On local existence results for generalized MHD equations

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Joint work with Yong Zhou (Sun Yat-Sen University, Zhuhai)

- Consider the Cauchy problem for the generalized MHD equations:

$$\begin{cases} u_t + (u \cdot \nabla)u + \nu\Lambda^{2\alpha}u + \nabla p = (b \cdot \nabla)b & \text{in } \mathbb{R}^d \times (0,\infty) \\ b_t + (u \cdot \nabla)b + \eta\Lambda^{2\beta}b = (b \cdot \nabla)u & \text{in } \mathbb{R}^d \times (0,\infty) \\ \operatorname{div} u = \operatorname{div} b = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(0) = u_0, \quad b(0) = b_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{1}$$

- Notations:

- $d \geq 2$: the spacial dimension, α, β: nonnegative constants

- $\nu \geq 0$: the viscosity constant, $\eta \geq 0$; the magnetic diffusivity

- $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$: the velocity field, $b : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$: the magnetic field

- $p : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$: a scalar pressure

- $\Lambda^s = (-\Delta)^{s/2}$: the fractional Laplacian of order $s \in \mathbb{R}$, defined via the Fourier transform by

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi)$$

- Sobolev spaces H^s: For $s \in \mathbb{R}$,

$$H^s = H^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}' \mid J^s f \in L^2 \right\},$$

where $J^s = (I - \Delta)^{s/2}$ is defined by

$$\widehat{J^s f}(\xi) = \left(1 + |\xi|^2\right)^{s/2} \hat{f}(\xi) \quad (f \in \mathcal{S}').$$

- H^s is a Hilbert space equipped with the inner product

$$(u, v)_{H^s} = (J^s u, J^s v) = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi,$$

with $(\cdot, \cdot)$ denoting the inner product on L^2.

- For $s \geq 0$, H^s may be equipped with the following equivalent norm:

$$\|u\|_{H^s} = \left(\|u\|^2 + \|\Lambda^s u\|^2\right)^{1/2} = \left[\int_{\mathbb{R}^d} \left(1 + |\xi|^{2s}\right) |\hat{u}(\xi)|^2\, d\xi\right]^{1/2},$$

where $\| \cdot \|$ denotes the usual L^2-norm.

- Energy identities in L^2:

Multiplying the equations in (1) by u and b, respectively, and using the divergence-free condition on u, we derive

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|^2\right) + \nu\|\Lambda^\alpha u\|^2 = ((b\cdot\nabla)b, u)$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\|b\|^2\right) + \eta\|\Lambda^\beta b\|^2 = ((b\cdot\nabla)u, b)\,.$$

Since b is divergence-free,

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \|b\|^2\right) + \nu\|\Lambda^\alpha u\|^2 + \eta\|\Lambda^\beta b(t)\|^2 = 0.$$

Hence setting $M_0 = \left(\|u_0\|^2 + \|b_0\|^2\right)^{1/2}$, we derive a global energy estimate

$$\|u(t)\|^2 + \|b(t)\|^2 + 2\nu\int_0^t \|\Lambda^\alpha u(\tau)\|^2\,d\tau + 2\eta\int_0^t \|\Lambda^\beta b(\tau)\|^2\,d\tau \le M_0^2$$

for all $t \ge 0$.

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- Energy identities in higher norms:

Multiplying the equations in (1) by $\Lambda^{2s_1}u$ and $\Lambda^{2s_2}b$, respectively, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{s_1}u\|^2\right) + \nu\|\Lambda^{s_1+\alpha}u\|^2 = -\left(\Lambda^{s_1}[(u\cdot\nabla)u], \Lambda^{s_1}u\right) + \left(\Lambda^{s_1}[(b\cdot\nabla)b], \Lambda^{s_1}u\right)$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{s_2}b\|^2\right) + \eta\|\Lambda^{s_2+\beta}b\|^2 = -\left(\Lambda^{s_2}[(u\cdot\nabla)b], \Lambda^{s_2}b\right) + \left(\Lambda^{s_2}[(b\cdot\nabla)u], \Lambda^{s_2}b\right).$$

