Macroscopic estimate of Boltzmann equation with mixed boundary

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May 13, 2024

Outline

Boltzmann equation

Modelling of a gas by a distribution function

$$F(t,x,v) \geq 0, \quad (t,x,v) \in [0,\infty) \times \Omega \times \mathbb{R}^3$$

- No interaction between particles, no external forces,

$$\partial_t F + v \cdot \nabla_x F = 0(\text{ free transport equation})$$

- Boltzmann equation consider interaction between particles:

$$\partial_t F + v \cdot \nabla_x F = Q(F,F)$$

  Collision operator

$$Q(F_1,F_2)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\omega)[F_1(v')F_2(u') - F_1(v)F_2(u)]\mathrm{d}\omega \mathrm{d}u$$

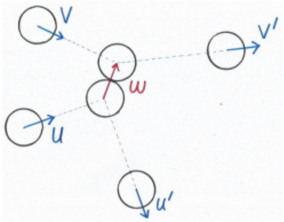
Collision operator

$$Q(F_1,F_2)(v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\omega)[F_1(v')F_2(u')-F_1(v)F_2(u)]\mathrm{d}\omega\mathrm{d}u$$

- Elastic collisions,

$$v'+u'=v+u \text{ (Conservation of Momentum)}$$
$$|v'|^2+|u'|^2=|v|^2+|u|^2 \text{ (Conservation of energy)}$$
$$\omega\in\mathbb{S}^2,\quad v'=v-[(v-u)\cdot\omega]\omega,\quad u'=u+[(v-u)\cdot\omega]\omega.$$

- Collision kernel: hard sphere $B(v-u,\omega)=|(v-u)\cdot\omega|$.



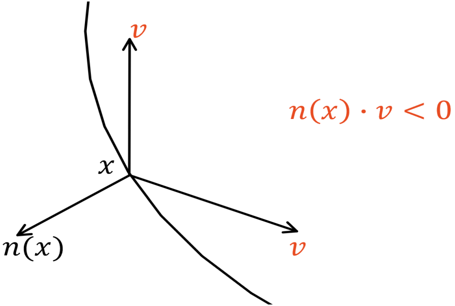
Boundary condition

Interaction with the boundary is described by the boundary condition.

- Incoming phase space

$$\gamma_- = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}$$

$n(x)$ is the outward normal vector at $x \in \partial\Omega$.



Classical boundary condition

- **Specular reflection boundary condition:**

$$F(t,x,v)\Big|_{\gamma_-} = F(t,x,R_x v),\ R_x = v - 2(n(x)\cdot v)n(x).$$

- **Diffuse reflection boundary condition:**

$$F(t,x,v)\Big|_{\gamma_-} = M_w(v)\int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\}\mathrm{d}u$$

$M_w(v)$ is the wall Maxwellian (isothermal) $M_w(v) = \frac{1}{2\pi}e^{-\frac{|v|^2}{2}}$.

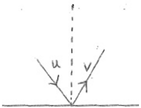


Figure 1: Specular reflection

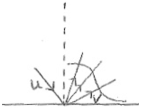


Figure 2: Diffuse reflection

Combination of diffuse and specular bc

- Maxwell boundary condition: accommodation coefficient $\alpha(x)$,

$$F(t,x,v)|_{\gamma_-} = (1-\alpha(x))F(t,x,R_x v) + \alpha(x)M_w(v)\int_{n(x)\cdot u>0} F(t,x,u)(n(x)\cdot u)\mathrm{d}u.$$

- Cercignani-Lampis boundary condition: accommodation coefficient $r_\parallel, r_\perp$,

$$\begin{aligned} F(t,x,v)|_{\gamma_-} &= \int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\} \\ &\times \frac{1}{2\pi r_\perp r_\parallel(2-r_\parallel)} I_0\Big(\frac{(1-r_\perp)^{1/2}v_\perp u_\perp}{r_\perp}\Big) \\ &\times \exp\Big(-\frac{1}{2}\Big[\frac{|v_\perp|^2+(1-r_\perp)|u_\perp|^2}{r_\perp} + \frac{|v_\parallel-(1-r_\parallel)u_\parallel|^2}{r_\parallel(2-r_\parallel)}\Big]\Big)\mathrm{d}u. \end{aligned}$$

$$v_\perp = v\cdot n(x),\ v_\parallel = v - v_\perp n(x).$$

Asymptotic stability

Equilibrium state is described by the global Maxwellian:

$$\mu(v) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|v|^2}{2}\right).$$

- $Q(\mu, \mu) = 0$. μ is the equilibrium state.
- **Natural question:** does $F(t) \to \mu$?

