DISCRIMINANT OF HYPERELLIPTIC CURVES

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Abstract. We prove the well-known smoothness criterion of a Weierstrass equation in terms of its discriminant.

It is well known that the smoothness of a hyperelliptic equation can be checked with its discriminant (Proposition 0.7) . In what follows we give a proof of this well known fact.

0.1. **Smoothness and resultants.** Let A be a ring and let $B = A[y]$ with $y^2 + Qy = P$ $(Q, P \in A)$. Then B is free of rank 2 over A, and we have the involution of B as A-algebra: $\sigma(y) = -y - Q$, the A-linear map trace $\mathrm{Tr}_{B/A}(b) = \sigma(b) - b$, the multiplicative map norm $\mathrm{N}_{B/A}(b) = \sigma(b)b$.

Lemma 0.1. *Let $b \in B$. Let $I = (2y + Q, b) \subseteq B$. Then we have*

$$\sqrt{I} = \sqrt{(F, \mathrm{N}_{B/A}(b))}.$$

Proof. The rhs is clearly contained in the lhs. Let $\mathfrak{p} \in \operatorname{Spec} B$ containing $(F, \mathrm{N}_{B/A}(b))$. We have to show that $\mathfrak{p} \supseteq I$.

First $\mathfrak{p} \ni 2y + Q = y - \sigma(y)$. This means that $\sigma(y) \equiv y \mod \mathfrak{p}$, therefore $\sigma(b) \equiv b \mod \mathfrak{p}$, so $b^2 \equiv \mathrm{N}_{B/A}(b) \equiv 0 \mod \mathfrak{p}$, hence $b \equiv 0 \mod \mathfrak{p}$. □

Now fix $g \geq 0$. Consider the polynomial ring

$$A_0 = R[b_0, ..., b_{g+1}, a_0, \dots, a_{2g+2}]$$

over a given ring R and $A = A_0[x]$. Let $Q = \sum_i b_i x^i, P = \sum_i a_i x^i \in A$ and $B := A[y]$, with $y^2 + Qy = P$. Then B is flat over A. The Jacobian criterion says that the primes $\mathfrak{p} \in \operatorname{Spec} B$ of non-smoothness over A_0 are those containing the ideal

$$I := (2y + Q, Q'y - P').$$

By the previous lemma,

$$\sqrt{I} = \sqrt{(F, G)} \subset B$$

where $F = -\mathrm{N}_{B/A}(2y + Q)$ and $G = \mathrm{N}_{B/A}(Q'y - P')$. We have

$$F = 4P + Q^2, \quad G = P'^2 - PQ'^2 + P'QQ' \in A_0[x].$$

Corollary 0.2. *Let $s \in \operatorname{Spec} A_0$. Then the fiber of* $\operatorname{Spec} B \to \operatorname{Spec} A_0$ *at s is singular (non-smooth) if and only if $F_s(x), G_s(x) \in k(s)[x]$ have a commun zero (in the algebraic closure of $k(s)$).*

Proof. Use the surjectivity of $\operatorname{Spec}(B \otimes_A k(s)) \to \operatorname{Spec} k(s)[x]$ to prove the if part. □

0.2. **Resultants and discriminant.** We see that the smoothness of the affine curve is controlled by $\operatorname{Res}(F, G)$. Next we relate it to a more common invariant, the discriminant of F. We have

$$16G = \mathrm{N}_{B/A}(4Q'y - 4P') = \mathrm{N}_{B/A}(2Q'(2y + Q) - F') = -4Q'^2F + F'^2$$

(note that $\mathrm{Tr}_{B/A}(2y + Q) = 0$.)

Take

$$R = \mathbb{Z}, \quad A_0 = \mathbb{Z}[a_i, b_j].$$

Then $\deg(-4Q'^2F + F'^2) = \deg F'^2$, so

$$\operatorname{Res}(F, 16G) = \operatorname{Res}(F, -4Q'^2F + F'^2) = \operatorname{Res}(F, F'^2) = \operatorname{Res}(F, F')^2$$

(wikipedia) where the first three resultants are in degrees $(2g+2, 4g+2)$, while the last one is in degrees $(2g + 2, 2g + 1)$. But

$$\operatorname{Res}(F, 16G) = 16^{\deg F}\operatorname{Res}(F, G) = 2^{8(g+1)}\operatorname{Res}(F, G).$$

This implies that

$$(\operatorname{Res}(F, F')/2^{4g+4})^2 = \operatorname{Res}(F, G) \in A_0.$$

Recall that if $V(x)$ is a polynomial of degree d with leading coeffcieint v_d, then $v_d \mid \operatorname{Res}(V, V')$ and by definition

$$\operatorname{disc}(V) = (-1)^{d(d-1)/2}v_d^{-1}\operatorname{Res}(V, V')$$

([2])

Denote by a, b the variables a_i, b_j. Let

$$c(a, b) := 4a_{2g+2} + b_{g+1}^2$$

be the leading coefficient of F. As $c(a, b)$ divides $\operatorname{Res}(F, F')$ and is irreducible and prime to 2 in $\mathbb{Z}[a, b]$, we have $c^2 \mid \operatorname{Res}(F, G)$. Hence

$$2^{-4(g+1)}\operatorname{disc}(F) \in A_0 = \mathbb{Z}[a_i, b_j].$$

Definition 0.3 With the above notation, define

$$\Delta_{2g+2}(a, b) = 2^{-4(g+1)}\operatorname{disc}(F) \in \mathbb{Z}[a, b].$$

