Dynamical Billiard and a long-time behavior of the Boltzmann equation in general 3D toroidal domains

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Introduction to the Boltzmann equation

Here, $F = F(t, x, v)$ stands for **the density distribution function of particles** with position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$ at time $t > 0$.

- **Boltzmann equation**

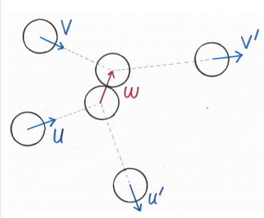
$$\partial_t F + v \cdot \nabla_x F = Q(F, F),$$

describes collisions among particle interactions.

- **Collision operator**

$$\begin{aligned} Q(F_1, F_2) &:= \iint_{u \in \mathbb{R}^3, \omega \in \mathbb{S}^2} B(v-u, \omega) \Big[F_1(u')F_2(v') - F_1(u)F_2(v) \Big] \, d\omega du \\ &:= Q_+(F_1, F_2) - Q_-(F_1, F_2), \end{aligned}$$

where $B(v-u, \omega)$ is a collision kernel for hard potential.



We assume these collisions to be elastic : Momentum and energy conservation

$$\begin{cases} v + u = v' + u' \\ |v|^2 + |u|^2 = |v'|^2 + |u'|^2, \end{cases}$$

where $u' = u + [(v-u)\cdot\omega]\omega, \quad v' = v - [(v-u)\cdot\omega]\omega.$

- **Collision kernel** $B(v-u,\omega)$

$$B(v-u,\omega) = |v-u|^\gamma b(\cos\theta), \quad 0 \le \gamma \le 1 \text{ (hard potential)},$$
$$0 \le b(\cos\theta) \le C|\cos\theta| \text{ (angular cut-off)},$$

where $\cos\theta = \langle \frac{v-u}{|v-u|}, \omega \rangle$.

Boundary conditions

We denote the phase boundary in the space $\Omega \times \mathbb{R}^3$ as $\gamma = \partial\Omega \times \mathbb{R}^3$, and split it into an outgoing boundary γ_+, an incoming boundary γ_-:

$$\gamma_+ := \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x)\cdot v > 0\},$$
$$\gamma_- := \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x)\cdot v < 0\}$$

where $n(x)$ is the outward normal vector at $x \in \partial\Omega$.

1. The in-flow boundary condition: for $(x,v) \in \gamma_-$,
$$F(t,x,v)|_{\gamma_-} = g(t,x,v)$$
2. The bounce-back boundary condition: for $x \in \partial\Omega$,
$$F(t,x,v)|_{\gamma_-} = F(t,x,-v)$$
3. **Specular reflection**: for $x \in \partial\Omega$,
$$F(t,x,v)|_{\gamma_-} = F(t,x,v-2(n(x)\cdot v)n(x)) = F(t,x,R(x)v)$$
4. Diffuse reflection: for $(x,v) \in \gamma_-$,
$$F(t,x,v)|_{\gamma_-} = c_\mu \mu(v) \int_{u\cdot n(x)>0} F(t,x,u)\{n(x)\cdot u\}du$$

Main Goal

- Does a solution have the **global well-posedness**?

  There exists a unique solution $F(t,x,v)$ which satisfies the system for any time $t > 0$ when an initial distribution function F_0 is given.

- Does a solution reach the **physical equilibrium**?

  In Stat. Physics, the Maxwellian $\mu(v) = \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{|v|^2}{2}}$ is regarded as an equilibrium state. We expected our solution reaches that.

$$\boxed{F(t,x,v) \to \mu(v) \text{ as } t \to \infty \text{ ?}}$$

Problem and main results

The Boltzmann equation near Maxwellian

Let $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v) \geq 0$. Then, the Boltzmann equation can be rewritten as

