Navier–Stokes limit of elastic bilinear kinetic equations

To appear in Annales Henri Lebesgue (ArXiv:2304.11698)

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24/06, Webinar France–Korea

Introduction

Collisional kinetic equation with no force :

$$(\partial_t + v \cdot \nabla_x) F(t,x,v) = \mathcal{C}\left[F(t,x,\cdot)\right](v)$$

Macroscopic elastic conservation (mass, momentum, energy):

$$\varphi(v) = 1, v_1, \ldots, v_d, |v|^2, \qquad \int_{\mathbb{R}^d} \mathcal{C}[F](v)\varphi(v)\mathrm{d}v = 0\,.$$

Microscopic dissipation (H-theorem) : For some convex $\Phi : \mathbb{R} \to \mathbb{R}$

$$\forall F, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v)\Phi(F(v))\mathrm{d}v \leq 0\,.$$

and, denoting $\boldsymbol{m} = \int_{\mathbb{R}^d} F(v)\left(1, v, |v|^2\right)\mathrm{d}v$ the conserved macroscopic quantities

$$\forall F, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v)\Phi(F(v))\mathrm{d}v = 0 \Leftrightarrow \mathcal{C}[F] = 0 \Leftrightarrow F(v) = \mathcal{M}_{\boldsymbol{m}}(v)\,.$$

Examples : (quantum) Boltzmann and Landau, BGK, non-linear Fokker-Planck

Introduction

Small fluctuations in diffusive regime : $F(t,x,v) = \mathcal{M}_{1,0,1}(v) + \varepsilon f^{\varepsilon}\left(\varepsilon^2 t, \varepsilon x, v\right)$

$$\boxed{\partial_t f^{\varepsilon} = \varepsilon^{-2}(\mathcal{L} - \varepsilon v \cdot \nabla_x) f^{\varepsilon} + \varepsilon^{-1}\mathcal{Q}(f^{\varepsilon}, f^{\varepsilon}) + o\left(\varepsilon^{-1}\right)}$$

where $\mathcal{C}[\mathcal{M}_{1,0,1} + \varepsilon g] = \varepsilon \mathcal{L} g + \varepsilon^2 \mathcal{Q}(g,g) + o\left(\varepsilon^2\right)$

Theorem (Bardos, Golse, Levermore, 1991 (for $\Phi(u) = -\log u$))

If $f^{\varepsilon} \to f^0$ and every term of the equation converges formally, then

$$f^0(t,x,v) = \left(u^0(t,x) \cdot v + \theta^0(t,x)(|v|^2 - cst.)\right) \mathcal{M}_{1,0,1}(v)$$

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 = \kappa_{inc} \Delta u^0 + \nabla p, \qquad \nabla \cdot u^0 = 0, \quad \kappa_{inc} = \kappa_{inc}(\mathcal{L}) > 0, \\ \partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = \kappa_{Fou} \Delta \theta^0, \qquad \kappa_{Fou} = \kappa_{Fou}(\mathcal{L}) > 0 \end{cases}$$

Weak convergence: For Boltzmann (Bardos, Golse, Levermore, Saint-Raymond, Arsénio)

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Quantitative convergence: For Boltzmann or Landau (Bardos, Ukai, Gallagher, Tristani, Carrapatoso, Rachid) using spectral studies (Ellis, Pinsky, Yang, Yu)

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Limitations on quantitative approach:

- Only $\Phi(u) = -\log u$ (exclude quantum equations)
- Equations treated separately (although common structure)
- Non-constructive spectral studies
- Restrictive integrability condition

Introduction

Our structural assumptions

Consider $V^1 \subset V := L^2\left(\mu^{-1}\mathrm{d}v\right)$ where $\langle v\rangle^k \mu \in L^1_+$ and $\mathcal{L} = \mathcal{L}^*$

(H1) Isotropy : $\forall O \in \mathbb{R}^{d\times d}$ orthogonal, denoting $(Of)(v) = f(Ov)$

$$O(\mathcal{L}f) = \mathcal{L}(Of), \qquad O\mathcal{Q}(f,f) = \mathcal{Q}(Of, Of),$$

(H2) Macroscopic conservation law:

$$\ker \mathcal{L} = \mathrm{Span}\left(\mu, v_1\mu, \dots, v_d\mu, |v|^2\mu\right) \subset V^1$$

$$\langle \mathcal{L}f, \varphi\rangle_V = \langle \mathcal{Q}(f,f), \varphi\rangle_V = 0, \qquad \varphi \in \ker(\mathcal{L})$$

(H3) Microscopic dissipation (linearized H-theorem):

$$\langle \mathcal{L}f, f\rangle_V \leq -\|\Pi_{(\ker\mathcal{L})^\perp} f\|^2_{V^1} \quad \text{and} \quad \left[\mathcal{L}, \langle v\rangle^{1 \text{ and } 2}\right] = \text{L.O.T.}$$

~~$\mathcal{L} = \text{dissipative} + \text{compact}$~~

(H4) Control of the collisions by the energy V and the dissipated entropy V^1:

$$\langle \mathcal{Q}(f,g), h\rangle_V \lesssim \|h\|_{V^1}\left(\|f\|_{V^1}\|g\|_V + \|f\|_V\|g\|_{V^1}\right)$$