Combining these with the L^2-energy identities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^{s_1}}^2\right) + \nu\|\Lambda^{\alpha}u\|_{H^{s_1}}^2 \\ &\quad = ((b\cdot\nabla)b, u) - \left(\Lambda^{s_1}[(u\cdot\nabla)u], \Lambda^{s_1}u\right) + \left(\Lambda^{s_1}[(b\cdot\nabla)b], \Lambda^{s_1}u\right),\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|b\|_{H^{s_2}}^2\right) + \eta\|\Lambda^{\beta}b\|_{H^{s_2}}^2 \\ &\quad = ((b\cdot\nabla)u, b) - \left(\Lambda^{s_2}[(u\cdot\nabla)b], \Lambda^{s_2}b\right) + \left(\Lambda^{s_2}[(b\cdot\nabla)u], \Lambda^{s_2}b\right),\end{aligned}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^{s_1}}^2+\|b\|_{H^{s_2}}^2\right)+\nu\|\Lambda^\alpha u\|_{H^{s_1}}^2+\eta\|\Lambda^\beta b\|_{H^{s_2}}^2$$
$$=-\left(\Lambda^{s_1}[(u\cdot\nabla)u],\Lambda^{s_1}u\right)+\left(\Lambda^{s_1}[(b\cdot\nabla)b],\Lambda^{s_1}u\right)$$
$$-\left(\Lambda^{s_2}[(u\cdot\nabla)b],\Lambda^{s_2}b\right)+\left(\Lambda^{s_2}[(b\cdot\nabla)u],\Lambda^{s_2}b\right).$$

• To estimate each term of the right hand sides in the energy identities, we need to estimate the trilinear form

$$(\Lambda^s[(u\cdot\nabla)v],\Lambda^s w)$$

under various assumptions on vector fields u, v, w, and a nonnegative number s.

An obvious way is to derive some *product estimates*, since

$$(\Lambda^s[(u\cdot\nabla)v],\Lambda^s w)\le\|\Lambda^s[(u\cdot\nabla)v]\|\|\Lambda^s w\|.$$

If u is divergence-free and $w=v$, then we may need *commutator estimates*, since

$$[(\Lambda^s[(u\cdot\nabla)v],\Lambda^s v)=(\Lambda^s[(u\cdot\nabla)v]-(u\cdot\nabla)\Lambda^s v,\Lambda^s v)$$
$$\le\|\Lambda^s[(u\cdot\nabla)v]-(u\cdot\nabla)\Lambda^s v\|\|\Lambda^s v\|.$$

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$$
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&\quad=-\left(\Lambda^{s_1}[(u\cdot\nabla)u],\Lambda^{s_1}u\right)+\left(\Lambda^{s_1}[(b\cdot\nabla)b],\Lambda^{s_1}u\right)\\
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Some known results in H^s

• Assuming that $\nu > 0$, $\eta > 0$, $\alpha > 0$, and $\beta > 0$, J. Wu (2003) proved global existence of a solution

$$u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^\alpha), \quad b \in L^\infty(0,T;L^2) \cap L^2(0,T;H^\beta)$$

for any divergence-free $(u_0, b_0) \in L^2 \times L^2$, where T is any finite time.

Moreover, if $\alpha, \beta \geq 1/2 + d/4$ and $(u_0, b_0) \in H^s \times H^s$ with $s \geq \max\{2\alpha, 2\beta\}$, then

$$u \in L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+\alpha}), \quad b \in L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+\beta}).$$

• Assuming that $\nu = \eta = 0$, P. G. Schmidt (1988) proved local existence of a unique solution

$$(u,b) \in L^\infty(0,T_*;H^m)$$

for $m \in \mathbb{N}$ with $m > 1 + d/2$.

Remark. (i) The integer m can be replaced by any real $s > 1 + d/2$.

(ii) A key tool is the following product estimate:

$$\|\Lambda^s[(u \cdot \nabla)v]\| \leq C\|u\|_{H^s}\|\nabla v\|_{H^s} \quad \text{if } s > \frac{d}{2}.$$

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• Assuming that $\nu > 0$, $\eta = 0$, and $\alpha = 1$, C. Fefferman, D.S. McCormick, J.C. Robinson, and J.L. Rodrigo (2014) proved local existence of a unique solution

$$u \in L^\infty(0, T_*; H^s) \cap L^2(0, T_*; H^{s+1}), \quad b \in L^\infty(0, T_*; H^s)$$

for $s > d/2$.

