STATIONARY FLOWS OF THE ES-BGK MODEL WITH THE CORRECT PRANDTL NUMBER

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ABSTRACT. The Ellipsoidal BGK model (ES-BGK) is a generalized version of the BGK model where the local Maxwellian in the relaxation operator of the BGK model is extended to an ellipsoidal Gaussian with a parameter $-1/2 \leq \nu < 1$, so that the correct Prandtl number can be computed in the Navier-Stokes limit. In this work, we consider steady rarefied flows arising from the evaporation and condensation process between two parallel condensed phases, which is formulated in this paper as the existence problem of stationary solutions to the ES-BGK model in a bounded interval with the mixed boundary conditions. One of the key difficulties arises in the uniform control of the temperature tensor from below. In the non-critical case $(-1/2 < \nu < 1)$, we utilize the property that the temperature tensor is equivalent to the temperature. In the critical case, $(\nu = -1/2)$, where such equivalence relation breaks down, we observe that the size of bulk velocity in x direction can be controlled by the discrepancy of boundary flux, which enables one to bound the temperature tensor from below.

1. INTRODUCTION

1.1. **Ellipsoidal BGK model:** The Boltzmann equation is a fundamental model that connects the particle regime and the fluid regime of rarefied gases. However, the practical application of the Boltzmann equation to various flow problems has been severely restricted by the intricate structure of the collision operator that requires a considerable amount of resources for numerical computations. The observation made by Bhatnaghar, Gross and Krook [12] in their attempts to overcome this difficulty is that the local equilibration occurs rather quickly so that the complicated process of collision can be successfully described by the relaxation process after a short time scale. The equation that was introduced based on this observation is now called the BGK model, and have enjoyed great popularity as a numerically amenable equation that provides qualitatively satisfactory results. There are, however, several shortcomings of the model. The most notable one is that the Prandtl number - the ratio between the thermal diffusivity and the viscosity, computed from the BGK model does not match the correct value computed from the Boltzmann equation, which means that the diffusivity and the viscosity in the Navier-Stokes limit cannot be correctly derived. In this regard, Holway proposed so-called the ellipsoidal BGK model (ES-BGK model), which generalizes the local Maxwellian of the BGK model to an ellipsoidal Gaussian endowed with an additional degree of freedom in adjusting the transport coefficients. ES-BGK model, however, was somewhat forgotten in the literature since it was not clear at the time whether the H-theorem holds for this model. This was resolved by Andries et al in [2] (and later in [16, 47]), which greatly popularized this model in the study of various problems in the rarefied gas dynamics. The existence result of the ES-model in the critical

case ($\nu = -1/2$), however, was never resolved so far except for the case where the solution lies close to equilibrium [68, 69], which is the main motivation of the current work.

In this paper, we are interested in the mathematical study of steady rarefied gas flows generated by the evaporation and condensation process between two condensed phases [3, 4, 56]. Using the ES-BGK model, this can be formulated as the boundary value problem as follows:

$$(1.1) \qquad v_1 \frac{\partial f}{\partial x} = \frac{1}{\kappa(1-\nu)} \left(\mathcal{M}_\nu(f) - f \right),$$

on a finite interval $[0, 1]$ where the boundary condition is given by the linear combination of the inflow boundary condition, the diffusive boundary condition, and the specular reflection ($\delta_1 + \delta_2 + \delta_3 = 1$):

$$(1.2)$$
$$f(0, v) = \delta_1 f_L(v) + \delta_2 \left(\int_{|v_1|>0} f(0, v)|v_1| dv \right) M_w(0, v) + \delta_3 f(0, Rv), \quad (v_1 > 0)$$

$$f(1, v) = \delta_1 f_R(v) + \delta_2 \left(\int_{|v_1|<0} f(1, v)|v_1| dv \right) M_w(1, v) + \delta_3 f(1, Rv). \quad (v_1 < 0)$$

Here M_w denotes the wall Maxwellians which, for a given wall temperature $T_w : \{0, 1\} \to \mathbb{R}_+$, is defined by

$$M_w(i, v) = \frac{1}{\sqrt{2\pi T_w(i)}} e^{-\frac{|v|^2}{2 T_w(i)}}. \qquad (i = 1, 2)$$

When there's no risk of confusion we denote both $M_w(0, v)$ and $M_w(1, v)$ by M_w. Rv denotes the reflection of v: $R(v_1, v_2, v_3) = (-v_1, v_2, v_3)$. We note that δ_1 term and δ_2 term corresponds to the condensation and the evaporation at the boundary [56].

The velocity distribution function $f(x, v)$ represents the number density of the gas molecules at the position $x \in [0, 1]$ with the microscopic velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. κ is the Knudsen number. The ellipsoidal Gaussian $\mathcal{M}_\nu(f)$ with the Prandtl parameter $\nu \in [-1/2, 1)$ reads

$$\mathcal{M}_\nu(f) = \frac{\rho}{\sqrt{\det(2\pi \mathcal{T}_\nu)}} \exp\left(-\frac{1}{2}(v - U)^\top \mathcal{T}_\nu^{-1}(v - U) \right).$$

The local density ρ, momentum U, temperature T and the stress tensor Θ are given by the following relations:

$$(1.3)$$
$$\rho(x) = \int_{\mathbb{R}^3} f(x, v) dv,$$
$$\rho(x)U(x) = \int_{\mathbb{R}^3} f(x, v) v \, dv,$$
$$3\rho(x)T(x) = \int_{\mathbb{R}^3} f(x, v)|v - U|^2 dv,$$
$$\rho(x)\Theta(x) = \int_{\mathbb{R}^3} f(x, v)(v - U) \otimes (v - U) dv,$$

and the temperature tensor $\mathcal{T}_\nu$ is defined as a linear combination of the temperature and the stress tensor:

$$\mathcal{T}_\nu = (1-\nu)T\mathbb{I} + \nu\Theta$$

$$= \begin{pmatrix} (1-\nu)T + \nu\Theta_{11} & \nu\Theta_{12} & \nu\Theta_{13} \\ \nu\Theta_{21} & (1-\nu)T + \nu\Theta_{22} & \nu\Theta_{23} \\ \nu\Theta_{31} & \nu\Theta_{32} & (1-\nu)T + \nu\Theta_{33} \end{pmatrix}$$

where $\mathbb{I}_3$ denotes the 3×3 identity matrix.

Note that this is not a convex combination since ν can take negative values. In the case $\nu = 0$, the ES-BGK model reduces to the original BGK model. The end-point $\nu = -1/2$ corresponds to ES-BGK model with the correct Prandtl number:

The Chapman Enskog expansion shows that the Prantl number of the Navier-Stokes equation derived from the ES-BGK model is $1/(1-\nu)$. Therefore, by choosing appropriate ν, we can derive the correct Prantl number in the hydrodynamic limit, which is achieved when $\nu = -1/2$. (See [2] for further discussion on this.)

For simplicity, we set $\tau = \kappa(1-\nu)$ throughout the paper and write (1.1) as

$$v_1\frac{\partial f}{\partial x} = \frac{1}{\tau}\big(\mathcal{M}_\nu(f) - f\big).$$

1.2. **Notations:** We first set up notational conventions and define norms:

- C denote generic constants. The value can change each line of computations, but it is explicitly computable in principle.
- $A \preceq B$ means that $A \leq CB$ for some constant C.
- $\mathbb{I}_3$ denotes the 3×3 identity matrix.
- We define f_{LR} and M_w by

$$f_{LR}(v) = f_L(v)1_{v_1>0} + f_R(v)1_{v_1<0}$$

  and

$$M_w(v) = M_w(0,v)1_{v_1>0} + M_w(1,v)1_{v_1<0}.$$

- We define $\sup_x \|\cdot\|_{L^1_2}$ by

$$\sup_x \|f\|_{L^1_2} = \sup_x \|f\|_{L^1_{2,+}} + \sup_x \|f\|_{L^1_{2,-}},$$

  where

$$\sup_x \|f\|_{L^1_{2,+}} = \sup_x \Big\{ \int_{v_1>0} |f(x,v)|(1+|v|^2)dv \Big\},$$

$$\sup_x \|f\|_{L^1_{2,-}} = \sup_x \Big\{ \int_{v_1<0} |f(x,v)|(1+|v|^2)dv \Big\}.$$

- We define the trace norm $\|\cdot\|_{L^1_{\gamma,|v_1|}}$ by

$$\|f\|_{L^1_{\gamma,|v_1|}} = \|f\|_{L^1_{\gamma,|v_1|,+}} + \|f\|_{L^1_{\gamma,|v_1|,-}},$$

  where the outward trace norm $\|f\|_{L^1_{\gamma,|v_1|,+}}$ and the inward trace norm $\|f\|_{L^1_{\gamma,|v_1|,-}}$ are given by

$$\|f\|_{L^1_{\gamma,|v_1|,+}} = \int_{v_1<0} |f(0,v)||v_1|dv + \int_{v_1>0} |f(1,v)||v_1|dv,$$

$$\|f\|_{L^1_{\gamma,|v_1|,-}} = \int_{v_1>0} |f(0,v)||v_1|dv + \int_{v_1<0} |f(1,v)||v_1|dv.$$

- Throughout the paper, we normalize the wall Maxwellian as follows:

$$\|M_w\|_{L^1_{\gamma,|v_1|,\pm}} = 1.$$

- Similarly, we define another trace norm $\|\cdot\|_{L^1_{\gamma,\langle v\rangle}}$ by

$$\|f\|_{L^1_{\gamma,\langle v\rangle}} = \|f\|_{L^1_{\gamma,\langle v\rangle,+}} + \|f\|_{L^1_{\gamma,\langle v\rangle,-}}.$$

  where the outward trace norm $\|\cdot\|_{L^1_{\gamma,\langle v\rangle,+}}$ and the inward trace norm $\|\cdot\|_{L^1_{\gamma,\langle v\rangle,-}}$ are given by

$$\|f\|_{L^1_{\gamma,\langle v\rangle,+}} = \int_{v_1<0} |f(0,v)|(1+|v|^2)dv + \int_{v_1>0} |f(1,v)|(1+|v|^2)dv,$$

$$\|f\|_{L^1_{\gamma,\langle v\rangle,-}} = \int_{v_1>0} |f(0,v)|(1+|v|^2)dv + \int_{v_1<0} |f(1,v)|(1+|v|^2)dv.$$

- Throughout the paper, $C_{LR,1}$, $C_{LR,2}$ denote

(1.4)
$$C_{LR,1} = \|f_{LM}\|_{L^1_{\gamma,\langle v\rangle}} \|M_w\|_{L^1_{\gamma,\langle v\rangle}},$$
$$C_{LR,2} = \|f_{LR}\|_{L^1_{\gamma,|v_1|}} + \|M_w\|_{L^1_{\gamma,\langle v\rangle}},$$

and $a_{\ell,1}$ and $a_{\ell,2}$ denote

(1.5)
$$a_{\ell,1} = \int_{\mathbb{R}^3} e^{-\frac{1}{|v_1|}} f_{LR}dv, \qquad a_{\ell,2} = \frac{1}{2}\int_{\mathbb{R}^3} e^{-\frac{1}{|v_1|}} M_w dv.$$

(P) Properties of boundary data: To avoid repetition in the statement of the theorem, we summarize here the assumptions to be imposed on f_{LR} later:

(P_1) The inflow boundary data $f_{LR} \geq 0$, not identically 0, has a finite trace norm:

$$\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} < \infty$$

(P_2) The inflow data does not induce vertical flows: :

$$\int_{\mathbb{R}^2} f_L v_i dv = \int_{\mathbb{R}^2} f_R v_3 dv = 0 \quad (i = 2, 3)$$

1.3. **Main result 1: inflow dominant case.** We now state our main results for the inflow dominant case ($\delta_1 \neq 0$). We first define the mild solution of (1.1) for the inflow dominant case as follows:

Definition 1.1. $f \in L^\infty\big([0,1]; L^1_2(\mathbb{R}^3)\big) \cap L^1_{\gamma,\langle v\rangle}(\mathbb{R}^3)$ is said to be a mild solution for (1.1) if it satisfies

(1.6)
$$f(x,v) = e^{-\frac{x}{\tau|v_1|}} f(0,v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{x-y}{\tau|v_1|}} \mathcal{M}_\nu(f)dy \quad \text{if } v_1 > 0$$

and

(1.7)
$$f(x,v) = e^{-\frac{1-x}{\tau|v_1|}\int_x^1 dy} f(1,v) + \frac{1}{\tau|v_1|}\int_x^1 e^{-\frac{y-x}{\tau|v_1|}} \mathcal{M}_\nu(f)dy \quad \text{if } v_1 < 0,$$

where $f(0, v)$ and $f(1, v)$ are defined in the trace sense as

$$f(0, v) = \delta_1 f_L(v) + \delta_2 \left(\int_{|v_1|<0} f(0, v)|v_1|dv \right) M_w(0) + \delta_3 f(0, Rv), \quad (v_1 > 0),$$

$$f(1, v) = \delta_1 f_R(v) + \delta_2 \left(\int_{|v_1|>0} f(1, v)|v_1|dv \right) M_w(1) + \delta_3 f(1, Rv), \quad (v_1 < 0).$$

We are now ready to state the main result of this paper:

Theorem 1.2. [Inflow dominant case: $\delta_1 \neq 0$]
(1) **Non-critical case** $(-1/2 < \nu < 1)$ **:** *Let* $-1/2 < \nu < 1$ *and* $\delta_1 \neq 0$. *Suppose* f_{LR} *satisfies* (P_1) *and* (P_2). *Then there exist* $\varepsilon > 0$ *depending on* f_{LR} *and* δ_i $(i = 1, 2, 3)$ *such that, if* $\tau > \varepsilon^{-1}$, *then there exists a unique mild solution* $f \geq 0$ *to the boundary value problem* *(1.1), (1.2) satisfying*

$$\int_{\mathbb{R}^3} f(x, v)dv \geq a_{\ell,1}, \quad \int_{\mathbb{R}^3} f(x, v)(1 + |v|^2)dv \leq 2C_{LR,1},$$

and

$$C_\nu^1 \delta_1^2 \frac{\gamma_{\ell,1}}{3C_{LR,1}^2} \leq \kappa^\top \left\{ \mathcal{T}_\nu \right\} \kappa \leq \frac{2}{3a_{\ell,1}\delta_1} C_\nu^2 C_{LR,1},$$

for all $\kappa \in \mathbb{R}^3$ *such that* $|\kappa| = 1$, *where* $\gamma_{\ell,1}$ *is defined by*

$$(1.8) \qquad \gamma_{\ell,1} = \left(\int_{v_1>0} e^{-\frac{1}{|v_1|}} f_L(v)|v_1|dv \right) \left(\int_{v_1<0} e^{-\frac{1}{|v_1|}} f_R(v)|v_1|dv \right) > 0.$$

(2) **Critical case** $(\nu = -1/2)$ **:** *Let* $\nu = -1/2$ *and* $\delta_1 \neq 0$. *Suppose* f_{LR} *satisfies* (P_1) *and* (P_2). *Then there exist* $\varepsilon > 0$ *depending on* f_{LR} *and* δ_i $(i = 1, 2, 3)$ *such that, if* $\tau > \varepsilon^{-1}$ *and*

$$\left| \int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv \right| \leq \varepsilon,$$

then there exists a unique mild solution $f \geq 0$ *to the boundary value problem (1.1), (1.2) satisfying*

$$\int_{\mathbb{R}^3} f(x, v)dv \geq \delta_1 a_{\ell,1}, \quad \int_{\mathbb{R}^3} f(x, v)|v|^2 dv \leq 2C_{LR,1},$$

and

$$\delta_1 \frac{a_{-1/2,1}}{2C_{LR,1}} \leq \kappa^\top \left\{ \mathcal{T}_{-1/2} \right\} \kappa \leq \frac{3}{2a_{\ell,1}} C_{LR,1},$$

for all $\kappa \in \mathbb{R}^3$ *such that* $|\kappa| = 1$, *where* $a_{-1/2,1}$ *denote*

$$(1.9) \qquad a_{-1/2,1} = \inf_{|\kappa|=1} \int_{\mathbb{R}^3} e^{-\frac{1}{|v_1|}} f_{LR} \left\{ |v|^2 - (v \cdot \kappa)^2 \right\} dv > 0.$$

Remark 1.3. (1). Since $\tau = \kappa(1 - \nu)$ in (1.1) is a parameter directly proportional to the Knudsen number, taking sufficiently large τ corresponds to the case when the gas is sufficiently rarefied.