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(H4) Control of the collisions by the energy V and the dissipated entropy V^1:

$$\|\mathcal{Q}(f,g)\|_{V^{-1}} \lesssim \|f\|_{V^1}\|g\|_V + \|f\|_V\|g\|_{V^1}, \quad V^{-1} := \left(V^1\right)' \text{w.r.t. V}$$

Strategy

Our problem: profile of small fluctuations f^ε around an equilibrium

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}\big(f^\varepsilon, f^\varepsilon\big) + \cancel{o\left(\varepsilon^{-1}\right)}, \quad f^\varepsilon(0,x,v) = f_{\text{in}}(x,v)$$

Goal: show that $\exists! f^\varepsilon$

$$f^\varepsilon(t,x,v) \xrightarrow{\varepsilon \to 0} \big(u(t,x) \cdot v + \theta(t,x) \left(|v|^2 - \text{cst.}\right)\big) \mu(v)$$

$$\begin{cases} \partial_t u + u \cdot \nabla_x u = \kappa_{\text{inc}} \Delta_x u, \quad \nabla_x \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa_{\text{Fou}} \Delta_x \theta. \end{cases}$$

and describe/quantify the convergence.

Strategy

Our problem: profile of small fluctuations f^ε around an equilibrium

$$f^\varepsilon(t) = U^\varepsilon(t) f_{\text{in}} + \Psi^\varepsilon\left(f^\varepsilon, f^\varepsilon\right)(t)$$

where

$$U^\varepsilon(t) := \exp\left(\frac{t}{\varepsilon^2}\left(\mathcal{L} - \varepsilon v \cdot \nabla_x\right)\right)$$

$$\Psi^\varepsilon(f, f)(t) := \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-\tau)\mathcal{Q}(f(\tau), f(\tau))\mathrm{d}\tau$$

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Strategy

Strategy for the existence of solutions

Identify some "hydrodynamic" and "kinetic" regimes:

$$\|f\|_{\mathcal{H}}^2 := \sup_{t\geq 0} \|f(t)\|_{H_x^\ell V_v^1}^2 + \int_0^\infty \|\nabla_x f(t)\|_{H_x^\ell V_v^1}^2 \mathrm{d}t, \quad \binom{\text{heat equation}}{\text{Navier-Stokes}}$$

$$\|f\|_{\mathcal{K}^\varepsilon}^2 := \sup_{t\geq 0} e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell V_v}^2 + \frac{1}{\varepsilon^2} \int_0^\infty e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell \boldsymbol{V_v^1}}^2 \mathrm{d}t, \quad \begin{pmatrix}\text{dissipative} \\ \text{equation}\end{pmatrix}$$

and a decomposition of the semigroup/nonlinearity:

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2}(\mathcal{L} - \varepsilon v \cdot \nabla_x)\right) = U^\varepsilon_{\text{hydro}} \oplus U^\varepsilon_{\text{kine}}$$

$$\Psi^\varepsilon(f,f)(t) = \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-\tau)\mathcal{Q}(f(\tau),f(\tau))\mathrm{d}\tau = \Psi^\varepsilon_{\text{hydro}}(f,f) + \Psi^\varepsilon_{\text{kine}}(f,f)$$

compatible with the corresponding regimes:

$$\|U^\varepsilon_{\text{hydro}} f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^\ell V_v}, \qquad \|U^\varepsilon_{\text{kine}} f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H_x^\ell V_v}$$

$$\left\|\Psi^\varepsilon_{\text{hydro}}(f,f)\right\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2, \qquad \|\Psi^\varepsilon_{\text{kine}}(f,f)\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{\mathcal{K}^\varepsilon}^2$$

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$$\|f\|_{\mathcal{K}^\varepsilon}^2 := \sup_{t\geq 0} e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell V_v}^2 + \frac{1}{\varepsilon^2}\int_0^\infty e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell \boldsymbol{V_v^1}}^2 \mathrm{d}t, \quad \begin{pmatrix}\text{dissipative} \\ \text{equation}\end{pmatrix}$$

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Consider an **arbitrary** system of equations

$$\left.\begin{aligned} f^\varepsilon_{\text{hydro}}(t) &= U^\varepsilon_{\text{hydro}}(t) f_{\text{in}} + \Psi^\varepsilon_{\text{hydro}}(f^\varepsilon_{\text{hydro}}, f^\varepsilon_{\text{hydro}}) + \dots \\ f^\varepsilon_{\text{kine}}(t) &= U^\varepsilon_{\text{kine}}(t) f_{\text{in}} + \Psi^\varepsilon_{\text{kine}}(f^\varepsilon_{\text{kine}}, f^\varepsilon_{\text{kine}}) + \dots \\ &\dots \end{aligned}\right\} \xrightarrow{\text{Picard}} \begin{aligned} f^\varepsilon &:= f^\varepsilon_{\text{hydro}} + f^\varepsilon_{\text{kine}} + \dots \\ &\in \mathcal{H} + \mathcal{K}^\varepsilon + \dots \end{aligned}$$

is **a** solution

Strategy

Strategy for the convergence

Kinetic convergence : OK because $\|f^\varepsilon_{\text{kine}}\|_{\mathcal{K}^\varepsilon} \lesssim 1 \Rightarrow f^\varepsilon_{\text{kine}} = \mathcal{O}\left(e^{-\sigma t/\varepsilon^2}\right)$