Remark. A key tool is the following commutator estimate:

$$\|\Lambda^s[(u \cdot \nabla)v] - (u \cdot \nabla)(\Lambda^s v)\| \le C\|\nabla u\|_{H^s}\|v\|_{H^s} \quad \text{if } s > \frac{d}{2},$$

which refines the classical one due to T. Kato and G. Ponce (1988):

$$\|\Lambda^s[(u \cdot \nabla)v] - (u \cdot \nabla)(\Lambda^s v)\| \le C\left(\|\nabla u\|_{H^s}\|v\|_{H^s} + \|u\|_{H^s}\|\nabla v\|_{H^s}\right)$$

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Remark. A key tool is the following estimate: if u is a solution of the heat equation

$$u_t - \nu\Delta u = g \quad \text{in } \mathbb{R}^d \times (0,T), \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d,$$

then

$$\int_0^T \|u(t)\|_{H^{s_2+1}}\, dt \le CT^{\frac{s_1+1-s_2}{2}} \|u_0\|_{H^{s_1}} + CT^{1-\frac{1}{r}} \|g\|_{L^r(0,T;H^{s_2-1})}.$$

for $1 < r < \infty$, provided that $s_2 > d/2$ and $s_2 - 1 < s_1 \le s_2$.

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Our result for $\nu > 0$ and $\eta = 0$

Theorem [K.-Zhou]. *Let $\nu > 0$ and $\eta = 0$. Suppose that $\alpha \geq 0, s_1 \geq 0$, and $s_2 > 0$ satisfy one of the following conditions:*

(i) $\alpha \geq 1$, $s_2 > d/2$, *and* $s_2 - \alpha < s_1 \leq s_2$.

(ii) $\alpha > 1$, $s_1 + \alpha > d/2 + 1$, *and* $s_1 \leq s_2 \leq s_1 + \alpha - 1$.

(iii) $0 \leq \alpha < 1$ *and* $s_1 = s_2 > d/2 + 1 - \alpha$.

Then for every $(u_0, b_0) \in H^{s_1} \times H^{s_2}$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, there exists $T_* > 0$ such that the Cauchy problem (1) has a solution

$$u \in L^\infty(0, T_*; H^{s_1}) \cap L^2(0, T_*; H^{s_1+\alpha}), \quad b \in L^\infty(0, T_*; H^{s_2}).$$

Remark. The conditions of the theorem are satisfied, in particular, for each of the following cases:

(i) If $\alpha > d/2 + 1$, then $s_1 \geq 0$, $s_2 > 0$, and $s_1 \leq s_2 \leq s_1 + \alpha - 1$.

(ii) If $d/2 < \alpha \leq d/2 + 1$, then $1 \leq s_1 \leq s_2 \leq s_1 + \alpha - 1$ or $0 \leq s_1 \leq d/2 < s_2 < \alpha$.

(iii) If $1 < \alpha \leq d/2$, then $d/2 \leq s_1 \leq s_2 \leq s_1 + \alpha - 1$ or $d/2 - 1 \leq s_1 \leq d/2 < s_2 < d/2 + \alpha - 1$.

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Theorem [K.-Zhou]. *Let $\nu > 0$ and $\eta > 0$. Suppose that $\alpha \geq 0, \beta \geq 0, s_1 \geq 0$, and $s_2 \geq 0$ satisfy one of the following conditions:*

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(iii) $\alpha \geq 1$, $(\alpha, \beta) \neq (1, 0)$, $s_1 + \alpha > d/2 + 1$, and $s_1 - \beta/2 < s_2 \leq s_1 + \alpha - 1$.

(iv) $\alpha + \beta \geq 2$, $s_1 + \alpha > d/2 + 1$, $s_2 + \beta > d/2 + 1$, and $s_1 + 1 - \beta \leq s_2 \leq s_1 + \alpha - 1$.

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(vi) $\beta \geq 1$, $1 < \alpha + \beta < 2$, $s_1 + \alpha > d/2 + 1$, and $s_1 + 1 - \beta \leq s_2 < s_1 + \alpha$.

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Then for every $(u_0, b_0) \in H^{s_1} \times H^{s_2}$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, there exists $T_* > 0$ such that the Cauchy problem (1) has a solution

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Remark. Assume that $\nu > 0$, $\eta > 0$, and $\beta = 1$. Then the conditions of the theorem are reduced as follows:

(i) $\alpha \geq 1$, $s_2 > d/2$, and $s_2 - \alpha < s_1 \leq s_2$.