(2). Our result states that the existence of the unique steady gas flow between two parallel condensed phases is guaranteed if the gas between the condensed phases is sufficiently rarefied. This makes sense in that, if the gas between the two phases are not sufficiently

rarefied compared to the condensed phases, the evaporation and condensation will not actively occur, and the gas flows between the two phases may not be generated. For example, if the gas is as dense as the condensed phases, no gas will flow between the two phases.

(3). By scaling argument, the boundary value problem (1.1), (1.2) in a unit interval can be reformulated as a boundary value problem of the ES-BGK model with the unit Knudsen number in a small interval of length κ. Therefore, our results also say that the stationary flow occurs between the condensed phases when the two condensed phases are sufficiently close.

1.4. **Main results 2: diffusive dominant case.** In the diffusive dominant case $(\delta_2 \neq 0)$, we impose the following flux control condition:

$$(1.10) \qquad \int_{v_1<0} f(0,v)|v_1|dv + \int_{v_1>0} f(1,v)|v_1|dv = 1.$$

which is adapted from the Maslova's argument [37]. This is because, without such additional assumptions, we generally don't have uniqueness for the boundary problem with diffusive boundary conditions (See the paragraphs following the Theorem 1.5.) Using (1.1), (1.2) and (1.10), we reformulate the boundary condition into (1.13), and we consider the following mild solution for the diffusive dominant case (Theorem 1.5). (See Section 7 for the detail of the reformulation.)

Definition 1.4. $f \in L^\infty\left([0,1]; L_2^1(\mathbb{R}^3)\right) \cap L^1_{\gamma,\langle v \rangle}(\mathbb{R}^3)$ is said to be a mild solution for (1.1) if it satisfies

$$(1.11) \qquad f(x,v) = e^{-\frac{x}{\tau|v_1|}} f(0,v) + \frac{1}{\tau|v_1|} \int_0^x e^{-\frac{x-y}{\tau|v_1|}} \mathcal{M}_\nu(f)dy \quad \text{if } v_1 > 0$$

and

$$(1.12) \qquad f(x,v) = e^{-\frac{1-x}{\tau|v_1|}} f(1,v) + \frac{1}{\tau|v_1|} \int_x^1 e^{-\frac{y-x}{\tau|v_1|}} \mathcal{M}_\nu(f)dy \quad \text{if } v_1 < 0,$$

where

$$(1.13) \qquad \begin{aligned} f(0,v) &= \delta_1 f_L(v) + \delta_2 \mathcal{S}_L(f) M_w(0) + \delta_3 f(0, Rv), & (v_1 > 0) \\ f(1,v) &= \delta_1 f_R(v) + \delta_2 \mathcal{S}_R(f) M_w(1) + \delta_3 f(1, Rv), & (v_1 < 0) \end{aligned}$$

and $\mathcal{S}_L(f)$, $\mathcal{S}_R(f)$ denote

$$\mathcal{S}_L(f) = \frac{1-\delta_1}{2-\delta_1} + \frac{\delta_1}{2-\delta_1} \int_{v_1<0} f_R|v_1|dv - \frac{1}{\tau(2-\delta_1)} \int_{v_1>0} \int_0^1 \mathcal{R}(y,v)dydv,$$

$$\mathcal{S}_R(f) = \frac{1-\delta_1}{2-\delta_1} + \frac{\delta_1}{2-\delta_1} \int_{v_1>0} f_L|v_1|dv - \frac{1}{\tau(2-\delta_1)} \int_{v_1<0} \int_0^1 \mathcal{R}(y,v)dydv,$$

with

$$\mathcal{R}(f)(x,v) = \mathcal{M}_\nu(f)(x,v) - f(x,v).$$

Theorem 1.5. [Diffusive dominant case: $\delta_2 \neq 0$]
(1) **Non-critical case** $(-1/2 < \nu < 1)$: Let $-1/2 < \nu < 1$ and $\delta_2 \neq 0$. Suppose f_{LR} satisfies (P_1). Then there exist $\varepsilon > 0$ depending on f_{LR} and δ_i $(i = 1,2,3)$ such that, if $\tau > \varepsilon^{-1}$, then there exists a unique mild solution $f \geq 0$ to the boundary value problem (1.1), (1.2) and (1.10) satisfying

$$\int_{\mathbb{R}^3} f(x,v)dv \geq a_{\ell,2}, \quad \int_{\mathbb{R}^3} f(x,v)(1+|v|^2)dv \leq 2C_{LR,2},$$

and

$$C_\nu^1 \delta_2^2 \frac{\gamma_{\ell,2}}{27 C_{LR,2}^2} \leq \kappa^\top \{\mathcal{T}_\nu\} \kappa \leq \frac{2}{3a_{\ell,2}} C_\nu^2 C_{LR,2},$$

for all $\kappa \in \mathbb{R}^3$ such that $|\kappa| = 1$, where $\gamma_{\ell,2}$ denotes

$$(1.14) \qquad \gamma_{\ell,2} = \left(\int_{v_1 > 0} e^{-\frac{1}{|v_1|}} M_w(0) |v_1| dv \right) \left(\int_{v_1 < 0} e^{-\frac{1}{|v_1|}} M_w(1) |v_1| dv \right) > 0.$$

(2) **Critical case** ($\nu = -1/2$): *Let $\nu = -1/2$ and $\delta_2 \neq 0$. Suppose f_{LR} satisfies (P_1). Then there exists $\varepsilon > 0$ depending on f_{LR} and δ_i ($i = 1, 2, 3$) such that, if $\tau > \varepsilon^{-1}$ and $\delta_1 < \varepsilon$, then there exists a unique mild solution $f \geq 0$ to the boundary value problem (1.1), (1.2) and (1.10) satisfying*

$$\int_{\mathbb{R}^3} f(x,v) dv \geq a_{\ell,2}, \quad \int_{\mathbb{R}^3} f(x,v) |v|^2 dv \leq 2 C_{LR,2},$$

and

$$\delta_2 \frac{a_{-1/2,2}}{4 C_{LR,2}} \leq \kappa^\top \left\{ \mathcal{T}_{-1/2} \right\} \kappa \leq \frac{3}{2a_{\ell,2}} C_{LR,2},$$

for all $\kappa \in \mathbb{R}^3$ such that $|\kappa| = 1$, where $a_{-1/2,2}$ denotes

$$(1.15) \qquad a_{-1/2,2} = \inf_{|\kappa|=1} \int_{\mathbb{R}^3} e^{-\frac{1}{|v_1|}} M_w \left\{ |v|^2 - (v \cdot \kappa)^2 \right\} dv > 0.$$

Remark 1.6. Note that we don't impose any smallness restriction on the discrepancy of the boundary flux for the critical case ($\nu = -1/2$) in the diffusive dominant case.

The basic strategy to prove our theorem is to rewrite the boundary value problem (1.1) - (1.2) in the mild form, set up an iteration scheme, and apply the contraction mapping theorem. Therefore, the most important step is to define an appropriate solution space in which the iteration map becomes a contraction. For this, we set up various estimates of the velocity distribution function and the macroscopic fields that are uniformly preserved through the iteration. Among others, the main difficulty lies in showing that the strict positive lower bound of the temperature tensor $\mathcal{T}_\nu$ is preserved through the iteration scheme.

In the non-critical case $-1/2 < \nu < 1$, the temperature tensor satisfies the following equivalence relation:

$$(1.16) \qquad \min\{1 - \nu, 1 + 2\nu\} T \mathbb{I}_3 \leq \mathcal{T}_\nu \leq \max\{1 - \nu, 1 + 2\nu\} T \mathbb{I}_3.$$

Therefore, it suffices to study the local temperature T, that can be shown to be bounded below by a quantity constructed from the boundary data. In the critical case, $\nu = -1/2$, however, the first inequality of (1.16) becomes trivial, giving no information on the strict positivity of the temperature tensor.

In the literature, the strict positivity of the temperature tensor is available only in the near-equilibrium regime, where the non-diagonal elements of the temperature tensor can be controlled arbitrarily small, enabling one to control the temperature tensor from below. In the current work, we establish an argument to prove such strict positive definiteness of the temperature tensor for general non-perturbative solutions.

Our main observation for this is that the bulk velocity in x direction can be controlled by the discrepancy of the boundary flux even without the smallness of δ_1 in the inflow dominance case:

$$(1.17) \qquad \left| U_1(x) \right| \leq \delta_1 \left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right| + O\Big(\delta_2, \delta_3, \tau^{-1} \Big),$$

which is physically relavent in that, if we don't have enough flux from both ends of the slab, we cannot expect fast flow inbetween. We then observe that the quadratic polynomial of $\mathcal{T}_{-1/2}$ can be expressed using the local temperature and the directional temperature in the critical case:

$$\kappa^\top \left\{ \mathcal{T}_{-1/2} \right\} \kappa = \frac{1}{\rho} \int_{\mathbb{R}^3} f |v - U|^2 dv - \frac{1}{\rho} \int_{\mathbb{R}^3} f \big\{ (v - U) \cdot \kappa \big\}^2 dv.$$

We mention that the concept of "directional temperature" was coined by Villani in [64], and was crucially used in the proof of entropy production estimates of the Boltzmann equation. This, with the use of (1.17), enables one to bound the temperature tensor in the critical case from below by a quantity defined only through the inflow boundary data and the inflow boundary flux:

$$\frac{1}{2} \inf_{|\kappa|=1} \int_{\mathbb{R}^3} e^{-\frac{2C_{LM,1}}{|v_1|}} f_{LR} \left\{ |v|^2 - (v \cdot \kappa)^2 \right\} dv - 2 \left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right|^2,$$

up to small error. The first term can roughly be interpreted as the difference of total energy minus the directional energy of the inflow boundary data away from zero, and the second term is the discrepancy of the flux at both ends. This enables one to bound the temperature tensor from below when δ_i $(i = 2, 3)$ and τ^{-1} are sufficiently small.

In the diffusive dominant case, similar argument is working but there is an important difference to be mentioned that, without additional assumption on the amount of flux given in (1.10), we cannot expect the uniqueness of the solutions. Consider the following simple boundary value problem with diffusive boundary condition:

$$v_1 \partial_x f = 0, \qquad f(i, v) = \left(\int_{(-1)^{i+1} v_1 > 0} f(i, v) |v_1| dv \right) M_w(i), \quad (i = 0, 1).$$

It can be easily checked that

$$f(x, v) = C_1 M_w(v, 0) 1_{v_1>0} + C_2 M_w(v, 1) 1_{v_1<0}$$

solves the problem for any $C_1, C_2 > 0$. In this regards, we impose the flux control condition (1.10) in this case.

Unlike the near-equilibrium problem, there is no diffusivity in our setting, therefore, the usual restriction such as fixing the mass of the gas is not sufficient to guarantee uniqueness. Some stronger restriction is required. We borrowed the flux control condition (1.10) from the work of Maslova [37]. We also mention that this is the main reason why we divide the proof into the inflow-dominant case and the diffusive-dominant case.

1.5. **Literature review.** We start with the results on the stationary problems of the BGK model in a slab, which is most relevent to the current work. The first existence theory for stationary BGK model can be found in [61], where Ukai applied the a version of Schauder fixed point theorem to solve the slab problem with inflow boundary condition. In [44], Nouri derived the existence of weak solutions for a quantum BGK model with a discretized condensation ansatz in a bounded interval. In [11], classical Banach fixed point argument

was developed to study the existence and uniqueness for slab problems for ES-BGK model. In [11], however, the boundary condition was limited to inflow boundary condition, and the case $\nu = -1/2$ is not treated, which is the main motivation of the current work. The argument of [11] was then applied to a relativistic BGK model [35] and to the quantum BGK model [10]. For the related results on the stationary Boltzmann equation, we refer to [6, 7, 8, 22, 23, 25, 26, 33] and references therein.

For the time dependent problems, it was Perthame who first obtained the existence of weak solutions [49] under the assumption of finite mass, momentum, energy and entropy. The unique mild solution was then found in [51] in a function space with sufficient decay in the velocity domain. Mischler extended this to the whole space in [43]. Zhang et al considered the L^p weak solution of the BGK model in [72]. For the asymptotic stability near global equilibriums, we refer to [9, 66]. Various macroscopic limit for the BGK type models, including the hydrodynamic limit at the Euler and Navier-Stokes limit, Diffusion limit, and fractional limit can be found in [24, 38, 39, 54, 55]. For the development or analysis of numerical schemes for BGK models, see [29, 30, 36, 40, 41, 52, 53] and rich references therein.

As was mentioned in the introduction, after the verification of H-theorem of ES-BGK model made in [2], the ES-BGK model got popularized a lot [1, 29, 31, 41, 73]. Brull et al developed a systematical way to derive of ES-BGK model and provided another proof of H-theorem in [16]. The entropy production estimate for ES-BGK model was obtained in [70]. For existence results, we refer to [45] for weak solutions, [67] for unique mild solution and [69] for the result in near-global-equilibrium regime. For related results for the ES-BGK model for polyatomic molecules, see [46, 47, 48, 68].

This paper is organized as follows. In Section 2, we set up an approximation scheme and the solution space for the inflow dominance case. In Section 3, we show that, under appropriate assumptions, the approximate solution stays in the solution space in each iteration. The lower bound estimate for the temperature tensor in the critical case ($\nu = -1/2$) is made. In Section 4, we prove the Cauchy estimate to complete the proof of Theorem 1.2. Section 5 is devoted to the proof of Theorem 1.5. Since many parts overlap with proof of Theorem 1.2, we focus on the difference of the argument.