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f,f),$$

where L is a linear operator

$$Lf = \nu(v) f - Kf = -\frac{1}{\sqrt{\mu}} \left[Q(\sqrt{\mu} f, \mu) + Q(\mu, \sqrt{\mu} f) \right],$$

$$\nu(v) = \frac{1}{\sqrt{\mu}} Q_-(\mu, \sqrt{\mu}) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v-u,\omega)\mu(u)\, d\omega du \sim (1+|v|)^\gamma,$$

$$Kf = \frac{1}{\sqrt{\mu}} [Q(\sqrt{\mu} f, \mu) + Q_+(\mu, \sqrt{\mu} f)] = \int_{\mathbb{R}^3} \mathbf{k}(v,u) f(u) du,$$

$$\mathbf{k}(v,u) \lesssim \left(\frac{1}{|v-u|} + |v-u| \right) e^{-\frac{1}{8}|v-u|^2 - \frac{1}{8}\frac{||v|^2-|u|^2|^2}{|v-u|^2}},$$

and Γ is a nonlinear operator

$$\Gamma(f,f) := \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} f) = \frac{1}{\sqrt{\mu}} Q_+(\sqrt{\mu} f, \sqrt{\mu} f) - \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \sqrt{\mu} f)$$
$$:= \Gamma_+(f,f) - \Gamma_-(f,f).$$

L^2 decay

For simplicity, let us consider a linearized Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + Lf = 0$$

We define the projection operator $\mathbf{P}$ onto the null space $N(L)$ of L

$$\mathbf{P}f(t,x,v) := \left\{ a_f(t,x) + b_f(t,x) \cdot v + c_f(t,x) \frac{|v|^2 - 3}{\sqrt{6}} \right\} \sqrt{\mu(v)}$$

- Semi-positivity of L: $(Lf, f)_{L^2} \gtrsim \|\sqrt{\nu}(\mathbf{I} - \mathbf{P})f\|_{L^2}^2$
- L^2 coercivity estimate: $\|\mathbf{P}f\|_{L^2_\nu} \lesssim \|(\mathbf{I} - \mathbf{P})f\|_{L^2_\nu} + (\text{Contributions from B.C.})$

In L^2 energy estimate for f, L^2 coercivity estimate yields L^2 exponential decay for linearized Boltzmann equation:

$$\|f(t)\|_{L^2} \lesssim e^{-\lambda t} \|f_0\|_{L^2}$$

Characteristics and Duhamel's principle

We define the characteristics associated with the specular boundary condition for given $t \geq s \geq 0$

$$X(s;t,x,v) := \text{Position of the particle at time } s, \text{which was at } (t,x,v),$$
$$V(s;t,x,v) := \text{Velocity of the particle at time } s, \text{which was at } (t,x,v),$$

which is determined by $\frac{dX(s)}{ds} = V(s)$, $\frac{dV(s)}{ds} = 0$, and $[X(t), V(t)] = [x, v]$.
Remind the linearized Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + \nu(v) f = Kf.$$

Along the characteristics

$$\frac{d}{ds}\left(e^{\nu(v)s} f(s, X(s;t,x,v), V(s;t,x,v))\right) = e^{\nu(v)s} Kf(s).$$

Taking the time-integration from 0 to t yields

$$f(t,x,v) = e^{-\nu(v)t} f_0(X(0;t,x,v), V(0;t,x,v)) + \int_0^t e^{-\nu(v)(t-s)} Kf(s) ds.$$

$L^2 - L^\infty$ bootstrap argument

We apply the Duhamel's principle for linearized Boltzmann equation

$$f(t,x,v) \lesssim e^{-t} f_0(X(0;t,x,v), V(0;t,x,v)) + \int_0^t e^{-(t-s)} \int_u \mathbf{k}(V(s;t,x,v),u) f(s, X(s;t,x,v),u)\,du\,ds$$

If we apply the Duhamel's principle for integrand $f(s, X(s;t,x,v),u)$ again, we obtain estimate

$$f(t,x,v) \lesssim (initial\ data) + O(\frac{1}{N}) + \int_0^t e^{-(t-s)} \int_0^s e^{-(s-s')} \iint_{|u|\le N, |u'|\le N} f(s', X(s'; s, X(s;t,x,v),u), u')\,du'\,du\,ds'\,ds$$