Hydrodynamic convergence : To prove the convergence of

$$f^\varepsilon_{\text{hydro}}(t) = U^\varepsilon_{\text{hydro}}(t) f_{\text{in}} + \Psi^\varepsilon_{\text{hydro}}(f^\varepsilon_{\text{hydro}}, f^\varepsilon_{\text{hydro}}) + \dots$$

(1) Show the convergence of the ‘‘hydrodynamic’’ operators:

$$U^\varepsilon_{\text{hydro}} = U^0_{\text{NS}} + \mathcal{O}(\varepsilon) + \text{dispersive} \quad \text{and} \quad \Psi^\varepsilon_{\text{hydro}}(f,f) = \Psi^0_{\text{NS}}(f,f) + \mathcal{O}(\varepsilon)$$

(2) Check from **explicit** expressions of U^0_{NS} and Ψ^0_{NS} that

$$f(t) = U^0_{\text{NS}}(t) f(0) + \Psi^0_{\text{NS}}(f,f)(t)$$

$$\Leftrightarrow \begin{cases} f(t,x,v) = \big(u(t,x)\cdot v + \theta(t,x)\left(|v|^2 - \text{cst.}\right)\big)\mu(v), \\ (u,\theta) \quad \text{is a mild solution of INSF} \end{cases}$$

Outline of the proof of the (non)linear bounds

Proposition (G, Lods)

The semigroup and Duhamelized nonlinearity split

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2}\left(\mathcal{L} - \varepsilon v \cdot \nabla_x\right)\right) = U^\varepsilon_{\text{hydro}}(t) \oplus U^\varepsilon_{\text{kine}}(t)$$

$$\frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-\tau)\mathcal{Q}(f(\tau), f(\tau))\mathrm{d}\tau = \Psi^\varepsilon_{\text{hydro}}(f,f)(t) + \Psi^\varepsilon_{\text{kine}}(f,f)(t)$$

and satisfy the continuity estimates

$$\|U^\varepsilon_{\text{hydro}} f\|_{\mathcal{H}} \lesssim \|f\|_{H^\ell_x V_v}, \qquad \|U^\varepsilon_{\text{kine}} f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H^\ell_x V_v}$$

$$\left\|\Psi^\varepsilon_{\text{hydro}}(f,f)\right\|_{\mathcal{H}} \lesssim \|f\|^2_{\mathcal{H}}, \qquad \|\Psi^\varepsilon_{\text{kine}}(f,f)\|_{\mathcal{K}^\varepsilon} \lesssim \varepsilon \|f\|^2_{\mathcal{K}^\varepsilon}$$

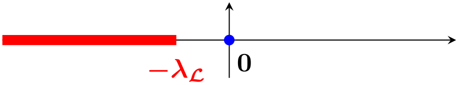
Difficulties:

1. Find the decomposition in stable subspaces $H^\ell_x V_v = \text{hydro}^\varepsilon \oplus \text{kine}^\varepsilon$
2. Compensate the stiff factor $\frac{1}{\varepsilon}$
3. Compensate the deregularizing effect $\mathcal{Q} : V^1 \to V^{-1}$

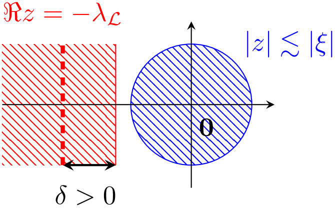
Outline of the proof of the (non)linear bounds

Finding stable subspaces of $\mathcal{L} - v \cdot \nabla_x$: localization of $\Sigma(\mathcal{L} - iv \cdot \xi)$

Localization for $\xi = 0$:

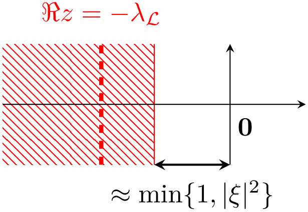


Localization for $|\xi| \ll 1$:



Factorization method inspired from [T] using $[\mathcal{L}, \langle v \rangle]$ = L.O.T. from **(H3)**

Localization for $|\xi| \gtrsim 1$:



Mode by mode hypocoercivity from [D]

Disjoint parts of $\Sigma(\mathcal{L} - iv \cdot \xi) \Rightarrow$ stable kinetic(ξ) and hydro(ξ) subspaces

[T] Tristani, 2016 (j.jfa.2015.09.025) [D] Duan, 2011 (0951-7715/24/8/003).