2. Approximation scheme and solution space for inflow dominant case

In the following, we aim to construct the solution f^n for (1.1). According to (1.3), ρ^n, U^n, T^n and Θ^n represent the hyrodynamic quantities associated to f^n. The approximate solution approximate scheme reads:

$$f^n(x,v) = f^n(x,v)1_{v_1>0} + f^n(x,v)1_{v_1<0},$$

where f^n and f^n are determined iteratively by

$$(2.1) \qquad f^{n+1}(x,v) = e^{-\frac{x}{\tau|v_1|}}f^{n+1}(0,v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{x-y}{\tau|v_1|}}\mathcal{M}_\nu(f^n)dy \quad \text{if } v_1 > 0$$

and

$$(2.2) \qquad f^{n+1}(x,v) = e^{-\frac{1-x}{\tau|v_1|}}f^{n+1}(1,v) + \frac{1}{\tau|v_1|}\int_x^1 e^{-\frac{y-x}{\tau|v_1|}}\mathcal{M}_\nu(f^n)dy \quad \text{if } v_1 < 0$$

where $f^{n+1}(0,v)$ and $f^{n+1}(1,v)$ are defined by

(2.3)
$$f^{n+1}(0,v) = \delta_1 f_L(v) + \delta_2 \left(\int_{v_1<0} f^n(0,v)|v_1|dv \right) M_w + \delta_3 f^n(0,Rv), \quad (v_1 > 0),$$

$$f^{n+1}(1,v) = \delta_1 f_R(v) + \delta_2 \left(\int_{v_1>0} f^n(1,v)|v_1|dv \right) M_w + \delta_3 f^n(1,Rv), \quad (v_1 < 0).$$

We will show that $\{f^n\}_n$ constructed from the above scheme satisfies several uniform-in-n estimates. To do this in a more systematical way, we define two solution spaces. First we define the following solution space for the non-critical case $(-1/2 < \nu < 1)$:

$$\Omega_1 = \left\{ f \in L^\infty \left([0,1]; L_2^1(\mathbb{R}_v^3)\right) \cap L_{\gamma,\langle v \rangle}^1(\mathbb{R}_v^3) \mid f \text{ satisfies } (\mathcal{A}_1),(\mathcal{B}_1),(\mathcal{C}_1),(\mathcal{D}_1) \right\}$$

where $(\mathcal{A}_1)$, $(\mathcal{B}_1)$, $(\mathcal{C}_1)$ and $(\mathcal{D}_1)$ denote

- $(\mathcal{A}_1)$ f is non-negative:
$$f(x,v) \geq 0 \text{ for } x,v \in [0,1] \times \mathbb{R}^3.$$

- $(\mathcal{B}_1)$ The macroscopic field is well-defined:
$$\int_{\mathbb{R}^3} f(x,v)dv \geq \delta_1 a_{\ell,1}, \quad \int_{\mathbb{R}^3} f(x,v)(1+|v|^2)dv \leq 2C_{LR,1}.$$

- $(\mathcal{C}_1)$ The temperature tensor is well-defined:
$$C_\nu^1 \delta_1^2 \frac{\gamma_{\ell,1}}{3C_{LR,1}^2} \leq \kappa^\top \{\mathcal{T}_\nu\} \kappa \leq \frac{2}{3a_{\ell,1}} C_\nu^2 C_{LR,1}.$$

- $(\mathcal{D}_1)$ The trace is well-defined:
$$\|f\|_{L_{\gamma,|v_1|,\pm}^1} \leq 2\|f_{LR}\|_{L_{\gamma,|v_1|}^1}, \quad \|f\|_{L_{\gamma,\langle v \rangle,\pm}^1} \leq 2C_{LR,1}.$$

For the critical case $\nu = -1/2$, we define

$$\Omega_2 = \left\{ f \in L^\infty \left([0,1]; L_2^1(\mathbb{R}_v^3)\right) \cap L_{\gamma,\langle v \rangle}^1(\mathbb{R}_v^3) \mid f \text{ satisfies } (\mathcal{A}_2),(\mathcal{B}_2),(\mathcal{C}_2),(\mathcal{D}_2) \right\}$$

where $(\mathcal{A}_2)$, $(\mathcal{B}_2)$, $(\mathcal{C}_2)$ and $(\mathcal{D}_2)$ denote

- $(\mathcal{A}_2)$ f is non-negative:
$$f(x,v) \geq 0 \text{ for } x,v \in [0,1] \times \mathbb{R}^3.$$

- $(\mathcal{B}_2)$ The macroscopic field is well-defined:
$$\int_{\mathbb{R}^3} f(x,v)dv \geq \delta_1 a_{\ell,1}, \quad \int_{\mathbb{R}^3} f(x,v)(1+|v|^2)dv \leq 2C_{LR}.$$

- $(\mathcal{C}_2)$ The temperature tensor is well-defined:
$$\delta_1 \frac{a_{-1/2}}{2C_{LR,1}} \leq \kappa^\top \{\mathcal{T}_{-1/2}\} \kappa \leq \frac{3}{2a_{\ell,1}} C_{LR,1}.$$

- $(\mathcal{D}_2)$ The trace satisfies:
$$\|f\|_{L_{\gamma,|v_1|,\pm}^1} \leq 2\|f_{LR}\|_{L_{\gamma,|v_1|}^1}, \quad \|f\|_{L_{\gamma,\langle v \rangle,\pm}^1} \leq 2C_{LR,1}.$$

Before we move on to the proof of uniform estimates for f^n, we record a few estimates that will be fruitfully used throughout the paper.

Lemma 2.1. (1) *Let $f \in \Omega_1$. Then there exists positive constants C_1, C_2 depending only on the quantities (1.4), (1.5) and $\gamma_{\ell,1}$ such that*

$$\mathcal{M}_\nu(f) \le C_1 e^{-C_2|v|^2}.$$

(2) *Let $f \in \Omega_2$. Then there exists positive constants C_1, C_2 depending only on the quantities (1.4), (1.5) and $a_{-1/2}$ such that*

$$\mathcal{M}_{-1/2}(f) \le C_1 e^{-C_2|v|^2}.$$

Proof. We only consider the proof of (2) to avoid repetition. We first note that the macroscopic velocity is well-defined in Ω_2:

$$(2.4) \qquad |U| = \frac{|\rho U|}{\rho} = \frac{\left| \int_{\mathbb{R}^3} f v dv \right|}{\int_{\mathbb{R}^3} f dv} \le \frac{C_{LR}}{a_{\ell,1}}.$$

On the other hand, $(\mathcal{C}_2)$ implies that

$$(2.5) \qquad -(v-U)^\top \{\mathcal{T}\}^{-1}_{-1/2}(v-U) \le -\frac{3}{2a_{\ell,1}} C_{LR,1}|v-U|^2,$$

and

$$(2.6) \qquad \det \mathcal{T}_{-1/2} = \lambda_1 \lambda_2 \lambda_3 \ge \left\{ \delta_1 \frac{a_{-1/2,1}}{2C_{LR,1}} \right\}^3,$$

where λ_i $(i = 1, 2, 3)$ to be the eigenvalues of $\mathcal{T}_\nu$. Note that $\mathcal{T}_\nu$ is diagonalizable since it's symmetric. The desired then estimate follows immediately from (2.4), (2.5) and (2.6). $\square$

The following lemma can be found in [11]. We present the detailed proof for the readers' convenience.

Lemma 2.2. *Let C be a fixed positive constants. Then we have*

$$\int_0^x \int_{v_1>0} \frac{1}{\tau|v_1|} e^{-\frac{x-y}{\tau|v_1|}} e^{-Cv_1^2} dv_1 dy \le C \left(\frac{\ln \tau + 1}{\tau} \right), \qquad x \in [0,1]$$

where $C > 0$ depends only on quantities in (1.4) and (1.5).

First, we divide the domain of integration as follows:

$$\left\{ \int_0^x \int_{|v_1|<\frac{1}{\tau}} + \int_0^x \int_{\frac{1}{\tau} \le |v_1| < \tau} + \int_0^x \int_{|v_1| \ge \tau} \right\} \frac{1}{\tau|v_1|} e^{-\frac{x-y}{\tau|v_1|}} e^{-Cv_1^2} dv_1 dy \equiv I_1 + I_2 + I_3.$$

(a) The estimate of I_1: For I_1, we integrate on y first to get

$$I_1 = \int_{|v_1|<\frac{1}{\tau}} \left\{ \int_0^x \frac{1}{\tau|v_1|} e^{-\frac{x-y}{\tau|v_1|}} dy \right\} e^{-Cv_1^2} dv_1$$

$$= \int_{|v_1|<\frac{1}{\tau}} \left\{ 1 - e^{-\frac{x}{\tau|v_1|}} \right\} e^{-Cv_1^2} dv_1$$

$$\le \int_{|v_1|<\frac{1}{\tau}} dv_1$$

$$\le \frac{1}{\tau}.$$

(b) The estimate of I_2: For this case, we find

$$I_2 \leq \frac{1}{a_{\ell,1}} \int_{\frac{1}{\tau} \leq |v_1| \leq \tau} 1 - e^{-\frac{a_{\ell,1}x}{\tau|v_1|}} dv_1,$$

and apply the Taylor expansion to $1 - e^{-\frac{1}{\tau|v_1|}}$ to get

$$I_2 \leq \int_{\frac{1}{\tau} < |v_1| < \tau} \left\{ \left(\frac{1}{\tau|v_1|} \right) - \frac{1}{2!} \left(\frac{1}{\tau|v_1|} \right)^2 + \frac{1}{3!} \left(\frac{1}{\tau|v_1|} \right)^3 + \cdots \right\} dv_1$$

$$\leq \left| \int_{\frac{1}{\tau}}^{\tau} \frac{1}{\tau r} dr \right| + \left| \int_{\frac{1}{\tau}}^{\tau} \frac{1}{2!} \left(\frac{1}{\tau r} \right)^2 dr \right| + \left| \int_{\frac{1}{\tau}}^{\tau} \frac{1}{3!} \left(\frac{1}{\tau r} \right)^3 dr \right| + \cdots$$

$$= \frac{1}{\tau} \ln \tau^2 + \frac{1}{2!} \frac{1}{\tau} \frac{\tau^2 - 1}{\tau^2} + \frac{1}{2 \cdot 3!} \frac{1}{\tau} \frac{\tau^4 - 1}{\tau^4} + \frac{1}{3 \cdot 4!} \frac{1}{\tau} \frac{\tau^6 - 1}{\tau^6} \cdots$$

$$\leq \frac{1}{\tau} \ln \tau^2 + \frac{e}{\tau}.$$

(c) We compute the remaining I_3 as

$$I_3 \leq \int_0^1 \int_{|v_1| > \tau} \frac{1}{\tau|v_1|} e^{-Cv_1^2} dv_1 dy \leq \frac{1}{\tau^2} \int_{|v_1| > \tau} e^{-Cv_1^2} dv_1 \leq C_{\ell,u} \frac{1}{\tau^2}.$$

Finally, we combine the estimates (a), (b), (c) to obtain the desired result.

3. Uniform-in-n estimates of f^n for inflow dominant case

The main result of this section is the following proposition

Proposition 3.1. (1) Let $-1/2 < \nu < 1$. Assume f_{LR} satisfies the conditions of Theorem 1.1 (1). Then $f^n \in \Omega_1$ for all n.
(2) Let $\nu = -1/2$. Assume f_{LR} satisfies the conditions of Theorem 1.1 (2). Then, $f^n \in \Omega_2$ for all n.

We divide the proof into Lemma 3.1, Lemma 3.3, and Lemma 3.5.

Lemma 3.1. Assume $f^n \in \Omega_1$ or Ω_2. Then we have

$$f^{n+1} \geq 0 \quad and \quad \int_{\mathbb{R}^3} f^{n+1} dv \geq \delta_1 a_{\ell,1}.$$

Proof. From (2.1) and (2.2), we find

$$f^{n+1} \geq \delta_1 e^{-\frac{x}{\tau|v_1|}} f_L(v) 1_{v_1 > 0} + \delta_1 e^{-\frac{x}{\tau|v_1|}} f_R(v) 1_{v_1 < 0} \geq \delta_1 e^{-\frac{1}{|v_1|}} f_{LR}.$$

Here, we assumed $\tau > 1$ without loss of generality. Integrating with respect to dv, we obtain the desired lower bound:

$$\int_{\mathbb{R}^3} f^{n+1} dv \geq \delta_1 \int_{\mathbb{R}^3} e^{-\frac{1}{|v_1|}} f_{LR} dv = \delta_1 a_{\ell,1}.$$

$\square$

Lemma 3.2. (1) Let $f^n \in \Omega_1$ or Ω_2. Then we have

$$\|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}}, \ \|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}} \leq 2\|f_{LR}\|_{L^1_{\gamma,|v_1|}}.$$

(2) Let $f^n \in \Omega_1$ or Ω_2. Then we have

$$\|f^{n+1}\|_{L^1_{\gamma,\langle v \rangle,+}}, \ \|f^{n+1}\|_{L^1_{\gamma,\langle v \rangle,-}} \leq 2\|M_w\|_{L^1_{\gamma,\langle v \rangle}} \|f_{LR}\|_{L^1_{\gamma,\langle v \rangle}}.$$

Proof. (1) • Estimate for outflux $\|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}}$: Using (2.2), we can write $f^{n+1}(0,v)$ for $v_1 < 0$ as

$$
\begin{aligned}
(3.1) \qquad f^{n+1}(0,v) &= \delta_1 f_R + \delta_2 \left(\int_{v_1>0} f(1,v)|v_1|dv \right) M_w(1) + \delta_3 f^n(1,Rv) \\
&\quad + \frac{1}{\tau|v_1|} \int_0^1 e^{-\frac{y}{\tau|v_1|}} \mathcal{M}_\nu(f^n)dy
\end{aligned}
$$

which, in view of Lemma 2.2, yields

$$
\begin{aligned}
(3.2) \qquad &\int_{v_1<0} f^{n+1}(0,v)|v_1|dv \\
&\leq \delta_1 \int_{v_1<0} f_R|v_1|dv + \delta_2 \int_{v_1>0} f^n(1,v)|v_1|dv + \delta_3 \int_{v_1>0} f^n(1,v)|v_1|dv \\
&\quad + C_{\ell,u} \left(\frac{\ln\tau + 1}{\tau} \right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(3.3) \qquad &\int_{v_1>0} f^{n+1}(1,v)|v_1|dv \\
&\leq \delta_1 \int_{v_1>0} f_L|v_1|dv + \delta_2 \int_{v_1<0} f^n(0,v)|v_1|dv + \delta_3 \int_{v_1<0} f^n(0,v)|v_1|dv \\
&\quad + C_{\ell,u} \left(\frac{\ln\tau + 1}{\tau} \right).
\end{aligned}
$$

From (3.2) and (3.3), we obtain

$$
\begin{aligned}
(3.4) \qquad \|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}} &= \int_{v_1<0} f^{n+1}(0,v)|v_1|dv + \int_{v_1>0} f^{n+1}(1,v)|v_1|dv \\
&\leq \delta_1 \|f_{LR}\|_{L^1_{\gamma,|v_1|}} + (\delta_2+\delta_3)\|f^n\|_{L^1_{\gamma,|v_1|,+}} \\
&\quad + C_{\ell,u} \left(\frac{\ln\tau + 1}{\tau} \right).
\end{aligned}
$$

Therefore, in view of $\mathcal{D}_i$ of Ω_i $(i=1,2)$, we see that

$$
\begin{aligned}
(3.5) \qquad \|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}} &\leq \delta_1 \|f_{LR}\|_{L^1_{\gamma,|v_1|}} + 2(\delta_2+\delta_3)\|f_{LR}\|_{L^1_{\gamma,|v_1|}} + C_{\ell,u} \left(\frac{\ln\tau + 1}{\tau} \right) \\
&= 2(\delta_1+\delta_2+\delta_3)\|f_{LR}\|_{L^1_{\gamma,|v_1|}} - \delta_1 \|f_{LR}\|_{L^1_{\gamma,|v_1|}} + C_{\ell,u} \left(\frac{\ln\tau + 1}{\tau} \right) \\
&\leq 2\|f_{LR}\|_{L^1_{\gamma,|v_1|}}
\end{aligned}
$$

for sufficiently large τ.

