall\, 1 \le k \le n-1,\ v\Big(\frac{(\rho_k y)^{q/2^{k+1}}}{2^{1+\cdots+2^{n-k-1}}}\Big) \ge 0, \tag{20}$$

thus $v(h+1) \ge 0$ and $v(2(h+1)) \ge v(2) > v(y)$. One checks in the same way that $v(\Sigma_1) > v(y)$. One has $v(\tilde{\Sigma}) \ge v(T) > v(y)$, so $v(h^2) = v(y)$ and $v_L(2y^{q/2}h) = s$.

To prove the second statement of **Step E**, we first remark that $\forall i \ge 1$ $f_\rho(y)^i = 1 + \sum_{k=1}^{i} \binom{i}{k} T^k = 1 + \Sigma_i$, whence $v(\Sigma_i) \ge v(T)$. Since, for all $0 \le k \le n-1$, $v(\rho_k y^{p^k}) > qv(y)$ one has mod $\pi_L^s \mathfrak{m}$

$$\frac{s_\rho^{q/2}(y)}{b_n} 2y^{q/2} \equiv \big[(2\rho_0 y)^{q/2} + \sum_{k=1}^{n-1} (\rho_k y^{2^k})^{q/2} + y^{q^2/2}\big] \frac{y^{q/2}}{2^{2+\cdots+2^{n-1}}}. \tag{21}$$

One also checks that $\forall i \ge 1$, $v_L(2y^{q/2}\Sigma_i) > s$, then according to (20), $\forall\ i \ge 1$ and $1 \le k \le n-1$

$$v_L\Big(\frac{(\rho_k y)^{q/2^{k+1}}}{2^{1+\cdots+2^{n-k-1}}} 2y^{q/2}\Sigma_i\Big) > s \text{ and one checks that } v_L\big(\frac{(2\rho_0)^{q/2} y^q}{2^{2+\cdots+2^{n-1}}}\big) > s. \tag{22}$$

Thus, applying relations (21), (22) and the definition of h, one has

$$\begin{aligned} 2hy^{q/2} \equiv &\big[\sum_{k=1}^{n-1} (\rho_k y^{2^k})^{q/2} + y^{q^2/2}\big] \frac{y^{q/2}}{2^{2+\cdots+2^{n-1}}} \\ &+ \sum_{k=1}^{n-1} \frac{(\rho_k y)^{q/2^{k+1}}}{2^{1+\cdots+2^{n-k-1}}} 2y^{q/2} - 2y^{q/2} \bmod \pi_L^s \mathfrak{m}. \end{aligned} \tag{23}$$

Finally one puts

$$u := 1 - y^{q/2} - \sum_{k=0}^{n-2} \frac{y^{2^k(1+q)}}{2^{1+\cdots+2^k}} + \sum_{i=1}^{n-1} \sum_{k=n-i-1}^{n-2} \frac{\rho_i^{2^k}}{2^{1+\cdots+2^k}} y^{2^k(1+2^i)} = 1 + \tilde{u},$$

and one checks that $v(\tilde{u}) = v(y^{q/2})$. From the equality

$$f_\rho(y)u^2 - 1 = \sum_{k=0}^{n-1} \rho_k y^{1+2^k} + y^q + y^{1+q} + (1+T)2\tilde{u} + (1+T)\tilde{u}^2,$$

taking into account that $v_L(2T\tilde{u}) > s$, $v_L(T\tilde{u}^2) > s$, $\forall\ 0 \leq k \leq n-2$, $v_L(\rho_k y^{1+2^k}) > s$ and expanding $\tilde{u}$ and $\tilde{u}^2$ one gets modulo $\pi_L^s \mathfrak{m}$

$$\begin{aligned} f_\rho(y)u^2 - 1 \equiv & \rho_{n-1} y^{1+q/2} - 2y^{q/2} + 2y^q - \sum_{k=1}^{n-2} \frac{2y^{2^k(1+q)}}{2^{1+\cdots+2^k}} + \sum_{k=1}^{n-1} \frac{y^{2^k(1+q)}}{2^{2+\cdots+2^k}} + \\ & + \sum_{i=1}^{n-1} \sum_{k=n-i-1}^{n-2} \frac{2\rho_i^{2^k} y^{2^k(1+2^i)}}{2^{1+\cdots+2^k}} + \sum_{i=1}^{n-1} \sum_{k=n-i}^{n-1} \frac{\rho_i^{2^k} y^{2^k(1+2^i)}}{2^{2+\cdots+2^k}}. \end{aligned} \tag{24}$$

Arranging the terms of (24) , taking into account that $v_L(2y^q) > s$ and for all $2 \leq i \leq n-1$ and $n-i \leq k \leq n-2$

$$v_L\left(\rho_i^{2^k} y^{2^k(1+2^i)} \frac{2}{2^{2+\cdots+2^k}}\right) > s,$$

and comparing with (23), one obtains $f_\rho(y)u^2 - 1 \equiv \rho_{n-1} y^{1+q/2} + 2hy^{q/2} \mod \pi_L^s \mathfrak{m}$.