Outline of the proof of the (non)linear bounds

Finding stable subspaces of $\mathcal{L} - v \cdot \nabla_x$: localization of $\Sigma(\mathcal{L} - iv \cdot \xi)$ for $|\xi| \ll 1$

Goal: find $\Re z \geq -\lambda_{\mathcal{L}} + \delta$ such that $(\mathcal{L} - iv \cdot \xi - z)^{-1}$ is bounded on V

Naive approach for localization:

$$(\mathcal{L} - iv \cdot \xi - z)^{-1} = \underbrace{(\mathcal{L} - z)^{-1}}_{\mathcal{O}(1/|z|)} + (\mathcal{L} - iv \cdot \xi - z)^{-1} \underbrace{(iv \cdot \xi)}_{\text{unbounded!}} \underbrace{(\mathcal{L} - z)^{-1}}_{\mathcal{O}(1/|z|)} \qquad (\star)$$

Enlargement method: $\mathcal{L} = \mathcal{B} + \mathcal{A}$ with $\mathcal{A} = M\mathbf{1}_{|v| \leq M} : V \to W := \langle \cdot \rangle^{-1} V$

$$[\mathcal{L}, \langle v \rangle] = \text{L.O.T. from } \textbf{(H3)} \Rightarrow \|(\mathcal{B} - iv \cdot \xi - z)^{-1}\| \lesssim 1 \text{ on } V \text{ and } W$$

$$\Rightarrow \|(\mathcal{L} - z)^{-1}\| \lesssim 1 + 1/|z| \text{ on } V \text{ and } W$$

Insert $(\star)$ in the right hand side of the expansion

$$(\mathcal{L} - iv \cdot \xi - z)^{-1} = (\mathcal{B} - iv \cdot \xi - z)^{-1} + (\mathcal{L} - iv \cdot \xi - z)^{-1} \mathcal{A}(\mathcal{B}_\xi - z)^{-1}$$

to use that $\mathcal{A}$ compensates the unboundedness of v:

$$(\mathcal{L} - iv \cdot \xi - z)^{-1} = \underbrace{(\mathcal{B} - iv \cdot \xi - z)^{-1} + (\mathcal{L} - z)^{-1} \mathcal{A}(\mathcal{B}_\xi - z)^{-1}}_{\text{bounded}}$$

$$+ (\mathcal{L} - iv \cdot \xi - z)^{-1} \underbrace{(iv \cdot \xi)(\mathcal{L} - z)^{-1} \mathcal{A}}_{\text{of order } |\boldsymbol{\xi}|(1 + 1/|\boldsymbol{z}|)} (\mathcal{B}_\xi - z)^{-1}$$

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$$\|(\mathcal{L} - iv \cdot \xi - z)^{-1}\| \left(1 - \text{cst.}|\xi| \left(1 + 1/|z|\right)\right) \lesssim 1 + \frac{1}{|z|}$$

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to use that $\mathcal{A}$ compensates the unboundedness of v:

$$\boxed{|\xi| \ll 1 \text{ and } |\xi| \ll |z| \Longrightarrow \|(\mathcal{L} - iv \cdot \xi - z)^{-1}\| \lesssim 1}$$

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to use that $\mathcal{A}$ compensates the unboundedness of v:

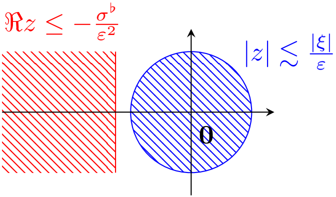
$$\boxed{|\xi| \ll 1 \Longrightarrow \Sigma(\mathcal{L} - iv \cdot \xi) \subset B(0, C|\xi|) \cup \{\Re z \geq -\lambda_{\mathcal{L}} + \delta\}}$$

Outline of the proof of the (non)linear bounds

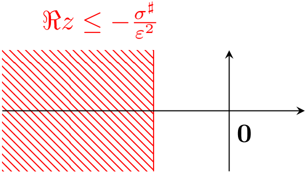
The kinetic regime

Scaling $(t,x) \to \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) \Rightarrow \mathcal{L} - i(v\cdot\xi) \to \frac{1}{\varepsilon^2}\left(\mathcal{L} - i\varepsilon(v\cdot\xi)\right)$

Localization for $|\xi| \ll \varepsilon^{-1}$:



Localization for $|\xi| \gtrsim \varepsilon^{-1}$:



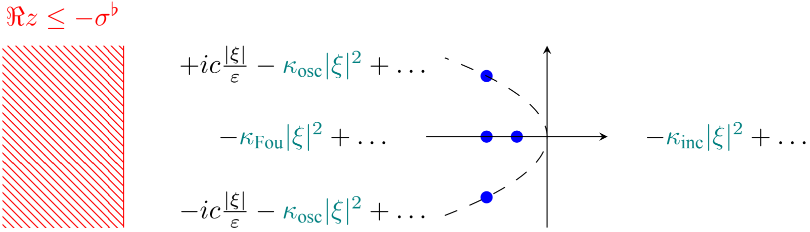
$$f \in V \xrightarrow{\text{localization}} U^\varepsilon_{\text{kine}} f \in L^\infty_t V_v\left(e^{t\frac{\sigma}{\varepsilon^2}}\mathrm{d}t\right) \xrightarrow[\text{using (H3)}]{\text{energy method}} \underbrace{\frac{1}{\varepsilon} U^\varepsilon_{\text{kine}} f \in L^2_t \boldsymbol{V^1_v}\left(e^{t\frac{\sigma}{\varepsilon^2}}\mathrm{d}t\right)}_{(\star)}$$

$$\Rightarrow \boxed{\|U^\varepsilon_{\text{kine}} f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H^\ell_x V_v}} \xrightarrow[\substack{= \text{double regularization} \\ \text{double smallness}}]{\text{convolution using } (\star)} \boxed{\underbrace{\left\| \frac{1}{\varepsilon} U^\varepsilon_{\text{kine}} *_t \mathcal{Q}(f,f) \right\|_{\mathcal{K}^\varepsilon}}_{= \|\Psi^\varepsilon_{\text{kine}}(f,f)\|_{\mathcal{K}^\varepsilon}} \lesssim \varepsilon \|f\|^2_{\mathcal{K}^\varepsilon}}$$