L^2 energy estimate

- Linearization $F = \mu + \sqrt{\mu} f$, linear equation:

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L} f = g, \ \mathcal{L} f = -\frac{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)}{\sqrt{\mu}},$$

$$\text{Null} \mathcal{L} f \in (1, v_i, |v|^2)\sqrt{\mu}.$$

- Macroscopic quantity: $\mathbf{P} f := \left(a + b \cdot v + c \frac{|v|^2 - 3}{2}\right)\sqrt{\mu}$.
- Energy estimate: $\langle \mathcal{L} f, f \rangle \geq C \|(\mathbf{I} - \mathbf{P}) f\|_2^2$. To obtain L^2 convergence, need to recover the macroscopic part: L^2 coercivity:

$$\|\mathbf{P} f\|_2 \lesssim \|(\mathbf{I} - \mathbf{P}) f\|_2 + \text{contribution of boundary}.$$

Convergence to equilibrium:

- (Guo 2010) $L^2 - L^\infty$ framework for both diffuse and specular boundary condition.
  - Assume that $\|F(0) - \mu\|_{L^\infty} \ll 1$, then $F(t, x, v) \to \mu(v)$ exponentially fast.
  - For specular bc, domain is assumed to be convex and analytic.
- (Esposito-Guo-Kim-Marra 2013) $L^2 - L^\infty$ with non-isothermal boundary.
- (Kim-Lee 2017) Specular bc: $L^2 - L^\infty$ with a triple iteration argument to remove the analytic assumption.

L^2 coercivity

- (Guo 2010) Contradiction argument.
- (Esposito-Guo-Kim-Marra 2013) Test function argument, constructive approach.
- (Bernou-Carrapatoso-Mischler-Tristani 2022) Constructive approach for Maxwell bc.

Outline

Hydrodynamic limit in incompressible regime.

$$\varepsilon\partial_t F + v \cdot \nabla_x F = \frac{Q(F,F)}{\varepsilon},\ F = \mu + \varepsilon\sqrt{\mu}f.$$

- (Bardos-Golse-Levermore 1990,1993) (Golse-Saint-Raymond 2004) DiPerna-Lions renormalized solution $f \to \left(\rho + u \cdot v + \theta\frac{|v|^2-3}{2}\right)\sqrt{\mu}$, u satisfies the Leray incompressible Navier-Stoke equation

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \kappa\Delta u,\ x \in \mathbb{R}^3.$$

- (Jiang-Masmoudi 2017) Bounded domain, Navier-Stokes-Fourier limit for DiPerna-Lions-Mischler renormalized solutions with Maxwell boundary condition.

$L^2 - L^6 - L^\infty$ argument in bounded domain

$$\varepsilon\partial_t f + v \cdot \nabla_x f + \frac{1}{\varepsilon}\mathcal{L}f = \Gamma(f,f).$$

- (Esposito-Guo-Kim-Marra 2016) Non-isothermal boundary, f converges to incompressible Navier Stoke Fourier system with no-slip boundary condition in both steady and dynamical setting.
- (Ouyang-Wu 2022) Higher order Hilbert expansion with $L^p - L^\infty$ and boundary layer correction, L^∞ convergence in ε towards INSF system.

