Multiphase formulation of plasma physics

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Outline

The Vlasov-Poisson equation and the kinetic Euler equation

Known results concerning linear and non-linear instability

New results: the measure-valued setting and the multiphase formulation

An application to incompressible optimal transport

The Vlasov-Poisson case

The Vlasov-Poisson equation and the kinetic Euler equation

The Vlasov-Poisson equation

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(VP)\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x U\cdot\nabla_v f = 0, \\ -\Delta_x U = \int f\,\mathrm{d}v - 1, \\ f|_{t=0} = f_0, \end{cases}
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- $f = f(t, x, v)$, $t \geq 0$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$ is the density of electrons in the phase-space $\mathbb{T}^d \times \mathbb{R}^d$,

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Global existence of classical solutions for $d = 2$ or 3 [Ukai, Okabe 78; Lions, Perthame 91; Pfaffelmoser 92].

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The parameter ε is the **Debye length**. It is typically very small w.r.t. the scale of observations ($\sim 10^{-3}$m in the ionosphere, $\sim 10^{-4}$m in a tokamak).

The kinetic Euler equation

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The pressure p has the same number of spatial derivatives as f. (Same scaling as in the **Vlasov-Benney equation** where p is replaced by the spatial density $\rho = \int f \, \mathrm{d}v$ [Jabin, Nouri 11; Bardos, Nouri 12]).

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[Grenier 96]: (kEu) is well-posed in spaces of analytic regularity.

Known results concerning linear and non-linear instability

Linearization of (VP) around homogeneous profiles

$$(VP) \begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) - \nabla_x U(t,x) \cdot \nabla_v f(t,x,v) = 0, \\ -\Delta_x U(t,x) = \int f(t,x,v)\,\mathrm{d}v - 1, \\ f\big|_{t=0} = f_0, \end{cases}$$

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We look for **exponential growing modes** (EGM):

$$f(t,x,v) = g(v)\exp(in \cdot x)\exp(\lambda t),$$

where $n \in \mathbb{Z}^d$ is the **frequency**, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ is the **growing rate**.

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If there exists an EGM, we say that μ is **unstable**.

Penrose instability criterion

Proposition (Penrose 1960)

Let μ be a smooth profile. Equation (L) admits an EGM of frequency n and growing rate λ iff the following ***Penrose condition (Pen)*** *holds:*

$$\boxed{\int \frac{in \cdot \nabla_v \mu(v)}{\lambda + in \cdot v} \, \mathrm{d}v = \begin{cases} \varepsilon^2 |n|^2, & \text{for } (VP_\varepsilon), \\ 0, & \text{for } (kEu). \end{cases}}$$

In that case:

$$g(v) \propto \frac{in \cdot \nabla_v \mu(v)}{\lambda + in \cdot v}.$$

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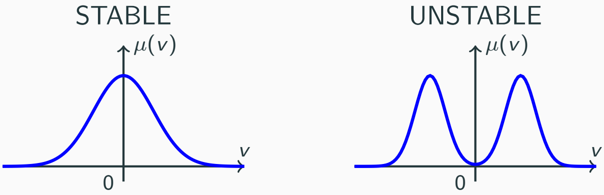
In dimension 1:



Toward non-linear instability

Take (λ, n) satisfying (Pen) and set the ansatz:

$$\begin{cases} f(t,x,v) = \mu(v) + \delta \Re\left(\dfrac{in \cdot \nabla_v \mu(v)}{\lambda + in \cdot v} \exp(\lambda t + in \cdot x) \right) + R^\delta(t,x,v), \\ R^\delta|_{t=0} = 0. \end{cases}$$

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Main question: Up to which time T_δ and in which norm $\| \bullet \|$ can you justify:

$$\forall t \in [0, T_\delta], \quad \|R^\delta(t)\| \ll \delta \exp\Big(\Re(\lambda) t\Big) \, ?$$

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Problem

$$\mu(v) + \delta \Re\left(\frac{in \cdot \nabla_v \mu(v)}{\lambda + in \cdot v} \exp(in \cdot x) \right)$$

needs to be sufficiently regular and nonnegative. It is hence needed to add assumptions on μ (regularity + cancellation conditions).

