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Network-Based Kinetic Models: Emergence of a Statistical Description of the Graph Topology

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Talk Based On

M. Nurisso, M. Raviola and A. Tosin. **"Network-based kinetic models: Emergence of a statistical description of the graph topology".** In: *European J. Appl. Math.* (2024), pp. 1–22. DOI: 10.1017/S0956792524000020

Motivations

- Interacting multi-agent systems are ubiquitous in both classical physics and socio-/econophysics
- In socio-/econophysics, unlike classical physics, interactions are often **networked**: agents do not interact "all-to-all" but according to some preferential **connections**
- Prototype: opinion formation on social networks
- **Large number** of networked agents $\rightsquigarrow$ need for a **statistical description** of the **network topology**

Some Related Contributions

- G. Toscani, A. Tosin and M. Zanella. **"Opinion modeling on social media and marketing aspects".** In: *Phys. Rev. E* 98.2 (2018), pp. 022315/1–15. DOI: 10.1103/PhysRevE.98.022315
- N. Loy, M. Raviola and A. Tosin. **"Opinion polarization in social networks".** In: *Philos. Trans. Roy. Soc. A* 380.2224 (2022), pp. 20210158/1–15. DOI: 10.1098/rsta.2021.0158
- B. Düring, J. Franceschi, M.-T. Wolfram and M. Zanella. **"Breaking consensus in kinetic opinion formation models on graphons".** Preprint. 2024
- M. Burger, N. Loy and A. Rossi. **"Stability of stationary solutions to Allen-Cahn type opinion formation models".** In preparation. 2024

Particle Description

- Agents are understood as the vertices of a graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$, $\mathcal{I} = \{1, \dots, N\}$
- A representative agent $X \in \mathcal{I}$ features an opinion $V_t \in \mathcal{O} \subset \mathbb{R}$ at time $t \geq 0$
- Opinion exchange algorithm in randomly selected pairs of agents:

$$\begin{aligned} V_{t+\Delta t} &= (1-\Theta)V_t + \Theta\Psi(V_t, V_t^*) \\ V_{t+\Delta t}^* &= (1-\Theta)V_t^* + \Theta\Psi_*(V_t^*, V_t), \end{aligned} \qquad \Theta \sim \text{Bernoulli}(B(X, X_*)\Delta t)$$

  in an interaction time step $0 < \Delta t \leq 1$
- The **interaction rate** B encodes the information on agents' connections:

$$B(X, X_*) = \begin{cases} 1 & \text{if } (X, X_*) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

- Ψ, $\Psi_* : \mathcal{O}^2 \to \mathcal{O}$ represent the post-interaction opinions in case of a successful interaction

Derivation of a Kinetic Description 1/2

- Let $(X,\, V_t) \sim f(x,v,t)$, $x \in \mathcal{I}$, $v \in \mathcal{O}$

$$f(x,v,t) = \frac{1}{N} \sum_{i \in \mathcal{I}} f_i(v,t) \otimes \delta(x-i)$$

  with $f_i : \mathcal{O} \times [0,\, +\infty) \to \mathbb{R}_+$ the pdf of the opinion of agent i at time t

- Taking the expectation of $\Phi(X, V_{t+\Delta t})$ and of $\Phi(X_*, V^*_{t+\Delta t})$, where Φ is an arbitrary scalar function, one obtains

$$\frac{d}{dt} \sum_{h \in \mathcal{I}} \int_{\mathcal{O}} \Phi(h,v) f_h(v,t)\, dv =$$

$$= \sum_{h,k \in \mathcal{I}} \iint_{\mathcal{O}^2} B(h,k) \frac{\Phi(h,v') + \Phi(k,v'_*) - \Phi(h,v) - \Phi(k,v_*)}{2N} f_h(v,t) f_k(v_*,t)\, dv\, dv_*$$

  where

$$v' = \Psi(v, v_*), \qquad v'_* = \Psi_*(v_*, v)$$

Derivation of a Kinetic Description 2/2

- Taking $\Phi(x,v) = \phi(x)\varphi(v)$ with

$$\phi(x) = \begin{cases} 1 & \text{if } x = i \in \mathcal{I} \\ 0 & \text{otherwise,} \end{cases} \qquad \varphi : \mathcal{O} \to \mathbb{R} \text{ arbitrary}$$

we get

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{O}} \varphi(v) f_i(v,t)\,dv = {} & \frac{1}{2N}\sum_{k\in\mathcal{I}} B(i,k) \iint_{\mathcal{O}^2} (\varphi(v') - \varphi(v)) f_i(v,t) f_k(v_*,t)\,dv\,dv_* \\ & + \frac{1}{2N}\sum_{h\in\mathcal{I}} B(h,i) \iint_{\mathcal{O}^2} (\varphi(v'_*) - \varphi(v)) f_h(v,t) f_i(v_*,t)\,dv\,dv_* \end{aligned}$$

