Classical transport theory for the collisional electronic M_1 model

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Abstract. The electronic M_1 model is widely used for electron transport studies in a hot collisional plasma. However, the moment extraction of the electron-electron collision operator from the kinetic collision operator, for this angular moments model, is challenging and some approximations are required. In this work a characterisation of the electron-electron and electron-ion collision operators is given and the electron plasma transport coefficients are derived. It is shown that in the high Z limit the electronic M_1 model and the Fokker-Planck-Landau equation coincide in the case of near equilibrium. Also, in general, the electron-electron collision operator proposed for the electronic M_1 model recovers accurate electron transport plasma coefficients.

Key words: electron M_1 model, collision operators, electron plasma transport coefficients.

1 Introduction

It was proposed to use laser pulses in order to compress a deuterium-tritium target and ignite the nuclear fusion reactions. In this process the energy is transported from the critical surface to denser parts of plasma by electrons. This process plays a key role in the understanding of plasma phenomena such as, parametric [31, 17] and hydrodynamic [39, 46, 12] instabilities, laser-plasma absorption [38, 21], wave damping [25, 11], energy redistribution and hot spot formation [7, 29]. Lasers produce a collisional ionised hot plasma, where the electron-ion mean free path is small compared to the plasma characteristic spatial size and the distribution function is close to the isotropic Maxwellian function. The physics of laser plasma interaction is described within the hydrodynamic plasma model. However, the moment extraction of the electron kinetic equation leads to an unclosed hydrodynamic set of equations. The closure of the system requires to express the fluxes in terms of the hydrodynamic variables and electron plasma transport coefficients. Spitzer and Härm first derived the electron plasma transport coefficients solving numerically the kinetic Fokker-Planck-Landau equation using the expansion of the electron mean free path over the temperature scale length. Their results have been reproduced in other works [6, 3, 40] using the early works of Chapman [8, 9] and Enskog [16] for neutral gases. However, the Spitzer-Härm theory is valid in the local regime where the electron flux is proportional to the temperature gradient. Indeed the electron transport plasma coefficient were derived in the case where the electron distribution function remains close to the isotropic Maxwellian function. However, in the context of inertial confinement fusion, the plasma particles may have an energy distribution far from the thermodynamic equilibrium so that the classical transport description is not adapted [34]. Moreover kinetic effects like the non local transport [7, 29], wave damping or the development of instabilities [12] can be important over time scales shorter than the collisional time so that fluid simulations are insufficient. Therefore, a kinetic description is more appropriate for the study of inertial confinement fusion processes. However such a kinetic description is computationally expensive for describing real physical applications. Kinetic codes

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are limited to time and length much shorter than those studied with fluid simulations. It is therefore an essential issue to describe kinetic effects by using reduced kinetic codes operating on fluid time scales.

Angular moments models can be seen as a compromise between kinetic and fluid models. On one hand, they have an advantage to be less computationally expensive than full kinetic models since less variables are involved and, on the other hand, they provide results with a higher accuracy than fluid models. Grad [18], initially proposed a moment closure hierarchy which leads to a hyperbolic set of equations for close equilibrium flows. The hierarchy proposed is based on a polynomial series expansion of a distribution function close to the Maxwellian equilibrium. However, the truncation of this expansion leads to a loss of the positivity of the distribution function and to unrealisable moments. In [26, 32, 33, 42, 1], closures based on entropy minimisation principles are investigated. It has been shown that this closure choice enables to preserve fundamental properties such as the positivity of the underlying distribution function, the hyperbolicity of the model and an entropy dissipation condition [19, 30, 26]. In this work, the moment model is based on an angular moments extraction. The kinetic equation is integrated only with respect to the velocity direction while the velocity modulus is kept as a variable. The closure used based on an entropy minimisation principle gives the angular M_1 model. The angular M_1 model is used in numerous applications such as radiative transfer [45, 5, 15, 44, 10, 36, 37], radiotherapy [35] or electron transport [27, 13, 21, 20]. This model satisfies fundamental properties and recovers the asymptotic diffusion equation in the limit of long time behaviour when collisions dominate [14].

The electronic M_1 model is derived integrating with respect to the velocity direction the Fokker-Planck-Landau equation. However, since the electron-electron collision operator is non-linear, the moments extraction is complex. A possibility could be to approximate the electron-electron collision operator assuming the main contribution of the distribution function comes from its isotropic part [4]. However, as mentioned in [28], the collisional electronic M_1 model obtained by angular integration does not ensure the preservation of the admissibility states, that is the angular moments derive from a positive underlying distribution function. Therefore, a new electron-electron collision operator was proposed in [28]. In this model, the angular integration leads to a electron-electron collision operator for the electronic M_1 model which preserves the admissible states. In this work, we start to recall the main results established in [27, 28] and complete them with an important result characterising the equilibrium states of the collision operators. Such fundamental properties make the model interesting for practical applications. In addition, to complete the validation of the considered collisional electronic M_1 model, we derive the electron transport coefficients. It is shown that in the high ion charge ($Z >> 1$) limit the electronic M_1 model and the Fokker-Planck-Landau equation coincide in the close-equilibrium case. The electron transport coefficients derived from the electron-electron collision operator used for the electronic M_1 model are compared with the ones obtained using the electron-electron collision operator for the Fokker-Planck-Landau equation.

The paper is organised as follows: first in Section 2, we introduce the collisional electronic M_1 model. The kinetic Fokker-Planck-Landau equation from which the model is derived is recalled. Then, the main properties of the collision operators are presented and completed by the characterisation of the equilibrium state. In Section 3, the electron transport coefficients are derived using the collisional electronic M_1 model and compared with the ones obtained from the Fokker-Planck-Landau equation. The strategy proposed, based on an expansion on the Laguerre polynomials [6, 9], is particularly efficient since the stiffness in $1/\zeta^3$ in the electron-ion collision operator is removed. It is shown that accurate electron plasma transport coefficients are obtained. Finally, Section 4 presents our conclusions.

2 Electronic M_1 model and collisional operators

This section provides a detailed description of the electronic M_1 model [27, 13], which is derived from the kinetic Fokker-Planck-Landau equation [11].

2.1 Kinetic model

The kinetic Fokker-Planck-Landau equation reads

$$\partial_t f(t,\vec{x},\vec{v}) + \vec{v}.\nabla_{\vec{x}} f(t,\vec{x},\vec{v}) + \frac{q}{m}\vec{E}(t,\vec{x}).\nabla_{\vec{v}} f(t,\vec{x},\vec{v}) = C_{ee}(f,f) + C_{ei}(f) \tag{1}$$

where f is the electron distribution function, $\vec{E}$ is the electric field, $q = -e$ and m are the charge and the mass of electron and C_{ee} and C_{ei} are the electron-electron and electron-ion collision operators. Their expression is given by

$$C_{ee}(f,f) = \alpha_{ee} div_{\vec{v}}\Big(\int_{\vec{v}'\in\mathbb{R}^3} S(\vec{v}-\vec{v}')[\nabla_{\vec{v}} f(\vec{v}) f(\vec{v}') - f(\vec{v})\nabla_{\vec{v}} f(\vec{v}')]d\vec{v}'\Big), \tag{2}$$

$$C_{ei}(f) = \alpha_{ei} div_{\vec{v}}\Big[S(\vec{v})\nabla_{\vec{v}} f(\vec{v})\Big], \tag{3}$$

where

$$S(\vec{u}) = \frac{1}{|\vec{u}|^3}(|\vec{u}|^2 Id - \vec{u}\otimes\vec{u}) \tag{4}$$

is the Landau tensor and Id is the unit tensor. The parameters α_{ee} and α_{ei} are positive physical parameters given by

$$\alpha_{ee} = \frac{e^4\Lambda}{8\pi\varepsilon_0^2 m^2}, \qquad \alpha_{ei} = \frac{Z n_0 e^4 \Lambda}{8\pi\varepsilon_0^2 m^2} \tag{5}$$

where Z is the ion ionisation degree and n_0 the ion density which is considered as a known function of space. The coefficients Λ and ε_0 are respectively the Coulombian logarithm and the vacuum permittivity. The force acting on electron from the magnetic field is not considered in this paper.