Study of the hydrodynamic operators

The hydrodynamic regime

Enlargement + matrix perturbation theory $\Rightarrow$ eigenmodes $(\lambda_\star(\xi), \mathcal{P}_\star(\xi))$



$$U^\varepsilon_{\text{hydro}} = U^\varepsilon_{\text{inc}} + U^\varepsilon_{\text{Fou}} + U^\varepsilon_{\text{osc}}, \qquad \widehat{U^\varepsilon_\star(t) f}(\xi) = e^{\varepsilon^{-2} t \lambda_\star(\varepsilon\xi)} \mathcal{P}_\star(\varepsilon\xi) \widehat{f}(\xi)$$

- $\Re(\lambda_\star(\xi)) \lesssim -|\xi|^2 \Rightarrow U^\varepsilon_\star(t)$ comparable to $e^{t\Delta_x} \Rightarrow$ parabolic estimate in (t, x)
- $\mathcal{P}_\star(\xi) : V^{-1} \to V^1 \Rightarrow$ regularization in v

$$\boxed{\|U^\varepsilon_\star f\|_{\mathcal{H}} \lesssim \|f\|_{H^\ell_x V^{-1}_v}}$$

- $\mathcal{Q} \perp \ker(\mathcal{L})$ by **(H2)** + expansion of $\mathcal{P} \Rightarrow \mathcal{P}(\varepsilon\xi)\mathcal{Q} = \cancel{\mathcal{P}^{(0)}(\xi/|\xi|)\mathcal{Q}} + \mathcal{O}(\varepsilon\xi)$

$$\Rightarrow \left\| \frac{1}{\varepsilon} U^\varepsilon_{\text{hydro}} *_t \mathcal{Q}(f, f) \right\|_{\mathcal{H}} = \boxed{\left\|\Psi^\varepsilon_{\text{hydro}}(f, f)\right\|_{\mathcal{H}} \lesssim \|f\|^2_{\mathcal{H}}}$$

Conclusion

Statement of the theorem

Previous estimates: $\|f^\varepsilon_{\text{hydro}}(0)\| \ll 1$ and $\varepsilon \ll 1 \xrightarrow{\text{Picard near } 0} \exists f^\varepsilon$.

Better: $\exists f^0_{\text{NS}}$ + norm such that $\|f^0_{\text{NS}}\|_{\mathcal{H}_{\left(f^0_{\text{NS}},\delta\right)}} \le \delta \ll 1 \xrightarrow{\text{Picard near } f^0_{\text{NS}}} \exists f^\varepsilon$ (as in [GT])

Theorem (G, Lods) — ArXiv:2304.11698

Consider any **(non-small)** $f_{\text{in}} \in H^\ell_x L^2_v\left(\mu^{-1}\mathrm{d}v\right)$, for any $\varepsilon \ll 1$, $\exists!$ solution to

$$\partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x f^\varepsilon = \varepsilon^{-2}\mathcal{L} + \varepsilon^{-1}\mathcal{Q}\left(f^\varepsilon, f^\varepsilon\right), \quad f^\varepsilon \in L^\infty_t H^\ell_x L^2\left(\mu^{-1}\mathrm{d}v\mathrm{d}x\right),$$

with the **same lifespan** as the Navier-Stokes limit f^0_{NS}, and for smooth initial data

$$f^\varepsilon = f^0_{\text{NS}} + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}\left(\sqrt{\varepsilon}\right) + \mathcal{O}\left(\left(\varepsilon/t\right)^{\frac{d-1}{2}}\right)$$

with **computable constants**, and for incompressible initial data

$$f^\varepsilon = f^0_{\text{NS}} + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}\left(\varepsilon\right) \quad \text{(optimal)}$$

[GT] Gallagher, Tristani, 2020 (10.5802/ahl.40).

Conclusion

Remarks and perspectives

Better integrability in v: similar assumptions in $W = L^2(m(v)\mathrm{d}v) \supset V$:

- $f_{\text{in}}(x,\cdot) \in W_v \Rightarrow f^{\varepsilon}_{\text{kine}}(t,x,\cdot) \in W_v$ but $f^{\varepsilon}_{\text{hydro}}(t,x,\cdot) \in V_v \subset W_v$
- Nonlinear theory more complicated (Boltzmann without cutoff/Landau):

$$\|\mathcal{Q}(f,g)\|_{W^{-1}} \not\lesssim \|f\|_W \|g\|_{W^1} + \|f\|_{W^1}\|g\|_W$$

Use of isotropy: adaptable to relativistic velocities $v \cdot \nabla_x \to \frac{v}{\langle v \rangle} \cdot \nabla_x$

Perspectives:

- OK for bilinear approximation of general models $\to$ include remainder $o(\varepsilon^{-1})$:
  - Quantum Boltzmann/Landau: $\mathcal{C}[F] = \mathcal{Q}(F,F) + \hbar^3 \mathcal{T}(F,F,F)$
  - BGK: $\mathcal{C}[F] = \mathcal{M}[F] - F$
  - Nonlinear Fokker-Planck: $\mathcal{C}[F] = T_F \nabla_v \cdot \left(\mathcal{M}[F]\nabla_v\left(\frac{F}{\mathcal{M}[F]}\right)\right)$
- Confining force $\nabla_x\phi \Rightarrow$ modified transport $v\cdot\nabla_x \to v\cdot\nabla_x - \nabla_x\phi\cdot\nabla_v$
- Fluctuation around inhomogeneous Maxwellian (stationary Euler sol. (R,U,T))

Conclusion

Remarks and perspectives

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Perspectives:

- OK for bilinear approximation of general models $\to$ include remainder $o(\varepsilon^{-1})$:

$$\partial_t f^{\varepsilon} + \varepsilon^{-1} v \cdot \nabla_x f^{\varepsilon} = \varepsilon^{-2}\mathcal{L} + \varepsilon^{-1}\mathcal{Q}(f^{\varepsilon}, f^{\varepsilon}) + o(\varepsilon^{-1}),$$

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Conclusion

Remarks and perspectives

Better integrability in v: similar assumptions in $W = L^2(m(v)\mathrm{d}v) \supset V$:

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Perspectives:

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- Confining force $\nabla_x \phi \Rightarrow$ modified transport $v \cdot \nabla_x \to v \cdot \nabla_x - \nabla_x \phi \cdot \nabla_v$

$$\partial_t f^\varepsilon + \varepsilon^{-1}(v \cdot \nabla_x f^\varepsilon - \nabla_x \phi \cdot \nabla_v f^\varepsilon) = \varepsilon^{-2} \underbrace{e^{-\phi} \mathcal{L}}_{\text{degraded coercivity}} + \varepsilon^{-1} e^{-\phi} \mathcal{Q}(f^\varepsilon, f^\varepsilon),$$

- Fluctuation around inhomogeneous Maxwellian (stationary Euler sol. (R,U,T))

Conclusion

Remarks and perspectives

Better integrability in v: similar assumptions in $W = L^2\left(m(v)\mathrm{d}v\right) \supset V$:

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- Fluctuation around inhomogeneous Maxwellian (stationary Euler sol. (R, U, T))

$$F(t,x,v) = \mathcal{M}_{R(\varepsilon x), U(\varepsilon x), T(\varepsilon x)} + \varepsilon f^{\varepsilon}\left(\varepsilon^2 t, \varepsilon x, \varepsilon v\right),$$

Thank you for your attention

Extra

Construction of the hydrodynamic modes

Expansion of the spectral hydrodynamic projector/Kato's isomorphism :

Resolvent bounds + enlargement method using $\left[\mathcal{L}, \langle v\rangle^2 \times\right]$ = L.O.T. from **(H3)**

$\Rightarrow$ expansion of hydrodynamic projector $\mathcal{P}(\xi) : V^{-1} \to$ hydro space$(\xi) \subset V^1$

$\Rightarrow$ expansion of Kato's isomorphism $\Phi(\xi) : \ker(\mathcal{L}) \to$ hydro space(ξ)

Diagonalization/expansion of the hydrodynamic part of $\mathcal{L} - iv \cdot \xi$ for $|\xi| \ll 1$:

Define the reduced matrix $L(\xi) \in \mathscr{B}(\ker(\mathcal{L})) \overset{\Phi(\xi)}{\sim} \left(\mathcal{L} - i(v \cdot \xi)\right)_{|\text{hydro. space}(\xi)}$

Isotropy $\Rightarrow L(\xi)_{|\xi|=0}$ diagonalizable $\Rightarrow$ diagonalization/expansion of $L(\xi)$

$\Rightarrow$ **expansion of spectral projections using expansion of $\Phi(\xi)$:**

$$\mathcal{P}(\xi) = \mathcal{P}_{\text{Fou}}(\xi) \oplus \mathcal{P}_{\text{inc}}(\xi) \oplus \mathcal{P}_{+\text{osc}}(\xi) \oplus \mathcal{P}_{-\text{osc}}(\xi),$$

$$\mathcal{P}_\star(\xi) = \mathcal{P}_\star^{(0)}\left(\xi/|\xi|\right) + \xi \cdot \mathcal{P}_\star^{(1)}\left(\xi/|\xi|\right) + \mathcal{O}\left(|\xi|^2\right) : V_v^{-1} \to V_v^1$$

$\Rightarrow$ **expansion of eigenvalues:**

$$\lambda_{\text{Fou}}(\xi) = -\kappa_{\text{Fou}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right), \qquad \lambda_{\text{inc}}(\xi) = -\kappa_{\text{inc}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right),$$

$$\lambda_{\pm\text{osc}}(\xi) = \pm ic|\xi| - \kappa_{\text{osc}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right)$$