L^6 estimate

$$\varepsilon\partial_t f + v\cdot\nabla_x f + \frac{1}{\varepsilon}\mathcal{L}f = \Gamma(f,f).$$

- Direct energy estimate: $\frac{1}{\varepsilon}\|(\mathbf{I}-\mathbf{P})f\|_2 \lesssim \|\Gamma(f,f)\|_2$. Do not have a good scale for the L^∞ estimate $\|f\|_\infty$.
- If $\|\mathbf{P}f\|_6 \lesssim 1$, then $\|\Gamma(\mathbf{P}f,\mathbf{P}f)\|_2 \lesssim \|\mathbf{P}f\|_3\|\mathbf{P}f\|_6 \lesssim 1$.
- L^6 estimate also leads to the L^∞ control as

$$\|f\|_\infty \lesssim \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}f\|_6 + \frac{1}{\sqrt{\varepsilon}}[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_2] \lesssim \frac{1}{\sqrt{\varepsilon}},\ \sqrt{\varepsilon}\|f\|_\infty \lesssim 1.$$

Outline

Mixed boundary condition

- Consider a mixed boundary condition, where a portion of boundary satisfies diffuse bc, and the portion satisfies the specular bc.
- This can be regarded as a Maxwell bc

$$F(t,x,v)|_{\gamma_-} = (1-\alpha(x))F(t,x,R_x v) + \alpha(x)\mu \int_{n(x)\cdot u>0} F(t,x,u)(n(x)\cdot u)\mathrm{d}u,$$

accommodation coefficient $\alpha(x) \in \{0,1\}$ is a discontinous piece-wise function.

- Denote

$$\partial\Omega_1 = \{x \in \partial\Omega : \alpha(x) = 0\}(\text{ Specular portion }),$$
$$\partial\Omega_2 = \{x \in \partial\Omega : \alpha(x) = 1\}(\text{ Diffuse portion }).$$

Convergence to equilibrium with mixed bc

Consider the linear equation

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L} f = g.$$

Theorem (C.-Duan 2024)

Let Ω be an arbitrary smooth and bounded domain, $\alpha(x)$ be a discontinuous piece-wise function. If the initial condition f_0 and source term satisfy

$$\|f_0\|_{L^2}^2 + \int_0^t \|e^{\lambda s} g(s)\|_2^2 \mathrm{d}s < \infty,$$

then there exists a unique solution f such that

$$\|f(t)\|_2^2 \lesssim e^{-\lambda t}\Big\{\|f_0\|_2^2 + \int_0^t \|e^{\lambda s} g(s)\|_2^2 \mathrm{d}s\Big\}.$$

- The L^2 estimate does not have convexity assumption.
- We also have L^∞ estimate to the nonlinear equation within a special geometry. The specular portion lies in two parallel plates.

Macroscopic control

Linear equation:

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L} f = g.$$

The diffuse bc at $\partial\Omega_2$ reads

$$f|_{\gamma_-} = P_\gamma f := \sqrt{\mu} \int_{n(x)\cdot u>0} f \sqrt{\mu}(n(x) \cdot u) \mathrm{d}u.$$

Lemma (C.-Duan 2024)

$$\int_s^t \|\mathbf{P} f(\tau)\|_2^2 \mathrm{d}\tau \lesssim \int_s^t \|(\mathbf{I} - \mathbf{P}) f\|_2^2 \mathrm{d}\tau + \int_s^t \|g(\tau)\|_2^2 \mathrm{d}\tau + \int_0^t \int_{\partial\Omega_2 \times \mathbb{R}^3} |(1 - P_\gamma) f|^2 |n(x) \cdot v| \mathrm{d}S_x \mathrm{d}v.$$

L^6 estimate with specular boundary condition

Linearization of the scaled Boltzman equation:

$$\varepsilon\partial_t f + v\cdot\nabla_x f + \frac{1}{\varepsilon}\mathcal{L}f = g.$$

Theorem (C.-Kim 2024)

Assume f solves the scaled linear Boltzmann equation with pure specular boundary condition. Also assume the total mass, energy, angular momentum(if domain is axis-symmetric) are 0*, then*

$$\|\mathbf{P}f\|_{L^6_{x,v}} \lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^6_{x,v}} + \frac{\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}}{\varepsilon} + \|\nu^{-1/2}(g-\varepsilon\partial_t f)\|_{L^2_{x,v}}.$$

- The specular bc preserves both mass and energy, but not momentum.
- When the domain is axis-symmetric, such as sphere, specular bc also preserves angular momentum.