The key idea is to control the integration part by $L^2_{x,v}$ norm of f via a change of variables $u \mapsto X(s'; s, X(s;t,x,v),u)$. Hence, the major challenge in $L^2 - L^\infty$ bootstrap argument is whether the following **non-degeneracy condition** holds

$$\det\left(\frac{\partial X(s'; s, X(s;t,x,v),u)}{\partial u}\right) \ge \delta > 0.$$

History

Small-amplitude problem

- Ukai (1974) : Global solution near Maxwellian in periodic box $\mathbb{T}^3$, $f(t) \in L^\infty_\beta(H^s_x),\ s > 3/2$.
- Y.Guo (2010) : Exponential decay to Maxwellian for boundary problems in analytic and uniformly convex domain.
- C.Kim and D.Lee (2018) : In general C^3 uniformly convex domain, exponential decay to Maxwellian in specular B.C.

Large-amplitude problem

- L.Devillettes and C.Villani (2005) : Almost exponential decay $(t^{-\infty})$ with large data under a priori assumption.
- R. Duan, F. Huang, Y. Wang, and T. Yang (2017) : Global well-posedness in a whole space $\mathbb{R}^3$ and periodic box $\mathbb{T}^3$.
- R. Duan, K, and D. Lee (2023) : Convergence to equilibrium in general C^3 uniformly convex domain under specular B.C.

To treat IBVP in a domain with **physical boundaries**, it seems impossible to construct global solutions of small amplitude within high-order Sobolev spaces due to **the presence of singularity**.

History

However, in general non-convex domains, establishing a global well-posedness of the Boltzmann equation with specular reflection boundary condition is still open problem!

Non-convex domains

- C.Kim and D.Lee (2018): They deal with the Boltzmann equation in a periodic cylindrical domain with non-convex analytic cross-section.

A general 3D toroidal domain Ω has $3D$ nontrivial non-convex structure. Actually, there are uncountably many grazing points in Ω, which makes it more difficult than previous research.

Main goal : For each $y \in \Omega$, construct "a bad set" $\mathcal{B}_y \subset \{|u| \leq N\}$ such that $\mu(\mathcal{B}_y) \ll 1$ and

$$\det\left(\frac{\partial X(s'; s, y, u)}{\partial u}\right) \gtrsim 1 \quad \text{except a small subset } \mathcal{B}_y \text{ of } u.$$

General 3D toroidal domains

Let $\bar{\xi} : xz$-plane $\to \mathbb{R}$ be a real-analytic and uniformly convex function and satisfies

$$\bar{\xi} = 0 \quad \text{on} \quad \partial\Omega \cap xz\text{-plane}$$

We set an indication function in $\mathbb{R}^3$, which is real-analytic and uniformly convex, by

$$\xi(x, y, z) := \bar{\xi}(\sqrt{x^2 + y^2}, z).$$

Now, we have

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 : \xi(x, y, z) := \bar{\xi}(\sqrt{x^2 + y^2}, z) < 0\},$$
$$\partial\Omega := \{(x, y, z) \in \mathbb{R}^3 : \xi(x, y, z) := \bar{\xi}(\sqrt{x^2 + y^2}, z) = 0\}.$$

We define the domain Ω as general 3D toroidal domains.

Example (Solid torus case) For fixed positive constants r, R with $r < R$,

$$\bar{\xi}(x, z) = (x - R)^2 + z^2 - r^2, \quad \xi(x, y, z) = (\sqrt{x^2 + y^2} - R)^2 + z^2 - r^2.$$

Main result (C.Kim, D.Lee and K, preprint)

Theorem

Assume that Ω *is a general 3D toroidal domain. Define a weighted function*

$$w(v) = (1 + |v|)^{\beta}$$

with $\beta > 5/2$*. We assume* $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$ *satisfies*

$$\iint_{\Omega\times\mathbb{R}^3} (F_0(x,v) - \mu)dxdv = 0, \quad \iint_{\Omega\times\mathbb{R}^3} |v|^2 (F_0(x,v) - \mu)dxdv = 0,$$

$$\iint_{\Omega\times\mathbb{R}^3} \{x \times \hat{z}\} \cdot vF_0(x,v)dxdv = 0.$$

