ASYMPTOTIC-PRESERVING WELL-BALANCED SCHEME FOR THE ELECTRONIC M_1 MODEL IN THE DIFFUSIVE LIMIT: PARTICULAR CASES. *

S. Guisset[1,2], S. Brull[1], E. d'Humières[2] and B. Dubroca[2]

Abstract. This work is devoted to the derivation of an asymptotic-preserving scheme for the electronic M_1 model in the diffusive regime. The case without electric field and the homogeneous case are studied. The derivation of the scheme is based on an approximate Riemann solver where the intermediate states are chosen consistent with the integral form of the approximate Riemann solver. This choice can be modified to enable the derivation of a numerical scheme which also satisfies the admissible conditions and is well-suited for capturing steady states. Moreover, it enjoys asymptotic-preserving properties and handles the diffusive limit recovering the correct diffusion equation. Numerical tests cases are presented, in each case, the asymptotic-preserving scheme is compared to the classical HLL [43] scheme usually used for the electronic M_1 model. It is shown that the new scheme gives comparable results with respect to the HLL scheme in the classical regime. On the contrary, in the diffusive regime, the asymptotic-preserving scheme coincides with the expected diffusion equation, while the HLL scheme suffers from a severe lack of accuracy because of its unphysical numerical viscosity.

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Introduction

In inertial confinement fusion, nanosecond laser pulses are used to ignite a deuterium-tritium target. An accurate description of this process is necessary for the understanding of laser-matter interactions and for target design. Numerous physical phenomena such as, parametric [35, 62] and hydrodynamic [31, 70, 77] instabilities, laser-plasma absorption [69], wave damping [53], energy redistribution [65] inside the plasma and hot spots formation [12, 60] from which the thermonuclear reactions propagates depend on the electron heat transport. The most popular electron heat transport theory was developed by Spitzer and Härm [72] who first solved the electron kinetic equation by using the expansion of the electron mean free path to the temperature scale length (denoted ε in this paper). Considering the distribution function of particles close to equilibrium, its deviation from the Maxwellian distribution function can be computed and the electron transport coefficients in a fully ionized plasma without magnetic field are derived. However, even if the electron heat transport is essential, it is not correctly described in large inertial confinement fusion tools. Indeed, when the electron mean free path exceeds about 2.10^{-3} times the temperature gradient length, the local electron transport model of

Spitzer and Härm fails. The transport coefficients were derived in the case where the isotropic part of the electron distribution function remains close to the Maxwellian function. The results of Spitzer and Härm have been reproduced in several approaches [4, 11, 71] which develop another technique of solution to the integral equation for the electron distribution function introduced many years before by Chapman and Enskog [22] for neutral gases. Therefore, kinetic approaches seem necessary in the context of inertial confinement fusion. In such multiscale issues, kinetic solvers are often very computationally expensive and usualy limited to time and length much shorter than those studied with hydrodynamic simulations. It is then a challenge to describe kinetic effects using reduced kinetic code on fluid time scales.

The angular moments models represent an alternative method situated between kinetic and fluid models. Their computational times are shorter than kinetics ones and provide results with a higher accuracy than fluid models. Originally, the moment closure hierarchy introduced by Grad [39] leads to hyperbolic set of equations for flows close to equilibrium but may suffer from closure breakdown and lead to unrealisable macroscopic moments. Grad hierarchy is derived from a truncated polynomial series expansion for the velocity distribution function near the Maxwellian equilibrium and does not ensure the postivity of the distribution function. Other moment closure approaches have been investigated based on entropy minimisation principles [2,57,63,64,73]. The distribution function derived, verifies a minimum entropy property and the consistency with the set of moments. Fundamental mathematical properties [41,61] such as positivity of the distribution function, hyperbolicity and entropy dissipation can be exhibited. Levermore [57] proposed a hierarchy of minimum-entropy closure where the lowest order closure are the Maxwellian and Gaussian closure. In the present case, the aim is different. Here the energy of particles constitutes a free parameter. Then we integrate only the kinetic equation with respect to the angle variable and we return only the energy of particles as kinetic variable. By using a closure defined from a minimization entropy principle, we obtain the M_1 model [32, 33, 58]. The M_1 model is largely used in various applications such as the radiative transfer [7, 23, 34, 67, 68, 75, 76] or electronic transport [32, 58]. The M_1 model is known to satisfy fondamental properties such as the positivity of the first angular moment, the flux limitation and conservation of total energy. Also, it correctly recovers the asymptotic diffusion equation in the limit of long time behavior with important collisions [33].