2.2 Collisional electronic M_1 model

The electronic M_1 model [27, 13] is derived performing an angular moment extraction from the Fokker-Planck-Landau equation (1). For the sake of clarity, we omit in the following, the $\vec{x}$ and t dependence of the distribution function. If S^2 is the unit sphere, $\vec{\Omega} = \vec{v}/|\vec{v}|$ represents the direction of propagation of the particle. By setting $\zeta = |\vec{v}|$, the distribution function f writes in the spherical coordinates in the phase space $f(\vec{\Omega},\zeta)$. Three first angular moments of the distribution function are given by

$$f_0(\zeta) = \zeta^2\int_{S^2} f(\vec{\Omega},\zeta)d\vec{\Omega}, \;\; \vec{f}_1(\zeta) = \zeta^2\int_{S^2} f(\vec{\Omega},\zeta)\vec{\Omega}d\vec{\Omega}, \;\; \bar{\bar{f}}_2(\zeta) = \zeta^2\int_{S^2} f(\vec{\Omega},\zeta)\vec{\Omega}\otimes\vec{\Omega}d\vec{\Omega}. \tag{6}$$

In [43, 13], the derivation of the transport part of the electronic M_1 model is detailed. The collisional operators studied here are introduced in [27, 28]. In this work, the following collisional electronic M_1 model is considered

$$\begin{cases} \partial_t f_0(\zeta) + \nabla_{\vec{x}}.(\zeta\vec{f}_1(\zeta)) + \dfrac{q}{m}\partial_\zeta(\vec{f}_1(\zeta).\vec{E}) = Q_0(f_0), \\ \partial_t \vec{f}_1(\zeta) + \nabla_{\vec{x}}.(\zeta\bar{\bar{f}}_2(\zeta)) + \dfrac{q}{m}\partial_\zeta(\bar{\bar{f}}_2(\zeta)\vec{E}) - \dfrac{q}{m\zeta}(f_0(\zeta)\vec{E} - \bar{\bar{f}}_2(\zeta)\vec{E}) = \vec{Q}_1(\vec{f}_1) + \vec{Q}_0(\vec{f}_1), \end{cases} \tag{7}$$

where the collisional operators Q_0 and Q_1 are given by

$$Q_0(f_0) = \frac{2\alpha_{ee}}{3}\partial_\zeta\Big(\zeta^2 A(\zeta)\partial_\zeta(\frac{f_0}{\zeta^2}) - \zeta B(\zeta) f_0\Big), \tag{8}$$

$$\vec{Q}_0(\vec{f}_1) = \frac{2\alpha_{ee}}{3}\partial_\zeta\Big(\zeta^2 A(\zeta)\partial_\zeta(\frac{\vec{f}_1}{\zeta^2}) - \zeta B(\zeta)\vec{f}_1\Big), \tag{9}$$

$$\vec{Q}_1(\vec{f}_1) = -\frac{2\alpha_{ei}}{\zeta^3}\vec{f}_1. \tag{10}$$

The coefficients $A(\zeta)$ and $B(\zeta)$ write

$$A(\zeta) = \int_0^\infty \min(\frac{1}{\zeta^3}, \frac{1}{\omega^3})\omega^2 f_0(\omega) d\omega, \tag{11}$$

$$B(\zeta) = \int_0^\infty \min(\frac{1}{\zeta^3}, \frac{1}{\omega^3})\omega^3 \partial_\omega(\frac{f_0(\omega)}{\omega^2}) d\omega. \tag{12}$$

Next we set,

$$F_0(\zeta) = \frac{f_0(\zeta)}{\zeta^2}, \qquad F_1(\zeta) = \frac{f_1(\zeta)}{\zeta^2}. \tag{13}$$

As remarked in [27], inserting expressions (11) and (12) into (8) and (10) gives the following equivalent expressions for $Q_0(f_0)$ and $\vec{Q}_0(\vec{f}_1)$

$$\begin{cases} Q_0(f_0) = \partial_\zeta\Big(\zeta\int_0^\infty J(\zeta,\zeta')\Big[\frac{F_0(\zeta')}{\zeta}\partial_\zeta F_0(\zeta) - \frac{F_0(\zeta)}{\zeta'}\partial_{\zeta'}F_0(\zeta')\Big]\zeta'^2 d\zeta'\Big), \\ \vec{Q}_0(\vec{f}_1) = \partial_\zeta\Big(\zeta\int_0^\infty J(\zeta,\zeta')\Big[\frac{F_0(\zeta')}{\zeta}\partial_\zeta \vec{F}_1(\zeta) - \frac{\vec{F}_1(\zeta)}{\zeta'}\partial_{\zeta'}F_0(\zeta')\Big]\zeta'^2 d\zeta'\Big), \end{cases} \tag{14}$$

with

$$J(\zeta,\zeta') = \frac{2\alpha_{ee}}{3}\min(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3})\zeta'^2\zeta^2. \tag{15}$$

In this work, both equivalent forms (11)-(12) and (14) are used.

The collisional electronic M_1 model (7) is not directly obtained by moment extraction of the kinetic equation (1). Indeed, the collisional operators (8) and (9) are not directly derived from the angular integration of (2). The moment extraction of the electron-electron collision operator (2) is complex because of its non-linearity. In [28], instead of using (2) the following electron-electron collision operator was proposed

$$Q_{ee}(f) = \frac{1}{\zeta^2}\partial_\zeta\Big(\zeta\int_0^\infty J(\zeta,\zeta')\Big[\frac{F_0(\zeta')}{\zeta}\partial_\zeta f(\zeta) - \frac{f(\zeta)}{\zeta'}\partial_{\zeta'}F_0(\zeta')\Big]\zeta'^2 d\zeta'\Big). \tag{16}$$

This operator satisfies mass and energy conservation properties and an entropy dissipation property. Also it preserves the realisability domain [28]. The angular integration of this operator leads to the definitions (14).

The fundamental point of the moments models is the definition of a closure, which writes the highest moment as a function of the lower ones. This closure relation corresponds to an approximation of the underlying distribution function, which the moments system is constructed from. In the M_1 model (7), we need to define $\bar{\bar{f}}_2$ as a function of f_0 and $\vec{f}_1$. The closure relation originates from an entropy minimisation principle [26, 32]. The underlying distribution function f is obtained as a solution of the following minimisation problem

$$\min_{f\geq 0}\{\ \mathcal{H}(f)\ /\ \forall\zeta\in\mathbb{R}^+,\ \ \zeta^2\int_{S^2} f(\vec{\Omega},\zeta)d\vec{\Omega}=f_0(\zeta),\ \ \zeta^2\int_{S^2} f(\vec{\Omega},\zeta)\vec{\Omega}d\vec{\Omega}=\vec{f_1}(\zeta)\ \},\tag{17}$$

where $\mathcal{H}(f)$ is the Boltzmann entropy defined by

$$\mathcal{H}(f)=\int_{\mathbb{S}^2}(f\ \ln\ f\ -\ f)d\vec{\Omega}.\tag{18}$$

The solution of (17) writes [14, 28]

$$f(\vec{\Omega},\zeta)=\exp(\ a_0(\zeta)+\vec{a}_1(\zeta)\,.\,\vec{\Omega}\ ),\tag{19}$$

where $a_0(\zeta)$ is a scalar and $\vec{a}_1(\zeta)$ a real valued vector. An important parameter is the anisotropy parameter $\vec{\alpha}$ defined with

$$\vec{\alpha}=\frac{\vec{f_1}}{f_0}.\tag{20}$$

Then the moment $\bar{\bar{f}}_2$ can be calculated [13, 15] as a function of f_0 and $\vec{f_1}$

$$\bar{\bar{f}}_2=f_0\Big(\frac{1-\chi(\vec{\alpha})}{2}\bar{\bar{I}}_d+\frac{3\chi(\vec{\alpha})-1}{2}\frac{\vec{f_1}}{|\vec{f_1}|}\otimes\frac{\vec{f_1}}{|\vec{f_1}|}\Big)\tag{21}$$

where $\chi(\vec{\alpha})$ is approximated [13] by

$$\chi(\vec{\alpha})=\frac{1+\vec{\alpha}^2+\vec{\alpha}^4}{3}.\tag{22}$$

The definition (21) enables to close the problem (7). The set of admissible states [13] is defined by

$$\mathcal{A}=\Big((f_0,\vec{f_1})\in\mathbb{R}\times\mathbb{R}^3,\ \ f_0\geq 0,\ \ \ |\vec{f_1}|<f_0\Big)\cup(0,0).\tag{23}$$

2.3 Properties of the collisional operators

In this part, we briefly recall important results established in [27, 28], then we characterise the equilibrium state of the collisional operators (8)-(10) which is given by an isotropic Maxwellian, similarly to the Landau collision operator. It is pointed out that this property is an important new result for the model. Firstly, it was demonstated in [27, 28] that the realisability domain $\mathcal{A}$ is conserved by the collisional operators (8)-(10). Secondly, the quantity $E=\alpha_0 f_0+\vec{\alpha}_1.\vec{f_1}$ is an entropy for the system in the case without electric field. More precisely, from system (7), in the case without electric field we can derive the following inequality

$$\partial_t E+\nabla_{\vec{x}}.\vec{F}\leq 0,\tag{24}$$

where $\vec{F}$ is the entropy flux given by $\vec{F}=\alpha_0\vec{f_1}+\bar{\bar{f}}_2\vec{\alpha}_1$.