Then, there exists $0 < \delta \ll 1$ *such that if*

$$\|wf_0\|_{L^\infty(\Omega\times\mathbb{R}^3)} \leq \delta,$$

then the Boltzmann equation with specular boundary condition has a unique global-in-time solution $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$*. Moreover, there exists* $\lambda > 0$ *such that*

$$\sup_{t\geq 0} e^{\lambda t} \|wf(t)\|_{L^\infty_{x,v}} \lesssim \|wf_0\|_{L^\infty_{x,v}}.$$

Sketch of proof

Grazing sets γ_0

Note that the non-degeneracy condition is closely related with uniform non-grazing

$$\frac{1}{|v \cdot \mathbf{n}(x)|} \geq \delta > 0.$$

Let us introduce the grazing set γ_0

$$\gamma_0 := \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : \quad n(x) \cdot v = 0\}.$$

Before we decompose the grazing set, we define a backward exit time and position as

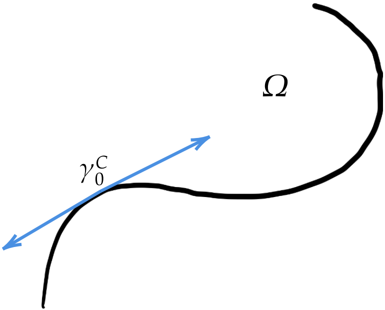
$$t_{\mathbf{b}}(x,v) := \sup\{\tau \geq 0 : x - sv \in \Omega \text{ for all } 0 \leq s \leq \tau\},\ x_{\mathbf{b}}(x,v) := x - t_{\mathbf{b}} v \in \partial\Omega.$$

Examples of grazing points

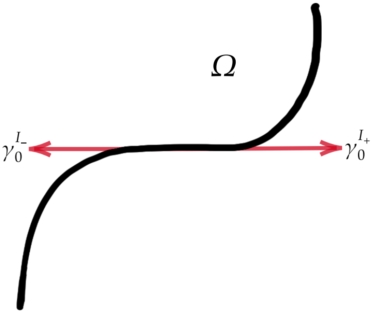
- Concave grazing $\gamma_0^C := \{(x,v) \in \gamma_0 :\ t_{\mathbf{b}}(x,v) \neq 0 \text{ and } t_{\mathbf{b}}(x,-v) \neq 0\}$

- Inflection grazing $\gamma_0^I := \gamma_0^{I+} \cup \gamma_0^{I-}$ where

$$\gamma_0^{I+} := \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) \neq 0, t_{\mathbf{b}}(x,-v) = 0, \text{ and } \exists \delta > 0 \text{ s.t. } x + sv \in \mathbb{R}^3 \backslash \overline{\Omega} \text{ for } s \in (0,\delta)\},$$
$$\gamma_0^{I-} := \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) = 0, t_{\mathbf{b}}(x,-v) \neq 0, \text{ and } \exists \delta > 0 \text{ s.t. } x + sv \in \mathbb{R}^3 \backslash \overline{\Omega} \text{ for } s \in (-\delta,0)\}.$$