• Estimate for influx: $\|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}}$: When $v_1 > 0$, we have from the boundary condition (2.1) that

$$
(3.6) \qquad f^{n+1}(0,v) = \delta_1 f_L + \delta_2 \left(\int_{v_1<0} f^n(0,v)|v_1|dv \right) M_w(0) + \delta_3 f^n(0,Rv).
$$

Integrate both sides with respect to $|v_1|dv$ on $v_1 > 0$ to get

(3.7)
$$\int_{v_1>0} f^{n+1}(0,v)|v_1|dv$$
$$= \delta_1 \int_{v_1>0} f_L |v_1|dv + \delta_2 \left(\int_{v_1>0} M_w(0)|v_1|dv \right) \left(\int_{v_1<0} f^n(0,v)|v_1|dv \right)$$
$$+ \delta_3 \int_{v_1>0} f^n(0,Rv)|v_1|dv$$
$$\leq \delta_1 \int_{v_1>0} f_L |v_1|dv + (\delta_2 + \delta_3) \int_{v_1<0} f^n(0,v)|v_1|dv,$$

where we used $\int_{v_1>0} M_w(0)|v_1|dv \leq 1$. Similarly, we estimate

(3.8)
$$\int_{v_1<0} f^{n+1}(1,v)|v_1|dv$$
$$= \delta_1 \int_{v_1<0} f_R |v_1|dv + \delta_2 \left(\int_{v_1<0} M_w(1)|v_1|dv \right) \left(\int_{v_1>0} f^n(1,v)|v_1|dv \right)$$
$$+ \delta_3 \int_{v_1>0} f^n(1,Rv)|v_1|dv$$
$$\leq \delta_1 \int_{v_1<0} f_R |v_1|dv + (\delta_2 + \delta_3) \int_{v_1>0} f^n(1,v)|v_1|dv,$$

Combining (3.7) and (3.8) gives

(3.9)
$$\|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}} = \int_{v_1>0} f^{n+1}(0,v)|v_1|dv + \int_{v_1<0} f^{n+1}(1,v)|v_1|dv$$
$$\leq \delta_1 \|f_{LR}\|_{L^1_{\gamma,|v_1|}} + (\delta_2 + \delta_3)\|f^n\|_{L^1_{\gamma,|v_1|,+}}.$$

Thanks to (3.5), we have

$$\|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}} \leq \delta_1 \|f_{LR}\|_{L^1_{\gamma,|v_1|}} + 2(\delta_2 + \delta_3)\|f_{LR}\|_{L^1_{\gamma,|v_1|}}$$
$$\leq 2\|f_{LR}\|_{L^1_{\gamma,|v_1|}}.$$

This completes the proof of (1). The proof of (2) is identical. We omit the proof. □

Lemma 3.3. *Let $f^n \in \Omega_1$ or Ω_2. Then we have*

$$\int_{\mathbb{R}^3} f^{n+1}(1+|v|^2)dv \leq 2\|f_{LR}\|_{L^1_{\gamma,\langle v \rangle}}\|M_w\|_{L^1_{\gamma,\langle v \rangle}}.$$

Proof. We integrate (2.1) w.r.t $(1+|v|^2)dv$ on $v_1 > 0$ to get

$$\int_{v_1>0} f^{n+1}(x,v)(1+|v|^2)dv = \int_{v_1>0} e^{-\frac{x}{\tau|v_1|}} f^{n+1}(0,v)(1+|v|^2)dv$$
$$+ \int_{v_1>0} \int_0^x \frac{1}{\tau|v_1|} e^{-\frac{x-y}{\tau|v_1|}} \rho^n(y)\mathcal{M}_\nu(f_n)(1+|v|^2)dydv.$$

We compute the boundary terms in $\int_{\mathbb{R}^3} f^{n+1}(1+|v|^2)dv$ as

$$\int_{v_1>0} e^{-\frac{x}{\tau|v_1|}} f^{n+1}(0,v)(1+|v|^2)dv$$

$$\leq \int_{v_1>0} f^{n+1}(0,v)(1+|v|^2)dv$$

$$\leq \delta_1 \int_{v_1>0} f_L(v)(1+|v|^2)dv + \delta_2 \left(\int_{v_1<0} f^n(0,v)|v_1|dv \right) \|M_w\|_{L^1_{\gamma,\langle v\rangle,-}}$$

$$+ \delta_3 \int_{v_1<0} f^n(0,v)(1+|v|^2)dv.$$

Thanks to Lemma 2.1, we can compute the source term as

$$\int_{v_1>0} \int_0^x \frac{1}{\tau|v_1|} e^{-\frac{x-y}{\tau|v_1|}} \mathcal{M}_\nu(f)(1+|v|^2)dydv$$

$$\leq C_{LM,1} \int_0^x \int_{v_1>0} \frac{1}{\tau|v_1|} e^{-\frac{x-y}{\tau|v_1|}} e^{-Cv_1^2} dv_1 dy$$

$$\equiv C_{LM,1} \left(\frac{\ln\tau+1}{\tau} \right).$$

Therefore,

$$\int_{v_1>0} f^{n+1}(x,v)(1+|v|^2)dv$$

$$= \delta_1 \int_{v_1>0} f_L(v)(1+|v|^2)dv + \delta_2 \left(\int_{v_1<0} f^n(0,v)|v_1|dv \right) \|M_w\|_{L^1_{\gamma,\langle v\rangle,-}}$$

$$+ \delta_3 \int_{v_1<0} f^n(0,v)(1+|v|^2)dv + C_{\ell,u} \left(\frac{\ln\tau+1}{\tau} \right).$$

We can have similar result for the case $v_1<0$:

$$\int_{v_1<0} f^{n+1}(x,v)(1+|v|^2)dv$$

$$= \delta_1 \int_{v_1<0} f_R(v)(1+|v|^2)dv + \delta_2 \left(\int_{v_1>0} f^n(1,v)|v_1|dv \right) \|M_w\|_{L^1_{\gamma,\langle v\rangle,+}}$$

$$+ \delta_3 \int_{v_1>0} f^n(1,v)(1+|v|^2)dv + C_{\ell,u} \left(\frac{\ln\tau+1}{\tau} \right).$$

Summing up and using $\mathcal{D}_i$ of Ω_i $(i=1,2)$, we get

$$\int_{\mathbb{R}^3} f^{n+1}(x,v)(1+|v|^2)dv$$

$$\leq \delta_1 \|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \delta_2 \|M_w\|_{L^1_{\gamma,\langle v\rangle}} \|f^n\|_{L^1_{\gamma,|v_1|,+}} + \delta_3 \|f^n\|_{L^1_{\gamma,\langle v\rangle,+}} + C_{\ell,u} \left(\frac{\ln\tau+1}{\tau} \right)$$

$$\leq \delta_1 \|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \delta_2 \|M_w\|_{L^1_{\gamma,\langle v\rangle}} \|f^n\|_{L^1_{\gamma,\langle v_1\rangle}} + 2\delta_3 \|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} \|M_w\|_{L^1_{\gamma,\langle v\rangle}}$$

$$+ C_{\ell,u} \left(\frac{\ln\tau+1}{\tau} \right).$$

Since
$$1 = \|M_w\|_{L^1_{\gamma,|v_1|}} \leq \|M_w\|_{L^1_{\gamma,\langle v\rangle}},$$
we have
$$\int_{\mathbb{R}^3} f^{n+1}(x,v)(1+|v|^2)dv$$
$$\leq \delta_1\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}} + \delta_2\|f^n\|_{L^1_{\gamma,\langle v_1\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}} + 2\delta_3\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}}$$
$$+ C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right)$$
$$\leq 2(\delta_1 + \delta_2 + \delta_3)\|f^n\|_{L^1_{\gamma,\langle v_1\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}} - (\delta_1 + \delta_3)\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}}$$
$$+ C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right)$$
$$\leq 2\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}}.$$
$$\square$$

In the following lemma, we show that (1) the bulk velocity in x direction is controlled by the discrepancy of the boundary flux, and (2) the bulk velocities in other directions can be taken arbitrarily small.

Lemma 3.4. *Let $f^n \in \Omega_1$ or Ω_2.*
(1) For $i = 1$, we have
$$\left|\int_{\mathbb{R}^3} f^{n+1}v_1 dv\right| \leq \delta_1\left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right| + 2\left(\delta_2 + \delta_3 + \frac{2}{\tau}\right)C_{LR,1}$$
where $C_{LR,1}$ denotes
$$C_{LR,1} = \|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}}.$$
(2) For $i = 2,3$, we have
$$\left|\int_{\mathbb{R}^3} f^{n+1}v_i dv\right| \leq 2\delta_3\|f_{LR}\|_{L^1_{\gamma,|v_1|}}\|M_w\|_{L^1_{\gamma,|v_1|}} + C_{\ell,u}\left(\frac{\ln\tau + 1}{\tau}\right).$$

Proof. (1) For $v_1 > 0$, (2.1) has the equivalent mild form:
$$f^{n+1}(x,v) = f^{n+1}(0,v) + \frac{1}{\tau|v_1|}\int_0^x \left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dy$$
$$= f^{n+1}(0,v) + \frac{1}{\tau v_1}\int_0^x \left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dy.$$
We multiply v_1 and integrate on $v_1 > 0$ to get
(3.10)
$$\int_{v_1>0} f^{n+1}(x,v)v_1 dv = \int_{v_1>0} f^{n+1}(0,v)|v_1|dv + \frac{1}{\tau}\int_0^x\int_{v_1>0}\left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dydv.$$
In the case $v_1 < 0$, we have from (2.2) that
$$f^{n+1}(x,v) = f^{n+1}(1,v) + \frac{1}{\tau|v_1|}\int_1^x \left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dy$$
$$= f^{n+1}(1,v) - \frac{1}{\tau v_1}\int_1^x \left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dy.$$

Integrating with respect to $v_1 dv$ on $v_1 < 0$:

$$\int_{v_1<0} f^{n+1}(x,v)v_1 dv = \int_{v_1<0} f^{n+1}(1,v)v_1 dv - \frac{1}{\tau}\int_1^x \int_{v_1<0}\left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dydv$$

(3.11)

$$= -\int_{v_1<0} f^{n+1}(1,v)|v_1|dv + \frac{1}{\tau}\int_x^1 \int_{v_1<0}\left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dydv.$$

From (3.10) and (3.11), we have

$$\int_{\mathbb{R}^3} f^{n+1}(x,v)v_1 dv = I + II,$$

where

$$I = \int_{v_1>0} f^{n+1}(0,v)|v_1|dv - \int_{v_1<0} f^{n+1}(1,v)|v_1|dv$$

and

$$II = \frac{1}{\tau}\int_0^x \int_{v_1>0}\left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dvdy + \frac{1}{\tau}\int_x^1 \int_{v_1<0}\left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dydv.$$

(a) The estimate of I: We observe from the boundary condition that

$$\int_{v_1>0} f^{n+1}(0,v)|v_1|dv$$

(3.12)
$$= \delta_1 \int_{v_1>0} f_L|v_1|dv + \delta_2 \left(\int_{v_1<0} f^n(0,v)|v_1|dv\right)\left(\int_{v_1>0} M_w|v_1|dv\right)$$

$$+ \delta_3 \int_{v_1>0} f^n(0,Rv)|v_1|dv$$

and

$$\int_{v_1<0} f^{n+1}(1,v)|v_1|dv$$

(3.13)
$$= \delta_1 \int_{v_1<0} f_R|v_1|dv + \delta_2 \left(\int_{v_1>0} f^n(1,v)|v_1|dv\right)\left(\int_{v_1<0} M_w|v_1|dv\right)$$

$$+ \delta_3 \int_{v_1>0} f^n(1,v)|v_1|dv.$$

Taking the difference of (3.12) and (3.13), we estimate I as

$$|I| = \left|\int_{v_1>0} f^{n+1}(0,v)|v_1|dv - \int_{v_1<0} f^{n+1}(1,v)|v_1|dv\right|$$

$$\leq \delta_1\left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right| + \delta_2\|f^n\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,|v_1|}} + \delta_3\|f^n\|_{L^1_{\gamma,\langle v\rangle}}$$

$$\leq \delta_1\left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right| + 2(\delta_2 + \delta_3)\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\|M_w\|_{L^1_{\gamma,\langle v\rangle}}.$$

In the last line, we used Lemma 5.4.

(b) The estimate for II: For II, we compute

$$
\begin{aligned}
|II| &\leq \frac{1}{\tau} \int_0^x \int_{v_1>0} \left| \mathcal{M}_\nu(f^n) - f^{n+1} \right| dv dy + \frac{1}{\tau} \int_x^1 \int_{v_1<0} \left| \mathcal{M}_\nu(f^n) - f^{n+1} \right| dy dv \\
&\leq \frac{1}{\tau} \int_0^1 \int_{v_1>0} \left\{ \mathcal{M}_\nu(f^n) + f^{n+1} \right\} dv dy + \frac{1}{\tau} \int_0^1 \int_{v_1<0} \left\{ \mathcal{M}_\nu(f^n) + f^{n+1} \right\} dy dv \\
&= \frac{1}{\tau} \int_0^1 \int_{\mathbb{R}^3} \left\{ \mathcal{M}_\nu(f^n) + f^{n+1} \right\} dv dy \\
&= \frac{1}{\tau} \int_0^1 \left\{ \rho^n + \rho^{n+1} \right\} dy \\
&\leq \frac{4}{\tau} \| f_{LR} \|_{L^1_{\gamma,\langle v \rangle}} \| M_w \|_{L^1_{\gamma,\langle v \rangle}}.
\end{aligned}
$$

In the last line, we used $(\mathcal{B}_i)$ $(i = 1, 2)$:

$$
\rho^n \leq 2 \| f_{LR} \|_{L^1_{\gamma,\langle v \rangle}} \| M_w \|_{L^1_{\gamma,\langle v \rangle}}
$$

and Lemma 3.3:

$$
\rho^{n+1} \leq 2 \| f_{LR} \|_{L^1_{\gamma,\langle v \rangle}} \| M_w \|_{L^1_{\gamma,\langle v \rangle}}.
$$

Now, we combine (a) and (b) to obtain the desired result.