Thirdly, the collisional operators (8)-(10) satisfy mass and energy conservation properties. Here, we complete these results characterising the equilibrium state of the collisional operators (8)-(10) which corresponds to an isotropic Maxwellian function.

Theorem 1. *The solution $(f_0, \vec{f_1})$ of the following system*

$$\begin{cases} Q_0(f_0) = 0, \\ \vec{Q}_0(\vec{f_1}) + \vec{Q}_1(\vec{f_1}) = \vec{0}, \end{cases} \tag{25}$$

is given by $f_0 = \zeta^2 K_1 \exp(-K_2\zeta^2)$ and $\vec{f_1} = \vec{0}$ where K_1 and K_2 are two positive real constants.

Proof. We first start to prove the following intermediate results

$$\int_0^{+\infty} \alpha_0 Q_0(f_0) d\zeta + \int_0^{+\infty} \vec{\alpha_1}.\vec{Q}_0(\vec{f_1}) d\zeta \leq 0, \tag{26}$$

and

$$\int_0^{+\infty} \vec{\alpha}_1.\vec{Q}_1(f_1) d\zeta \leq 0. \tag{27}$$

The definition of $\vec{Q}_1(\vec{f_1})$ and the fact that $\vec{\alpha}_1.\vec{f_1} \geq 0$, (see [27]), directly lead to (27). Next, to prove (26) we use a Green formula in the expression of $\int_0^{+\infty} \alpha_0 Q_0(f_0) d\zeta$ to obtain

$$\begin{aligned} &\int_0^{+\infty} \partial_\zeta \left[\zeta \int_0^{+\infty} J(\zeta,\zeta') \left(\frac{f^0(\zeta')}{\zeta'^2} \frac{1}{\zeta} \partial_\zeta (\frac{f^0(\zeta)}{\zeta^2}) - \frac{f^0(\zeta)}{\zeta^2} \frac{1}{\zeta'} \partial_{\zeta'} (\frac{f^0(\zeta')}{\zeta'^2}) \right) \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. (\zeta')^2 \, d\zeta' \right] \alpha_0 d\zeta \\ &= -\int_0^{+\infty} \int_0^{+\infty} J(\zeta,\zeta') \left(\frac{1}{\zeta} F^0(\zeta') \partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta) \partial_{\zeta'} F^0(\zeta') \right) \partial_\zeta \alpha_0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \zeta \, (\zeta')^2 \, d\zeta d\zeta'. \end{aligned} \tag{28}$$

Next we compute $\frac{1}{\zeta} F^0(\zeta')\partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta)\partial_{\zeta'} F^0(\zeta')$. From (13) and (6), we get the relation

$$\partial_\zeta F^0(\zeta) = \int_{S^2} \partial_\zeta \alpha_0(\zeta) \exp(\alpha_0(\zeta) + \vec{\alpha}_1(\zeta).\vec{\Omega} d\vec{\Omega} + \int_{S^2} \vec{\Omega}.\partial_\zeta \vec{\alpha}_1(\zeta) \exp(\alpha_0(\zeta) + \vec{\alpha}_1(\zeta).\vec{\Omega}) d\vec{\Omega}. \tag{29}$$

The expressions of F^0 and $\partial_\zeta F^0$ give

$$\begin{aligned} \frac{1}{\zeta} F^0(\zeta')\partial_\zeta F^0(\zeta) - \frac{1}{\zeta'} F^0(\zeta)\partial_{\zeta'} F^0(\zeta') &= \int_{S^2} \int_{S^2} \exp(\alpha_0(\zeta) + \vec{\alpha}_1(\zeta).\vec{\Omega}) \exp(\alpha_0(\zeta') + \vec{\alpha}_1(\zeta').\vec{\Omega}') \\ &\left(\frac{\partial_\zeta \alpha_0(\zeta)}{\zeta} + \frac{\vec{\Omega}}{\zeta}.\partial_\zeta \vec{\alpha}_1(\zeta) - \frac{\partial_{\zeta'} \alpha_0(\zeta')}{\zeta'} - \frac{\vec{\Omega}'}{\zeta'}.\partial_{\zeta'} \vec{\alpha}_1(\zeta') \right) d\vec{\Omega} d\vec{\Omega}'. \end{aligned}$$

Next by setting

$$K(\zeta,\zeta',\vec{\Omega},\vec{\Omega}') = J(\zeta,\zeta') \, \zeta^2 \zeta'^2 \exp(\alpha_0(\zeta) + \vec{\alpha}_1(\zeta).\vec{\Omega}) \exp(\alpha_0(\zeta') + \vec{\alpha}_1(\zeta').\vec{\Omega}'), \tag{30}$$

$$\delta(\zeta) = \frac{\partial_\zeta \alpha_0(\zeta)}{\zeta}, \quad \vec{\beta}(\zeta) = \frac{\partial_\zeta \vec{\alpha}_1(\zeta)}{\zeta}. \tag{31}$$

and by using equality (30) in (28) we get

$$
\begin{aligned}
-\int_0^{+\infty}\zeta^2\int_0^{+\infty}&\zeta'^2 J(\zeta,\zeta')(\frac{1}{\zeta}F^0(\zeta')\partial_\zeta F^0(\zeta)-\frac{1}{\zeta'}F^0(\zeta)\partial_{\zeta'}F^0(\zeta'))\frac{\partial_\zeta\alpha_0(\zeta)}{\zeta}\,d\zeta d\zeta'\\
&=-\int_0^{+\infty}\int_0^{+\infty}\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\delta(\zeta)-\delta(\zeta')\right)\delta(\zeta)d\zeta d\zeta' d\vec\Omega d\vec\Omega'\\
&+\int_0^{+\infty}\int_0^{+\infty}\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\vec\Omega.\vec\beta(\zeta)-\vec\Omega'.\vec\beta(\zeta')\right)\delta(\zeta)d\zeta d\zeta' d\vec\Omega d\vec\Omega'.
\end{aligned}
$$

The change of variables $(\zeta,\zeta')\mapsto(\zeta',\zeta)$ leads to

$$
\begin{aligned}
-\int_0^{+\infty}\zeta^2\int_0^{+\infty}&\zeta'^2 J(\zeta,\zeta')(\frac{1}{\zeta}F^0(\zeta')\partial_\zeta F^0(\zeta)-\frac{1}{\zeta'}F^0(\zeta)\partial_{\zeta'}F^0(\zeta'))\frac{\partial_\zeta\alpha_0(\zeta)}{\zeta}\,d\zeta d\zeta'\\
&=-\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\delta(\zeta)-\delta(\zeta')\right)^2 d\zeta d\zeta' d\vec\Omega d\vec\Omega' \qquad (32)\\
+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}&\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\vec\Omega.\vec\beta(\zeta)-\vec\Omega'.\vec\beta(\zeta')\right)\left(\delta(\zeta)-\delta(\zeta')\right)d\zeta d\zeta' d\vec\Omega d\vec\Omega'.
\end{aligned}
$$

Next, for the remaining term

$$
\begin{aligned}
\int_0^{+\infty}\vec Q_0(\vec f_1).\vec\alpha_1(\zeta)d\zeta=-\int_0^{+\infty}\zeta^2\int_0^{+\infty}&J(\zeta,\zeta')(\frac{1}{\zeta}F^0(\zeta')\partial_\zeta\vec F^1(\zeta)-\frac{1}{\zeta'}\vec F^1(\zeta)\partial_{\zeta'}F^0(\zeta'))\\
&.\frac{\partial_\zeta(\vec\alpha_1)}{\zeta}(\zeta')^2\,d\zeta d\zeta'\ ,
\end{aligned}
$$

we proceed as previously. The expression of $\vec F^1$ given in (13) leads to

$$
\partial_\zeta\vec F^1(\zeta)=\int_{S^2}\vec\Omega\partial_\zeta\exp(\alpha_0(\zeta)+\vec\alpha_1(\zeta).\vec\Omega)d\vec\Omega+\int_{S^2}\vec\Omega^2\partial_\zeta\vec\alpha_1(\zeta)\,\exp(\alpha_0(\zeta)+\vec\alpha_1(\zeta).\vec\Omega)d\vec\Omega. \qquad (33)
$$