(a) Concave grazing



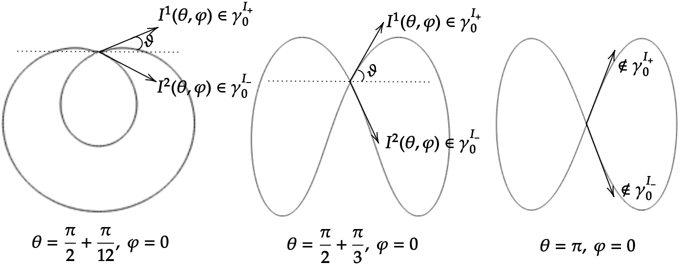
(b) Inflection grazing

Structure of inflection grazing

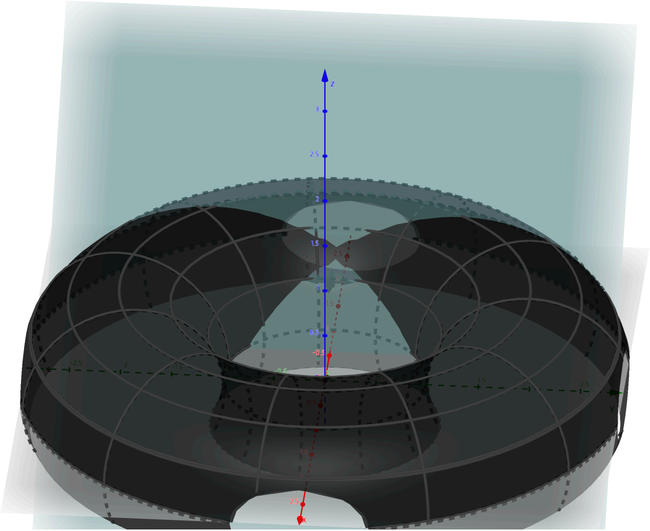
Let us consider standard solid torus Ω with inner radius r and revolving radius R. Then, boundary $\partial\Omega$ can be parametrized by

$$\sigma(\theta,\varphi) := ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta),\ 0 \le \theta \le 2\pi, 0 \le \varphi \le 2\pi$$

We visualize the intersection of tangent plane $T_p(\partial\Omega)$ at $p = \sigma(\theta, 0) \in \partial\Omega$ and the torus($\partial\Omega$) for each $\theta = \frac{\pi}{2} + \frac{\pi}{12}, \frac{\pi}{2} + \frac{\pi}{3}$, and π.



Moreover, we explicitly find inflection directions satisfying $\tan\vartheta = \sqrt{\frac{-r\cos\theta}{R + r\cos\theta}}$.


z
3
2.5
2
1.5

Finite number of bounce away from inflection grazing

Lemma

Let us fix a point $x \in \overline{\Omega}$ and velocity v with unit speed. If there is no inflection grazing (i.e. $[x^i(x,v), v^i(x,v)] \notin \gamma_0^I$ for all $i \in \mathbb{N}$), then

$$\sum_{i=1}^{\infty} |x^i(x,v) - x^{i-1}(x,v)| = \infty,$$

by excluding some cases.

Key idea of proof) We used a contradiction argument. We suppose that

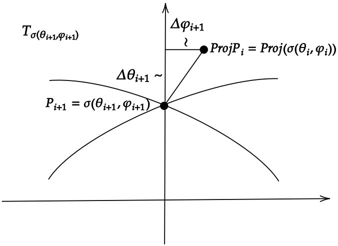
$$\sum_{i=1}^{\infty} |x^i(x,v) - x^{i-1}(x,v)| < \infty.$$

Then, $x^i(x,v)(= \sigma(\theta_i, \varphi_i)) \to x^\infty(x,v)(= \sigma(\theta_\infty, \varphi_\infty))$. Using axial symmetric of Ω, specular boundary condition, and convexity of the cross section, we derive the following relation:

$$\Delta\theta_{i+1} - \Delta\theta_i = \mathcal{O}(\Delta\theta_{i+1}^2) + \mathcal{O}(\Delta\theta_i^2) + \mathcal{O}(\Delta\varphi_{i+1}^2) + \mathcal{O}(\Delta\varphi_i^2)$$

However, we also obtained the following relation between $\Delta\theta_i$ and $\Delta\varphi_i$ ($|\Delta\theta_i|, |\Delta\varphi_i| \ll 1$) through the figure:

$$|\Delta\varphi_i| \lesssim_\varepsilon |\Delta\theta_i|$$



Using two relations above, we have

$$\begin{aligned}
\sum_{i=1}^{N} |\Delta\theta_{i+1}| &\geq \sum_{i=1}^{N} (|\Delta\theta_i| - C(|\Delta\theta_{i+1}|^2 + |\Delta\theta_i|^2 + |\Delta\varphi_{i+1}|^2 + |\Delta\varphi_i|^2)) \\
&\geq \sum_{i=1}^{N} (|\Delta\theta_i| - C_\varepsilon|\Delta\theta_{i+1}|^2 - C_\varepsilon|\Delta\theta_i|^2) \\
&\vdots \\
&\gtrsim \sum_{i=1}^{N} |\Delta\theta_1|
\end{aligned}$$