Outline

Momentum b estimate

- Finite dimensional set of rigid

$$\mathcal{R}(\Omega) := x \in \Omega \to Ax \in \mathbb{R}^3,$$
$$A \text{ is skew-symmetric } : A + A^T = 0.$$

- Infinitesimal rigid displacement fields preserving Ω

$$\mathcal{R}_\Omega = \{G \in \mathcal{R}(\Omega) : G(x) \cdot n(x) = 0 \text{ for any } x \in \partial\Omega\}.$$

Korn's inequality

For a smooth vector field $u \in \mathbb{R}^3$, denote $\nabla^{\text{sym}} u$ and $\nabla^{\text{a}} u$ as the symmetric and anti-symmetric part of ∇u:

$$(\nabla u)_{ij} := \frac{\partial u_i}{\partial x_j},\ (\nabla^{\text{sym}} u)_{ij} := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),\ (\nabla^{\text{a}} u)_{ij} := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\Big).$$

Theorem (Desvillettes-Villani 2002, Bernou-Carrapatoso-Mischler-Tristani 2022)

Assume Ω is a C^1 bounded domain, u is tangent to the boundary

$$u(x) \cdot n(x) = 0,\ x \in \partial\Omega.$$

Then

$$\|u\|_{H^1_x}^2 \lesssim \|\nabla^{sym} u\|_{L^2_x}^2 + \Big|P_\Omega\Big(\int_\Omega \nabla^a u \mathrm{d}x\Big)\Big|^2.$$

P_Ω denotes the orthogonal projection onto the set $\mathcal{A}_\Omega = \{A + A^T = 0; Ax \in \mathcal{R}_\Omega\}$.
Assume $\alpha \not\equiv 0$, then

$$\|u\|_{H^1_x}^2 \lesssim \|\nabla^{sym} u\|_{L^2_x}^2 + \Big\|\sqrt{\frac{\alpha}{(2-\alpha)}}u\Big\|_{L^2(\partial\Omega)}^2.$$

Symmetric Poisson system for mixed boundary

$$\mathcal{X} := \{u \in H^1(\Omega),\ u(x)\cdot n(x) = 0 \text{ on } \partial\Omega\}.$$

Korn's inequality leads to the construction of the following elliptic system

Lemma (Bernou-Carrapatoso-Mischler-Tristani 2022)

Let $h \in L^2_x(\Omega)$, there exists a unique solution $u \in \mathcal{X}$ to the variational formulation

$$\int_\Omega \nabla^{sym} u : \nabla^{sym} v \mathrm{d}x + \int_{\partial\Omega} \frac{\alpha(x)}{2-\alpha(x)} u \cdot v \mathrm{d}S_x = \int_\Omega v \cdot h \mathrm{d}x \ \ \textit{for all } v \in \mathcal{X},$$

Suppose $\alpha(x)$ is Lipshitz on $\partial\Omega$, then $u \in H^2_x(\Omega)$ and satisfies

$$\begin{cases} div(\nabla^{sym} u) = h \textit{ in } \Omega \\ u \cdot n = 0 \textit{ on } \partial\Omega \\ (2-\alpha(x))[\nabla^{sym} u n - (\nabla^{sym} u : n \otimes n) n] + \alpha(x) u = 0 \textit{ on } \partial\Omega, \end{cases}$$

with

$$\|u\|_{H^2_x} \lesssim \|h\|_{L^2_x}.$$

Difficulty in momentum b estimate

$\alpha(x)$ is assumed to be Lipshitz continuous to achieve H_x^2 regularity.

- For discontinuous accommodation coefficient $\alpha(x) \not\equiv 0$ and $\alpha \in \{0, 1\}$, choose smooth $\beta(x) \not\equiv 0$ and $\beta(x) \leq \alpha(x)$
- Consider the following system

$$\begin{cases} -\operatorname{div}(\nabla^{\text{sym}}\phi) = b \text{ in } \Omega \\ \phi \cdot n = 0 \text{ on } \partial\Omega \\ (2 - \beta(x))\left[\nabla^{\text{sym}}\phi n - (\nabla^{\text{sym}}\phi : n \otimes n)n\right] + \beta(x)\phi = 0 \text{ on } \partial\Omega. \end{cases}$$