One challenging issue is to derive numerical schemes satisfying fondamental properties. For example, the classical HLL scheme [43] ensures the positivity of the first angular moment and the flux limitation property. However, this scheme fails in recovering the correct limit diffusion equation in the asymptotic regime [3]. Then, numerous numerical schemes have been derived over the last 20 years to recover the correct asymptotic limit. These schemes are able to handle multiscales situations and are called asymptotic-preserving (AP) scheme. They are consistent with the macroscopic model when ε tends to zero and are uniformly stable with respect to ε. AP schemes also avoid the coupling of different scales equations where the coupling conditions at the interface can be difficult to obtain. Early works on AP schemes have been performed in [45, 46, 54, 55]. These works have been largely extended in the frame of kinetic equations in fluid and diffusive regimes [24, 47, 51, 66]. The time stiffness induced by the collisional operator led to propose a decomposition of the distribution function between an equilibrium and a deviation [5, 14, 18, 44, 48, 50, 52, 56]. In [13], a two steps method based on a relaxation scheme and a well-balanced scheme step is proposed, (see [9, 49] for more details on the relaxation scheme framework). The derivation of well-balanced schemes also help to design AP schemes [37, 38] (see also [1, 9, 10, 17, 20, 21, 36, 40] for details on well-balanced schemes in different framework). The AP frame was also largely extended to the quasi-neutral limit [25–29, 42]. In [7], an $HLLC$ scheme is proposed to solve the M_1 model of radiative transfer in two space dimensions. The $HLLC$ approximate Riemann solver considered and relevant numerical approximations of the extreme wavespeeds give the asymptotic-preserving property. Close ideas were also developed in [6], where a relaxation scheme is exhibited. In order to derive situable schemes pertinent for transport and diffusion regimes, different authors proposed modified Godunov-type schemes in order to include sources terms [40]. The numerical viscosity is modified in [15, 16, 37, 38] to correctly recover the expected diffusion regimes but extensions seem challenging issues. In [8], the approximate HLL Riemann solver is modified to include collisional source term. The resulting numerical scheme satisfies all the fundamental properties and a clever correction enables to recover the good diffusion equation in the asymptotic limit.

In this paper, we consider the M_1 model for the electronic transport [32,58,59] in a Lorentzian plasma where ions are supposed fixed. The moment system studied writes

$$
\begin{cases}
\partial_t f_0(t,x,\zeta) + \zeta\partial_x f_1(t,x,\zeta) + E(x)\partial_\zeta f_1(t,x,\zeta) = 0, \\[2mm]
\partial_t f_1(t,x,\zeta) + \zeta\partial_x f_2(t,x,\zeta) + E(x)\partial_\zeta f_2(t,x,\zeta) - \dfrac{E(x)}{\zeta}(f_0(t,x,\zeta) - f_2(t,x,\zeta)) \\[3mm]
\hspace{7cm} = -\dfrac{2\alpha_{ei}(x)f_1(t,x,\zeta)}{\zeta^3},
\end{cases}
\tag{1}
$$

where f_0, f_1 and f_2 are the first three angular moments of the electron distribution function f. Omitting the x and t dependancy, they are given by

$$
f_0(\zeta) = \zeta^2 \int_{-1}^{1} f(\mu,\zeta)d\mu, \ \ f_1(\zeta) = \zeta^2 \int_{-1}^{1} f(\mu,\zeta)\mu d\mu, \ \ f_2(\zeta) = \zeta^2 \int_{-1}^{-1} f(\mu,\zeta)\mu^2 d\mu.
\tag{2}
$$

The coefficient α_{ei} is a positive physical function which may depend of x, E represents the electrostatic field as function of x and ζ the velocity modulus. The fundamental point of the moments models is the definition of the closure which writes the highest moment as a function of the lower ones. This closure relation corresponds to an approximation of the underlying distribution function, which the moments system is constructed from. In the M_1 problem we need to define f_2 as a function of f_0 and f_1. The closure relation originates from an entropy minimization principle [57,63]. The moment f_2 can be calculated [32,34] as a function of f_0 and f_1

$$
f_2(t,x,\zeta) = \chi\Big(\frac{f_1(t,x,\zeta)}{f_0(t,x,\zeta)}\Big)f_0(t,x,\zeta), \quad \text{with} \quad \chi(\alpha) \approx \frac{1+\alpha^2+\alpha^4}{3}.
\tag{3}
$$

The set of admissible states [32] is defined by

$$
\mathcal{A} = \Big((f_0,f_1) \in \mathbb{R}^2, \ f_0 \geq 0, \ |f_1| \leq f_0\Big).
\tag{4}
$$

A challenging issue is to derive a numerical scheme for the electron M_1 model (1) satisfying all the fondamental properties and which handles correctly the diffusive limit recovering the good diffusion equation. Such a scheme could then have a direct access to all the nonlocal regimes and their related physical effects described above while the others numerical schemes breakdown in such regimes. Different complications arise when considering such an issue. Firstly, the electronic M_1 model (1) is nonlinear. Because, of the entropic closure, the angular moment f_2 is a nonlinear function of f_0 and f_1. Secondly, the approach undertaken must be sufficiently general to correctly take into account the source term $-E(x)(f_0(t,x,\zeta) - f_2(t,x,\zeta))/\zeta$. One must notice, that this term is closely related to the term $E\partial_\zeta f_2(t,x,\zeta)$, it plays an important role for low energies and can not be treated as a collisionnal source term. Thirdly, for the purpose of realistic physical applications, one may require to correctly capture steady states. In the case of near-equilibrium configurations such a well-balancing property is then desired. Also, the physical parameter α_{ei} is a function of x and cannot be treated as a constant. Finally, the space and energy dependencies of the angular moments, lead to a very complexe diffusion equation in the asymptotic limit with mixed derivatives.