Therefore by using expressions (29) and (33), we get

$$
\begin{aligned}
-\int_0^{+\infty}\int_0^{\infty}&J(\zeta,\zeta')\,\zeta^2\zeta'^2\left(F^0(\zeta')\frac{1}{\zeta}\partial_\zeta\vec F^1(\zeta)-\vec F^1(\zeta)\frac{1}{\zeta'}\partial_{\zeta'}F^0(\zeta')\right).\frac{\partial_\zeta(\vec\alpha_1)}{\zeta}d\zeta d\zeta'\\
&=\int_0^{+\infty}\int_0^{\infty}\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\delta(\zeta)-\delta(\zeta')\right)\vec\Omega.\vec\beta(\zeta)\,d\zeta d\zeta' d\vec\Omega d\vec\Omega'\\
&+\int_0^{+\infty}\int_0^{\infty}\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\vec\beta(\zeta').\vec\Omega'-\vec\beta(\zeta).\vec\Omega\right)\vec\Omega.\vec\beta(\zeta)\,d\zeta d\zeta' d\vec\Omega d\vec\Omega'.
\end{aligned}
$$

Then the change of variables $(\zeta,\zeta')\mapsto(\zeta',\zeta)$ gives

$$
\begin{aligned}
-\int_0^{+\infty}\int_0^{\infty}&J(\zeta,\zeta')\,\zeta^2\zeta'^2\left(F^0(\zeta')\frac{1}{\zeta}\partial_\zeta\vec F^1(\zeta)-\vec F^1(\zeta)\frac{1}{\zeta'}\partial_{\zeta'}F^0(\zeta')\right).\frac{\partial_\zeta(\vec\alpha_1)}{\zeta}d\zeta d\zeta'\\
=\frac{1}{2}\int_0^{+\infty}\int_0^{\infty}&\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\delta(\zeta)-\delta(\zeta')\right)\left(\vec\Omega.\vec\beta(\zeta)-\vec\Omega'.\vec\beta(\zeta')\right)d\zeta d\zeta' d\vec\Omega d\vec\Omega'\\
&-\frac{1}{2}\int_0^{+\infty}\int_0^{\infty}\int_{S^2}\int_{S^2}K(\zeta,\zeta',\vec\Omega,\vec\Omega')\left(\vec\beta(\zeta').\vec\Omega'-\vec\beta(\zeta).\vec\Omega\right)^2 d\zeta d\zeta' d\vec\Omega d\vec\Omega'. \qquad (34)
\end{aligned}
$$

Finally, we add the right-hand sides of (32) and (34) and by using the inequality

$$
(\delta(\zeta)-\delta(\zeta'))(\vec\beta(\zeta).\vec\Omega-\vec\beta(\zeta').\vec\Omega')\le\frac{1}{2}((\delta(\zeta)-\delta(\zeta'))^2+(\vec\beta(\zeta).\vec\Omega-\vec\beta(\zeta').\vec\Omega')^2), \qquad (35)
$$

we obtain (26).

Next, multiplying the first equation of (25) by α_0 and projecting the second on $\vec{\alpha}_1$, adding the two equalities and integrating over ζ gives

$$\int_0^{+\infty} \alpha_0 Q_0(f_0) d\zeta + \int_0^{+\infty} \vec{\alpha}_1.\vec{Q}_0(\vec{f}_1) d\zeta + \int_0^{+\infty} \vec{\alpha}_1.\vec{Q}_1(\vec{f}_1) d\zeta = 0.$$

Since, we proved (26) and (27), it comes

$$\vec{\alpha}_1.\vec{Q}_1(\vec{f}_1) = 0.$$

It follows that $\vec{f}_1 = 0$.

Multiplying the first equation of (25) by $\ln(F_0)$ and integrating over ζ gives

$$\int_0^{+\infty} \partial_\zeta (\zeta \int_0^{+\infty} J(\zeta,\zeta') \Big[\frac{\partial_\zeta F_0(\zeta)}{F_0(\zeta)\zeta} - \frac{\partial_{\zeta'} F_0(\zeta')}{F_0(\zeta')\zeta'}\Big] \zeta'^2 F_0(\zeta) F_0(\zeta') d\zeta' \ln(F_0(\zeta)) d\zeta = 0.$$

By integration by part, it comes

$$-\int_0^{+\infty}\int_0^{+\infty} K(\zeta,\zeta') \Big[\frac{\partial_\zeta F_0(\zeta)}{F_0(\zeta)\zeta} - \frac{\partial_{\zeta'} F_0(\zeta')}{F_0(\zeta')\zeta'}\Big] \frac{\partial_\zeta F_0(\zeta)}{F_0(\zeta)\zeta} d\zeta' d\zeta = 0.$$

with $K(\zeta,\zeta') = \zeta^2 \zeta'^2 F_0(\zeta) F_0(\zeta')$.

The change of variables $(\zeta,\zeta') \mapsto (\zeta',\zeta)$ leads to

$$-\int_0^{+\infty}\int_0^{+\infty} K(\zeta,\zeta') \Big[\frac{\partial_\zeta F_0(\zeta')}{F_0(\zeta')\zeta'} - \frac{\partial_\zeta F_0(\zeta)}{F_0(\zeta)\zeta}\Big] \frac{\partial_{\zeta'} F_0(\zeta')}{F_0(\zeta')\zeta'} d\zeta' d\zeta = 0.$$

Summing the two previous equations gives

$$\int_0^{+\infty}\int_0^{+\infty} K(\zeta,\zeta') \Big[\frac{\partial_\zeta F_0(\zeta')}{F_0(\zeta')\zeta'} - \frac{\partial_\zeta F_0(\zeta)}{F_0(\zeta)\zeta}\Big]^2 d\zeta' d\zeta = 0.$$

It follows that

$$F_0(\zeta) = K_1 \exp(-K_2\zeta^2), \qquad \text{and so} \qquad f_0(\zeta) = \zeta^2 K_1 \exp(-K_2\zeta^2).$$

Since the integral of f_0 in ζ must be positive and finite, K_1 and K_2 are positive real constants.

□

These results demonstrate that the electron-electron collisional operator used for the electronic M_1 model satisfies fundamental properties. In the next section, the derivation of the plasma transport coefficients using this operator is investigated in the framework of the classical transport theory.

3 Derivation of the electronic transport coefficients

3.1 Electron collisional hydrodynamics

It has been shown that the equilibrium state of system (25) is given by an isotropic Maxwellian distribution function. Therefore, in this analytical derivation we consider a distribution function close to the equilibrium

$$f(t,\vec{x},\zeta,\vec{\Omega}) = M_f(\zeta, T_e(t,\vec{x}), n_e(t,\vec{x})) + \varepsilon F(t,\vec{x},\zeta,\vec{\Omega}) \tag{36}$$

where the Maxwellian distribution function reads

$$M_f(\zeta, T_e(t,\vec{x}), n_e(t,\vec{x})) = n_e(t,\vec{x})\Big(\frac{m_e}{2\pi T_e(t,\vec{x})}\Big)^{3/2} \exp\Big(-\frac{m_e\zeta^2}{2T_e(t,\vec{x})}\Big) \tag{37}$$

and the Knudsen number $\varepsilon = \lambda_{ei}/L$ is a small parameter which corresponds to the ratio between the mean free path λ_{ei} and the macroscopic scale lenght L. The perturbation F is seeked under the form

$$F(t,\vec{x},\zeta,\vec{\Omega}) = F_0(t,\vec{x},\zeta) + \vec{F}_1(t,\vec{x},\zeta).\vec{\Omega} \tag{38}$$

According to the Chapman-Enskog approach, the density and temperature macroscopic quantities are defined as

$$n_e(t,\vec{x}) = 4\pi \int_0^{+\infty} f(t,\vec{x},\zeta,\vec{\Omega})\zeta^2 d\zeta, \tag{39}$$

$$T_e(t,\vec{x}) = \frac{4\pi m_e}{3n_e k_B}\int_0^{+\infty} f(t,\vec{x},\zeta,\vec{\Omega})\zeta^4 d\zeta. \tag{40}$$