- Elliptic estimate

$$\|\phi\|_{H^2} \lesssim \|b\|_{L^2}.$$

Weak formulation for linear Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + \mathcal{L} f = g.$$

Weak formulation with test function ψ :

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^3} \{\psi f(t) - \psi f(s)\} \mathrm{d}x\mathrm{d}v - \int_s^t \int_{\Omega\times\mathbb{R}^3} f \partial_\tau \psi \mathrm{d}x\mathrm{d}v\mathrm{d}\tau \\ &= \underbrace{\int_s^t \int_{\Omega\times\mathbb{R}^3} v \cdot \nabla_x \psi f \mathrm{d}x\mathrm{d}v\mathrm{d}\tau}_{\text{transport}} - \underbrace{\int_s^t \int_\gamma \psi f \mathrm{d}\gamma\mathrm{d}\tau}_{\text{bdr}} \\ &- \int_s^t \int_{\Omega\times\mathbb{R}^3} (\mathcal{L} f) \psi \mathrm{d}v\mathrm{d}x\mathrm{d}\tau + \int_s^t \int_{\Omega\times\mathbb{R}^3} g \psi \mathrm{d}x\mathrm{d}v\mathrm{d}\tau. \end{aligned}$$

Transport contribution

Choose test function

$$\psi := (v\cdot\nabla\phi\cdot v)\mu^{1/2} - (\nabla\cdot\phi)\mu^{1/2} = \sum_{i,j=1}^{3}\partial_j\phi_i v_i v_j\mu^{1/2} - \sum_{i=1}^{3}\partial_i\phi_i\mu^{1/2}.$$

Transport operator on ψ :

$$\begin{aligned}
-v\cdot\nabla_x\psi &= -\sum_{i,j,k=1}^{3}\partial_{kj}\phi_i v_i v_j v_k\mu^{1/2} + \sum_{i,k=1}^{3} v_k\partial_{ki}\phi_i\mu^{1/2}\\
&= -\sum_{i,j,k=1}^{3}\partial_{kj}\phi_i(\mathbf{I}-\mathbf{P})(v_i v_j v_k\mu^{1/2}) - \sum_{i,j,k=1}^{3}\partial_{kj}\phi_i\mathbf{P}(v_i v_j v_k\mu^{1/2}) + \sum_{i,k=1}^{3} v_k\partial_{ki}\phi_i\mu^{1/2}.
\end{aligned}$$

Transport contribution

Macroscopic part, $\chi_i := v_i\sqrt{\mu}$.

$$-\sum_{i,j,k=1}^{3}\partial_{kj}\phi_i\mathbf{P}(v_iv_jv_k\mu^{1/2}) = -\sum_{i,j,k,l}\partial_{kj}\phi_i v_iv_jv_k\chi_l\mu^{1/2}$$

$$= -3\sum_{i=1}^{3}\partial_{ii}\phi_i\chi_i - \sum_{j\neq i}\partial_{jj}\phi_i\chi_i - \sum_{i\neq k}\partial_{ki}\phi_i\chi_k - \sum_{i\neq j}\partial_{ij}\phi_i\chi_j$$

$$\sum_{i,k=1}^{3}v_k\partial_{ki}\phi_i\mu^{1/2} = \sum_{i=1}^{3}\chi_i\partial_{ii}\phi_i + \sum_{i\neq k}v_k\partial_{ki}\phi_i\mu^{1/2}.$$

Sum and obtain

$$-\sum_{i=1}^{3}\partial_{ii}\phi_i\chi_i - \sum_{j\neq i}\partial_{jj}\phi_i\chi_i - \sum_{i=1}^{3}\partial_{ii}\phi_i\chi_i - \sum_{i\neq j}\partial_{ij}\phi_i\chi_j$$

$$= -\sum_{i=1}^{3}\chi_i\sum_{j=1}^{3}\partial_{jj}\phi_i - \sum_{j=1}^{3}\chi_j\sum_{i=1}^{3}\partial_{ij}\phi_i = -\sum_{i=1}^{3}\chi_i[\Delta\phi_i + \partial_i\mathrm{div}\,(\phi)] = \sum_{i=1}^{3}\chi_i b_i.$$