Hence, $\sum_{i=1}^{N} |\Delta\theta_{i+1}| \to \infty$ which contradicts to the hypothesis $\theta_i \to \theta_\infty$.

Uniform number of bounce for γ_0^I

To consider backward in time trajectory which belongs γ_0^I before finite travel length $L < \infty$, we define the set

$$B_L^\varepsilon := \left\{ (x,v) : \exists k \in \mathbb{N} \text{ such that } (x^k(x,v), v^{k-1}(x,v)) \in \gamma_0^{I-}, \text{ and } \sum_{j=1}^{k} |x^j(x,v) - x^{j-1}(x,v)| \leq L \right\}.$$

Lemma

If we define the number of bounce as

$$\mathcal{N}(x,v,L) := \sup \left\{ k \in \mathbb{N} : (x^j(x,v), v^{j-1}(x,v)) \notin \gamma_0^{I-}, \ \forall 1 \leq j \leq k \text{ and } \sum_{j=1}^{k} |x^j(x,v) - x^{j-1}(x,v)| \leq L \right\}$$

then we have the uniform finite number of bounce

$$\sup_{(x,v)\in B_L^\varepsilon} \mathcal{N}(x,v,L) < K_{\varepsilon,L}.$$

Note that the backward in time trajectory from $(x,v) \in B_L^\varepsilon$ does not generate inflection grazing after its $K_{\varepsilon,L}$.

Measure zero of grazing sets

Motivated by this fact, We construct the following sets:

$$
\begin{aligned}
(G_1)_x &:= \{v \in \mathbb{S}^2 : \ (x^1(x,v), v^0(x,v)) \notin \gamma_0^C \cup \gamma_0^I\}, \\
(B_1)_x &:= \{v \in \mathbb{S}^2 : \ (x^1(x,v), v^0(x,v)) \in \gamma_0^C \cup \gamma_0^I\}, \\
&\vdots \\
(G_j)_x &:= \{v \in (G_{j-1})_x : (x^j(x,v), v^{j-1}(x,v)) \notin \gamma_0^C \cup \gamma_0^I\}, \\
(B_j)_x &:= \{v \in (G_{j-1})_x : (x^j(x,v), v^{j-1}(x,v)) \in \gamma_0^C \cup \gamma_0^I\},
\end{aligned}
$$

for all $1 \leq j \leq K_{\varepsilon,L}$, for a fixed point $x \in \overline{\Omega}$. Let us treat γ_0^I case which is easier than γ_0^C case.

Our main goal is to prove

$$
\mathfrak{m}_2((B_j)_x) = 0, \quad \forall 1 \leq j \leq K_{\varepsilon,L}.
$$

To obtain our goal, we will use the following lemmas related to analyticity:

Lemma (Zero set of analytic function)

Suppose that f is a non-constant real-analytic function on a connected open domain $D \subset \mathbb{R}^n$. Then, the zero set

$$Z_f := \{x \in D : f(x) = 0\}$$

has zero n-dimensional Lebesgue measure.

Lemma (Lusin's property)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function. Then, $f(E)$ has measure zero in $\mathbb{R}^n$ for a measure zero set $E \subset \mathbb{R}^n$.

Sketch of proof) By one-to-one correspondence between $(\sigma(\theta,\varphi), I^2(\theta,\varphi))$ and $(x, v) \in B_L^{\varepsilon}$, we define

$$F_j(\theta,\varphi) := X(s(\theta,\varphi); 0, \sigma(\theta,\varphi), I^2(\theta,\varphi)) - x,$$

when forward in time trajectory from $(\sigma(\theta,\varphi), I^2(\theta,\varphi))$ passes S_0 ($\varphi = 0$ cross-section) after j-th bouncing. Here, $s(\theta,\varphi)$ is arrival time which measure traveling time from $(\sigma(\theta,\varphi), I^2(\theta,\varphi))$ to S_0.