Extra

Hydrodynamic regime $(\star = \textbf{inc}, \textbf{Fou}, \pm\textbf{osc})$

Linear estimate:

$$\lambda_\star(\xi) \approx i\alpha_\star|\xi| - \kappa_\star|\xi|^2 \Rightarrow \Re\left(\frac{\lambda_\star(\varepsilon\xi)}{\varepsilon^2}\right) \leq -\frac{\kappa_\star}{2}|\xi|^2 \quad \text{and} \quad \mathcal{P}_\star(\xi) : V_v^{-1} \to V_v^1$$

$$\Rightarrow \underbrace{\left\| e^{t\frac{\lambda_\star(\varepsilon\nabla_x)}{\varepsilon^2}} \mathcal{P}_\star(\varepsilon\nabla_x) f \right\|_{H_x^\ell V_v^1}}_{\|U_\star^\varepsilon(t) f\|_{H_x^\ell V_v^{-1}}} \lesssim \left\| e^{t\frac{\kappa_\star}{2}\Delta_x} f \right\|_{H_x^\ell V_v^{-1}} \Rightarrow \boxed{\|U_{\text{hydro}}^\varepsilon f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^\ell V_v^{-1}}}$$

<u>Reminder:</u> $\mathcal{H} = L_t^\infty H_x^\ell \cap L_t^2 \dot{H}_x^{\ell+1}$ (parabolic regularity)

Nonlinear estimate:

$$\mathcal{P}^{(0)}\Pi_{\ker(\mathcal{L})} = \mathcal{P}^{(0)} \quad \text{and} \quad \mathcal{Q} \perp \ker(\mathcal{L}) \text{ from } \textbf{(H2)} \Rightarrow \mathcal{P}^{(0)}\mathcal{Q} = 0$$

$$\Rightarrow \frac{\mathbf{1}}{\boldsymbol{\varepsilon}}\mathcal{P}_\star(\boldsymbol{\varepsilon}\xi)\mathcal{Q} \approx \mathbf{0} + \boldsymbol{\xi} \cdot \mathcal{P}^{(1)}(\xi/|\xi|)\,\mathcal{Q}(f,g) \in \boldsymbol{V_v^1}$$

$$\Rightarrow \left\| \frac{\mathbf{1}}{\boldsymbol{\varepsilon}} U_{\text{hydro}}^\varepsilon *_t \mathcal{Q}(f,f) \right\|_{\mathcal{H}} = \boxed{\left\|\Psi_{\text{hydro}}^\varepsilon(f,f)\right\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2}$$

Extra

Rough idea of the proof of hydrodynamic convergence

Refining the previous step, one gets for $\star = \mathrm{Fou}, \mathrm{inc}$

$$U_\star^\varepsilon(t) = U_\star^0(t) + \mathcal{O}(\varepsilon) \quad \text{and} \quad \Psi_\star^\varepsilon(t) = \Psi_\star^0(t) + \mathcal{O}(\varepsilon)$$

where each limit writes

$$U^0(t) = e^{t\kappa_\star \Delta_x} \mathcal{P}^{(0)}(\nabla_x)$$

$$\Psi_\star^0(f,f)(t) = \int_0^t e^{(t-\tau)\kappa_\star \Delta_x} \mathcal{P}^{(1)}(\nabla_x) \mathcal{Q}\left(f(\tau), f(\tau)\right) \mathrm{d}\tau$$

which is related to the mild formulation of Navier-Stokes-Fourier:

$$u(t) = e^{t\kappa_{\mathrm{inc}}\Delta_x} u(0) + \int_0^t e^{(t-\tau)\kappa_{\mathrm{inc}}\Delta_x} \mathbb{P}\left[\nabla_x \cdot (u(\tau) \otimes u(\tau))\right] \mathrm{d}\tau$$

$$\theta(t) = e^{t\kappa_{\mathrm{Fou}}\Delta_x} \theta(0) + \int_0^t e^{(t-\tau)\kappa_{\mathrm{Fou}}\Delta_x} \left[\nabla_x \cdot (\theta(\tau) u(\tau))\right] \mathrm{d}\tau$$

And for the oscillating terms:

$$U_{\pm\mathrm{osc}}^\varepsilon(t) = \mathcal{O}\left(\frac{\varepsilon}{t}\right) \quad \text{and} \quad \Psi_{\pm\mathrm{osc}}^\varepsilon(f,f) = \mathcal{O}(\varepsilon)\,.$$

Extra

Kato's reduction process: eigen. pbm. in Banach $\to$ eigen. pbm. in finite dimension

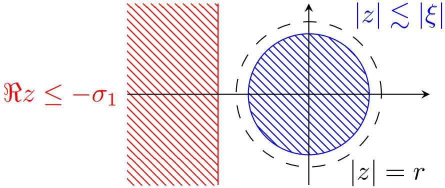


Figure: Localization of $\mathrm{Spec}\big(\mathcal{L} - i(v\cdot\xi)\big)$ for $|\xi| \ll 1$

Goal: expansion of eigenvalues and eigenfunctions of $\big(\mathcal{L} - i(v\cdot\xi)\big)_{|\text{hydro. space}(\xi)}$

Difficulty: hydro. space(ξ) depends on ξ

Extra

Kato's reduction process: eigen. pbm. in Banach → eigen. pbm. in finite dimension

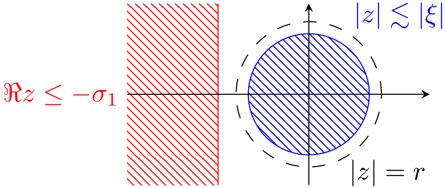


Figure: Localization of $\mathrm{Spec}\big(\mathcal{L} - i(v \cdot \xi)\big)$ for $|\xi| \ll 1$