Transport contribution

$$
\begin{aligned}
&\int_s^t \int_{\Omega \times \mathbb{R}^3} v \cdot \nabla_x \psi f \, \mathrm{d}x \mathrm{d}v \mathrm{d}\tau \\
&= \int_s^t \int_{\Omega \times \mathbb{R}^3} \sum_{i=1}^{3} \chi_i b_i f - \sum_{i,j,k=1}^{3} \partial_{kj} \phi_i (\mathbf{I} - \mathbf{P}) (v_i v_j v_k \mu^{1/2}) f \, \mathrm{d}x \mathrm{d}v \mathrm{d}\tau \\
&= \|b\|_2^2 + (\mathbf{I} - \mathbf{P}) f \text{ contribution} .
\end{aligned}
$$

Boundary contribution

$$\int_{\partial\Omega\times\mathbb{R}^3}(n\cdot v)\Big[\underbrace{\sum_{i,j=1}^{3}\partial_j\phi_i v_i v_j\mu^{1/2}}_{bdr_1}-\underbrace{\sum_{i=1}^{3}\partial_i\phi_i\mu^{1/2}}_{bdr_2}\Big]f\mathrm{d}v\mathrm{d}S_x.$$

bdr_2 :

- Specular portion $\partial\Omega_1$: vanishes from change of variable $v\to v-2n(x)(n(x)\cdot v)$.
- Diffuse portion $\partial\Omega_2$:

$$\int_{\partial\Omega_2}\Big[\int_{n(x)\cdot v>0}+\int_{n(x)\cdot v<0}\Big]\mathrm{div}\,\phi\mu^{1/2}[(1-P_\gamma)f+P_\gamma f](n\cdot v)\mathrm{d}v\mathrm{d}S_x$$
$$\lesssim|\mathbf{1}_{x\in\partial\Omega_2}(1-P_\gamma)f|_{2,+}^2.$$

Boundary contribution

$$\int_{\partial\Omega\times\mathbb{R}^3}(n\cdot v)\Big[\sum_{i,j=1}^{3}\partial_j\phi_i v_i v_j \mu^{1/2}\Big] f \mathrm{d}v\mathrm{d}S_x = \int_{\partial\Omega\times\mathbb{R}^3}(n\cdot v)\big[\nabla^{\mathrm{sym}}\phi : (v\otimes v)\big]\mu^{1/2} f \mathrm{d}v\mathrm{d}S_x.$$

- Specular portion: from $\beta(x)\le\alpha(x)=0$ on $\partial\Omega_1$, change of variable: $v\to v-2n(x)(n(x)\cdot v)$

$$\begin{aligned}&4\int_{\partial\Omega_1}\int_{n(x)\cdot v>0}|n\cdot v|^2\mu^{1/2}f(x,v)\big[\nabla^{\mathrm{sym}}\phi : (n\otimes v) - \nabla^{\mathrm{sym}}\phi : (n\otimes n)(n\cdot v)\big]\\ &= -4\int_{\partial\Omega_1}\int_{n(x)\cdot v>0}|n\cdot v|^2\mu^{1/2}f(x,v)\frac{\beta(x)}{2-\beta(x)}(\phi\cdot v)=0.\end{aligned}$$

$$\begin{cases} -\mathrm{div}(\nabla^{\mathrm{sym}}\phi) = b \text{ in } \Omega \\ \phi\cdot n = 0 \text{ on } \partial\Omega \\ (2-\beta(x))\big[\nabla^{\mathrm{sym}}\phi n - (\nabla^{\mathrm{sym}}\phi : n\otimes n)n\big] + \beta(x)\phi = 0 \text{ on } \partial\Omega. \end{cases}$$

Boundary contribution

$$\int_{\partial\Omega\times\mathbb{R}^3}(n\cdot v)\Big[\sum_{i,j=1}^{3}\partial_j\phi_i v_i v_j \mu^{1/2}\Big] f \mathrm{d}v\mathrm{d}S_x = \int_{\partial\Omega\times\mathbb{R}^3}(n\cdot v)\big[\nabla^{\mathrm{sym}}\phi : (v\otimes v)\big]\mu^{1/2} f \mathrm{d}v\mathrm{d}S_x.$$