Step F: *The ramification filtration of M/L is*

$$H_0 = H_1 = \cdots = H_{1+q} \supsetneq \{1\}.$$

One has to show that $v_M(\mathcal{D}_{M/L}) = q+2$, we will use freely results from [Ser79] IV. If $\rho_{n-1} = 0$, then according to **Step E**, one has

$$f_\rho(y)u^2 = 1 + 2y^{q/2}h + r,$$

and one concludes using [CM11] Lemma 2.1. Else, if $\rho_{n-1} \neq 0$, one has

$$\max_{u \in L^\times} v_L(f_\rho(y)u^2 - 1) \geq v_L(\rho_{n-1} y^{1+q/2}),$$

then [LRS93] Lemma 6.3 implies that $v_M(\mathcal{D}_{M/L}) \leq q+3$. Using **Step B**, **Step D** and [Ser79] IV §2 Proposition 11, one has that the break in the ramification filtration of M/L is congruent to $1 \bmod 2$, i.e. $v_M(\mathcal{D}_{M/L}) \leq q+2$.

According to **Step D** and lemma 2.1, the break t of M/L is in $1+q\mathbb{N}$. If $t=1$ then $G_2=\{1\}$ and $G_1/G_2=G/G_2\simeq G$ would be abelian, so $t\geq 1+q$, i.e. $v_M(\mathcal{D}_{M/L})\geq q+2$.

Step G : *Computations of conductors.*
For $l\neq 2$ a prime number, the G-modules $\mathrm{Jac}(C)[l]$ and $\mathrm{Jac}(\mathcal{C}_k)[l]$ being isomorphic one has that for $i\geq 0$:

$$\dim_{\mathbb{F}_l}\mathrm{Jac}(C)[l]^{G_i}=\dim_{\mathbb{F}_l}\mathrm{Jac}(\mathcal{C}_k)[l]^{G_i}.$$

Moreover, for $0\leq i\leq 1+q$ one has $\mathrm{Jac}(\mathcal{C}_k)[l]^{G_i}\subseteq \mathrm{Jac}(\mathcal{C}_k)[l]^{\mathrm{Z}(G)}$, then from $\mathcal{C}_k/\mathrm{Z}(G)\simeq\mathbb{P}^1_k$ and lemma 2.2 it follows that for $0\leq i\leq 1+q$, $\dim_{\mathbb{F}_l}\mathrm{Jac}(\mathcal{C}_k)[l]^{G_i}=0$. Since $g(C)=q/2$ one gets $f(\mathrm{Jac}(C)/K)=2q+1$ and $\mathrm{sw}(\mathrm{Jac}(C)/\mathbb{Q}_2^{\mathrm{ur}})=1$. □

Example : Magma codes are available on the author webpage. Let $K:=\mathbb{Q}_2^{\mathrm{ur}}(2^{1/5})$ and $f(X):=1+2^{6/5}X^2+2^{4/5}X^3+X^4+X^5\in K[X]$, one checks that the smooth, projective, integral curve birationally given by $Y^2=f(X)$ has the announced properties, that is the wild monodromy M/K has degree 32 and one can describe its ramification filtration. The first program checks that **Step A** and **Step D** hold for this example. The second program checks **Step F** and is due to Guardia, J., Montes, J. and Nart, E. (see [GMN11]) and computes $v_M(\mathcal{D}_{M/\mathbb{Q}_2^{\mathrm{ur}}})=194$. Using [Ser79] III §4 Proposition 8, one finds that $v_M(\mathcal{D}_{M/K})=66$, which was the announced result in Theorem 3.1 *3.*

Remarks :

1. The above example was the main motivation for **Step F** since it shows that one could expect the correct behaviour for the ramification filtration of $\mathrm{Gal}(M/K)$ when $p=2$.

2. The naive method to compute the ramification filtration of M/K in the above example fails. Indeed, in this case Magma needs a huge precision when dealing with 2-adic expansions to get the correct discriminant.

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