- Introducing the adjacency matrix $\mathbf{M} := (B(i,j))_{i,j\in\mathcal{I}} \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$:

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{O}} \varphi(v)\mathbf{f}(v,t)\,dv = {} & \frac{1}{2N}\iint_{\mathcal{O}^2} (\varphi(v') - \varphi(v))\mathbf{f}(v,t) \odot \mathbf{M}\mathbf{f}(v_*,t)\,dv\,dv_* \\ & + \frac{1}{2N}\iint_{\mathcal{O}^2} (\varphi(v'_*) - \varphi(v))\mathbf{M}^T\mathbf{f}(v,t) \odot \mathbf{f}(v_*,t)\,dv\,dv_* \end{aligned}$$

with $\mathbf{f}(v,t) = (f_i(v,t))_{i\in\mathcal{I}}$ and $\odot$ = Hadamard's vector product

Interlude

- The system of kinetic equations

$$\frac{d}{dt}\int_{\mathcal{O}} \varphi(v)\mathbf{f}(v,t)\,dv = \frac{1}{2N}\iint_{\mathcal{O}^2} (\varphi(v') - \varphi(v))\mathbf{f}(v,t) \odot \mathbf{M}\mathbf{f}(v_*,t)\,dv\,dv_* + \frac{1}{2N}\iint_{\mathcal{O}^2} (\varphi(v'_*) - \varphi(v))\mathbf{M}^T\mathbf{f}(v,t) \odot \mathbf{f}(v_*,t)\,dv\,dv_*$$

  for the array $\mathbf{f}$ of opinion distribution functions is valid on **whatever graph**

- **Problem**: it requires a "pointwise" description of the graph connections, which gets readily unfeasible when the size N of $\mathcal{G}$ grows

Towards a Statistical Description of the Graph Connections

- Global opinion distribution (v-marginal of f):

$$F(v,t) := \int_{\mathcal{I}} f(x,v,t)\,dx = \frac{1}{N}\sum_{i\in\mathcal{I}} f_i(v,t) = \frac{1}{N}\mathbf{1}^T\mathbf{f}(v,t)$$

  with $\mathbf{1} = (1,\,\ldots,\,1) \in \mathbb{R}^N$

- The equation for F is not closed in general:

$$\frac{d}{dt}\int_{\mathcal{O}} \varphi(v)F(v,t)\,dv = \\ = \frac{1}{2N^2}\iint_{\mathcal{O}^2} (\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*))\mathbf{f}^T(v,t)\mathbf{M}\mathbf{f}(v_*,t)\,dv\,dv_*$$

- A preliminary idea is to see whether this equation closes at least for special classes of interaction rules

Polarised Memory Interactions

- We say that an interaction rule $v' = \Psi(v, v_*)$ is of **polarised memory** type if Ψ depends only on either v or v_*
  - If $v' = \Psi(v)$ we say that the interaction rule has **perfect memory**
  - If $v' = \Psi(v_*)$ we say that the interaction rule is **memoryless**
- To fix the ideas, in the following we will focus on the case

$$v' = \Psi(v), \qquad v'_* = \Psi_*(v),$$

  i.e. v' has perfect memory whereas v'_* is memoryless
- In this case:

$$\frac{d}{dt}\int_{\mathcal{O}} \varphi(v)F(v,t)\,dv =$$
$$= \frac{1}{N^2}\int_{\mathcal{O}} \left((\mathbf{w}^+)^T \frac{\varphi(v') + \varphi(v'_*)}{2} - \frac{(\mathbf{w}^-)^T + (\mathbf{w}^+)^T}{2}\varphi(v) \right) \mathbf{f}(v,t)\,dv$$

  with $\mathbf{w}^-$, $\mathbf{w}^+$ vectors of **incoming** and **outgoing degrees** of the vertices of $\mathcal{G}$
- Notice: information about $\mathbf{M}$ is *lumped* in $\mathbf{w}^-$, $\mathbf{w}^+$