In this paper, the case without electric field and the homogeneous case are studied. The extension to the general case is beyond the scope of this paper and postponed to another paper. However, the generalisation to the general problem requires a deep understanding of the two configurations studied here. The approach retained is noticeably different with [6, 7, 43]. The derivation of the scheme is based on an approximate Riemann solver where the intermediate states are chosen consistent with the integral form of the approximate Riemann solver. This choice can be modified to enable the derivation of a scheme which also satisfies the admissibility conditions (4) and is well-suited for capturing steady states. Moreover, it enjoys asymptotic-preserving properties and correctly handles the diffusive limit recovering the good diffusion equation.

We first introduce the model without electrostatic field and its diffusive limit in Section 2. The limits of the classical HLL scheme [43] are briefly recalled before introducing the derivation of the new numerical

scheme. The asymptotic-preserving property is exhibited. Then, Section 3 is devoted to the homogeneous case with an electric field. We point out the great difficulties encountered when using a relaxation approach in order to include the source term $-E(x)(f_0(t, x, \zeta) - f_2(t, x, \zeta))/\zeta$. Then, the derivation of an asymptotic-preserving scheme following the method introduced in the previous section is detailed and the well-balanced and asymptotic-presering properties are analysed. In Section 4, different numerical tests are presented to highlight the efficiency of the present method. We conclude the paper in Section 5.

1. CASE WITHOUT ELECTROSTATIC FIELD

The first simplified case we consider is given by system (1) without electrostatic field E. In this case the M_1 model (1) writes

$$\begin{cases} \partial_t f_0 + \zeta \partial_x f_1 = 0, \\ \partial_t f_1 + \zeta \partial_x f_2 = -\dfrac{2\alpha_{ei}}{\zeta^3} f_1. \end{cases} \tag{5}$$

A very similar system was considered in [6] in the frame of radiative transfer and a relaxation scheme was proposed. The same procedure could be applied in this case, however we introduce a different approach based on approximate Riemann solvers.

1.1. Model and diffusive limit

Introducing a diffusive scaling, time is scaled by t^*, space by x^*, velocity by v_{th}^* and electric field by v_{th}^2/x^*. These parameters are chosen such that $\tau_{ei}/t^* = \varepsilon^2$, $\lambda_{ei}/x^* = \varepsilon$ where the electron-ion collisionnal period is given by $\tau_{ei} = v_{th}^3/(\alpha_{ei}\sigma)$ and the mean free path by $\lambda_{ei} = v_{th}\tau_{ei}$. The coefficient σ is a positive function of x and ε a positive parameter, which is devoted to tend to zero. In that case, system (5) rewrites

$$\begin{cases} \varepsilon \partial_t f_0^\varepsilon + \zeta \partial_x f_1^\varepsilon = 0, \\ \varepsilon \partial_t f_1^\varepsilon + \zeta \partial_x f_2^\varepsilon = -\dfrac{2\sigma}{\zeta^3} \dfrac{f_1^\varepsilon}{\varepsilon}. \end{cases} \tag{6}$$

Inserting the following Hilbert expansion of f_0^ε and f_1^ε

$$\begin{cases} f_0^\varepsilon = f_0^0 + \varepsilon f_0^1 + O(\varepsilon^2), \\ f_1^\varepsilon = f_1^0 + \varepsilon f_1^1 + O(\varepsilon^2), \end{cases} \tag{7}$$

into the second equation of (6) leads to

$$f_1^0 = 0. \tag{8}$$

Using the definition (3), it follows that

$$f_2^0 = f_0^0/3. \tag{9}$$

So, the second equation of (6) gives

$$f_1^1 = -\frac{\zeta^4}{6\sigma} \partial_x f_0^0. \tag{10}$$

Using the previous equation and the first equation of (6) finally leads to the diffusion equation for f_0^0

$$\partial_t f_0^0(t, x) - \zeta \partial_x \Big(\frac{\zeta^4}{6\sigma(x)} \partial_x f_0^0(t, x) \Big) = 0. \tag{11}$$

1.2. **The numerical method**

In this part, we first recall the limit of the HLL scheme, usually used for the electronic M_1 model, for the diffusive limit.