Lyapounov instability for (VP)

This question has been widely studied, see e.g. [Guo, Strauss 95; Han-Kwan, Hauray 15; Han-Kwan, Nguyen 16].

Theorem (Han-Kwan, Nguyen 16)

Let μ be smooth, Penrose unstable and satisfying cancellation conditions. For all $s, m \in \mathbb{N}$, there exist solutions f^δ up to time $T_\delta > 0$ of (VP) such that:

- *Convergence at the initial time:*

$$\left\| \left(1+|v|^2\right)^{m/2} \left\{ f_0^\delta - \mu \right\} \right\|_{H^s(\mathbb{T}^d \times \mathbb{R}^d)} = \mathcal{O}(\delta),$$

- *No convergence at time $T_\delta = \mathcal{O}\big(|\log \delta|\big)$:*

$$\liminf_{\delta \to 0} \|f^\delta(T_\delta) - \mu\|_{L^2(\mathbb{T}^d \times \mathbb{R}^d)} > 0.$$

Ill-posedness for (kEu)

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If μ is unstable, if (n, λ) satisfies (Pen) for (kEu) and if $k \in \mathbb{N}^*$, then $(kn, k\lambda)$ also satisfies (Pen).

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Let μ be analytic, Penrose unstable and satisfying cancellation conditions. For all $s \in \mathbb{N}$, $\alpha \in (0, 1]$, there exist solutions f^δ up to T_δ with $T_\delta \to 0$ and

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New results: the measure-valued setting and the multiphase formulation

The measure-valued setting

We call a **measure-valued solution** of (kEu) any f associating to (t, x) a probability measure $f(t, x, \bullet) \in \mathcal{P}(\mathbb{R}^d)$ in such a way that:

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Do there exist unstable solutions in the neighbourhood of these unstable measures?

Ill-posedness for (kEu) around measures

Theorem (B. 2019)

Take μ an unstable ***measure****, $\varphi_1, \ldots, \varphi_N \in C_c^\infty(\mathbb{R}^d)$, $s \in \mathbb{N}$ and $\alpha \in (0, 1]$.*

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- *we have:*

$$\frac{\|p_\delta\|_{L^1([0,T_\delta)\times\mathbb{T}^d)}}{\sum_{i=1}^N \|\langle f_0^\delta, \varphi_i\rangle - \langle \mu, \varphi_i\rangle\|^\alpha_{W^{s,\infty}(\mathbb{T}^d)}} \underset{\delta\to 0}{\longrightarrow} +\infty.$$

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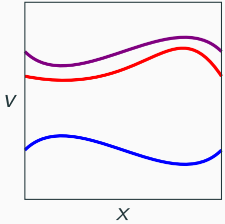

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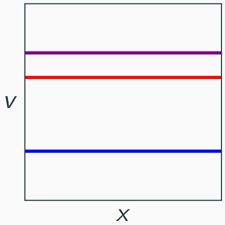

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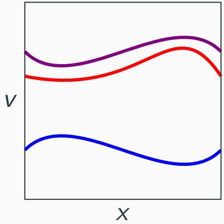

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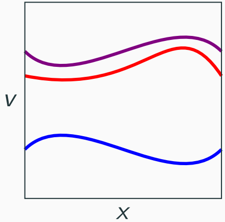

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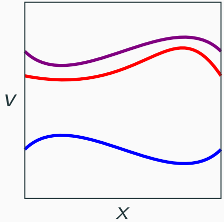

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The stationary solution $(1, w)_{w\in\mathbb{R}^d}$ is linearly unstable **if and only if μ is Penrose unstable**.