- Diffuse portion $\partial\Omega_2$: $v \to v - 2n(x)(n(x)\cdot v)$: we obtain

$$\int_{\partial\Omega_2}\int_{n(x)\cdot v>0}(n\cdot v)\mu^{1/2}\big[\nabla^{\mathrm{sym}}\phi : (v\otimes v)\big](1-P_\gamma)f$$

and

$$\begin{aligned} & 4\int_{\partial\Omega_2}\int_{n(x)\cdot v>0}|n\cdot v|^2\mu^{1/2}P_\gamma f\big[\nabla^{\mathrm{sym}}\phi : (n\otimes v) - \nabla^{\mathrm{sym}}\phi : (n\otimes n)(n\cdot v)\big] \\ &= -4\int_{\partial\Omega_2}\int_{n(x)\cdot v>0}|n\cdot v|^2\mu^{1/2}P_\gamma f\frac{\beta(x)}{2-\beta(x)}\phi\cdot v \\ &= -4\int_{\partial\Omega_2}\int_{n(x)\cdot v>0}|n\cdot v|^2\mu^{1/2}P_\gamma f\frac{\beta(x)}{2-\beta(x)}\phi\cdot\big(\underbrace{v-(n\cdot v)n}_{\text{odd}} + \underbrace{(n\cdot v)n}_{\phi\cdot n}\big) = 0. \end{aligned}$$

Korn's inequality for pure specular bc

Theorem (Desvillettes-Villani 2002, Bernou-Carrapatoso-Mischler-Tristani 2022)

Assume Ω is a C^1 bounded domain, u is tangent to the boundary

$$u(x) \cdot n(x) = 0, \ x \in \partial\Omega.$$

Then

$$\|u\|_{H_x^1}^2 \lesssim \|\nabla^{sym} u\|_{L_x^2}^2 + \Big| P_\Omega \Big(\int_\Omega \nabla^a u \mathrm{d}x \Big) \Big|^2$$

P_Ω denotes the orthogonal projection onto the set $\mathcal{A}_\Omega = \{A \in A_3(\mathbb{R}); Ax \in \mathcal{R}_\Omega\}$.

$$\mathcal{R}_\Omega = \{R \in \mathcal{R}(\Omega) : R(x) \cdot n(x) = 0 \textit{ for any } x \in \partial\Omega\}.$$

$$\mathcal{X} := \{u : \Omega \to \mathbb{R}^3 : u \in H_x^1, \ u \cdot n = 0 \textit{ on } \partial\Omega, \ P_\Omega \Big(\int_\Omega \nabla^a u \mathrm{d}x \Big) = 0\}.$$

Symmetric Poisson system for specular bc

Lemma (Bernou-Carrapatoso-Mischler-Tristani 2022)

There exists a unique solution $u \in \mathcal{X}$ to

$$\int_\Omega \nabla^{sym} u : \nabla^{sym} v \mathrm{d}x = \int_\Omega h \cdot v \mathrm{d}x \ \textit{for all} \ \ v \in \mathcal{X}.$$

Furthermore, suppose h satisfies the compatibility condition:

$$\int_\Omega Ax \cdot h(x) \mathrm{d}x = 0 \ \textit{for any} \ \ Ax \in \mathcal{R}_\Omega.$$

Then the variational solution satisfies

$$\begin{aligned} -div\left(\nabla^{sym} u\right) &= h \ \ in \ \ \Omega, \\ u \cdot n &= 0 \ \ on \ \ \partial\Omega, \\ \nabla^{sym} u \, n &= (\nabla^{sym} u : n \otimes n) n \ \ on \ \ \partial\Omega. \end{aligned}$$

and $u \in H^2_x$ with

$$\|u\|_{H^2_x} \lesssim \|h\|_{L^2_x}.$$

Remark

$$\begin{aligned} -\mathrm{div}\left(\nabla^{\mathrm{sym}} u\right) &= h \ \text{ in } \ \Omega, \\ u \cdot n &= 0 \ \text{ on } \ \partial\Omega, \\ \nabla^{\mathrm{sym}} u\, n &= (\nabla^{\mathrm{sym}} u : n \otimes n) n \ \text{ on } \ \partial\Omega. \end{aligned}$$