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In addition, if $\alpha = 1$, then these conditions are reduced as:

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Note that the condition (i) is exactly the same as the case $\eta = 0$ and (ii) is a new one, due to the parabolicity of the equation for the magnetic field b.

Our result for $\nu > 0$ and $\eta > 0$

Remark. Assume that $\nu > 0$, $\eta > 0$, and $\beta = 1$. Then the conditions of the theorem are reduced as follows:

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• Sobolev embedding results:

Lemma.

(i) *If* $0 \leq s < d/2$, *then* $H^s \hookrightarrow L^{2d/(d-2s)}$.

(ii) *If* $s > d/2$, *then* $H^s \hookrightarrow L^\infty$.

(iii) *If* $s > 0$, $2 \leq p < \infty$, *and* $1/p \geq 1/2 - s/d$, *then* $H^s \hookrightarrow L^p$.

Proof of (iii). Suppose that $s > 0$, $2 \leq p < \infty$, and $1/p \geq 1/2 - s/d$.

If $s < d/2$, then

$$\|u\|_{L^p} \leq \|u\|_{L^2}^{1-\theta} \|u\|_{L^{\frac{2d}{d-2s}}}^{\theta} \leq C\|u\|_{H^s},$$

where $0 \leq \theta \leq 1$ is defined by $1/p = 1/2 - \theta s/d$.

If $s \geq d/2$, then choosing $0 \leq s_0 < d/2$ with $1/p = 1/2 - s_0/d$, we have

$$\|u\|_{L^p} \leq C\|u\|_{H^{s_0}} \leq C\|u\|_{H^s}.$$

□

• Sobolev embedding results:

Lemma.

(i) *If* $0 \leq s < d/2$, *then* $H^s \hookrightarrow L^{2d/(d-2s)}$.

(ii) *If* $s > d/2$, *then* $H^s \hookrightarrow L^\infty$.

(iii) *If* $s > 0$, $2 \leq p < \infty$, *and* $1/p \geq 1/2 - s/d$, *then* $H^s \hookrightarrow L^p$.

Proof of (iii). Suppose that $s > 0$, $2 \leq p < \infty$, and $1/p \geq 1/2 - s/d$.

If $s < d/2$, then

$$\|u\|_{L^p} \leq \|u\|_{L^2}^{1-\theta} \|u\|_{L^{\frac{2d}{d-2s}}}^{\theta} \leq C\|u\|_{H^s},$$

where $0 \leq \theta \leq 1$ is defined by $1/p = 1/2 - \theta s/d$.

If $s \geq d/2$, then choosing $0 \leq s_0 < d/2$ with $1/p = 1/2 - s_0/d$, we have

$$\|u\|_{L^p} \leq C\|u\|_{H^{s_0}} \leq C\|u\|_{H^s}.$$

□

Tools of the proof: embedding results

- A joint embedding result:

Lemma. *Suppose that* $s_1 \geq 0$, $s_2 \geq 0$, *and* $s_1 + s_2 > d/2$. *Then there exists a pair* (p, q) *with* $2 \leq p, q \leq \infty$ *and* $1/p + 1/q = 1/2$ *such that*

$$H^{s_1} \hookrightarrow L^p \quad \textit{and} \quad H^{s_2} \hookrightarrow L^q.$$

Proof. If $s_1 = 0$ or $s_2 = 0$, then the lemma follows from the previous embedding lemma (ii) by taking $(p, q) = (2, \infty)$ or $(p, q) = (\infty, 2)$.

Suppose that $s_1 > 0$ and $s_2 > 0$. Then since $d/2 < s_1 + s_2$, there exists $2 < p < \infty$ such that

$$\max\left\{\frac{1}{2} - \frac{s_1}{d}, 0\right\} < \frac{1}{p} < \min\left\{\frac{1}{2}, \frac{s_2}{d}\right\}.$$

If $q = 2p/(p-2)$, then

$$2 < q < \infty \quad \text{and} \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{p} > \frac{1}{2} - \frac{s_2}{d}.$$

Hence the desired estimates immediately follow from the embedding lemma (iii). □

- A joint embedding result:

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Hence the desired estimates immediately follow from the embedding lemma (iii). □

- The classical estimates due to Kato and Ponce (1988, 1991):

Theorem. *Let* $s \geq 0$ *and* $1 < p < \infty$. *Suppose that* $1 < p_1, p_2, q_1, q_2 \leq \infty$ *satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Then for all $f, g \in \mathcal{S}$,

$$\|\Lambda^s(fg)\|_{L^p} \leq C\left(\|f\|_{L^{p_1}}\|J^s g\|_{L^{q_1}} + \|J^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right)$$

and

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C\left(\|\nabla f\|_{L^{p_1}}\|J^{s-1} g\|_{L^{q_1}} + \|J^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right),$$

where $C = C(d, s, p, p_1, p_2)$.