(2) We only prove the case $i{=}2$. For $v_1 > 0$, we integrate (2.1) with respect to $v_2 dv_2 dv_3$ to get

$$
\begin{aligned}
\int_{\mathbb{R}^2} f^{n+1}(x, v) v_2 dv_2 dv_3 &= e^{-\frac{x}{\tau |v_1|}} \int_{\mathbb{R}^2} f^{n+1}(0, v) v_2 dv_2 dv_3 \\
&+ \frac{1}{\tau |v_1|} \int_0^x e^{-\frac{x-y}{\tau |v_1|}} \int_{\mathbb{R}^2} \mathcal{M}_\nu(f^n) v_2 dv_2 dv_3 dy
\end{aligned}
$$

From (P_2) of f_L that no vertical flow is induced from f_{LR}, and the fact that $M_w v_2$ is odd in v_2, we have

$$
\int_{\mathbb{R}^2} f^{n+1}(0, v) v_2 dv_2 dv_3 = \delta_3 \int_{v_1<0} f^n(0, v) dv
$$

which, together with Property $(\mathcal{B}_1)$ or $(\mathcal{B}_2)$ and Lemma 2.1 gives

$$
\begin{aligned}
\int_{\mathbb{R}^2} f^{n+1}(x, v) v_2 dv_2 dv_3 &\leq \delta_3 \int_{\mathbb{R}^2} f^n(0, v) v_2 dv_2 dv_3 \\
&+ C_{\ell,u} \frac{1}{\tau |v_1|} \int_{\mathbb{R}^6} \int_0^x e^{-\frac{x-y}{\tau |v_1|}} e^{-C|v|^2} dy dv_2 dv_3.
\end{aligned}
$$

Now, integrating on $v_1 > 0$ and recalling Lemma 2.2, we get

$$
\left| \int_{v_1>0} f^{n+1}(x, v) v_2 dv \right| \leq \delta_3 \left| \int_{v_1<0} f^n(0, v) v_2 dv \right| + C_{\ell,u} \left(\frac{\ln \tau + 1}{\tau} \right).
$$

Applying the same type of argument to (2.2), we can derive

$$
\left| \int_{v_1<0} f^{n+1}(x, v) v_2 dv \right| \leq \delta_3 \left| \int_{v_1>0} f^n(1, v) v_2 dv \right| + C_{\ell,u} \left(\frac{\ln \tau + 1}{\tau} \right).
$$

We sum these up them to obtain

$$\left| \int_{v_1 < 0} f^{n+1}(x,v) v_2 dv \right| \le 2\delta_3 \|f_{LR}\|_{L^1_{\gamma,\langle v_1 \rangle}} \|M_w\|_{L^1_{\gamma,\langle v_1 \rangle}} + C_{\ell,u} \left(\frac{\ln \tau + 1}{\tau} \right).$$

Here, we used Lemma 5.4 (2). □

In the following lemma, we show that the quadratic polynomial of the temperature tensor can be controlled from below and above. We mention that the critical case ($\nu = -1/2$) has never been treated in the literature so far, except for the near-global-equilibrium regime [68, 69].

Lemma 3.5. (1) *Let* $-1/2 < \nu < 1$. *Assume* $f^n \in \Omega_1$. *Then, for sufficiently large* τ*, we have*

$$C_\nu^1 \delta_1^2 \frac{\gamma_{\ell,1}}{3C_{LR,1}^2} \le \kappa^\top \left\{ \mathcal{T}_\nu^{n+1} \right\} \kappa \le \frac{2}{3a_{\ell,1}\delta_1} C_\nu^2 C_{LR,1}$$

for any $\kappa \in \mathbb{R}$ *and* $|\kappa| = 1$.
(2) *Let* $\nu = -1/2$ *and* $f \in \Omega_2$. *Then, for sufficiently large* τ*, we have*

$$\delta_1 \frac{a_{-1/2,1}}{2C_{LR,1}} \le \kappa^\top \left\{ \mathcal{T}_{-1/2}^{n+1} \right\} \kappa \le \frac{3}{2a_{\ell,1}\delta_1} C_{LR,1}$$

for any $\kappa \in \mathbb{R}$ *and* $|\kappa| = 1$.

Remark 3.6. We recall that where $\gamma_{\ell,1}$ and $a_{-1/2,1}$ are defined in (1.8) and (1.9) respectively.

Proof. (1) We recall the following equivalence estimate [11, 69] which holds for $-1/2 < \nu < 1$:

$$(3.14) \qquad C_\nu^1 T^{n+1} \mathbb{I}_3 \le \mathcal{T}_\nu^{n+1} \le C_\nu^2 T^{n+1} \mathbb{I}_3,$$

where $C_\nu^1 = \min\{1 - \nu, 1 + 2\nu\}$ and $C_\nu^2 = \max\{1 - \nu, 1 + 2\nu\}$. Therefore, it is enough to derive the lower and upper bound of T^{n+1}. The upper bound follows easily from Lemma 3.1 and Lemma 3.3:

$$(3.15) \qquad T^{n+1} \le \frac{1}{3\rho^{n+1}} \int_{\mathbb{R}^3} f^{n+1} |v|^2 dv \le \frac{2}{3\delta_1 a_{\ell,1}} \|f_{LR}\|_{L^1_{\gamma,\langle v \rangle}} \|M_w\|_{L^1_{\gamma,\langle v \rangle}}.$$

Now we turn to the lower bound of T^{n+1}. Since $f^{n+1} \ge 0$ and $|v| \ge |v_1|$, we have from the Cauchy-Schwarz inequality that

$$3\{\rho^{n+1}\}^2 T^{n+1} = \left(\int_{\mathbb{R}^3} f^{n+1} dv \right) \left(\int_{\mathbb{R}^3} f^{n+1} |v|^2 dv \right) - \left| \int_{\mathbb{R}^3} f^{n+1} v dv \right|^2$$

$$\ge \left(\int_{\mathbb{R}^3} f^{n+1} |v_1| dv \right)^2 - \left| \int_{\mathbb{R}^3} f^{n+1} v dv \right|^2.$$

Then we decompose according to whether they contain vertical flow or not:

$$3\{\rho^{n+1}\}^2 T^{n+1} \ge \left(\int_{\mathbb{R}^3} f^{n+1} |v_1| dv \right)^2 - \left\{ \sum_{1 \le i \le 3} \left| \int_{\mathbb{R}^3} f^{n+1} v_i dv \right| \right\}^2$$

$$(3.16) \qquad\qquad = \left(\int_{\mathbb{R}^3} f^{n+1} |v_1| dv \right)^2 - \left(\int_{\mathbb{R}^3} f^{n+1} v_1 dv \right)^2 - R$$

$$\equiv I - R,$$

where

$$R = \sum_{(i,j) \neq (1,1)} \left| \int_{\mathbb{R}^3} f^{n+1} v_i dv \right| \left| \int_{\mathbb{R}^3} f^{n+1} v_j dv \right|.$$

Since $f^n \in \Omega_1$, we see from Lemma 3.4 that R can be taken to be arbitrarily small by taking τ sufficiently large:

$$R = O\left(\delta_3, (\ln \tau + 1)\tau^{-1}\right).$$

For I, we use $a^2 - b^2 = (a-b)(a+b)$ to compute

$$\begin{aligned}
I &\geq \left\{ \int_{\mathbb{R}^3} f^{n+1}(|v_1| + v_1)dv \right\} \left\{ \int_{\mathbb{R}^3} f^{n+1}(|v_1| - v_1)dv \right\} \\
&= 4 \left\{ \int_{v_1 > 0} f^{n+1}|v_1|dv \right\} \left\{ \int_{v_1 < 0} f^{n+1}|v_1|dv \right\} \\
&\geq 4\delta_1^2 \left(\int_{v_1 > 0} e^{-\frac{1}{|v_1|}} f_L |v_1| dv \right) \left(\int_{v_1 < 0} e^{-\frac{1}{|v_1|}} f_R |v_1| dv \right) \\
&= 4\delta_1^2 \gamma_{\ell,1}.
\end{aligned}$$

In the last line, we used (2.1) as

$$\begin{aligned}
f^{n+1} &\geq \delta_1 e^{-\frac{x}{\tau|v_1|}} f_L 1_{v_1 > 0} + \delta_1 e^{-\frac{1-x}{\tau|v_1|}} f_R 1_{v_1 < 0} \\
&\geq \delta_1 e^{-\frac{1}{|v_1|}} f_L 1_{v_1 > 0} + \delta_1 e^{-\frac{1}{|v_1|}} f_R 1_{v_1 < 0}
\end{aligned}$$

and $\tau > 1$. Therefore, for sufficiently large τ, we get

$$(3.17) \qquad T^{n+1} \geq \frac{1}{3\{\rho^{n+1}\}^2} \left\{ 4\delta_1^2 \gamma_{\ell,1} - C\left(\frac{\ln \tau + 1}{\tau}\right) \right\} \geq \delta_1^2 \frac{\gamma_{\ell,1}}{3C_{LR,1}^2}.$$

Thanks to Lemma 3.3. Finally, we put (3.15) and (3.17) into (3.14) to get the desired result.

(2) In this critical case, the l.h.s of the equivalence type estimate (3.14) become trivial, and does not give any meaningful information about the positivity of the temperature tensor. Therefore, we need to take a more careful look in the structure of the temperature tensor directly. For this, we observe that the quadratic polynomial of the temperature tensor can be written in terms of the local energy and the directional local energy in the critical case $\nu = -1/2$:

$$\begin{aligned}
(3.18) \qquad \left[\kappa^\top \left\{ \mathcal{T}_{-1/2}^{n+1} \right\} \kappa \right] &= \frac{1}{\rho^{n+1}} \int_{\mathbb{R}^3} f^{n+1} \left\{ |v|^2 - (v \cdot \kappa)^2 \right\} dv \\
&\quad - \frac{1}{\rho^{n+1}} \left\{ \rho^{n+1}|U^{n+1}|^2 - \rho^{n+1}(U^{n+1} \cdot \kappa)^2 \right\} \\
&\equiv I - II,
\end{aligned}$$

for $|\kappa| = 1$

(i) Upper bound of $I + II$: Since,

$$|v|^2 - (v \cdot \kappa)^2 \leq |v|^2 \quad \text{and} \quad \rho^{n+1}|U^{n+1}|^2 - \rho^{n+1}(U^{n+1} \cdot \kappa)^2 \geq 0,$$

We can ignore II and employ Lemma 3.1 and Lemma 3.3 for I to get

$$\kappa^\top \left\{ \mathcal{T}^{n+1}_{-1/2} \right\} \kappa \leq \frac{1}{\rho^{n+1}} \int_{\mathbb{R}^3} f^{n+1} |v|^2 dv \leq \frac{2}{a_{\ell,1}\delta_1} \|f_{LR}\|_{L^1_{\gamma,\langle v \rangle}} \|M_w\|_{L^1_{\gamma,\langle v \rangle}}.$$

(ii) Lower bound of $I + II$: For this, we derive the lower bound for the quadratic polynomial of $\mathcal{T}_{-1/2}$ by combining the lower bound of I and smallness of II:

$(ii\text{-}a)$ Lower bound of I: Since $|v|^2 - (v \cdot \kappa)^2 \geq 0$, we observe from (2.1), (2.2) and Lemma 3.3 that

$$\begin{aligned}
\rho^{n+1} I &= \int_{\mathbb{R}^3} f^{n+1} \left\{ |v|^2 - (v \cdot \kappa)^2 dv \right\} \\
&\geq \delta_1 \int_{\mathbb{R}^3} \left\{ e^{-\frac{x}{\tau|v_1|}} f_L 1_{v_1>0} + e^{-\frac{1-x}{\tau|v_1|}} f_R 1_{v_1<0} \right\} \left\{ |v|^2 - (v \cdot \kappa)^2 \right\} dv \\
&\geq \delta_1 \inf_{|\kappa|=1} \int_{\mathbb{R}^3} e^{-\frac{1}{\tau|v_1|}} f_{LR} \left\{ |v|^2 - (v \cdot \kappa)^2 \right\} dv.
\end{aligned}$$

Since we are assuming τ is sufficiently large, we assume without loss of generality that $\tau > 1$, so that

$$\rho^{n+1} I \geq \delta_1 \inf_{|\kappa|=1} \int_{\mathbb{R}^3} e^{-\frac{1}{|v_1|}} f_{LR} \left\{ |v|^2 - (v \cdot \kappa)^2 \right\} dv = \delta_1 a_{-1/2}.$$

$(ii\text{-}b)$ Smallness of II: Thanks to Lemma 3.4, we see that II can be controlled, up to small error, by the discrepancy of the boundary flux:

$$\begin{aligned}
II &\leq \frac{|\rho^{n+1} U^{n+1}|^2}{\rho^{n+1}} \leq \frac{1}{\delta_1 a_{\ell,1}} \left| \int_{\mathbb{R}^3} f^{n+1} v\, dv \right|^2 \leq \frac{1}{\delta_1 a_{\ell,1}} \sum_{i=1}^{3} \left| \int_{\mathbb{R}^3} f^{n+1} v_i\, dv \right|^2 \\
&\leq \frac{2\delta_1}{a_{\ell,1}} \left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right|^2 \\
&\quad + \frac{8}{\delta_1 a_{\ell,1}} \left(\delta_2 + \delta_3 + \frac{1}{\tau} \right)^2 C^2_{LR,1} + C_{\ell,u} \left(\frac{\ln \tau + 1}{\tau} \right)^2.
\end{aligned}$$

From the estimates in $(ii\text{-}a)$, $(ii\text{-}b)$, we have

$$\begin{aligned}
\kappa^\top \left\{ \mathcal{T}^{n+1}_{-1/2} \right\} \kappa &\geq \frac{1}{\rho^{n+1}} \{ I - |II| \} \\
&\geq \frac{1}{a_{\ell,1}} \left\{ a_{-1/2,1} - 2 \left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right|^2 - O\left(\delta_2, \delta_3, \tau^{-1} \right) \right\}.
\end{aligned}$$

Therefore, if δ_2, δ_3, τ^{-1} and the flux discrepency:

$$\left| \int_{v_1>0} f_L |v_1| dv - \int_{v_1<0} f_R |v_1| dv \right|$$

are sufficiently small, we get the desired lower bound. $\square$

4. Cauchy estimate for f^n

The goal of this section is to show that $\{f^n\}$ forms a Cauchy sequence in an appropriate norm. First, we consider the continuity property of the ellipsoidal Gaussian:

Proposition 4.1. *Let f, g be elements of Ω_1 ($-1/2 < \nu < 1$) or Ω_2 ($\nu = -1/2$). Then the non-isotropic Gaussian $\mathcal{M}_\nu$ satisfies*

$$|\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)| \leq C \sup_x \|f - g\|_{L^1_2} e^{-C|v|^2}.$$

Proof. The case of $-1/2 < \nu < 1$ is covered in [11]. Here we only consider the case $\nu = -1/2$. Throughout this proof, ν is fixed to be $-1/2$. We apply Taylor expansion to $\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)$ as

(4.1)
$$\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)$$
$$= (\rho_f - \rho_g) \int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \rho} d\theta + (U_f - U_g) \int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial U} d\theta + (\mathcal{T}_f - \mathcal{T}_g) \int_0^1 \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \mathcal{T}_\nu} d\theta$$
$$\equiv (\rho_f - \rho_g) I_1 + (U_f - U_g) I_2 + (\mathcal{T}_f - \mathcal{T}_g) I_3,$$

where we used abbreviated notation:

$$\frac{\partial \mathcal{M}_\nu(\theta)}{\partial X} = \frac{\partial \mathcal{M}_\nu}{\partial X}(\rho_\theta, U_\theta, \mathcal{T}_\theta)$$

and $(\rho_\theta, U_\theta, \mathcal{T}_\theta) = (1-\theta)(\rho_f, U_f, \mathcal{T}_f) + \theta(\rho_g, U_g, \mathcal{T}_g)$. Since the transitional macroscopic fields $(\rho_\theta, U_\theta, \mathcal{T}_\theta)$ are all linear combinations of macroscopic fields of f and g, all the estimates for the macroscopic fiels given in $(\mathcal{A}_2)$, $(\mathcal{B}_2)$, $(\mathcal{C}_2)$ and $(\mathcal{D}_2)$ ($i = 1, 2$) hold the same for the transitional macroscopic fields too. Therefore, we will refer to the corresponding properties of Ω_i ($i = 1, 2$) for ρ, U, $\mathcal{T}_\nu$, whenever such estimates are needed for $(\rho_\theta, U_\theta, \mathcal{T}_\theta)$.
(a) Estimate for I_1: Since

$$\frac{\partial \mathcal{M}_\nu(\theta)}{\partial \rho} = \frac{1}{\rho_\theta} \mathcal{M}_\nu(\theta),$$

we see from $(\mathcal{B}_2)$ of Ω_2 and Lemma 2.1 that

(4.2)
$$I_1 \leq \frac{1}{\delta_1} C_{\ell,u} e^{-C_{\ell,u}|v|^2}.$$

(b) Estimate for I_2: An explicit computation gives

$$\frac{\partial \mathcal{M}_\nu(\theta)}{\partial U} = -\frac{1}{2} \Big\{ (v - U_\theta)^\top \mathcal{T}_\theta^{-1} + \mathcal{T}_\theta^{-1}(v - U_\theta) \Big\} \mathcal{M}_\nu(\theta).$$