Lemma (Analyticity of non-grazing trajectory)

Assume that $(x,v) \in \Omega \times \mathbb{R}^3$ *and* $\mathcal{N}(x,v,N(t-s)) := M_1 < \infty$. *If*

$$(x^i(x,v), v^{i-1}(x,v)) \notin \gamma_0 \quad \text{for all } 1 \le i \le M_1,$$
$$\text{resp, } (x^i(x,v), v^{i-1}(x,v)) \notin \gamma_0^I \quad \text{for all } 1 \le i \le M_1,$$

then
(a) $(t^i(t,x,v), x^i(x,v), v^i(x,v))$ *is locally analytic function of* (x,v) *(resp, locally continuous function of* (x,v)*).*

(b) If $s \notin t^i(t,x,v)$ *for any* $1 \le i \le M_1$, *then* $(X(s;t,x,v)), V(s;t,x,v))$ *is locally analytic (resp, locally continous) function of* (x,v) *for fixed* s.

(c) There exsists $\delta_{x,v} \ll 1$ *such that if* $|(y,u)-(x,v)| < \delta_{x,v}$, *then* $N(y,u,N(t-s)) \le M_1$ *(resp,* $N(y,u,N(t-s)) \le M_1$*).*

By applying (b) of Lemma above to forward in time trajectory, the function $F_j(\theta,\varphi)$ is analytic on $P_L^{\varepsilon,j}$ where

$$P_L^{\varepsilon,j} := \{(\theta,\varphi) : (\sigma(\theta,\varphi), I^2(\theta,\varphi)) = (x^j(y,u), v^{j-1}(y,u)) \in \gamma_0^I \text{ for some } (y,u) \in X^{\varepsilon}, u \in (G_{j-1})_y\}.$$

Once we prove that F_j is real-analytic, we have the following dichotomy:

(a) (F_j is identically zero) Fortunately, we can exclude such case, away from small sets, by using axial symmetric of Ω. More explicitly, we observe that

$$\frac{\partial X(s;0,\sigma(\theta,\varphi),I^2(\theta,\varphi))}{\partial\varphi} \parallel \hat{\varphi}(0) \ (y\text{-axis}).$$

However, direction of $V(s(\theta,\varphi);0,\sigma(\theta,\varphi),I^2(\theta,\varphi))$ must satisfy some specific direction except $\hat{\varphi}(0)$.

(b)(F_j is not identically zero) Then, from analyticity, zero set of F_j has measure zero in (θ,φ) space. And, by Lusin's property, we have

$$\{v\in\mathbb{S}^2 : v = V(s(\theta,\varphi);0,\sigma(\theta,\varphi),I^2(\theta,\varphi)) \ (\theta,\varphi)\in Z_{F_j}\}$$

has also measure zero in $\mathbb{S}^2$.

Proposition

Let S_0 be $\varphi=0$ cross-section and $\varepsilon>0$. There exists a compacat set $\mathcal{Y}^\varepsilon\subset\bar{S_0}\times\mathbb{S}^2$ such that

$$[x^j(x,v),v^{j-1}(x,v)]\notin\gamma_0^I\cup\gamma_0^C, \quad \forall 1\le j\le M,$$

and

$$\inf_{1\le i\le M}|v^{i-1}(x,v)\cdot n(x^i(x,v))|\ge C^*_{\varepsilon,L}>0,$$

for $(x,v)\in\mathcal{Y}^\varepsilon$. In addition, for $x\in\bar{S_0}$, $\mu((\{\bar{S_0}\times\mathbb{S}^2\}\backslash\mathcal{Y}^\varepsilon)_x)\le\varepsilon$, where $A_x:=\{v:(x,v)\in A\}(\subset\mathbb{S}^2)$.

Thank you!