Remark. Assume that $s = \gamma > d/2$. Then since $H^\gamma \hookrightarrow L^\infty$, we have

$$\|\Lambda^\gamma(fg)\| \leq C\left(\|f\|_{L^\infty}\|J^\gamma g\|_{L^2} + \|J^\gamma f\|_{L^2}\|g\|_{L^\infty}\right) \leq C\|f\|_{H^\gamma}\|g\|_{H^\gamma}$$

and

$$\|\Lambda^\gamma(fg) - f\Lambda^\gamma g\| \leq C\left(\|\nabla f\|_{H^\gamma}\|g\|_{H^{\gamma-1}} + \|f\|_{H^\gamma}\|g\|_{H^\gamma}\right).$$

Tools of the proof: product and commutator estimates

• The classical estimates due to Kato and Ponce (1988, 1991):

Theorem. *Let* $s \geq 0$ *and* $1 < p < \infty$. *Suppose that* $1 < p_1, p_2, q_1, q_2 \leq \infty$ *satisfy*

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Then for all $f, g \in \mathcal{S}$,

$$\|\Lambda^s(fg)\|_{L^p} \leq C\left(\|f\|_{L^{p_1}}\|J^s g\|_{L^{q_1}} + \|J^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right)$$

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$$\|\Lambda^\gamma(fg)\| \leq C\left(\|f\|_{L^\infty}\|J^\gamma g\|_{L^2} + \|J^\gamma f\|_{L^2}\|g\|_{L^\infty}\right) \leq C\|f\|_{H^\gamma}\|g\|_{H^\gamma}$$

and

$$\|\Lambda^\gamma(fg) - f\Lambda^\gamma g\| \leq C\left(\|\nabla f\|_{H^\gamma}\|g\|_{H^{\gamma-1}} + \|f\|_{H^\gamma}\|g\|_{H^\gamma}\right).$$

- Our product and commutator estimates:

Lemma. *Let* $\gamma > d/2$

(i) *Assume that* $0 \le s \le \gamma$. *Then for all* $f \in H^s$ *and* $g \in H^\gamma$,

$$\|\Lambda^s(fg)\| \le C\|f\|_{H^s}\|g\|_{H^\gamma},$$

where $C = C(d, \gamma, s)$.

(ii) *Assume that* $0 \le s \le \gamma + 1$. *Then for all* $f \in H^s$ *and* $g \in H^\gamma$,

$$\|\Lambda^s(fg) - f\Lambda^s g\| \le C\|f\|_{H^s}\|g\|_{H^\gamma},$$

where $C = C(d, \gamma, s)$.

(iii) *Assume that* $1 \le s \le \gamma + 1$. *Then for all* $f \in H^{\gamma+1}$ *and* $g \in H^{s-1}$,

$$\|\Lambda^s(fg) - f\Lambda^s g\| \le C\|f\|_{H^{\gamma+1}}\|g\|_{H^{s-1}},$$

where $C = C(d, \gamma, s)$.

Remark. Taking $s = \gamma$ in (ii) and (iii), respectively, we obtain

$$\|\Lambda^\gamma(fg) - f\Lambda^\gamma g\| \le C \min\{\|f\|_{H^\gamma}\|g\|_{H^\gamma}, \|f\|_{H^{\gamma+1}}\|g\|_{H^{\gamma-1}}\}.$$

- Our product and commutator estimates:

Lemma. *Let* $\gamma > d/2$

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(iii) *Assume that* $1 \leq s \leq \gamma + 1$. *Then for all* $f \in H^{\gamma+1}$ *and* $g \in H^{s-1}$,

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Tools of the proof: product and commutator estimates

Proof of (iii): Assume that $1 \le s \le \gamma + 1$.