Put $X = v - U_\theta$ and recall the property $(\mathcal{C}_2)$ of Ω_2 to compute

(4.3)
$$|X^\top \mathcal{T}_\theta^{-1}| = \sup_{|Y|=1} X^\top \{\mathcal{T}_\theta\}^{-1} Y$$
$$= \frac{1}{2} \sup_{|Y|=1} \Big\{ (X+Y)^\top \{\mathcal{T}_\theta\}^{-1}(X+Y) - X^\top \{\mathcal{T}_\theta\}^{-1} X - Y^\top \{\mathcal{T}_\theta\}^{-1} Y \Big\}$$
$$\leq \frac{2a_{\ell,1}}{\delta_1 a_{-1/2,1}} \sup_{|Y|=1} \Big(|X+Y|^2 + |X|^2 + |Y|^2 \Big)$$
$$\leq \frac{8a_{\ell,1}}{\delta_1 a_{-1/2,1}} (1 + |X|^2),$$

so that

$$(4.4) \qquad |\{\mathcal{T}_\theta\}^{-1}(v - U_\theta)| \le C \frac{8 a_{\ell,1}}{\delta_1 a_{-1/2,1}}(1 + |v|^2).$$

Therefore, in view of Lemma 2.1, we have

$$\left| \frac{\partial \mathcal{M}_\nu(\theta)}{\partial U} \right| \le C_{\ell,u} e^{-C_{\ell,u}|v|^2}.$$

(c) Estimate for I_3: We compute

$$(4.5) \qquad \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \mathcal{T}_{ij}} = \frac{1}{2} \left[-\frac{1}{\det \mathcal{T}_\theta} \frac{\partial \det \mathcal{T}_\theta}{\partial \mathcal{T}_{\theta ij}} + (v - U_\theta)^\top \mathcal{T}_\theta^{-1} \left(\frac{\partial \mathcal{T}_\theta}{\partial \mathcal{T}_{ij}} \right) \mathcal{T}_\theta^{-1}(v - U_\theta) \right] \mathcal{M}_\nu(\theta)$$

and observe that, for each pair of (i,j), $\frac{\partial \mathcal{T}_\theta}{\partial \mathcal{T}_{\theta ij}}$ is either 1 or 0, so that

$$(4.6) \qquad \left| (v - U_\theta)^\top \mathcal{T}_\theta^{-1} \left(\frac{\partial \mathcal{T}_\theta}{\partial \mathcal{T}_{\theta ij}} \right) \mathcal{T}_\theta^{-1}(v - U_\theta) \right| \le \left| (v - U_\theta)^\top \mathcal{T}_\theta^{-1} \right| \left| \mathcal{T}_\theta^{-1}(v - U_\theta) \right|$$

$$\le C_{\ell,u}^2 (1 + |v|^2).$$

Here, we used (4.4). On the other hand, to estimate the derivatives of the determinant, we write $\frac{\partial \det \mathcal{T}_\theta}{\partial \mathcal{T}_{\theta ij}}$ as follows:

$$\sum_{i,j,m,n} C_{ijmn} \mathcal{T}_{\theta ij} \mathcal{T}_{\theta mn}$$

for some constants C_{ijmn}. We then note that Lemma 3.5 (2) implies

$$\mathcal{T}_{\theta ij} \le \frac{8}{2 a_{\ell,1} \delta_1} C_{LR,1}.$$

Indeed, as in (4.3), we see that

$$|\mathcal{T}_{\theta ij}| = |e_i^\top \mathcal{T}_\theta e_j|$$

$$= \frac{1}{2} \sup_{|Y|=1} \left| \left\{ (e_i + e_j)^\top \mathcal{T}_\theta (e_i + e_j) - e_i^\top \mathcal{T}_\theta e_i - e_j^\top \mathcal{T}_\theta e_j \right\} \right|$$

$$\le \frac{2 a_{\ell,1}}{\delta_1 a_{-1/2,1}} \left(|e_i + e_j|^2 + |e_i|^2 + |e_j|^2 \right)$$

$$= \frac{8 a_{\ell,1}}{\delta_1 a_{-1/2,1}}.$$

Therefore, we can estimate

$$\left| \frac{\partial \det \mathcal{T}_\theta}{\partial \mathcal{T}_{\theta ij}} \right| \le \left(\frac{3}{2 a_{\ell,1} \delta_1} C_{LR,1} \right)^2.$$

Finally, we recall $(\mathcal{B}_2)$ of Ω_2 and Lemma 3.5 (2) to find

$$(4.7) \qquad \det \mathcal{T}_\theta \ge \left(\delta_1 \frac{a_{-1/2}}{2 C_{LR,1}} \right)^3, \qquad (\nu = -1/2)$$

We plug (4.6), (4.8) and (4.7) into (4.5), and employ Lemma 2.1 to get

$$(4.8) \qquad \left| \frac{\partial \mathcal{M}_\nu(\theta)}{\partial \mathcal{T}_{\theta ij}} \right| \le C \delta_1^{-8} (1 + |v|^2) \mathcal{M}_\nu(\theta) \le C e^{-C|v|^2}.$$

(Note that δ_1 is not small in the inflow dominant case.) We now turn back to (4.1) with these estimates to derive

$$(4.9) \qquad |\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)| \le C\Big\{|\rho_f - \rho_g| + |U_f - U_g| + |\mathcal{T}_f - \mathcal{T}_g|\Big\}e^{-C|v|^2}.$$

For the difference of macroscopic fields, we treat as

$$|\rho_f - \rho_g| = \int_{\mathbb{R}^3} |f - g| dv \le C \sup_x \|f - g\|_{L^1_2},$$

$$|U_f - U_g| \le \frac{1}{\rho_f}|\rho_f U_f - \rho_g U_g| + \frac{1}{\rho_f}|\rho_f - \rho_g||U_g| \le C_{\ell,u}\delta_1^{-1} \sup_x \|f - g\|_{L^1_2},$$

and

$$|\mathcal{T}_f - \mathcal{T}_g| \le \frac{1}{\rho_f}|\rho_f \mathcal{T}_f - \rho_g \mathcal{T}_g| + \frac{1}{\rho_f}|\rho_f - \rho_g||\mathcal{T}_g| = C_{\ell,u}\delta_1^{-1} \sup_x \|f - g\|_{L^1_2}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4.2. *Suppose f^n, $f^{n+1} \in \Omega_i$. ($i = 1, 2$) Then, under the assumption of Theorem 1.2, we have*

$$\sup_x \|f^{n+1} - f^n\|_{L^1_2} + \|f^{n+1} - f^n\|_{L^1_{\gamma,|v_1|}} + \|f^{n+1} - f^n\|_{L^1_{\gamma,\langle v\rangle}}$$

$$\preceq \left(\frac{\ln \tau + 1}{\tau}\right) \sup_x \|f^n - f^{n-1}\|_{L^1_2} + (\delta_2 + \delta_3)\|f^n - f^{n-1}\|_{L^1_{\gamma,|v_1|}} + \delta_3\|f^n - f^{n-1}\|_{L^1_{\gamma,\langle v\rangle}}$$

Proof. (1) Estimates in the trace norm $\|\cdot\|_{L^1_{\gamma,|v_1|}}$: First, we note from our boundary condition that, for $v_1 > 0$

$$f^{n+1}(0, v) - f^n(0, v) = \delta_2 \left(\int_{v_1 < 0} \Big\{f^n(0, v) - f^{n-1}(0, v)\Big\} |v_1| dv\right) M_w(0)$$

$$+ \delta_3 \Big\{f^n(0, Rv) - f^{n-1}(0, Rv)\Big\}.$$

Taking integration w.r.t $|v_1|dv$,

$$\int_{v_1 > 0} |f^{n+1}(0, v) - f^n(0, v)||v_1|dv$$

$$(4.10) \qquad \begin{aligned} &\le \delta_2 \left(\int_{v_1 < 0} |f^n(0, v) - f^{n-1}(0, v)||v_1|dv\right) \int_{v_1 > 0} M_w(0)|v_1|dv \\ &+ \delta_3 \int_{v_1 > 0} |f^n(0, Rv) - f^{n-1}(0, Rv)||v_1|dv \\ &\le (\delta_2 + \delta_3) \int_{v_1 < 0} |f^n(0, v) - f^{n-1}(0, v)||v_1|dv. \end{aligned}$$

On the other hand, for $v_1 < 0$, we have from (2.2)

$$f^{n+1}(0, v) = I(f^n) + II(f^n),$$

where

$$I(f) = e^{-\frac{1}{\tau|v_1|}} f(1, v), \quad II(f) = \frac{1}{\tau|v_1|} \int_0^1 e^{-\frac{y}{\tau|v_1|}} \mathcal{M}_\nu(f) dy.$$

$(1 - i)$ The estimate for $I(f^n) - I(f^{n-1})$: Since

$$I(f^n) - I(f^{n-1}) = \delta_2 e^{-\frac{1}{\tau|v_1|}} M_w(1) \left(\int_{v_1>0} \left\{ f^n(1,v) - f^{n-1}(1,v) \right\} |v_1| dv \right)$$

$$+ \delta_3 e^{-\frac{1}{\tau|v_1|}} \left\{ f^n(1, Rv) - f^{n-1}(1, Rv) \right\},$$

we have

$$\int_{v_1<0} |I(f^n) - I(f^{n-1})| |v_1| dv$$

(4.11)
$$\leq \delta_2 \left\{ \int_{v_1<0} e^{-\frac{1}{\tau|v_1|}} M_w(v) |v_1| dv \right\} \left\{ \int_{v_1>0} |f^n(1,v) - f^{n-1}(1,v)| |v_1| dv \right\}$$

$$+ \delta_3 \left\{ \int_{v_1>0} |f^n(1,v) - f^{n-1}(1,v)| |v_1| dv \right\}$$

$$\leq (\delta_2 + \delta_3) \left\{ \int_{v_1>0} |f^n(1,v) - f^{n-1}(1,v)| |v_1| dv \right\}$$

• The estimate for $II(f) - II(g)$:
We recall Lemma 2.2 and Proposition 4.1 to estimate

$$\int_{v_1>0} |II(f^n) - II(f^{n-1})| |v_1| dv$$

(4.12)
$$\leq \int_{v_1>0} \frac{1}{\tau|v_1|} \int_0^x e^{-\frac{y}{\tau|v_1|}} |\mathcal{M}_\nu(f^n) - \mathcal{M}_\nu(f^{n-1})| |v_1| dy dv$$

$$\leq C \left\{ \int_0^x \int_{v_1>0} \frac{1}{\tau|v_1|} e^{-\frac{y}{\tau|v_1|}} e^{-C_{\ell,u}|v|^2} dv_1 dy \right\} \sup_x \|f^n - f^{n-1}\|_{L_2^1}$$

$$\leq C \left(\frac{\ln \tau + 1}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L_2^1}.$$

We sum up (4.10), (4.11) and (4.12) to obtain

$$\|f^{n+1} - f^n\|_{L_{\gamma,|v_1|,-}^1} \preceq \left(\frac{\ln \tau + 1}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L_2^1} + (\delta_2 + \delta_3) \|f^n - f^{n-1}\|_{L_{\gamma,|v_1|,+}^1}.$$

In an almost identical manner, we can derive

$$\|f^{n+1} - f^n\|_{L_{\gamma,|v_1|,+}^1} \preceq \left(\frac{\ln \tau + 1}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L_2^1} + (\delta_2 + \delta_3) \|f^n - f^{n-1}\|_{L_{\gamma,|v_1|,-}^1}.$$

Therefore,

(4.13) $$\|f^{n+1} - f^n\|_{L_{\gamma,|v_1|}^1} \preceq \left(\frac{\ln \tau + 1}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L_2^1} + (\delta_2 + \delta_3) \|f^n - f^{n-1}\|_{L_{\gamma,|v_1|}^1}.$$

(2) Estimates in the trace norm $\| \cdot \|_{L^1_{\gamma, \langle v_1 \rangle}}$: We only estimate the boundary term I, since the estimates for II are almost identical. By an almost identical calculation, we arrive at

$$\int_{v_1 < 0} |I(f^n) - I(f^{n-1})| \langle v \rangle dv$$

$$\leq \delta_2 \left\{ \int_{v_1 < 0} e^{-\frac{a_{\ell,1}}{\tau |v_1|}} M_w(1) \langle v \rangle dv \right\} \left\{ \int_{v_1 > 0} |f^n(1,v) - f^{n-1}(1,v)| |v_1| dv \right\}$$

$$+ \delta_3 \left\{ \int_{v_1 > 0} |f^n(1,v) - f^{n-1}(1,v)| |v_1| dv \right\}$$

$$\preceq (\delta_2 + \delta_3) \int_{v_1 > 0} |f^n(1,v) - f^{n-1}(1,v)| \langle v \rangle dv.$$

where we used $\|M_w\|_{L^1_{\gamma, \langle v \rangle}} \leq C(T_w)$. Then, through similar computaitons for II terms, (we omit the proof to avoid repetitions.) we can obtain the estimates in $\| \cdot \|_{L^1_{\gamma, \langle v \rangle}}$:

$$(4.14) \qquad \begin{aligned} \|f^{n+1} - f^n\|_{L^1_{\gamma, \langle v \rangle}} &\preceq \left(\frac{\ln t + 1}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L^1_2} \\ &\quad + \delta_2 \|f^n - f^{n-1}\|_{L^1_{\gamma, |v_1|}} + \delta_3 \|f^n - f^{n-1}\|_{L^1_{\gamma, \langle v \rangle}}. \end{aligned}$$

The estimates in $\sup_x \| \cdot \|_{L^1_2}$ can be derived similarly:

$$(4.15) \qquad \begin{aligned} \|f^{n+1} - f^n\|_{L^1_2} &\preceq \left(\frac{\ln t + 1}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L^1_2} \\ &\quad + \delta_2 \|f^n - f^{n-1}\|_{L^1_{\gamma, |v_1|}} + \delta_3 \|f^n - f^{n-1}\|_{L^1_{\gamma, \langle v \rangle}}. \end{aligned}$$

The estimates (4.13), (4.14) and (4.15) give the desired result.   □

5. Proof of Theorem 1.5: The diffusive boundary condition

We now turn to the proof of Theorem 1.5. Since many parts overlap with the proof of Theorem 1.2, we focus on the difference of the proof. We start with the reformulation of the problem.