- If $\dim \mathcal{R}_\Omega = 0$, no need the compatibility condition.
- If $\dim \mathcal{R}_\Omega \in \{1, 2\}$, need to create the compatibility condition, and set the angular momentum to be 0:

$$\int_\Omega R(x) \cdot b(x) \mathrm{d}x = 0, \ \text{for } R(x) \in \mathcal{R}_\Omega.$$

$W^{2,\frac{6}{5}}(\Omega)$ estimate

Lemma (C.-Kim 2024)

Let the source term $h \in L^{\frac{6}{5}}(\Omega)$ satisfy compatibility condition, the solution to the symmetric Poisson system satisfies

$$\|u\|_{W_x^{2,\frac{6}{5}}} \lesssim \|h\|_{L_x^{6/5}}.$$

- Complementing boundary condition(Agmon-Douglis-Nirenberg) as a-priori estimate.

$$\|u\|_{W_x^{2,\frac{6}{5}}} \lesssim \|h\|_{L_x^{6/5}} + \|u\|_{L_x^{6/5}}.$$

- Contradiction argument to show

$$\|u\|_{L_x^{6/5}} \lesssim \|h\|_{L_x^{6/5}}.$$

Test function: axis-symmetric domain

- Compatibility condition: let $R_1(x)$ and $R_2(x)$ be the orthonormal basis of $\mathcal{R}_\Omega$. Define

$$h(x) = b^5(x) - \frac{\int R_1 \cdot b^5 \mathrm{d}x}{\int |R_1|^2 \mathrm{d}x} R_1(x) - \frac{\int R_2 \cdot b^5 \mathrm{d}x}{\int |R_2|^2 \mathrm{d}x} R_2(x), \ b^5 = (b_1^5, b_2^5, b_3^5),$$

then for $Ax = C_1 R_1(x) + C_2 R_2(x)$,

$$\int_\Omega h(x) \cdot Ax \mathrm{d}x = \int_\Omega C_1 h(x) \cdot R_1(x) + C_2 h(x) \cdot R_2(x) = 0.$$

- Symmetric Poisson system:

$$\begin{cases} -\mathrm{div}\,(\nabla^{\mathrm{sym}} \phi) = h(x) \text{ in } \Omega \\ \phi \cdot n = 0 \text{ on } \partial\Omega \\ \nabla^{\mathrm{sym}} \phi n = (\nabla^{\mathrm{sym}} \phi : n \otimes n) n \text{ on } \partial\Omega. \end{cases}$$

- Elliptic $W^{2,\frac{6}{5}}(\Omega)$ estimate:

$$\|\phi\|_{W_x^{2,\frac{6}{5}}} \lesssim \|b^5\|_{L_x^{\frac{6}{5}}} \lesssim \|b\|_{L_x^6}^5.$$

Transport contribution in weak formulation

$$\psi := \sum_{i,j=1}^{3} \partial_j \phi_i v_i v_j \mu^{1/2} - \sum_{i=1}^{3} \partial_i \phi_i \mu^{1/2}.$$

$$-v \cdot \nabla_x \psi = -\sum_{i,j,k=1}^{3} \partial_{kj} \phi_i (\mathbf{I} - \mathbf{P})(v_i v_j v_k \mu^{1/2}) - \sum_{i=1}^{3} \chi_i [\Delta \phi_i + \partial_i \mathrm{div}\,(\phi)].$$

- Zero angular momentum leads to

$$\int_{\Omega \times \mathbb{R}^3} b \cdot \left[b^5(x) - \frac{\int R_1 \cdot b^5 \mathrm{d}x}{\int |R_1|^2 \mathrm{d}x} R_1(x) - \frac{\int R_2 \cdot b^5 \mathrm{d}x}{\int |R_2|^2 \mathrm{d}x} R_2(x) \right] = \|b\|_6^6.$$

Thank You

Merci

감사합니다