Statistical Distribution of the Degrees

- To obtain a kinetic formulation free from references to single vertices we **augment the space of microscopic states** by including also information on the connections via the incoming and outgoing degrees:

$$g_N(v, w^-, w^+, t) := \frac{1}{N} \sum_{\substack{i \in \mathcal{I} \\ \text{indeg}(i) = w^- \\ \text{outdeg}(i) = w^+}} f_i(v, t), \qquad w^-, w^+ \in \{0, \ldots, N\}$$

- Then:

$$F(v, t) = \sum_{w^-, w^+ = 0}^{N} g_N(v, w^-, w^+, t), \qquad (\mathbf{w}^{\pm})^T \mathbf{f}(v, t) = N \sum_{w^-, w^+ = 0}^{N} w^{\pm} g_N(v, w^-, w^+, t)$$

whence we deduce a **closed equation** for g_N:

$$\frac{d}{dt} \sum_{w^-, w^+ = 0}^{N} \int_{\mathcal{O}} \varphi(v) g_N(v, w^-, w^+, t) \, dv =$$

$$= \frac{1}{N} \sum_{w^-, w^+ = 0}^{N} \int_{\mathcal{O}} \left(w^+ \frac{\varphi(v') + \varphi(v'_*)}{2} - \frac{w^- + w^+}{2} \varphi(v) \right) g_N(v, w^-, w^+, t) \, dv$$

Formal Limit of Growing Graph ($N \to \infty$) 1/2

- Scaling:

$$\tilde{w}^{\pm} := \frac{w^{\pm}}{N} \in \mathcal{W}_N := \left\{ \frac{n}{N},\, n = 0,\, \dots,\, N \right\}, \qquad \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t) := N^2 g_N(v, N\tilde{w}^-, N\tilde{w}^+, t)$$

- Introduce the steps $\Delta\tilde{w}^{\pm} := \frac{1}{N}$ so that

$$\sum_{\tilde{w}^-, \tilde{w}^+ \in \mathcal{W}_N} \int_{\mathcal{O}} \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t)\, dv\, \Delta\tilde{w}^-\, \Delta\tilde{w}^+ = \sum_{w^-, w^+ = 0}^{N} \int_{\mathcal{O}} g_N(v, w^-, w^+, t)\, dv = 1$$

  and the r.h.s. may be understood as a **Riemann sum** approximating, for every N, the integral of the pdf $\int_{\mathcal{O}} \tilde{g}\, dv$ on the square mesh of $[0,\, 1]^2$ produced by the grid $\mathcal{W}_N \times \mathcal{W}_N \rightsquigarrow$ cf. a **graphon**

- Moreover:

$$\frac{d}{dt} \sum_{\tilde{w}^-, \tilde{w}^+ \in \mathcal{W}_N} \int_{\mathcal{O}} \varphi(v) \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t)\, dv\, \Delta\tilde{w}^-\, \Delta\tilde{w}^+ =$$

$$= \sum_{\tilde{w}^-, \tilde{w}^+ \in \mathcal{W}_N} \int_{\mathcal{O}} \left(\tilde{w}^+ \frac{\varphi(v') + \varphi(v'_*)}{2} - \frac{\tilde{w}^- + \tilde{w}^+}{2} \varphi(v) \right) \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t)\, dv\, \Delta\tilde{w}^-\, \Delta\tilde{w}^+$$

Formal Limit of Growing Graph ($N \to \infty$) 2/2

- Passing formally to the limit $N \to \infty$, the Riemann sums w.r.t. $\tilde{w}^{\pm}$ become integrals:

$$\frac{d}{dt}\iint_{[0,\,1]^2}\int_{\mathcal{O}}\varphi(v)\tilde{g}(v,\tilde{w}^-,\tilde{w}^+,t)\,dv\,d\tilde{w}^-\,d\tilde{w}^+ =$$
$$= \iint_{[0,\,1]^2}\int_{\mathcal{O}}\left(\tilde{w}^+\frac{\varphi(v')+\varphi(v'_*)}{2}-\frac{\tilde{w}^-+\tilde{w}^+}{2}\varphi(v)\right)\tilde{g}(v,\tilde{w}^-,\tilde{w}^+,t)\,dv\,d\tilde{w}^-\,d\tilde{w}^+$$

- This is a single kinetic equation in which the **pointwise information** on the graph topology encoded in $\mathbf{M}$ has been **replaced** asymptotically by the **statistical distribution** of the (normalised) incoming and outgoing degrees of the vertices