Therefore the isotropic part of the perturbation verifies the following relations

$$\int_0^{+\infty} F_0(t,\vec{x},\zeta)\zeta^2 d\zeta = 0 \quad \text{and} \quad \int_0^{+\infty} F_0(t,\vec{x},\zeta)\zeta^4 d\zeta = 0. \tag{41}$$

Equation for the density and temperature are following from the integration over ζ of the electronic M_1 model (7) and definitions (39-40)

$$\left\{ \begin{array}{l} \dfrac{\partial n_e}{\partial t} + \nabla_{\vec{x}}.(n_e\vec{u}_e) = 0, \\ \dfrac{\partial T_e}{\partial t} + \vec{u}_e.\nabla_{\vec{x}}(T_e) + \dfrac{2}{3}T_e\nabla_{\vec{x}}.(\vec{u}_e) + \dfrac{2}{3n_e}\nabla_{\vec{x}}.(\vec{q}) = \dfrac{2}{3n_e}\vec{j}.\vec{E} \end{array} \right. \tag{42}$$

where we retained only linear terms in the Knudsen number ε. The temporal evolution of n_e and T_e in these equations is driven by the fluxes of the particles and energy that are expressed through the electric current density and the electron heat flux defined by

$$\vec{j} = -en_e\vec{u}_e = -\frac{4\pi e\varepsilon}{3}\int_0^{+\infty} \vec{F}_1\zeta^3 d\zeta, \quad \vec{q} = \frac{2\pi m_e\varepsilon}{3}\int_0^{+\infty} \vec{F}_1\zeta^5 d\zeta. \tag{43}$$

In order to close the hydrodynamic system (42), one needs to express the electric current and the heat flux (43) in terms of the macroscopic variables n_e, T_e. More precisely, the term $\vec{F}_1$ should be derived explicitly in terms of the gradients of n_e and T_e, then definitions (43) give the electric current and the heat flux. In the quasi-stationary case ($\partial/\partial t << \nu_{ei}$) the second equation of the electronic M_1 model (7) reads

$$\nabla_{\vec{x}}.(\zeta\bar{\bar{f}}_2) + \frac{q}{m}\partial_\zeta(\bar{\bar{f}}_2\vec{E}) - \frac{q}{m\zeta}(f_0\vec{E} - \bar{\bar{f}}_2\vec{E}) = \vec{Q}_1(\vec{f}_1) + \vec{Q}_0(\vec{f}_1). \tag{44}$$

Using the fact that $\bar{\bar{f}}_2 = f_0/3\bar{\bar{I}}_d$ according to equation (38), the previous equation leads to

$$\frac{\zeta}{3}\nabla_{\vec{x}}(f_0) - \frac{e\vec{E}}{3m_e}\frac{\partial f_0}{\partial\zeta} + \frac{2e\vec{E}}{3m\zeta}f_0 = \vec{Q}_1(\vec{f}_1) + \vec{Q}_0(\vec{f}_1) \tag{45}$$

which also rewrites

$$\frac{\zeta}{3}\nabla_{\vec{x}}f_0 - \frac{e\vec{E}\zeta^2}{3m_e}\frac{\partial}{\partial\zeta}\Big(\frac{f_0}{\zeta^2}\Big) = \vec{Q}_1(\vec{f}_1) + \vec{Q}_0(\vec{f}_1). \tag{46}$$

Then using in the place of f_0 the Maxwellian distribution (37), the previous equation gives

$$M_f\zeta\Big[\frac{e\vec{E}^*}{T_e} + \frac{1}{2T_e}\nabla_{\vec{x}}(T_e)(\frac{m_e\zeta^2}{T_e} - 5)\Big] = -\frac{2\alpha_{ei}\varepsilon}{\zeta^3}\vec{F}_1 + \frac{\varepsilon}{\zeta^2}\vec{Q}_0(\zeta^2\vec{F}_1), \tag{47}$$

with $\vec{E}^* = \vec{E} + (1/en_e)\nabla_{\vec{x}}(n_eT_e)$. In the following we note α_{ei} and α_{ee} instead of $\alpha_{ei}\varepsilon$ and $\alpha_{ee}\varepsilon$. In the dimensionless case a parameter $1/\varepsilon$ appears in front of the collisional operators, therefore considering the development (36), the parameter ε vanishes.

In order to obtain $\vec{F}_1$, one should solve the integro-differential equation (47). The resolution of this equation is challenging, however it is a linear equation in $\vec{F}_1$ and the form of the left hand side indicates that the solution is a linear combination of terms proportional to the generalized forces $\vec{E}^*$ and $\nabla(T_e)/T_e$ which can be represented as follows

$$\vec{F}_1 = \zeta\Big(\frac{e\vec{E}^*}{T_e}\phi^E + \nabla_{\vec{x}}(\ln T_e)\phi^Q\Big)M_f \tag{48}$$

where ϕ^E and ϕ^Q are defined below. Inserting this expression into (43) one obtains the following relations [3]

$$\vec{j} = \sigma\vec{E}^* + \alpha\nabla_{\vec{x}}T_e, \tag{49}$$

$$\vec{q} = -\alpha T_e\vec{E}^* - \chi\nabla_{\vec{x}}T_e \tag{50}$$

where α, σ and χ are called the plasma transport coefficients defined by

$$\sigma = -\frac{4\pi e^2}{3T_e}\int_0^\infty \zeta^4\phi^E M_f d\zeta, \quad \chi = \frac{2\pi}{3}\int_0^\infty \zeta^4(5 - \frac{m_e\zeta^2}{T_e})\phi^Q M_f d\zeta, \tag{51}$$

$$\alpha = -\frac{4\pi e}{3T_e}\int_0^\infty \zeta^4\phi^Q M_f d\zeta = \frac{2\pi e}{3T_e}\int_0^\infty \zeta^4(5 - \frac{m_e\zeta^2}{T_e})\phi^E M_f d\zeta. \tag{52}$$

The coefficients α, σ and χ are respectively called the electrical conductivity, the thermoelectric coefficient and the thermal conductivity. In the case of a homogeneous plasma (with no density nor temperature gradients) relation (49) simplifies into the Ohm's law $\vec{j} = \sigma\vec{E}$ and equation (50) leads to $\vec{q} = -\alpha T_e\vec{E}$. One can define the heat conductivity coefficient κ, which is a combination of the other three coefficients

$$\kappa = \chi - \alpha^2 T_e/\sigma. \tag{53}$$

Equation (47) has been established from the collisional electronic M_1 model (7). This equation is identical to the one obtained using the full Fokker-Planck-Landau equation (1), (see [3]) with the exception of the electron-electron collisional operator. Therefore, the possible differences in the plasma transport coefficients between the collisional electronic M_1 model (7) and the Fokker-Planck-Landau equation (1) are due to the electron-electron collisional operator. More precisely, the approximations made to derive the electron-electron collisional operator (8)-(9) for the electronic M_1 model (7) may lead to different plasma transport coefficients. The aim of the following subsections, is to derive the plasma transport coefficients using the collisional electronic M_1 model (7) and to compare them to the ones obtained using the Fokker-Planck-Landau equation (1).

3.2 Transport theory in Lorentzian plasma

In the case of a Lorentzian plasma the ions are highly charged therefore one can neglect the electron-electron collision operator in equation (47). As explained in the previous section, in this case ($Z >> 1$), the plasma transport coefficients are the same in the collisional electronic M_1 model (7) and in the Fokker-Planck-Landau equation (1). An explicit expression of $\vec{F}_1$ and the basic functions ϕ_E and ϕ_Q are easily derived

$$\vec{F}_1 = \zeta M_f \Big[\frac{e\vec{E}^*}{T_e}\Big(-\frac{\zeta^3}{2\alpha_{ei}}\Big) + \nabla_{\vec{x}}(\ln(T_e))\frac{\zeta^3}{4\alpha_{ei}}\Big(5 - \frac{m_e\zeta^2}{v_{T_e}^2}\Big)\Big], \tag{54}$$

and

$$\phi^E = -\frac{\zeta^3}{2\alpha_{ei}}, \qquad \phi^Q = \frac{\zeta^3}{4\alpha_{ei}}\Big(5 - \frac{m_e\zeta^2}{v_{T_e}^2}\Big). \tag{55}$$

Inserting (55) into expressions (51) and (52) gives the transport coefficients for a high Z plasma [3]

$$\sigma_0 = \frac{32}{3\pi}\frac{e^2 n_e}{m_e\nu_{ei}}, \qquad \alpha_0 = \frac{16}{\pi}\frac{en_e}{m_e\nu_{ei}}, \qquad \chi_0 = \frac{200}{3\pi} n_e v_{T_e}\lambda_{ei}. \tag{56}$$

Here the subscript 0 corresponds to the high Z limit. In Figure 1, the electric current and heat flux are displayed in terms of $y = v/v_{T_e}$ using the definition (54).