5.1. **Reformulation of the problem.** Consider the following mild formulation of (1.1):

$$f(x,v) = \delta_1 f_L + \delta_2 \left(\int_{v_1 < 0} f(0,v) |v_1| dv \right) M_w(0) + \delta_3 f(0, Rv)$$

$$+ \frac{1}{\tau |v_1|} \int_0^x \mathcal{R}(y,v) dy, \quad (v_1 > 0)$$

$$f(x,v) = \delta_1 f_R + \delta_2 \left(\int_{v_1 > 0} f(1,v) |v_1| dv \right) M_w(1) + \delta_3 f(1, Rv)$$

$$+ \frac{1}{\tau |v_1|} \int_x^1 \mathcal{R}(y,v) dy, \quad (v_1 < 0)$$

so that

$$f(1,v) = \delta_1 f_L + \delta_2 \left(\int_{v_1<0} f(0,v)|v_1|dv \right) M_w(0) + \delta_3 f(0, Rv)$$

$$+ \frac{1}{\tau|v_1|} \int_0^1 \mathcal{R}(y,v)dy, \quad (v_1 > 0)$$

$$f(0,v) = \delta_1 f_R + \delta_2 \left(\int_{v_1>0} f(1,v)|v_1|dv \right) M_w(1) + \delta_3 f(1, Rv)$$

$$+ \frac{1}{\tau|v_1|} \int_0^1 \mathcal{R}(y,v)dy, \quad (v_1 < 0).$$

Integrating with respect to $|v_1|dv$:

(5.1)
$$\int_{v_1>0} f(1,v)|v_1|dv = \delta_1 \int_{v_1>0} f_L|v_1|dv + (\delta_2 + \delta_3) \int_{v_1<0} f(0,v)|v_1|dv$$

$$+ \frac{1}{\tau} \int_{v_1>0} \int_0^1 \mathcal{R}(y,v)dydv,$$

$$\int_{v_1<0} f(0,v)|v_1|dv = \delta_1 \int_{v_1<0} f_R|v_1|dv + (\delta_2 + \delta_3) \int_{v_1>0} f(1,v)|v_1|dv$$

$$+ \frac{1}{\tau} \int_{v_1<0} \int_0^1 \mathcal{R}(y,v)dydv,$$

where

$$\mathcal{R}(f)(x,v) = \mathcal{M}_\nu(f)(x,v) - f(x,v),$$

throughout this section. Inserting (1.10) into $(5.1)_1$, we get

$$\int_{v_1<0} f(0,v)|v_1|dv = \frac{1-\delta_1}{2-\delta_1} + \frac{\delta_1}{2-\delta_1} \int_{v_1<0} f_R|v_1|dv - \frac{1}{\tau(2-\delta_1)} \int_{v_1>0} \int_0^1 \mathcal{R}(y,v)dydv.$$

Similarly, from (1.10) and $(5.1)_2$, we get

$$\int_{v_1>0} f(1,v)|v_1|dv = \frac{1-\delta_1}{2-\delta_1} + \frac{\delta_1}{2-\delta_1} \int_{v_1>0} f_L|v_1|dv - \frac{1}{\tau(2-\delta_1)} \int_{v_1<0} \int_0^1 \mathcal{R}(y,v)dydv.$$

From this, we derive the new formulation of the problem given in Definition 1.3.

5.2. **Approximation scheme and solution spaces.** We construct the solution for (1.1) from the following approximate scheme:

(5.2) $$f^{n+1}(x,v) = e^{-\frac{x}{\tau|v_1|}} f^n(0,v) + \frac{1}{\tau|v_1|} \int_0^x e^{-\frac{x-y}{\tau|v_1|}} \mathcal{M}_\nu(f^n)dy, \quad \text{if } v_1 > 0$$

and

(5.3) $$f^{n+1}(x,v) = e^{-\frac{x}{\tau|v_1|}} f^n(1,v) + \frac{1}{\tau|v_1|} \int_x^1 e^{-\frac{x-y}{\tau|v_1|}} \mathcal{M}_\nu(f^n)dy, \quad \text{if } v_1 < 0$$

where

(5.4)
$$f^{n+1}(0,v) = \delta_1 f_L(v) + \delta_2 \mathcal{S}_L(f^n) M_w(0) + \delta_3 f^n(0, Rv), \quad (v_1 > 0)$$
$$f^{n+1}(1,v) = \delta_1 f_R(v) + \delta_2 \mathcal{S}_R(f^n) M_w(1) + \delta_3 f^n(1, Rv), \quad (v_1 < 0)$$

and

$$\mathcal{S}_L(f^n) = \frac{1 - \delta_1}{2 - \delta_1} + \frac{\delta_1}{2 - \delta_1} \int_{v_1 < 0} f_R |v_1| dv - \frac{1}{\tau(2 - \delta_1)} \int_{v_1 > 0} \int_0^1 \mathcal{R}^n(y, v) dy dv$$

$$\mathcal{S}_R(f^n) = \frac{1 - \delta_1}{2 - \delta_1} + \frac{\delta_1}{2 - \delta_1} \int_{v_1 > 0} f_L |v_1| dv - \frac{1}{\tau(2 - \delta_1)} \int_{v_1 < 0} \int_0^1 \mathcal{R}^n(y, v) dy dv$$

with

$$\mathcal{R}^n(y, v) = \rho^n \left\{ \mathcal{M}_\nu(f^n) - f^n \right\}.$$

As in the inflow dominant case, we define two function spaces. First we define the function space for the non-critical case $-1/2 < \nu < 1$:

$$\Omega_3 = \left\{ f \in L^\infty \left([0, 1]; L_2^1(\mathbb{R}^3) \right) \cap L_{\gamma, \langle v \rangle}^1(\mathbb{R}^3) \mid f \text{ satisfies } (\mathcal{A}_3), (\mathcal{B}_3), (\mathcal{C}_3), (\mathcal{D}_3) \right\}$$

where $(\mathcal{A}_3)$, $(\mathcal{B}_3)$, $(\mathcal{C}_3)$ and $(\mathcal{D}_3)$ denote

- $(\mathcal{A}_3)$ f is non-negative:

$$f(x, v) \geq 0 \text{ for } x, v \in [0, 1] \times \mathbb{R}^3.$$

- $(\mathcal{B}_3)$ The macroscopic field is well-defined:

$$\int_{\mathbb{R}^3} f(x, v) dv \geq a_{\ell, 2}, \quad \int_{\mathbb{R}^3} f(x, v)(1 + |v|^2) dv \leq 2C_{LR, 2}.$$

- $(\mathcal{C}_3)$ The temperature tensor is well-defined:

$$C_\nu^1 \delta_1^2 \frac{\gamma_{\ell, 2}}{3 C_{LR, 2}^2} \leq \kappa^\top \left\{ \mathcal{T}_\nu \right\} \kappa \leq \frac{2}{3 a_{\ell, 2}} C_\nu^2 C_{LR, 2}.$$

- $(\mathcal{D}_3)$ The inflow data satisfies:

$$\|f\|_{L_{\gamma, |v_1|, \pm}^1} \leq 2 \left(1 + \|f_{LR}\|_{L_{\gamma, |v_1|}^1} \right), \quad \|f\|_{L_{\gamma, \langle v \rangle}^1} \leq 2C_{LR, 2}.$$

For the critical case $\nu = -1/2$, we define

$$\Omega_4 = \left\{ f \in L^\infty \left([0, 1]; L_2^1(\mathbb{R}^3) \right) \cap L_{\gamma, \langle v \rangle}^1(\mathbb{R}^3) \mid f \text{ satisfies } (\mathcal{A}_4), (\mathcal{B}_4), (\mathcal{C}_4), (\mathcal{D}_4) \right\}$$

where $(\mathcal{A}_4)$, $(\mathcal{B}_4)$, $(\mathcal{C}_4)$ and $(\mathcal{D}_4)$ denote

- $(\mathcal{A}_4)$ f is non-negative:

$$f(x, v) \geq 0 \text{ for } x, v \in [0, 1] \times \mathbb{R}^3.$$

- $(\mathcal{B}_4)$ The macroscopic field is well-defined:

$$\int_{\mathbb{R}^3} f(x, v) dv \geq a_{\ell, 2}, \quad \int_{\mathbb{R}^3} f(x, v)(1 + |v|^2) dv \leq 2C_{LR, 2}.$$

- $(\mathcal{C}_4)$ The temperature tensor is well-defined:

$$\delta_2 \frac{a_{-1/2}}{2 C_{LR, 2}} \leq \kappa^\top \left\{ \mathcal{T}_{-1/2} \right\} \kappa \leq \frac{3}{2 a_{\ell, 2}} C_{LR, 2}.$$

- $(\mathcal{D}_4)$ The inflow data satisfies:

$$\|f\|_{L_{\gamma, |v_1|, \pm}^1} \leq 2 \left(1 + \|f_{LR}\|_{L_{\gamma, |v_1|}^1} \right), \quad \|f\|_{L_{\gamma, \langle v \rangle, \pm}^1} \leq 2C_{LR, 2}.$$

Before we move on to the proof of uniform estimates for f^n, we recored a few estimates that will be fruitfully used throughout the paper. The proof for the Lemma 5.1 is almost identical to the corresponding estimates in Lemma 2.1, and we omit the proof.

Lemma 5.1. (1) *Let $f \in \Omega_3$. Then there exist positive constants C depending only on the quantities (1.5) and $\gamma_{\ell,2}$ such that*

$$\mathcal{M}_\nu(f) \le Ce^{-C|v|^2}.$$

(2) *Let $f \in \Omega_4$. Then there exists positive constants C depending only on the quantities (1.5) and $a_{-1/2,2}$ such that*

$$\mathcal{M}_\nu(f) \le Ce^{-C|v|^2}.$$

5.3. $f^n \in \Omega_i$ $(i = 3, 4)$ **for all** n. The main result of this section is the following proposition

Proposition 5.1. (1) *Let $-1/2 < \nu < 1$. Assume f_{LR} satisfies the conditions of Theorem 1.5 (1). Then $f^n \in \Omega_3$ for all n.*
(2) *Let $\nu = -1/2$. Assume f_{LR} satisfies the conditions of Theorem 1.5 (2). Then, $f^n \in \Omega_4$ for all n.*

We divide the proof into Lemma 5.2, 5.3, 5.5, and Lemma 5.7.

Lemma 5.2. *Let $f^n \in \Omega_3$ or Ω_4. Then, for sufficiently small δ_1 and sufficiently large τ, we have*

$$f^{n+1} \ge 0.$$

Proof. Since $f^n \in \Omega_3$, we have from Lemma 2.2

$$
\begin{aligned}
\mathcal{S}_L(f^n) &= \frac{1-\delta_1}{2-\delta_1} + \frac{\delta_1}{2-\delta_1} \int_{v_1>0} f_L|v_1|dv - \frac{1}{\tau(2-\delta_1)} \int_{v_1>0} \int_0^1 \mathcal{R}^n(y,v)dydv \\
&\ge \frac{1-\delta_1}{2-\delta_1} + \delta_1 \int_{v_1>0} f_L|v_1|dv - \frac{1}{\tau} \int_{\mathbb{R}} \int_0^1 \left(\mathcal{M}_\nu(f^n) + f^n\right)(1+|v|^2)dydv \\
&\ge \frac{1}{3}
\end{aligned}
$$

for sufficiently small δ_1 and τ^{-1}. Similarly, $\mathcal{S}_R(f^n) \ge 1/3$. Therefore,

$$
\begin{aligned}
f^{n+1} &\ge \frac{1}{3}\delta_2 e^{-\frac{x}{\tau|v_1|}} M_w(0)1_{v_1>0} + \frac{1}{3}\delta_2 e^{-\frac{1-x}{\tau|v_1|}} M_w(1)1_{v_1<0} \\
&\ge \frac{1}{3}\delta_2 e^{-\frac{1}{|v_1|}} M_w \ge 0
\end{aligned}
$$

for $v_1 > 0$. The case for $v_1 < 0$ is the same. □

Lemma 5.3. *Assume $f \in \Omega_3$ or Ω_4. Then we have*

$$\int_{\mathbb{R}^3} f^{n+1}dv \ge \delta_2 a_{\ell,2}.$$

Proof. We only prove the second one. Recall from the previous proof that

$$f^{n+1} \ge \frac{1}{3}\delta_2 e^{-\frac{1}{|v_1|}} M_w.$$

Integrating with respect to v, we obtain the desired lower bound. □

Lemma 5.4. (1) *Let $f^n \in \Omega_1$ or Ω_2. Then we have*

$$\|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}}, \ \|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}} \le 2\big(1 + \|f_{LR}\|_{L^1_{\gamma,|v_1|}}\big).$$

(2) *Let $f^n \in \Omega_1$ or Ω_2. Then we have*

$$\|f^{n+1}\|_{L^1_{\gamma,\langle v\rangle,+}}, \ \|f^{n+1}\|_{L^1_{\gamma,\langle v\rangle,-}} \le 2\big(\|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \|M_w\|_{L^1_{\gamma,\langle v\rangle}}\big).$$

Proof. (1) • Estimate for outflux $\|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}}$: Using (5.2), we can write $f^{n+1}(0,v)$ for $v_1 < 0$ as

$$(5.5) \qquad f^{n+1}(0,v) = \delta_1 f_R + \delta_2 \mathcal{S}_R M_w(1) + \delta_3 f^n(1,Rv) + \frac{1}{\tau|v_1|} \int_0^1 e^{-\frac{y}{\tau|v_1|}} \mathcal{M}_\nu(f^n) dy$$

which, in view of Lemma 2.2, yields

$$(5.6) \qquad \begin{aligned} &\int_{v_1<0} f^{n+1}(0,v)|v_1|dv \\ &\leq \delta_1 \int_{v_1<0} f_R|v_1|dv + \delta_2 \int_{v_1>0} M_w(1)|v_1|dv + \delta_3 \int_{v_1>0} f^n(1,v)|v_1|dv \\ &+ C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right). \end{aligned}$$

Here, we used $\mathcal{S}_L \leq 1$, which follows directly from the smallness of δ_1 and Lemma 2.2. Similarly,

$$(5.7) \qquad \begin{aligned} &\int_{v_1>0} f^{n+1}(1,v)|v_1|dv \\ &\leq \delta_1 \int_{v_1>0} f_L|v_1|dv + \delta_2 \int_{v_1<0} M_w(0)|v_1|dv + \delta_3 \int_{v_1<0} f^n(0,v)|v_1|dv \\ &+ C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right). \end{aligned}$$

From (5.6) and (5.7), we obtain

$$(5.8) \qquad \begin{aligned} \|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}} &= \int_{v_1<0} f^{n+1}(0,v)|v_1|dv + \int_{v_1>0} f^{n+1}(1,v)|v_1|dv \\ &\leq \delta_1\|f_{LR}\|_{L^1_{\gamma,|v_1|}} + \delta_2\|M_w\|_{L^1_{\gamma,|v_1|,+}} + \delta_3\|f^n\|_{L^1_{\gamma,|v_1|,+}} \\ &+ C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right) \\ &\leq \delta_1\|f_{LR}\|_{L^1_{\gamma,|v_1|}} + \delta_2 + \delta_3\|f^n\|_{L^1_{\gamma,|v_1|,+}} \\ &+ C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right). \end{aligned}$$

Therefore, in view of $\mathcal{D}_i$ of Ω_i $(i=1,2)$, we see that

$$(5.9) \qquad \begin{aligned} \|f^{n+1}\|_{L^1_{\gamma,|v_1|,+}} &\leq \delta_1\|f_{LR}\|_{L^1_{\gamma,|v_1|}} + \delta_2 + 2\delta_3\big(1+\|f_{LR}\|_{L^1_{\gamma,|v_1|}}\big) + C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right) \\ &= 2(\delta_1+\delta_2+\delta_3)\big(1+\|f_{LR}\|_{L^1_{\gamma,|v_1|}}\big) - (\delta_1+\delta_2)\big(1+\|f_{LR}\|_{L^1_{\gamma,|v_1|}}\big) \\ &+ C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right) \\ &\leq 2\big(1+\|f_{LR}\|_{L^1_{\gamma,|v_1|}}\big) \end{aligned}$$

for sufficiently large τ.