1.2.1. *Limit of the HLL scheme*

Introduce a uniform mesh with constant space step $\Delta x = x_{i+1/2} - x_{i-1/2}$, $i \in Z$ and a time step Δt. We consider a piecewise constant approximate solution $U^h(x, t^n) \in \mathbb{R}^2$ at time t^n

$$U^h(x, t^n) = U_i^n \quad if \quad x \in [x_{i-1/2}, x_{i+1/2}] \tag{12}$$

with $U_i^n = {}^t(f_{0i}^n, f_{1i}^n)$. The classical HLL scheme [43] for the system (11), in the case where the minimum and maximum velocity waves involved in the approximate Riemann solver are chosen equal to $-\zeta$ and ζ, writes

$$\begin{cases} \varepsilon \dfrac{f_{0i}^{n+1,\varepsilon} - f_{0i}^{n,\varepsilon}}{\Delta t} + \zeta \dfrac{f_{1i+1}^{n,\varepsilon} - f_{1i-1}^{n,\varepsilon}}{2\Delta x} - \zeta \Delta x \dfrac{f_{0i+1}^{n,\varepsilon} - 2f_{0i}^{n,\varepsilon} + f_{0i-1}^{n,\varepsilon}}{\Delta x^2} = 0, \\[3mm] \varepsilon \dfrac{f_{1i}^{n+1,\varepsilon} - f_{1i}^{n,\varepsilon}}{\Delta t} + \zeta \dfrac{f_{2i+1}^{n,\varepsilon} - f_{2i-1}^{n,\varepsilon}}{2\Delta x} - \zeta \Delta x \dfrac{f_{1i+1}^{n,\varepsilon} - 2f_{1i}^{n,\varepsilon} + f_{1i-1}^{n,\varepsilon}}{\Delta x^2} = -\dfrac{2\sigma_i}{\zeta^3} \dfrac{f_{1i}^{n,\varepsilon}}{\varepsilon}. \end{cases} \tag{13}$$

We introduce the discrete Hilbert expansions

$$\begin{cases} f_{0i}^\varepsilon = f_{0i}^{n,0} + \varepsilon f_{0i}^{n,1} + O(\varepsilon^2), \\ f_{1i}^{n,\varepsilon} = f_{1i}^{n,0} + \varepsilon f_{1i}^{n,1} + O(\varepsilon^2). \end{cases} \tag{14}$$

At the order ε^{-1}, the second equation of (13) gives

$$f_{1i}^{n,0} = 0 \tag{15}$$

and using the definition (3), it follows that

$$f_{2i}^{n,0} = f_{0i}^{n,0}/3. \tag{16}$$

At the order ε^0, the second equation of (13) gives

$$f_{1i}^{n,1} = -\frac{\zeta^3}{3\sigma_i} \frac{f_{0i+1}^{n,0} - f_{0i-1}^{n,0}}{2\Delta x}. \tag{17}$$

However, because of the diffusive part of the HLL scheme, the first equation of (13) also leads to

$$\frac{f_{0i+1}^{n,0} - 2f_{0i}^{n,0} + f_{0i-1}^{n,0}}{\Delta x^2} = 0 \tag{18}$$

which is not the diffusion equation expected for f_0^0. The diffusive part of the HLL scheme gives an unphysical numerical viscosity and leads to the wrong asymptotic behavior.

1.2.2. *Derivation of the scheme*

The ideas introduced in [6, 8, 9, 40] in order to include the contribution of source terms, urge to consider approximate Riemann solvers which owns a stationnary discontinuity (0-contact discontinuity). Therefore, we introduce the following approximate Riemann solvers at each cells interface, denoted by $U_\mathcal{R}(x/t, U^L, U^R)$, defined by

$$U_\mathcal{R}(x/t, U^L, U^R) = \begin{cases} U^L & if \quad x/t < -a_x, \\ U^{L*} & if \quad -a_x < x/t < 0, \\ U^{R*} & if \quad 0 < x/t < a_x, \\ U^R & if \quad a_x < x/t, \end{cases} \tag{19}$$

where $U^{L*} = {}^t(f_0^{L*}, f_1^*)$, $U^{R*} = {}^t(f_0^{R*}, f_1^*)$ and the minimum and maximum velocity waves $-a_x$ and a_x. Note, we choose the two velocity waves to be opposite. The structure solution of the approximate Riemann problem is displayed on Figure 1. At the interface $x_{i+\frac{1}{2}}$, the quantites U^L and U^R stand for $U_i = {}^t(f_{0i}, f_{1i})$ and $U_{i+1} = {}^t(f_{0i+1}, f_{1i+1})$. Contrarily to the classical HLL scheme [74] two intermediate states U^{L*} and U^{R*} are introduced. The second components of the two intermediate states are chosen equal, ie $f_1^{L*} = f_1^{R*} = f_1^*$.
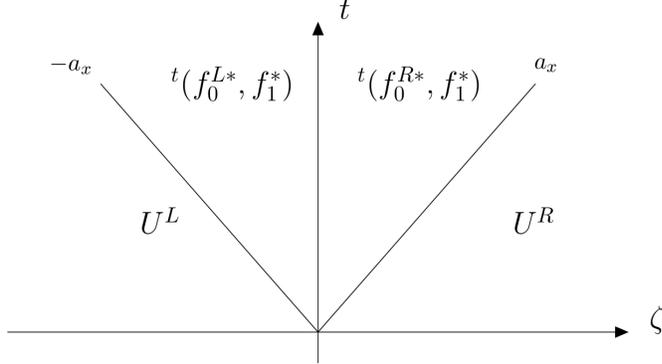
FIGURE 1. Structure solution of the approximate Riemann problem.