3.3 Transport theory with electron-electron collisions

In the case of low Z plasmas the calculation presented in the previous section overestimates the transport coefficients because the electron-electron collision operator is not taken into account. In this case, one should solve the full equation (47). Spitzer and Härm [41] solved it numerically in the case of the Fokker-Planck-Landau equation (1). Braginskii [6] derived an approximate analytical solution by expanding $\vec{F}_1$ onto a series of the Laguerre polynomials following ideas used in the kinetic theory of neutral gases [9]. In the present work, we apply the latter method for the case of the electronic M_1 model (7). Following (48), using a decomposition of $\vec{f}_1$ with the two functions ϕ^E and ϕ^Q in equation (47) reads

$$\frac{1}{\zeta^2}\vec{Q}_0(\zeta^2\zeta M_f\vec{\phi}^A) - \frac{2\alpha_{ei}}{\zeta^2}M_f\vec{\phi}^A = \zeta M_f\vec{S}^A \tag{57}$$

where

$$\vec{S}^A = \Big[\frac{e\vec{E}^*}{T_e}S^E - \nabla_{\vec{x}}\ln(T_e)S^Q\Big], \tag{58}$$

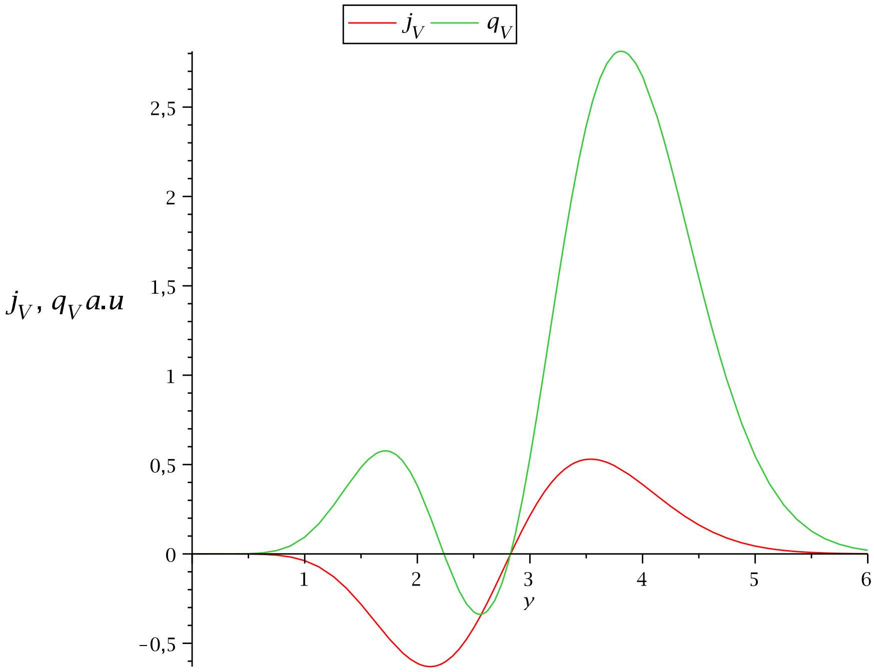

Figure 1: Representation of the velocity-dependent particle flux, $\vec{j}_V = -\zeta^3 \vec{f}_1$ in red and the electron energy flux $\vec{q}_V = m_e \vec{f}_1 \zeta^5 - 5T_e \vec{f}_1 \zeta^3$ in green in the case $Z >> 1$ (Lorentzian approximation).

with

$$S^E = 1, \qquad S^Q = \frac{1}{2}\Big(\frac{\zeta^2}{v_{T_e}^2} - 5\Big). \tag{59}$$

Following Chapman [9] and Braginskii [6], we expand $\vec{F}_1$ over the Laguerre polynomials [2] $L_n^{(3/2)}(x)$, with $x = \zeta^2/2v_{T_e}^2$. Indeed, the source term in the right hand side of (57) is a combination of the two first Laguerre polynomials $S^E = L_0^{3/2}(x)$ and $S^Q = -L_1^{3/2}(x)$. We represent the basic function ϕ^A as

$$\vec{\phi}^A(\zeta) = \sum_{m=0}^{+\infty} \vec{\phi}_m^A L_m^{(3/2)}(\zeta^2/2v_{T_e}^2), \tag{60}$$

multiply (57) by $\zeta^3 L_n^{(3/2)}(\zeta^2/2v_{T_e}^2)$ and integrate over ζ. The electron-ion collision term gives

$$\int_0^{+\infty} -\frac{2\alpha_{ei}}{\zeta^2} M_f \vec{\phi}^A \zeta^3 L_n^{(3/2)}(\zeta^2/2v_{T_e}^2) d\zeta = -2\alpha_{ei} \sum_{m=0}^{+\infty} \vec{\phi}_m^A \int_0^{+\infty} M_f v_{T_e}^2 L_m^{(3/2)}(x) L_n^{(3/2)}(x) dx.$$

Using the definition (37), it comes

$$\int_0^{+\infty} -\frac{2\alpha_{ei}}{\zeta^2} M_f \vec{\phi}^A \zeta^3 L_n^{(3/2)}(\zeta^2/2v_{T_e}^2) d\zeta = -2\alpha_{ei} \frac{n_e}{v_{T_e}(2\pi)^{3/2}} \sum_{m=0}^{+\infty} \vec{\phi}_m^A \int_0^{+\infty} L_m^{(3/2)}(x) L_n^{(3/2)}(x) e^{-x} dx.$$

The computation for the source term reads

$$\int_0^{+\infty} \zeta M_f \vec{S}^A \zeta^3 L_n^{(3/2)}\Big(\frac{\zeta^2}{2v_{T_e}^2}\Big) d\zeta = \frac{n_e v_{T_e}^2}{\pi\sqrt{\pi}} \int_0^{\infty} x\sqrt{x} e^{-x} \Big(\frac{e\vec{E}^*}{T_e} + \frac{1}{T_e} \nabla_{\vec{x}}(T_e)(x - \frac{5}{2})\Big) L_n^{(3/2)}(x) dx$$

and using the orthogonality of the Laguerre polynomials, the previous equation reads

$$\int_0^{+\infty} \zeta M_f \vec{S}^A \zeta^3 L_n^{(3/2)}\Big(\frac{\zeta^2}{2v_{T_e}^2}\Big) d\zeta = \frac{n_e v_{T_e}^2}{\pi}\Big(\frac{3}{4}\frac{e\vec{E}^*}{T_e}\delta_{0n} - \frac{15}{8}\frac{1}{T_e}\nabla_{\vec{x}}(T_e)\delta_{1n}\Big) L_n^{(3/2)}(x)dx.$$

A similar derivation applies to the electron-electron collision operator

$$\int_0^{+\infty} \frac{1}{\zeta^2}\vec{Q}_0(\zeta^2\zeta M_f \vec{\phi}^A)\zeta^3 L_n^{(3/2)}\Big(\frac{\zeta^2}{2v_{T_e}^2}\Big) d\zeta = \frac{n_e v_{T_e}^2}{\pi\sqrt{\pi}} \sum_{m=0}^{+\infty} \vec{\phi}_m^A \int_0^{+\infty} L_n^{(3/2)}(x) Q_0(x\sqrt{x}e^{-x} L_m^{(3/2)}(x))dx.$$

A direct calculation finally gives the following set of equations

$$Z^{-1}\sum_{m=0}^{+\infty} ce_{nm}\vec{\phi}_m^A - \sum_{m=0}^{+\infty} ci_{nm}\vec{\phi}_m^A = \nu_{ei}^{-1}\vec{S}_n^A. \tag{61}$$

Here, ce_{nm} and ci_{nm} are the matrices of the integrals of the electron-electron and electron-ion collision operators. They are defined by

$$ci_{nm} = \int_0^{+\infty} L_n^{(3/2)}(x) L_m^{(3/2)}(x) e^{-x} dx, \tag{62}$$

$$ce_{nm} = \frac{2^{(3/2)} v_{T_e}^3}{Y_{ee}} \int_0^{+\infty} L_n^{(3/2)}(x) Q_0(x\sqrt{x}e^{-x} L_m^{(3/2)}(x))dx, \tag{63}$$

with $Y_{ee} = Z^{-1}Y_{ei}$ and $Y_{ei} = (3\pi/2)\nu_{ei} v_{T_e}^3$.