Goal: expansion of eigenvalues and eigenfunctions of $\big(\mathcal{L} - i(v \cdot \xi)\big)_{|\text{hydro. space}(\xi)}$

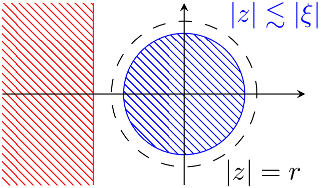
Difficulty: hydro. space(ξ) depends on ξ

Solution: Rectify $\mathcal{L} - i(v \cdot \xi)$ to a matrix $L(\xi)$ by conjugating with

$$\Phi(\xi) : \text{hydro. space}(\xi) \xrightarrow{\text{iso.}} \ker \mathcal{L} \approx \mathbb{C}^{d+2}$$

Extra

Kato's reduction process



Projection on the hydrodynamic spectrum(ξ) = R$(\mathcal{P}(\xi))$:

$$\mathcal{P}(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} \left(z - \mathcal{L} + i(v\cdot\xi)\right)^{-1} \mathrm{d}z \in \mathscr{B}\left(V^{-1} \to V^{1}\right)$$

Kato's isomorphism:

$$\frac{\mathcal{P}(0)\mathcal{P}(\xi) + \mathcal{P}(0)^{\perp}\mathcal{P}(\xi)^{\perp}}{\sqrt{\mathrm{Id} - (\mathcal{P}(\xi) - \mathcal{P}(0))^2}} =: \Phi(\xi) \ : \ \text{hydro. space}(\xi) \xrightarrow{\text{iso}} \ker(\mathcal{L})$$

$$\Phi(\xi) = \mathrm{Id} + |\xi|\Phi^{(1)} + |\xi|^2\Phi^{(2)}, \qquad \Phi^{(j)} \in \mathscr{B}\left(V^{-1} \to V^{1}\right)$$

Rectified operator:

$$L(\xi) := \Phi(\xi)\big(\mathcal{L} - i(v\cdot\xi)\big)\Phi(\xi)^{-1} \in \mathscr{B}\left(\ker \mathcal{L}\right) \sim \mathbb{C}^{(d+2)\times(d+2)}$$

Extra

Diagonalization of $L(\xi) = \Phi(\xi)^{-1}\left(\mathcal{L} - i(v\cdot\xi)\right)\Phi(\xi)$

Problem: $|\xi|^{-1}L(\xi) \xrightarrow{\xi\to 0}$ matrix with **non-simple** eigenvalues

Solution: Isotropy of $\mathcal{L} \Rightarrow$ block representation of $L(\xi)$:

$$L(\xi) = \begin{pmatrix} \lambda_{\text{inc}}(\xi)\text{Id} & 0 \\ 0 & |\xi| M(\xi) \end{pmatrix}, \qquad \lambda_{\text{inc}}(\xi) = -\kappa_{\text{inc}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right),$$

in the decomposition

$$\{u\cdot v\mu \,|\, u \perp \xi\} \oplus \left\{\varrho\mu + \alpha\xi\cdot v\mu + e|v|^2\mu \,:\, (\varrho,\alpha,e)\in\mathbb{R}^3\right\}$$

where

$$M(\xi) = \begin{pmatrix} +ic & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -ic \end{pmatrix} + |\xi| \begin{pmatrix} -\kappa_{\text{osc}} & * & * \\ * & -\kappa_{\text{Fou}} & * \\ * & * & -\kappa_{\text{osc}} \end{pmatrix} + \mathcal{O}\left(|\xi|^2\right)$$

Conclusion: $L(\xi)$ diagonalizable + expansion of eigenprojector and eigenvalues:

$$\lambda_{\pm\text{osc}}(\xi) = \pm ic|\xi| - \kappa_{\text{osc}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right), \qquad \lambda_{\text{Fou}}(\xi) = -\kappa_{\text{Fou}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right)$$

$$L(\xi) = \sum_\star \lambda_\star(\xi)P_\star(\xi), \qquad P_\star(\xi) = P_\star^{(0)} + |\xi|P_\star^{(1)} + |\xi|^2 P_\star^{(2)}$$

Extra

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Consequence: expansion of $\Phi(\xi) \Rightarrow$ expansion of $\left(\mathcal{L} - i(v\cdot\xi)\right)_{|\text{hydro. space}(\xi)}$:

$$\left(\mathcal{L} - i(v\cdot\xi)\right)_{|\text{hydro. space}(\xi)} = \Phi(\xi)^{-1}L(\xi)\Phi(\xi)$$

Extra

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$$\left(\mathcal{L} - i(v \cdot \xi)\right)_{|\text{hydro. space}(\xi)} = \sum_{\star} \lambda_\star(\xi) \mathcal{P}_\star(\xi)$$

$$\mathcal{P}_\star(\xi) = \Phi(\xi)^{-1} P_\star(\xi) \Phi(\xi) = \mathcal{P}_\star^{(0)} + |\xi| \mathcal{P}_\star^{(1)} + |\xi|^2 \mathcal{P}_\star^{(2)}, \qquad \mathcal{P}_\star^{(j)} : V^{-1} \to V^1$$