By construction,

$$c(a, b)^2\Delta_{2g+2}^2(a, b) = \operatorname{Res}(F, G) \in \mathbb{Z}[a, b]. \tag{1}$$

Consider now generic polynomials $\deg Q = g$ and $\deg P = 2g+1$. Similarly to the previous case we have

$$(\mathrm{Res}(F, F')/2^{4g+2})^2 = \mathrm{Res}(F, G) \in \mathbb{Z}[a_i, b_j]_{0 \le i \le 2g+1, 0 \le j \le g}.$$

The leading coefficient of F is $4a_{2g+1}$. We have

$$a_{2g+1}^2 (\mathrm{disc}(F)/2^{4g})^2 = \mathrm{Res}(F, G),$$

hence $a_{2g+1}^2 \mid \mathrm{Res}(F, G)$ in $\mathbb{Z}[a, b]$.

Definition 0.4 We put

$$\Delta_{2g+1}(a, b) = 2^{-4g}\mathrm{disc}(F) \in \mathbb{Z}[a, b].$$

We have

$$a_{2g+1}^2 \Delta_{2g+1}(a, b) = \mathrm{Res}(F, G) \in \mathbb{Z}[a, b]. \tag{2}$$

Next we relate $\Delta_{2g+2}(a, b)$ to $\Delta_{2g+1}(a, b)$.

Proposition 0.5. *Let* $A(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots \in \mathbb{Z}[a_0, ..., a_d]$. *Then*

$$\mathrm{disc}(A)(a_0, ..., a_{d-1}, 0) = a_{d-1}^2 \mathrm{disc}(a_{d-1}x^{d-1} + \cdots + a_0).$$

Proof. See [1], A.IV.80, Corollaire 2. □

Corollary 0.6. *Denote by* (a, b) *the variables* $a_0, ..., a_{2g+2}, b_0, ..., b_{g+1}$ *and by* $\hat{a}, \hat{b}$ *the variables after removing* a_{2g+2} *and* b_{g+1}. *Then we have*

$$\Delta_{2g+2}(a, b)|_{a_{2g+2}=b_{g+1}=0} = a_{2g+1}^2 \Delta_{2g+1}(\hat{a}, \hat{b}).$$

Let $\hat{F} = 4(\sum_{i \le 2g+1} a_i x^i) + (\sum_{j \le g} b_j x^j)^2$. *Then*

$$\Delta_{2g+2}(a, b)|_{a_{2g+2}=b_{g+1}=0} = 2^{-4(g+1)}(4a_{2g+1})^2 \mathrm{disc}(\hat{F}).$$

Proposition 0.7. *Let* K *be a field. Consider the affine curve* C *over* K *defined by an equation*

$$y^2 + (\sum_{j \le g+1} t_j x^j) y = \sum_{i \le 2g+2} s_i x^i$$

with coefficients in K. *Denote by* $q(x) = \sum_j t_j x^j$ *and* $p(x) = \sum_i s_i x^i$. *Let* $\hat{C}$ *be the completion of* C *by gluing it with the affine curve* C_∞ *defined by the equation*

$$z^2 + (\sum_j t_j u^{g+1-j}) z = \sum_i s_i u^{2g+2-i}, \quad u = 1/x, z = y/x^{g+1}.$$

Then $\Delta_{2g+2}(s, t) \neq 0$ *if and only if* $\hat{C}$ *is smooth.*

Proof. We can suppose K algebraically closed, and $\mathrm{char}(K) = 2$ (otherwise the proof is easier by reducing to the case $q(x) = 0$.) Translating y by $\sqrt{s_{2g+2}}x^{g+1}$ (in C_∞, z is translated by $\sqrt{s_{2g+2}}$), we can suppose that $s_{2g+2} = 0$.

(1) Suppose $t_{g+1} \neq 0$. Then $C_\infty \to \operatorname{Spec} K[1/x]$ is étale above $x = \infty$, and $\hat{C}$ is smooth at ∞. We have

$$t_{g+1}^4 \Delta_{2g+2}(s,t)^2 = \mathrm{Res}(F,G)(s,t).$$

Let r be the degree of $G(s,t)(x) \in K[x]$. As $\deg F(s,t)(x) = 2g+2$ with leading coefficient t_{g+1}^2, we have

$$t_{g+1}^{2k} \mathrm{Res}(F(s,t)(x), G(s,t)(x)) = \mathrm{Res}(F,G)(s,t)$$

where $k = \deg G(x) - r$ (wikipedia). Therefore $\Delta_{2g+2}(s,t) \neq 0$ if and only if C is smooth.

(2) Suppose $t_{g+1} = 0$. We have

$$\Delta_{2g+2}(s,t) = s_{2g+1}^2 \Delta_{2g+1}(\hat{s}, \hat{t}).$$

If $\Delta_{2g+2}(s,t) \neq 0$, then $s_{2g+1} \neq 0$. Similarly to the previous case, the smoothness of C is then equivalent to $\Delta_{2g+1}(\hat{s}, \hat{t}) \neq 0$. Finally the condition $s_{2g+1} \neq 0$ is equivalent (when $s_{2g+2} = t_{g+1} = 0$) to the smoothness at ∞. This proves the statement when $t_{g+1} = 0$. □

References

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