The Case of General Interaction Rules

- For general interaction rules:

$$v' = \Psi(v, v_*), \qquad v'_* = \Psi_*(v,\, v_*)$$

  the kinetic equation for F can be written, using $\tilde{g}$, as:

$$\begin{aligned}
&\frac{d}{dt} \sum_{\tilde{w}^-, \tilde{w}^+ \in \mathcal{W}_N} \int_{\mathcal{O}} \varphi(v) \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t)\, dv\, \Delta\tilde{w}^- \,\Delta\tilde{w}^+ = \\
&= \frac{1}{2N^2} \iint_{\mathcal{O}^2} \big(\varphi(v') + \varphi(v'_*)\big) \mathbf{f}^T(v,t) \mathbf{M} \mathbf{f}(v_*, t)\, dv\, dv_* \\
&\quad - \frac{1}{2} \sum_{\tilde{w}^-, \tilde{w}^+ \in \mathcal{W}_N} \int_{\mathcal{O}} \big(\tilde{w}^- + \tilde{w}^+\big) \varphi(v) \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t)\, dv\, \Delta\tilde{w}^- \,\Delta\tilde{w}^+
\end{aligned}$$

- The first term on the r.h.s. cannot be closed in terms of the statistics of the graph connections only. In general, it requires a pointwise knowledge of the connections

Rank-one Approximation of M

- $$\mathbf{M} \approx \frac{\mathbf{w}^+(\mathbf{w}^-)^T}{M_N}, \qquad M_N := \sum_{i\in\mathcal{I}} \mathrm{indeg}(i) = \sum_{i\in\mathcal{I}} \mathrm{outdeg}(i)$$

  is a natural rank-one approximation of $\mathbf{M}$ with given incoming/outgoing degrees
- Within this approximation it results:

$$\frac{1}{2N^2}\iint_{\mathcal{O}^2}\big(\varphi(v') + \varphi(v'_*)\big)\mathbf{f}^T(v,t)\mathbf{M}\mathbf{f}(v_*,t)\,dv\,dv_* \approx$$

$$\approx \frac{N^2}{2M_N}\sum_{\tilde{w}_*^-,\tilde{w}_*^+\in\mathcal{W}_N}\ \sum_{\tilde{w}^-,\tilde{w}^+\in\mathcal{W}_N}\iint_{\mathcal{O}^2}\tilde{w}^+\tilde{w}_*^-\big(\varphi(v') + \varphi(v'_*)\big)\tilde{g}(v,\tilde{w}^-,\tilde{w}^+,t)$$

$$\times\,\tilde{g}(v_*,\tilde{w}_*^-,\tilde{w}_*^+,t)\,dv\,dv_*\,\Delta\tilde{w}^- \ldots \Delta\tilde{w}_*^+$$

- Moreover, it can be shown that

$$\frac{M_N}{N^2} \xrightarrow{N\to\infty} m := \iint_{[0,1]^2}\int_{\mathcal{O}}\tilde{w}^{\pm}\tilde{g}(v,\tilde{w}^-,\tilde{w}^+,t)\,dv\,d\tilde{w}^-\,d\tilde{w}^+$$

General Formal Limit of Growing Graph ($N \to \infty$)

- Within the rank-one approximation of $\mathbf{M}$, the equation for $\tilde{g}$ converges formally to:

$$\frac{d}{dt} \iint_{[0,\,1]^2} \int_{\mathcal{O}} \varphi(v) \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t)\, dv\, d\tilde{w}^-\, d\tilde{w}^+ =$$

$$= \frac{1}{2} \iiiint_{[0,\,1]^4} \iint_{\mathcal{O}^2} \frac{\tilde{w}^+ \tilde{w}_*^-}{m} \big(\varphi(v') + \varphi(v_*') - \varphi(v) - \varphi(v_*)\big)$$

$$\times\, \tilde{g}(v, \tilde{w}^-, \tilde{w}^+, t) \tilde{g}(v_*, \tilde{w}_*^-, \tilde{w}_*^+, t)\, dv\, dv_*\, d\tilde{w}^- \,\ldots\, d\tilde{w}_*^+$$

- Interestingly, this is a **classical Boltzmann-type equation** for the distribution function $\tilde{g}$ on the space state $\mathcal{O} \times [0,\,1]^2$ with

$$\frac{\tilde{w}^+ \tilde{w}_*^-}{m}$$

as **collision kernel**