The approximate solution at time $t^n + \Delta t$ is chosen as

$$U^h(x, t^n + \Delta t) = U_{\mathcal{R}}\left(\frac{x - x_{i+1/2}}{t^n + \Delta t}, U_i, U_{i+1}\right) \text{ if } x \in [x_i, x_{i+1}]. \tag{20}$$

As the following CFL condition is respected

$$\Delta t \leq \frac{\Delta x}{2a_x}, \tag{21}$$

the piecewise constant approximate solution is then obtained

$$U_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U^h(x, t^{n+1})dx. \tag{22}$$

The intermediate states f_0^{L*}, f_0^{R*} and f_1^* must be defined. Integrating the first equation of (5) on $[-a_x\Delta t, a_x\Delta t] \times [0, \Delta t]$ and multiplying by $\frac{1}{2a_x\Delta t}$, gives the following consistency condition

$$\frac{f_0^{L*} + f_0^{R*}}{2} = \frac{f_0^L + f_0^R}{2} - \frac{1}{2a_x}[\zeta f_1^R - \zeta f_1^L]. \tag{23}$$

The unknowns f_0^{L*} and f_0^{R*} will be chosen in order to satisfy this consistency condition (23). The same procedure using the second equation of (5) gives

$$f_1^* = \frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x}(\zeta f_2^R - \zeta f_2^L) - \frac{2}{\zeta^3}\frac{1}{2a_x\Delta t}\int_{-a_x\Delta t}^{a_x\Delta t}\int_0^{\Delta t}\sigma(x)f_1(x, t)dtdx. \tag{24}$$

The following approximation is made

$$\frac{1}{2a_x\Delta t}\int_{-a_x\Delta t}^{a_x\Delta t}\int_0^{\Delta t}\sigma(x)f_1(x, t)dtdx = \bar{\sigma}\Delta t f_1^*, \tag{25}$$

with $\bar{\sigma} = \sigma(0)$. Using (25) in (24), it follows that

$$f_1^* = \frac{\zeta^3}{\zeta^3 + 2\bar{\sigma}\Delta t}\left[\frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x}(\zeta f_2^R - \zeta f_2^L)\right]. \tag{26}$$

Finally the following definition of f_1^* is chosen

$$f_1^* = \frac{2a_x\zeta^3}{2a_x\zeta^3 + 2\bar{\sigma}\Delta x}\left[\frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x}(\zeta f_2^R - \zeta f_2^L)\right]. \tag{27}$$

It will be shown in the next part, that this choice enables to obtain the good asymptotic-preserving property. Also, this definition recovers the formalism introduced in [7, 8].

In order to respect the consistency relation (23), the unknown f_0^{L*} and f_0^{R*} are defined by

$$\begin{cases} f_0^{L*} = \tilde{f}_0 - \Gamma, \\ f_0^{R*} = \tilde{f}_0 + \Gamma, \end{cases} \tag{28}$$

with

$$\tilde{f}_0 = \frac{f_0^L + f_0^R}{2} - \frac{1}{2a_x}[\zeta f_1^R - \zeta f_1^L] \tag{29}$$

and the coefficient Γ is calculated using the classical Rankine-Hugoniot conditions

$$\begin{cases} f_0^{L*} = f_0^L - \dfrac{\zeta}{a_x}(f_1^* - f_1^L), \\ f_0^{R*} = f_0^R - \dfrac{\zeta}{a_x}(f_1^R - f_1^*). \end{cases} \tag{30}$$

It follows that

$$\Gamma = \frac{1}{2}[f_0^R - f_0^L - \frac{\zeta}{a_x}(f_1^L - 2f_1^* + f_1^R)]. \tag{31}$$

In order to satisfy the admissibility conditions (4), we propose to modify the states f_0^{L*} and f_0^{R*} such that

$$\begin{cases} f_0^{L*} = \tilde{f}_0 - \Gamma\theta, \\ f_0^{R*} = \tilde{f}_0 + \Gamma\theta, \end{cases} \tag{32}$$

where $\theta \in [0, 1]$ is fixed to ensure the admissibility conditions.

Remark 1.1. In the case $\theta = 0$, the admissibility requirements (4) are fulfilled.