If $2 \le p, q \le \infty$ and $1/p + 1/q = 1/2$, then by the Kato-Ponce commutator estimate,

$$\begin{aligned} \|\Lambda^s(fg) - f\Lambda^s g\| &\le C\left(\|\nabla f\|_{L^\infty}\|J^{s-1}g\| + \|J^s f\|_{L^q}\|g\|_{L^p}\right) \\ &\le C\left(\|f\|_{H^{\gamma+1}}\|g\|_{H^{s-1}} + \|J^s f\|_{L^q}\|g\|_{L^p}\right). \end{aligned}$$

Applying the joint embedding lemma to $s_1 = s - 1$ and $s_2 = \gamma + 1 - s$, we can find $2 \le p, q \le \infty$ with $1/p + 1/q = 1/2$ such that

$$H^{s-1} \hookrightarrow L^p \quad \text{and} \quad H^{\gamma+1-s} \hookrightarrow L^q.$$

Then

$$\|g\|_{L^p} \le C\|g\|_{H^{s-1}}$$

and

$$\|J^s f\|_{L^q} \le C\|J^s f\|_{H^{\gamma+1-s}} \le C\|f\|_{H^{\gamma+1}}.$$

□

Tools of the proof: product and commutator estimates

Proof of (iii): Assume that $1 \le s \le \gamma + 1$.

If $2 \le p, q \le \infty$ and $1/p + 1/q = 1/2$, then by the Kato-Ponce commutator estimate,

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□

Lemma. *Let* $\gamma > d/2$.

(i) *Assume that* $0 \le s \le \gamma$. *Then for all* $u \in H^\gamma$ *and* $v \in H^{s+1}$,

$$\|\Lambda^s[(u \cdot \nabla)v]\| \le C\|u\|_{H^\gamma}\|\nabla v\|_{H^s}.$$

(ii) *Assume that* $0 \le s \le \gamma$. *Then for all* $u \in H^s$ *and* $v \in H^{\gamma+1}$,

$$\|\Lambda^s[(u \cdot \nabla)v]\| \le C\|u\|_{H^s}\|\nabla v\|_{H^\gamma}.$$

(iii) *Assume that* $0 \le s \le \gamma + 1$. *Then for all* $u \in H^s$ *and* $v \in H^{\gamma+1}$,

$$\|\Lambda^s[(u \cdot \nabla)v] - (u \cdot \nabla)(\Lambda^s v)\| \le C\|u\|_{H^s}\|\nabla v\|_{H^\gamma}.$$

(iv) *Assume that* $1 \le s \le \gamma + 1$. *Then for all* $u \in H^{\gamma+1}$ *and* $v \in H^s$,

$$\|\Lambda^s[(u \cdot \nabla)v] - (u \cdot \nabla)(\Lambda^s v)\| \le C\|u\|_{H^{\gamma+1}}\|\nabla v\|_{H^{s-1}}.$$

Remark. Essentially the same estimates as (iv) has been obtained by Fefferman et al. (2014) for $s = \gamma$ and by Jiang et al. (preprint) for $1 < s \le \gamma$. In fact, the estimate can be proved for all $0 \le s \le \gamma + 1$.

• Using the Leray projection, we can remove the pressure term in the Navier-Stokes equations. We then need to consider the following Cauchy problem for the fractional heat equation:

$$\begin{cases} u_t + \nu\Lambda^{2\alpha}u = g & \text{in } \mathbb{R}^d \times (0,T) \\ u(0) = u_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2}$$

where $\nu > 0$, $\alpha > 0$, and $0 < T < \infty$.

• The solution formula via the Fourier transform:

A regular function $u = u(x,t)$ is a solution of (2) if and only if its Fourier transform $\hat{u} = \hat{u}(\xi,t)$ satisfies

$$\begin{cases} \hat{u}_t + \nu|\xi|^{2\alpha}\hat{u} = \hat{g} & \text{in } \mathbb{R}^d \times (0,T) \\ \hat{u}(0) = \hat{u}_0 & \text{in } \mathbb{R}^d. \end{cases}$$

Solving this ODE problem, we derive

$$\hat{u}(\xi,t) = e^{-\nu|\xi|^{2\alpha}t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^{2\alpha}(t-\tau)}\,\hat{g}(\xi,\tau)\,d\tau.$$

- Our estimates for solutions:

Lemma. *Assume that* $u_0 = 0$, $g \in L^r(0,T;H^{s-\alpha})$, $s \in \mathbb{R}$, *and* $1 < r < \infty$. *Then the Cauchy problem (2) has a unique solution* $u \in L^r(0,T;H^{s+\alpha})$. *Moreover, we have*

$$\|u\|_{L^r(0,T;H^{s+\alpha})} \leq C\,(1+T)\,\|g\|_{L^r(0,T;H^{s-\alpha})}.$$

Proof. Define

$$\Phi(\xi,t) = \begin{cases} |\xi|^{2\alpha} e^{-\nu|\xi|^{2\alpha}t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then

$$\widehat{\Lambda^{2\alpha} u}(\xi,t) = \int_{\mathbb{R}} \Phi(\xi,t-\tau)\hat{g}(\xi,\tau)\,d\tau$$

and

$$\Lambda^{2\alpha} u(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} K(x-y,t-\tau)g(y,\tau)\,dyd\tau,$$

where $K(\cdot,t)$ is the inverse Fourier transform of $\Phi(\cdot,t)$. By the parabolic Calderon-Zygmund result due to I. Kim, S. Lim, and K. Kim (2016),

$$\|\Lambda^{2\alpha} u\|_{L^r(0,T;L^2)} \leq C(d,\nu,\alpha,r)\|g\|_{L^r(0,T;L^2)}.$$

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$$\|\Lambda^{2\alpha}u\|_{L^r(0,T;L^2)} \leq C(d,\nu,\alpha,r)\|g\|_{L^r(0,T;L^2)}.$$

□

Lemma. *Assume that $u_0 \in H^s$, $g = 0$, and $s \in \mathbb{R}$. Then the Cauchy problem (2) has a unique solution*

$$u \in L^2(0,T;H^{s+\alpha}) \quad \textit{with} \quad \sqrt{t}u \in L^2(0,T;H^{s+2\alpha}).$$

Moreover, we have

$$\int_0^T \left(\|u(t)\|^2_{H^{s+\alpha}} + t\|u(t)\|^2_{H^{s+2\alpha}}\right) dt \le C\,(1+T)^2\,\|u_0\|^2_{H^s}.$$

In addition, if $0 < \varepsilon < \alpha$, then

$$u \in L^1\left(0,T;H^{s+2\alpha-\varepsilon}\right)$$

and

$$\int_0^T \|u(t)\|_{H^{s+2\alpha-\varepsilon}}\,dt \le C(1+T)T^{\frac{\varepsilon}{2\alpha}}\|u_0\|_{H^s}.$$

Proof. A smooth solution u is given via the Fourier transform by

$$\hat{u}(\xi,t) = e^{-\nu|\xi|^{2\alpha}t}\hat{u}_0(\xi).$$

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$$\hat{u}(\xi,t) = e^{-\nu|\xi|^{2\alpha}t}\hat{u}_0(\xi).$$

Hence

$$\begin{aligned}\int_0^T \|\Lambda^\alpha u(t)\|^2\,dt &= \int_0^T \int_{\mathbb{R}^d} |\xi|^{2\alpha}|\hat{u}(\xi,t)|^2\,d\xi dt\\ &= \int_{\mathbb{R}^d}\int_0^T |\xi|^{2\alpha}e^{-2\nu|\xi|^{2\alpha}t}|\hat{u}_0(\xi)|^2\,dtd\xi\\ &\le \frac{1}{2\nu}\int_{\mathbb{R}^d}|\hat{u}_0(\xi)|^2\,d\xi\\ &= \frac{1}{2\nu}\|u_0\|^2.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_0^T t\|\Lambda^{2\alpha} u(t)\|^2\,dt &= \int_0^T \int_{\mathbb{R}^d} t|\xi|^{4\alpha}|\hat{u}(\xi,t)|^2\,d\xi dt\\ &= \int_{\mathbb{R}^d}\int_0^T t|\xi|^{4\alpha}e^{-2\nu|\xi|^{2\alpha}t}|\hat{u}_0(\xi)|^2\,dtd\xi\\ &\le \frac{1}{2\nu}\int_{\mathbb{R}^d}\int_0^T |\xi|^{2\alpha}e^{-2\nu|\xi|^{2\alpha}t}|\hat{u}_0(\xi)|^2\,dtd\xi\\ &\le \frac{1}{(2\nu)^2}\|u_0\|^2.\end{aligned}$$