• Estimate for influx: $\|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}}$: When $v_1 > 0$, we have from the boundary condition (5.4) that

$$f^{n+1}(0, v) = \delta_1 f_L + \delta_2 \mathcal{S}_L M_w(0) + \delta_3 f^n(0, Rv).$$

Integrate both sides with respect to $|v_1| dv$ on $v_1 > 0$ to get

(5.10) $$\int_{v_1>0} f^{n+1}(0,v)|v_1|dv = \delta_1 \int_{v_1>0} f_L|v_1|dv + \delta_2 + \delta_3 \int_{v_1>0} f^n(0, Rv)|v_1|dv.$$

where we used $\mathcal{S}_L < 1$ and $\int_{v_1>0} M_w(0)|v_1|dv = 1$. Similarly, we estimate

(5.11) $$\int_{v_1<0} f^{n+1}(1,v)|v_1|dv = \delta_1 \int_{v_1<0} f_R|v_1|dv + \delta_2 + \delta_3 \int_{v_1>0} f^n(1, Rv)|v_1|dv.$$

Combining (5.10) and (5.11) gives

(5.12) $$\|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}} \leq \delta_1 \|f_{LR}\|_{L^1_{\gamma,|v_1|}} + \delta_2 + \delta_3 \|f^n\|_{L^1_{\gamma,|v_1|,+}},$$

which, thanks to (5.9), gives

$$\|f^{n+1}\|_{L^1_{\gamma,|v_1|,-}} \leq 2\big(1 + \|f_{LR}\|_{L^1_{\gamma,|v_1|}}\big).$$

This completes the proof of (1). The proof of (2) is identical. We omit the proof. □

Lemma 5.5. (1) Let $f^n \in \Omega_3$ or Ω_4. For sufficiently large $\tau > 0$, we have

$$\int_{\mathbb{R}^3} f^{n+1}(1 + |v|^2)dv \leq 2\left(\|f_{LR}\|_{L^1_{\gamma,|v_1|}} + \|M_w\|_{L^1_{\gamma,\langle v\rangle}}\right).$$

Proof. The proof is almost identical to Lemma 3.3. We omit it. □

Lemma 5.6. Let $f^n \in \Omega_3$ or Ω_4.
(1) For $i = 1$, we have

$$\left|\int_{\mathbb{R}^3} f^{n+1}v_1 dv\right| \leq \delta_1 \left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right| + \frac{2}{\tau}C_{LM,2}$$

where $C_{LR,2}$ denotes

$$C_{LM,2} = \|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}} + \|M_w\|_{L^1_{\gamma,\langle v\rangle}}.$$

(2) For $i = 2, 3$, we have

$$\left|\int_{\mathbb{R}^3} f^{n+1}v_i dv\right| \leq 2\delta_3 C_{LR,2} + C\left(\frac{\ln\tau + 1}{\tau}\right).$$

Proof. (1) Recall from the proof of Lemma 3.4 that

$$\int_{\mathbb{R}^3} f^{n+1}(x,v)v_1 dv = I + II,$$

where

$$I = \int_{v_1>0} f^{n+1}(0,v)|v_1|dv - \int_{v_1<0} f^{n+1}(1,v)|v_1|dv$$

and

$$II = \int_0^x \int_{v_1>0} \frac{\rho^n}{\tau}\left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dvdy + \frac{1}{\tau}\int_x^1 \int_{v_1<0} \left(\mathcal{M}_\nu(f^n) - f^{n+1}\right)dydv.$$

(a) The estimate of I: We observe from our boundary condition that

$$|I| \leq \left(\delta_1 + \frac{\delta_1\delta_2}{2-\delta_1}\right)\left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right|$$

$$+ \frac{1}{\tau(2-\delta_2)}\int_{\mathbb{R}^3}\int_0^1 \{\mathcal{M}_\nu(f^n) + f^n\}(1+|v|^2)dydv$$

$$\leq 2\delta_1\left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right| + \frac{2}{\tau}C_{LM,2}.$$

In the last line, we used Lemma 5.4.

(b) The estimate for II: The argument for this part is identical except that we use Lemma 5.5 instead of Lemma 3.3. Now, we combine (a) and (b) to obtain the desired result.

(2) The proof is identical to the inflow dominant case, since the δ_2 contribution vanishes:

$$\int_{\mathbb{R}^2}\mathcal{S}(f^n)M_w v_2 dv_2 dv_3 = 0.$$

We omit the proof. $\qquad\qquad\square$

Lemma 5.7. (1) *Let $-1/2 < \nu < 1$. Assume $f^n \in \Omega_3$. Then, for sufficiently large τ, we have*

$$C_\nu^1\delta_2^2\frac{\gamma_{\ell,2}}{27C_{LR,2}^2} \leq \kappa^\top\{\mathcal{T}_\nu^{n+1}\}\kappa \leq \frac{2}{3a_{\ell,2}}C_\nu^2 C_{LM,2}$$

(2) *Let $\nu = -1/2$ and $f \in \Omega_4$. Then, for sufficiently large τ, we have*

$$\delta_1\frac{a_{-1/2}}{4C_{LR,2}} \leq \kappa^\top\{\mathcal{T}_{-1/2}^{n+1}\}\kappa \leq \frac{3}{2a_{\ell,2}}C_{LR,2}$$

for any $\kappa \in \mathbb{R}$ and $|\kappa| = 1$. We recall that

$$C_{LM,2} = \|f_{LR}\|_{L_{\gamma,\langle v\rangle}^1} + \|M_w\|_{L_{\gamma,\langle v\rangle}^1}.$$

Proof. (1) The proof is identical to the inflow dominant case, except for the computation of I, where we bound it from below using δ_2 and $\gamma_{\ell,2}$, instead of using δ_1 and $\gamma_{\ell,1}$.

$$I = 4\left\{\int_{v_1>0} f^{n+1}|v_1|dv\right\}\left\{\int_{v_1<0} f^{n+1}|v_1|dv\right\}$$

$$\geq \delta_2^2\left(\int_{v_1>0} e^{-\frac{1}{|v_1|}}M_w(0)|v_1|dv\right)\left(\int_{v_1<0} e^{-\frac{1}{|v_1|}}M_w(1)|v_1|dv\right)$$

$$= \delta_2^2\gamma_{\ell,2}.$$

In the last line, we used

$$f^{n+1} \geq \frac{1}{3}\delta_2 e^{-\frac{1}{\tau|v_1|}}M_w(0)1_{v_1>0} + \frac{1}{3}\delta_2 e^{-\frac{1}{\tau|v_1|}}M_w(1)1_{v_1<0}$$

and $\tau > 1$.

(2) We recall from the inflow dominant case that

$$\rho^{n+1}\kappa^\top \left\{\mathcal{T}_{-1/2}^{n+1}\right\}\kappa$$

(5.13)
$$= \int_{\mathbb{R}^3} f^{n+1}\left\{|v|^2 - (v\cdot\kappa)^2\right\}dv - \left\{\rho^{n+1}|U^{n+1}|^2 - \rho^{n+1}(U^{n+1}\cdot\kappa)^2\right\}$$

$$\equiv I + II,$$

for $|\kappa| = 1$

(i) Upper bound: We have from Lemma 5.5 that

$$\kappa^\top\left\{\mathcal{T}_{-1/2}^{n+1}\right\}\kappa \leq \frac{1}{\rho^{n+1}}\int_{\mathbb{R}^3} f^{n+1}|v|^2 dv \leq \frac{1}{a_{\ell,2}\delta_2}\left(1 + \|f_{LR}\|_{L^1_{\gamma,\langle v\rangle}}\right)\|M_w\|_{L^1_{\gamma,\langle v\rangle}}.$$

(ii) Lower bound: For this, we estimate the lower bound of I and the smallness of II:

(ii-a) Lower bound of I: The proof is the same, except that we bound it using $a_{-1/2,2}$ this time:

$$I \geq \frac{1}{3}\delta_2 \inf_{|\kappa|=1}\int_{\mathbb{R}^3} e^{-\frac{1}{|v_1|}}M_w\left\{|v|^2 - (v\cdot\kappa)^2\right\}dv = \frac{1}{3}\delta_2 a_{-1/2,2}.$$

(ii-b) Smallness of of II: The estimate for this case is the same either, except that we use Lemma 5.6, instead of Lemma 3.4:

$$II \leq \frac{|\rho^{n+1}U^{n+1}|^2}{\rho^{n+1}} \leq \frac{1}{a_{\ell,1}}\left|\int_{\mathbb{R}^3} f^{n+1}v dv\right|^2 \leq \frac{1}{a_{\ell,1}}\sum_{i=1}^3\left|\int_{\mathbb{R}^3} f^{n+1}v_i dv\right|^2$$

$$\leq 4\delta_1^2\left|\int_{v_1>0} f_L|v_1|dv - \int_{v_1<0} f_R|v_1|dv\right|^2 + O\left(\delta_3,\tau^{-1}\right).$$

The by exactly the same argument, we get the desired result. Note that, since we can take δ_1 arbitrariliy small in this case, we don't need to assume that the discrepancy of the flux from the inflow data is small. $\square$

5.4. **Cauchy estimate for f^n.**

Proposition 5.2. *Let f, g be elements of Ω_3 $(-1/2 < \nu < 1)$ or Ω_4 $(\nu = -1/2)$. Then the non-isotropic Gaussian $\mathcal{M}_\nu$ satisfies*

$$|\mathcal{M}_\nu(f) - \mathcal{M}_\nu(g)| \leq C\sup_x \|f - g\|_{L^1_2}e^{-C|v|^2}.$$

Proof. The proof is almost identical with the one given for Proposition 4.1. We omit it. $\square$

Proposition 5.3. *Suppose f^n, $f^{n+1} \in \Omega_i$ $(i = 3, 4)$. Then, under the assumption of Theorem 1.5, we have*

$$\sup_x \|f^{n+1} - f^n\|_{L^1_2} + \|f^{n+1} - f^n\|_{L^1_{\gamma,|v_1|}} + \|f^{n+1} - f^n\|_{L^1_{\gamma,\langle v\rangle}}$$

$$\preceq \left(\frac{\ln\tau + 1 + \delta_2}{\tau}\right)\sup_x \|f^n - f^{n-1}\|_{L^1_2} + \delta_3\|f^n - f^{n-1}\|_{L^1_{\gamma,|v_1|}} + \delta_3\|f^n - f^{n-1}\|_{L^1_{\gamma,\langle v\rangle}}.$$

Remark 5.8. We note that, unlike in Proposition 4.2, K does not have $\|f_{LR}|v|^{-1}\|_{L^1_{\gamma,\langle v\rangle}}$ term in this case. This is why we don't need the no-concentration assumption (P_2) in Theorem 1.5.

Proof. We only consider the boundary terms in $\| \cdot \|_{L^1_{\gamma,|v_1|}}$ estimate. We note from our boundary condition that, for $v_1 > 0$

$$\int_{v_1>0} |f^{n+1}(0,v) - f^n(0,v)||v_1|dv$$

(5.14)
$$\leq \delta_2 |\mathcal{S}^+(f^n) - \mathcal{S}(f^{n-1})| \int_{v_1>0} M_w(0)|v_1|dv$$
$$+ \delta_3 \int_{v_1>0} |f^n(0,Rv) - f^{n-1}(0,Rv)||v_1|dv$$
$$\leq C\frac{\delta_2}{\tau} \sup_x \|f^n - f^{n-1}\|_{L^1_2} + \delta_3 \int_{v_1<0} |f^n(0,v) - f^{n-1}(0,v)||v_1|dv,$$

where we used Proposition 5.2 as

$$|\mathcal{S}^+(f^n) - \mathcal{S}(f^n)| = \frac{1}{\tau(2-\delta_1)} \int_{v_1>0} \int_0^1 \left| \mathcal{R}^n(y,v) - \mathcal{R}^{n-1}(y,v) \right| dydv$$
$$\leq \frac{C}{\tau} \sup_x \|f^n - f^{n-1}\|_{L^1_2}.$$

On the other hand, for $v_1 < 0$, we have from (5.3)

$$f^{n+1}(0,v) = I(f^n) + II(f^n),$$

where

$$I(f) = e^{-\frac{1}{\tau|v_1|}\int_0^x \rho_f(y)dy} f(1,v), \quad II(f) = \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{x-y}{\tau|v_1|}} \mathcal{M}_\nu(h)dy.$$

Since

$$I(f^n) - I(f^{n-1}) + \delta_2 e^{-\frac{x}{\tau|v_1|}} \left\{ \mathcal{S}^-(f^n) - \mathcal{S}^-(f^{n-1}) \right\} M_w(1)$$
$$+ \delta_3 e^{-\frac{x}{\tau|v_1|}} \left\{ f^n(1,Rv) - f^{n-1}(1,Rv) \right\},$$

we have

$$\int_{v_1<0} |I(f^n) - I(f^{n-1})||v_1|dv$$

(5.15)
$$\leq C\frac{\delta_2}{\tau} \sup_x \|f^n - f^{n-1}\|_{L^1_2} + \delta_3 \left\{ \int_{v_1>0} |f^n(1,v) - f^{n-1}(1,v)||v_1|dv \right\}.$$

Now, through an almost identical computations as in the inflow dominant case, we get the following estimates:

(5.16) $\|f^{n+1} - f^n\|_{L^1_{\gamma,|v_1|}} \preceq \left(\frac{\ln\tau + 1 + \delta_2}{\tau} \right) \sup_x \|f^n - fn-1\|_{L^1_2} + \delta_3 \|f^n - f^{n-1}\|_{L^1_{\gamma,|v_1|}}.$

Estimates in $\| \cdot \|_{L^1_{\gamma,\langle v \rangle}}$ and $\| \cdot \|_{L^1_2}$ can be obtained similarly:

(5.17) $\|f^{n+1} - f^n\|_{L^1_{\gamma,\langle v \rangle}} \preceq \left(\frac{\ln\tau + 1 + \delta_2}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L^1_2} + \delta_3 C \|f^n - f^{n-1}\|_{L^1_{\gamma,\langle v \rangle}}$

and

(5.18) $\|f^{n+1} - f^n\|_{L^1_2} \preceq \left(\frac{\ln\tau + 1 + \delta_2}{\tau} \right) \sup_x \|f^n - f^{n-1}\|_{L^1_2} + \delta_3 \|f^n - f^{n-1}\|_{L^1_{\gamma,\langle v \rangle}}.$

The estimates (5.16), (5.17) and (5.18) give the desired result. $\square$

Data availability: No data was used for the research described in the article.

Conflict of interest: The authors declare that they have no conflict of interest.

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