Indeed, in this case system (32) gives $f_0^{R*} = f_0^{L*} = \tilde{f}_0$ and f_1^* is given by (27). Since $2a_x\zeta^3/(2a_x\zeta^3 + \sigma\Delta x) \leq 1$ it follows that $f_1^* \leq f_0^{R*} = f_0^{L*}$. Then the parameter θ is computed as the largest possible such that

$$\begin{cases} f_0^{R*} - |f_1^*| \geq 0, \\ f_0^{L*} - |f_1^*| \geq 0, \\ f_0^{R*} \geq 0 \text{ and } f_0^{L*} \geq 0. \end{cases} \tag{33}$$

Equations (31), (32) and (33) lead to the following condition

$$\tilde{\theta} = \frac{\tilde{f}_0 - |f_1^*|}{|\Gamma|} \geq 0. \tag{34}$$

Finally, θ is chosen as $\theta = \min(\tilde{\theta}, 1)$.

Therefore the unknowns f_{0i}^{n+1} and f_{1i}^{n+1} are computed using (22)

$$\begin{cases} f_{0i}^{n+1} = \dfrac{a_x \Delta t}{\Delta x} f_{0i-1/2}^{R*} + (1 - \dfrac{2a_x \Delta t}{\Delta x}) f_{0i}^n + \dfrac{a_x \Delta t}{\Delta x} f_{0i+1/2}^{L*}, \\[3mm] f_{1i}^{n+1} = \dfrac{a_x \Delta t}{\Delta x} f_{1i-1/2}^{*} + (1 - \dfrac{2a_x \Delta t}{\Delta x}) f_{1i}^n + \dfrac{a_x \Delta t}{\Delta x} f_{1i+1/2}^{*}. \end{cases} \tag{35}$$

The wavespeed a_x is fixed using the ideas introduced in [6]. It is known the electronic M_1 model without electric field is hyperbolic symmetrizable [57] and the eigenvalues of the Jacobian matrix always belong in the interval $[-\zeta, \zeta]$. Therefore, we set $a_x = \zeta$.

1.3. Asymptotic-preserving properties

In this part we prove the asymptotic-preserving property of the scheme (27)-(32)-(35). It is shown that when ε tends to zero, the scheme (27)-(32)-(35) is consistent with the limit diffusion equation (11).

Theorem 1.2. *When ε tends to zero, the unknown $f_{0i}^{n+1,0}$ given by the numerical scheme (35)-(32)-(27) satisfies the following discrete equation*

$$\frac{f_{0i}^{n+1,0} - f_{0i}^{n,0}}{\Delta t} - \frac{\zeta}{\Delta x} \left[\frac{\zeta^3}{6\bar{\sigma}_{i+1/2} \Delta x} \left[(\zeta f_{0i+1}^{n,0} - \zeta f_{0i}^{n,0}) \right] - \frac{\zeta^3}{6\bar{\sigma}_{i-1/2} \Delta x} \left[(\zeta f_{0i}^{n,0} - \zeta f_{0i-1}^{n,0}) \right] \right] = 0. \tag{36}$$

Proof. Following the same approach as in [7,8], using the diffusive scaling and equation (35) leads to

$$\begin{cases} \varepsilon \dfrac{f_{0i}^{n+1} - f_{0i}^n}{\Delta t} = \dfrac{a_x}{\Delta x} f_{0i+1/2}^{L*} - \dfrac{2a_x}{\Delta x} f_{0i}^n + \dfrac{a_x}{\Delta x} f_{0i-1/2}^{R*}, \\[3mm] \varepsilon \dfrac{f_{1i}^{n+1} - f_{1i}^n}{\Delta t} = \dfrac{a_x}{\Delta x} f_{1i+1/2}^{*} - \dfrac{2a_x}{\Delta x} f_{1i}^n + \dfrac{a_x}{\Delta x} f_{1i-1/2}^{*}, \end{cases} \tag{37}$$

where the intermediate states f_0^{L*} and f_0^{R*} are given by (32) and (27) rewrites

$$f_1^* = \frac{2a_x \zeta^3}{2a_x \zeta^3 + 2\bar{\sigma} \Delta x / \varepsilon} \left[\frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x} (\zeta f_2^R - \zeta f_2^L) \right]. \tag{38}$$

As soon as ε tends to zero, we obtain $f_1^* = 0$. We now suppose that $f_{1i}^n = 0$ in the limit ε tends to zero. In this case, the definition (34) leads to

$$\tilde{\theta} = \frac{f_0^L + f_0^R}{|f_0^L - f_0^R|} \geq 1. \tag{39}$$

Then the parameter θ is equal to 1.

Remark 1.3. In the diffusive regime when ε tends to zero, no limitation on the intermediates states (32) is required.