Ill-posedness in the multiphase formulation

Recap:
1. Each stationary profile is a multiphase solution;
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$$\frac{\|p_\delta\|_{L^1([0,T_\delta)\times\mathbb{T}^d)}}{\sup\limits_{w\in\mathbb{R}^d}\left\{\|\rho_0^{\delta,w} - 1\|_{W^{s,\infty}}^\alpha + \|u_0^{\delta,w} - w\|_{W^{s,\infty}}^\alpha\right\}} \underset{\delta\to 0}{\longrightarrow} +\infty.$$

An application to incompressible optimal transport

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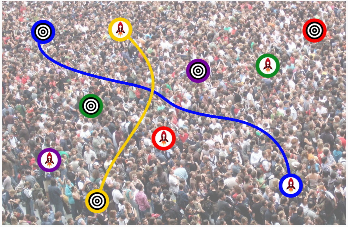

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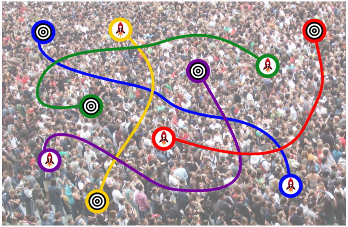

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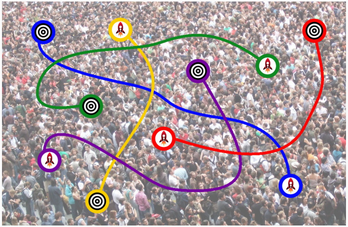

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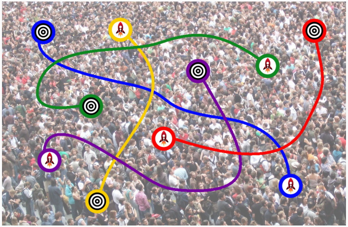

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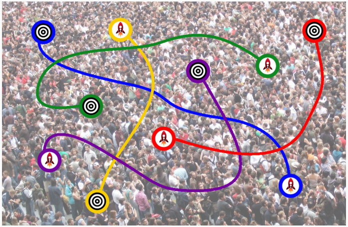

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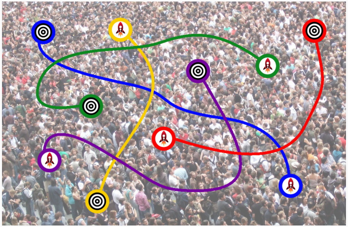

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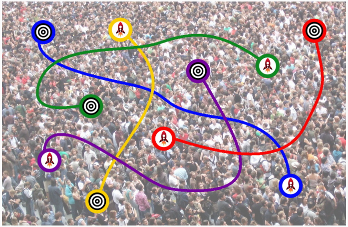

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By our ill-posedness result: $\boldsymbol{p}$ **is not a smooth function of** γ [B. 2019].

The Vlasov-Poisson case

Lyapounov instability in the Vlasov-Poisson case

Ongoing work with D. Han-Kwan.

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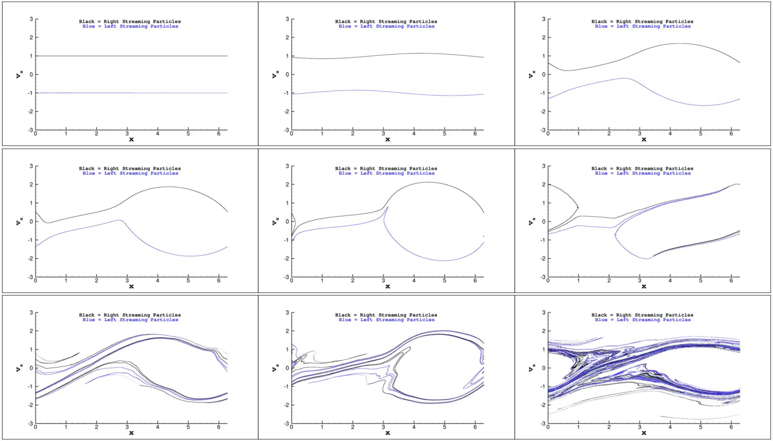
- **A generalization of their proof in higher dimension and general μ would provide a proof of non-linear instability for (VP) in a measure-valued setting.**

Thank you!



Pictures from Frans Ebersohn, PEPL, University of Michigan.