The term $\vec{S}_n^A$ reads

$$\vec{S}_n^A = \frac{e\vec{E}^*}{T_e}\delta_{0n} - \frac{5}{2}\frac{1}{T_e}\nabla_{\vec{x}}(T_e)\delta_{1n}. \tag{64}$$

The vector S_n^A has only two non-zero components. Therefore, only two first expansion coefficients ϕ_0^A and ϕ_1^A contribute to the transport coefficients (51)-(52)

$$\sigma = -\frac{e^2 n_e}{m_e}\phi_0^E, \qquad \alpha = -\frac{en_e}{m_e}\phi_0^Q = \frac{5}{2}\frac{en_e}{m_e}\phi_1^E, \tag{65}$$

$$\chi = \frac{5}{2} n_e v_{T_e}^2 \phi_1^Q, \qquad \kappa = \frac{5}{2} n_e v_{T_e}^2 (\phi_1^Q - \phi_0^Q \phi_1^E/\phi_0^E). \tag{66}$$

In the limit $Z >> 1$, the first term in (61) vanishes and the model simplifies into the case of a Lorentzian plasma. In this case the first expansion coefficients read $\phi_0^E = -32/3\pi\nu_{ei}$, $\phi_1^E = 32/5\pi\nu_{ei}$, $\phi_2^E = -32/35\pi\nu_{ei}$, $\phi_0^Q = \phi_2^Q = -16/\pi\nu_{ei}$ and $\phi_1^Q = 80/3\pi\nu_{ei}$.

Multiplying (47) by ζ^3 one obtains an equation more suitable for numerical integration. Indeed, the term $1/\zeta^3$ in the electron-ion collision operator makes the equation (47) very stiff when ζ becomes close to zero.

The computation of ci_{nm} using (62) is straightforward. However, the derivation of ce_{nm} using (63) is more challenging. The coefficients $A(\zeta)$ and $B(\zeta)$ in (11) and (12) are involved in the definition of the electron-electron collision operator Q_0. Using the variable $x = \zeta^2/2v_{T_e}^2$ a straight calculation gives

$$A(x) = \frac{n_e}{2\sqrt{\pi}x\sqrt{x}v_{T_e}}\Big[\frac{3\sqrt{\pi}}{\sqrt{2}}\,\mathrm{erf}(\sqrt{x}) - e^{-x}(3\sqrt{2x} + 2\sqrt{2}x\sqrt{x})\Big] + \sqrt{\frac{2}{\pi}}\frac{n_e}{v_{T_e}}e^{-x}, \tag{67}$$

$$B(x) = -\frac{3n_e}{4\sqrt{\pi}v_{T_e}^3 x\sqrt{x}}\Big[\sqrt{2\pi}\,\mathrm{erf}(\sqrt{x}) - 2\sqrt{2x}e^{-x}\Big] - \sqrt{\frac{2}{\pi}}\frac{n_e}{v_{T_e}^3}e^{-x}, \tag{68}$$

where erf is the error function. Next, inserting the definition of Q_0 (8) and expressions (67) and (68) into (63) a long but straight calculation leads to the following expression for ce_{nm}

$$\begin{aligned} ce_{nm} = \int_0^{+\infty} L_n^{(3/2)}(x)\sqrt{x}\partial_x\Big(\Big(2\,\mathrm{erf}(\sqrt{x}) - \frac{4\sqrt{x}}{\sqrt{\pi}}e^{-x}\Big)\partial_x g(x) \\ + \Big(2\,\mathrm{erf}(\sqrt{x}) - \frac{e^{-x}}{\sqrt{\pi}}[4\sqrt{x} - \frac{8}{3}x\sqrt{x}\,]\Big)g(x)\Big)dx \end{aligned} \tag{69}$$

where $g(x) = \sqrt{x}e^{-x}L_m^{(3/2)}(x)$. Using definitions (62) and (69), each component of the matrices ci_{nm} and ce_{nm} can be computed numerically and the set of equations (61) can be solved.

The accuracy of the solution of (61) increases with the number of coefficients ϕ_n^A chosen. The minimum number is two since the first two coefficients ϕ_0 and ϕ_1 contribute to the transport coefficients. Such a two polynomial approximation was considered by Braginskii [6] for the Fokker-Planck-Landau equation (1). The four-polynomial approximation provides results beyond the need of experimental plasma physics. Kaneko [22] used 6 Laguerre polynomials and the high accuracy of transport coefficients he obtained was confirmed in [23] and [24] with 50 Laguerre polynomials. In this work, 6 Laguerre polynomials were used to ensure a high accuracy of the transport coefficients. The sixth polynomial expansion leads to the following approximations

$$\phi_0^E \approx -\nu_{ei}^{-1}\frac{670.4256Z + 4467.7933Z^2 + 3306.3497Z^3 + 851.0715Z^4 + 90.4477Z^5 + 3.3952Z^6}{173.6923 + 2826.2811Z + 3603.5560Z^2 + 1604.8485Z^3 + 320.2840Z^4 + 29.3133Z^5 + Z^6},$$

$$\phi_0^Q \approx -\frac{5}{2\nu_{ei}}\frac{29.3839Z + 1611.9336Z^2 + 1595.3372Z^3 + 462.0396Z^4 + 52.2678Z^5 + 2.0373Z^6}{173.6923 + 2826.2811Z + 3603.5560Z^2 + 1604.8485Z^3 + 320.2840Z^4 + 29.3133Z^5 + Z^6},$$

$$\phi_1^E \approx \nu_{ei}^{-1}\frac{-86.0931Z + 1177.6149Z^2 + 1414.6187Z^3 + 437.3817Z^4 + 51.1892Z^5 + 2.0373Z^6}{173.6923 + 2826.2811Z + 3603.5560Z^2 + 1604.8485Z^3 + 320.2840Z^4 + 29.3133Z^5 + Z^6},$$

$$\phi_1^Q \approx \frac{5}{2\nu_{ei}}\frac{163.9843Z + 2155.5776Z^2 + 2263.5819Z^3 + 702.4687Z^4 + 83.7720Z^5 + 3.3950Z^6}{173.6923 + 2826.2811Z + 3603.5560Z^2 + 1604.8485Z^3 + 320.2840Z^4 + 29.3133Z^5 + Z^6}.$$

The velocity-dependent flux functions presented in Fig. 2 show that the electron-electron contribution decreases as Z increases. We introduce the following dimensionless coefficients $\gamma_\sigma, \gamma_\alpha, \gamma_\chi, \gamma_\kappa$ defined by

$$\gamma_\sigma = \sigma/\sigma_0, \quad \gamma_\alpha = \alpha/\alpha_0, \quad \gamma_\chi = \chi/\chi_0, \quad \gamma_\kappa = \kappa/\kappa_0 \tag{70}$$

where the index 0 denotes the case of the Lorentzian approximation ($Z >> 1$). The computation of these coefficients shows that all of them are inferior to 1, that is, the Lorentzian approximation ($Z >> 1$) overestimates the electron transport coefficients for low-Z plasmas. The coefficients $\gamma_\sigma, \gamma_\alpha, \gamma_\chi, \gamma_\kappa$ are displayed in Figs. 3 and 4 in function of Z for the electron-electron Landau collision operator C_{ee} given in (4) and for the electron-electron M_1 collision operator (8)-(9) using six Laguerre polynomials.