Using the definition (32), it follows that the intermediate states f_0^{L*} and f_0^{R*} are given by

$$\begin{cases} f_0^{L*} = f_0^L - \dfrac{\zeta}{a_x} (f_1^* - f_1^L), \\[3mm] f_0^{R*} = f_0^R - \dfrac{\zeta}{a_x} (f_1^R - f_1^*). \end{cases} \tag{40}$$

The discrete Hilbert expansions (14) are now used. Inserting the previous expressions in the first equation of (37), considered at the order ε^0, gives no information since the terms cancel each other out. However, at the order ε^1, the expressions (40), (38) and the first equation of (37) lead to

$$\begin{cases} f_0^{L*,1} = f_0^{L,1} - \dfrac{\zeta}{a_x}(f_1^{*,1} - f_1^{L,1}), \\[2mm] f_0^{R*,1} = f_0^{R,1} - \dfrac{\zeta}{a_x}(f_1^{R,1} - f_1^{*,1}), \end{cases} \tag{41}$$

with

$$f_1^{*,1} = -\frac{\zeta^3}{6\bar{\sigma}\Delta x}\left(\zeta f_0^{R,n,0} - \zeta f_0^{L,n,0}\right) \tag{42}$$

and

$$\frac{f_{0i}^{n+1,0} - f_{0i}^{n,0}}{\Delta t} = \frac{a_x}{\Delta x}f_{0i+1/2}^{*,1} - \frac{2a_x}{\Delta x}f_{0i}^{n,1} + \frac{a_x}{\Delta x}f_{0i-1/2}^{*,1}. \tag{43}$$

Inserting expressions (41) into (43) leads to equation (36) which is consistent with the limit diffusion equation (11).

To complete the proof, it is necessary to show that f_1^n tends to zero, when ε tends to zero. Equation (22) gives

$$\Delta x U_i^{n+1} = \int_{x_{i-1/2}}^{x_i} U_{\mathcal{R}}dx + \int_{x_i}^{x_{i+1/2}} U_{\mathcal{R}}dx, \tag{44}$$

where $U_{\mathcal{R}}$ is computed with the approximate Riemann problem (19). Then

$$\int_{x_{i-1/2}}^{x_i} f_1(x, \Delta t)dx = a_x\Delta t f_{1i-1/2}^* + (\frac{\Delta x}{2} - a_x\Delta t)f_{1i}^n \tag{45}$$

and

$$\int_{x_i}^{x_{i+1/2}} f_1(x, \Delta t)dx = (\frac{\Delta x}{2} - a_x\Delta t)f_{1i}^n + a_x\Delta t f_{1i+1/2}^*. \tag{46}$$

A long but classical calculation [8] leads to

$$\frac{f_{1i}^{n+1} - f_{1i}^n}{\Delta t} + \frac{1}{\Delta x}\left[\frac{2a_x}{2a_x + \bar{\sigma}_{i+1/2}\Delta x}\mathcal{F}_{i+1/2} - \frac{2a_x}{2a_x + \bar{\sigma}_{i-1/2}\Delta x}\mathcal{F}_{i-1/2}\right] \tag{47}$$
$$+ \frac{1}{\Delta x}\left[\frac{\Delta x\bar{\sigma}_{i+1/2}}{2a_x + \bar{\sigma}_{i+1/2}\Delta x}(-a_x f_{1i}^n - \zeta f_{2i}^n) + \frac{\Delta x\bar{\sigma}_{i-1/2}}{2a_x + \bar{\sigma}_{i-1/2}\Delta x}(-a_x f_{1i}^n + \zeta f_{2i}^n)\right] = 0,$$

with

$$\mathcal{F}_{i+1/2} = \frac{1}{2}\left[\zeta f_{2i+1}^n + \zeta f_{2i}^n - a_x(f_{1i+1}^n + f_{1i}^n)\right]. \tag{48}$$

Using the diffusive scaling we obtain that f_{1i}^n tends to zero as soon as ε tends to zero. $\qquad\square$

2. Homogeneous case with electric field

The second simplified model studied, is given by (1) without space dependency but considering an electric field. In this section, the difficulties encountered when using a relaxation-type method to include the source term $-\frac{E}{\zeta}(f_0-f_2)$ are highlighted. Following the same procedure as in the case without electric field, a numerical scheme is proposed and the source term $-\frac{E}{\zeta}(f_0-f_2)$ is taken into account. The scheme presented, satifies a well-balanced property and is asymptotic-preserving. The coefficient σ is a function of x and is then constant in the present case. However, the method proposed here, is able to handle the case where σ depends on ζ. Without spatial dependency, the model (1) simplifies into

$$\begin{cases} \partial_t f_0 + E\partial_\zeta f_1 = 0, \\ \partial_t f_1 + E\partial_\zeta f_2 - \frac{E}{\zeta}(f_0 - f_2) = -\frac{2\sigma f_1}{\zeta^3}. \end{cases} \tag{49}$$

Using the Hilbert expansions (7) as in the previous case, the following diffusion equation is obtained

$$\partial_t f_0^0(t,\zeta) - E\partial_\zeta\left(\frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0(t,\zeta) - \frac{E\zeta^2}{3\sigma}f_0^0(t,\zeta)\right) = 0. \tag{50}$$