Tools of the proof: estimates for the fractional heat equation

Suppose that $0 < \varepsilon < \alpha$. Then by an interpolation inequality for H^s,

$$\begin{aligned}\|w\|_{H^{s+2\alpha-\varepsilon}} &\leq C\|w\|_{H^{s+\alpha}}^{\theta}\|w\|_{H^{s+2\alpha}}^{1-\theta} \\ &= C\|w\|_{H^{s+\alpha}}^{\theta}\left(\sqrt{t}\|w\|_{H^{s+2\alpha}}\right)^{1-\theta} t^{-(1-\theta)/2},\end{aligned}$$

where $0 < \theta = \varepsilon/\alpha < 1$. Hence by Hölder's inequality,

$$\begin{aligned}\int_0^T \|w\|_{H^{s+2\alpha-\varepsilon}}\,dt &\leq C\left(\int_0^T \|w(t)\|_{H^{s+\alpha}}^2\,dt\right)^{\frac{\theta}{2}} \\ &\quad\times\left(\int_0^T t\|w(t)\|_{H^{s+2\alpha}}^2\,dt\right)^{\frac{1-\theta}{2}}\left(\int_0^T t^{-(1-\theta)}\,dt\right)^{\frac{1}{2}} \\ &\leq C(1+T)\|u_0\|_{H^s}T^{\frac{\theta}{2}}.\end{aligned}$$

□

Lemma. *Assume that $u_0 \in H^{s_1}$, $g \in L^r(0,T;H^{s_2-\alpha})$, $s_1 \leq s_2 < s_1+\alpha$, and $1 < r < \infty$. Then the Cauchy problem (2) has a unique solution*

$$u \in L^1\left(0,T;H^{s_2+\alpha}\right).$$

Moreover, we have

$$\frac{1}{1+T}\int_0^T \|u(t)\|_{H^{s_2+\alpha}}\,dt \leq CT^{\frac{s_1+\alpha-s_2}{2\alpha}}\|u_0\|_{H^{s_1}} + CT^{1-\frac{1}{r}}\|g\|_{L^r(0,T;H^{s_2-\alpha})}.$$

Suppose that $0 < \varepsilon < \alpha$. Then by an interpolation inequality for H^s,

$$\begin{aligned}\|w\|_{H^{s+2\alpha-\varepsilon}} &\le C\|w\|_{H^{s+\alpha}}^{\theta}\|w\|_{H^{s+2\alpha}}^{1-\theta} \\ &= C\|w\|_{H^{s+\alpha}}^{\theta}\left(\sqrt{t}\|w\|_{H^{s+2\alpha}}\right)^{1-\theta} t^{-(1-\theta)/2},\end{aligned}$$

where $0 < \theta = \varepsilon/\alpha < 1$. Hence by Hölder's inequality,

$$\begin{aligned}\int_0^T \|w\|_{H^{s+2\alpha-\varepsilon}}\,dt &\le C\left(\int_0^T \|w(t)\|_{H^{s+\alpha}}^2\,dt\right)^{\frac{\theta}{2}} \\ &\quad \times \left(\int_0^T t\|w(t)\|_{H^{s+2\alpha}}^2\,dt\right)^{\frac{1-\theta}{2}}\left(\int_0^T t^{-(1-\theta)}\,dt\right)^{\frac{1}{2}} \\ &\le C(1+T)\|u_0\|_{H^s}T^{\frac{\theta}{2}}.\end{aligned}$$

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Lemma. *Assume that $u_0 \in H^{s_1}$, $g \in L^r(0,T;H^{s_2-\alpha})$, $s_1 \le s_2 < s_1+\alpha$, and $1 < r < \infty$. Then the Cauchy problem (2) has a unique solution*

$$u \in L^1\left(0,T;H^{s_2+\alpha}\right).$$

Moreover, we have

$$\frac{1}{1+T}\int_0^T \|u(t)\|_{H^{s_2+\alpha}}\,dt \le CT^{\frac{s_1+\alpha-s_2}{2\alpha}}\|u_0\|_{H^{s_1}} + CT^{1-\frac{1}{r}}\|g\|_{L^r(0,T;H^{s_2-\alpha})}.$$

Thank you very much for your attention!