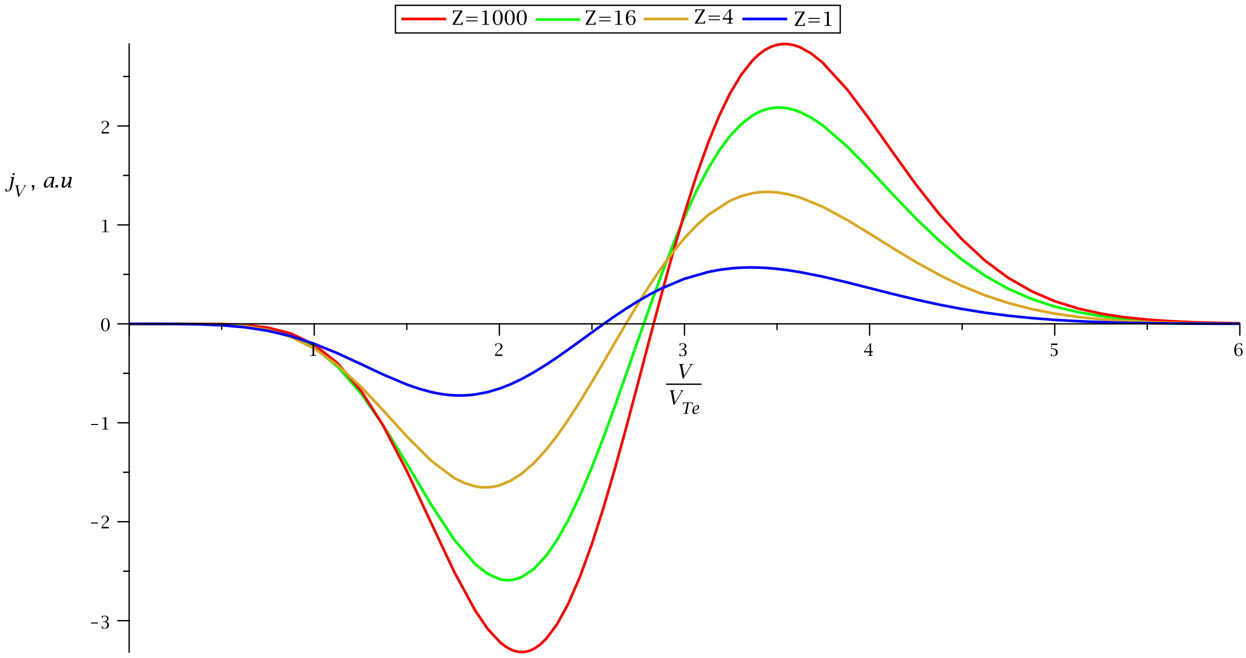

Figure 2: Representation of the velocity-dependent particle flux, $\vec{j}_V = -\zeta^3 \vec{f}_1$, in the case $Z = 1$ (blue), $Z = 4$ (yellow), $Z = 16$ (green) and $Z >> 1$ (Lorentzian approximation) in red.

According to Fig. 3, the electron-electron collision operator (8)-(9) used for the electronic M_1 model underestimates the thermoelectric coefficient σ. In the large Z limit (Lorentzian approximation), the collisional M_1 model and the Fokker-Planck-Landau equation coincide. However, despite the correct tendency, the curve obtained using the M_1 collisional model underestimates the thermoelectric coefficient σ with a largest error of 43% in the case $Z = 1$. Also, the two curves of γ_α, obtained with the M_1 model and the Fokker-Planck-Landau equation, as a function of Z are very close. In Figure 4, one observes that the curves representing the coefficients γ_χ and γ_κ overlap. The electron-electron collisional operator (8)-(9) recovers the correct χ and κ plasma transport coefficients.

In conclusion, the electron-electron collisional operator (8)-(9) used for the electronic M_1 model recovers the correct χ and κ plasma transport coefficients and is very accurate for the coefficient α. The main error is made with the coefficient σ with a maximum error of 43% in the case $Z = 1$. These results demonstrate the correct behaviour of the electron-electron collision operator (8)-(9) which can be used for practical applications.

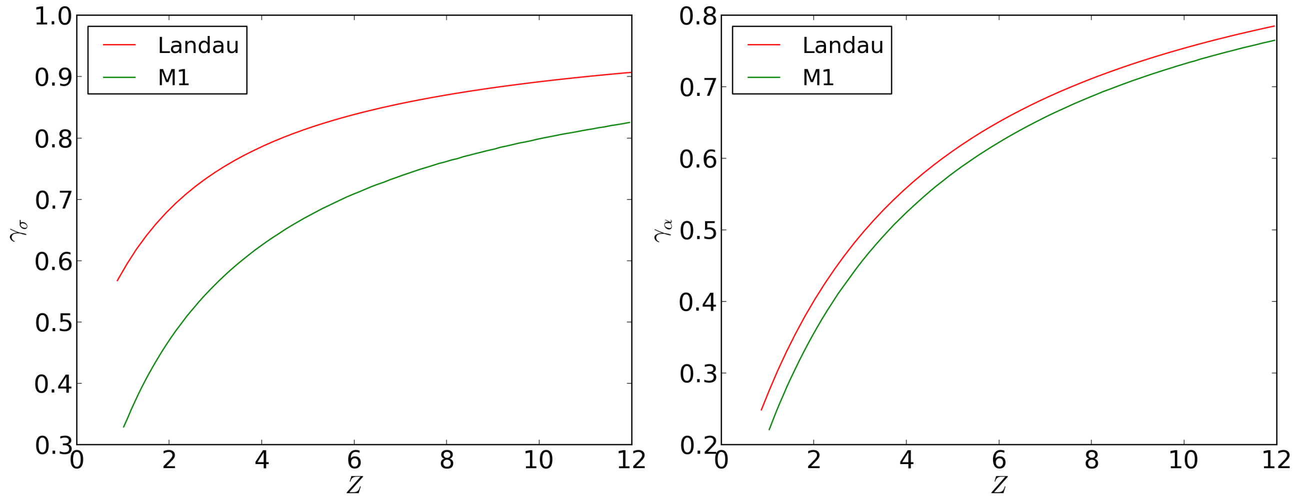

Figure 3: Representation of γ_σ (left) and γ_α (right) as a function of Z for the Landau (red) and the M_1 (green) collision operators using six Laguerre polynomials.

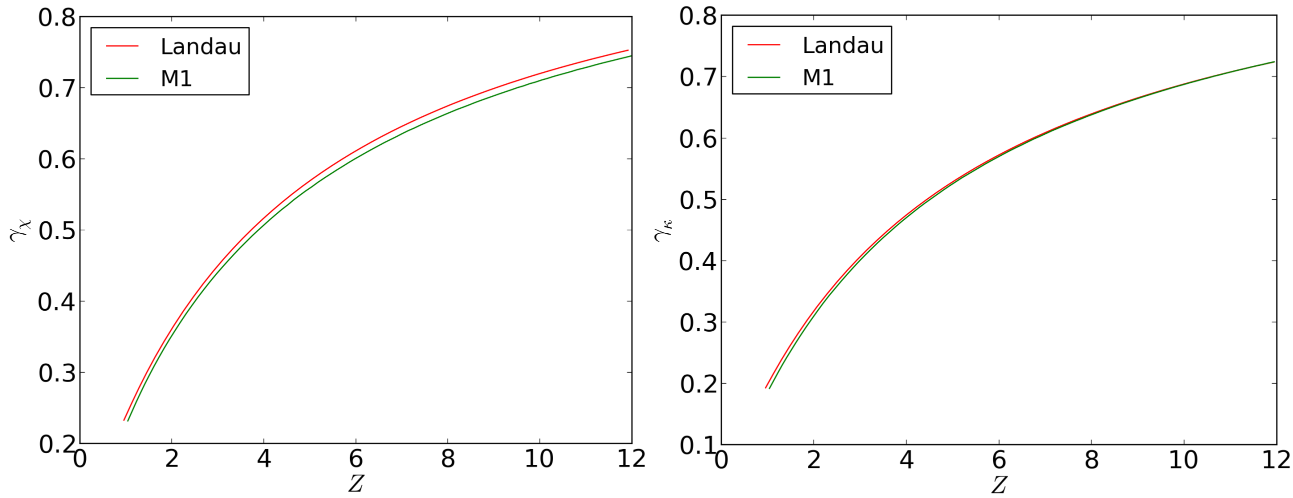

Figure 4: Representation of γ_χ (left) and γ_κ (right) as a function of Z for the Landau (red) and the M_1 (green) collision operators using six Laguerre polynomials.

4 Conclusion

In this work, the fundamental properties of the electron-electron and electron-ion collision operators used for the electronic M_1 model have been studied. It is shown that their equilibrium states is given by an isotropic Maxwellian distribution function. In addition, in the Lorentzian approximation, the electronic M_1 model and the Fokker-Planck-Landau equation coincide. The electron transport coefficients are derived using the electron-electron collision operators proposed for the electronic M_1 model. Despite, the approximations used, accurate plasma transport coefficients have been obtained. The correct χ and κ plasma transport coefficients are recovered and the coefficient α is very close to the one obtained with the Fokker-Planck-Landau equation. The main error is made with the thermoelectric coefficient σ in the case $Z = 1$. In spite of this error, these results show that the electron-electron collision operator is a good candidate for physical applications. It may be possible to improve this operator in order to obtain a more accurate σ coefficient. However, since the angular extraction of the kinetic electron-electron collision operator is complex, such an issue seems challenging.

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