2.1. Limit of the relaxation approach

Using the ideas introduced in [6], one could think in deriving a relaxation scheme for system (49). Even if the approach is similar, the relaxation scheme involved would be significantly different with the one proposed in [6] since the source term $-\frac{E}{\zeta}(f_0 - f_2)$ should be added. To assess such an issue, we first consider the collisionless case

$$\begin{cases} \partial_t f_0 + E\partial_\zeta f_1 = 0, \\ \partial_t f_1 + E\partial_\zeta f_2 - \frac{E}{\zeta}(f_0 - f_2) = 0. \end{cases} \tag{51}$$

Setting $\partial_\zeta z(\zeta) = 1/\zeta$, we propose the following relaxation model

$$\begin{cases} \partial_t f_0 + E\partial_\zeta \phi - E(f_1 - \phi)z'(\zeta) = 0, \\ \partial_t \phi + E\partial_\zeta f_0 - 2Ef_0 z'(\zeta) = \mu(f_1 - \phi), \\ \partial_t f_1 + E\partial_\zeta \pi - E(f_0 - \pi)z'(\zeta) = 0, \\ \partial_t \pi + E\partial_\zeta f_1 - 2Ef_1 z'(\zeta) = \mu(f_2 - \pi), \\ \partial_t z = 0, \end{cases} \tag{52}$$

where ϕ and π are relaxation variables. In the case $\mu = 0$, the previous system is hyperbolic, the eigenvalues are $-E$, 0, E and are associated with linearly degenerate fields. Hence, the Riemann problem can be solved.

Eigenvalue	Multiplicity	Riemann Invariants	Eigenvectors
E	2	$f_0 + \phi$, $f_1 + \pi$, z	$^t(0,0,1,1,0)$, $^t(1,1,0,0,0)$
$-E$	2	$-f_0 + \phi$, $-f_1 + \pi$, z	$^t(0,0,-1,1,0)$, $^t(-1,1,0,0,0)$
0	1	$\frac{f_1}{\zeta^2}$, $\frac{f_0}{\zeta^2}$, $\zeta(\pi - f_0/3)$, $\zeta(\phi - f_1/3)$	$^t(2f_0, f_1 - \phi, 2f_1, f_0 - \pi, 1)$

TABLE 1. Features of the Riemann problem

In order to be consistent with the notations [6], we introduce

$$w = {}^t(f_0, \phi, f_1, \pi, z), \quad \mathcal{U} = {}^t(f_0, f_1), \quad \mathcal{F}(\mathcal{U}) = {}^t\big(Ef_1, Ef_2(f_0, f_1)\big) \tag{53}$$

Lemma 2.1. *Let $w_{L,R}$ be equilibrium constant states with $\phi^{L,R} = f_1^{L,R}$ and $\pi^{L,R} = f_2^{L,R}$. Defining the initial condition of (52) by $w_0(x) = w_L$ if $x < 0$ and $w_0(x) = w_R$ if $x > 0$ for $\mu = 0$, the solution of (52) writes*

$$w(x,t) = \begin{cases} w^L & if \ x/t < -E, \\ w^{L*} & if \ -E < x/t < 0, \\ w^{R*} & if \ 0 < x/t < E, \\ w^R & if \ E < x/t, \end{cases} \tag{54}$$

with

$$f_0^{L*,R*} = \frac{3(\zeta^{L,R})^2}{4(2(\zeta^R)^6 + 2(\zeta^L)^6 + 5(\zeta^R)^3(\zeta^L)^3)}\Big((-f_2^R - 2f_1^R + 3f_0^R)(\zeta^R)^4 + (-f_2^L + 2f_1^L + 3f_0^L)(\zeta^L)^4$$
$$+(f_2^L + 4f_1^L + 3f_0^L)(\zeta^R)^3(\zeta^L) + (f_2^R - 4f_1^R + 3f_0^R)(\zeta^R)(\zeta^L)^3\Big),$$

$$f_1^{L*,R*} = \frac{3(\zeta^{L,R})^2}{4(2(\zeta^R)^6 + 2(\zeta^L)^6 + 5(\zeta^R)^3(\zeta^L)^3)}\Big((3f_2^R - 2f_1^R - f_0^R)(\zeta^R)^4 + (-3f_2^L - 2f_1^L + f_0^L)(\zeta^L)^4$$
$$+(-3f_2^L - 4f_1^L - f_0^L)(\zeta^R)^3(\zeta^L) + (3f_2^R - 4f_1^R + f_0^R)(\zeta^R)(\zeta^L)^3\Big),$$

$$z^{L*,R*} = z^{L,R},$$
$$\phi^{L*} = f_0^L + f_1^L - f_0^{L*}, \qquad \phi^{R*} = -f_0^R + f_1^R + f_0^{R*},$$
$$\pi^{L*} = f_1^L + f_2^L - f_1^{L*}, \qquad \pi^{R*} = -f_1^R + f_2^R + f_1^{R*},$$

and $\mathcal{U}^{L*,R*} = {}^t(f_0^{L*,R*}, f_1^{L*,R*})$ satisfy the admissibility conditions (4).

The computation of the intermediate states $\mathcal{U}^{L*,R*}$ is straightfoward using the Riemann invariants given in Table 1. A long but easy calculation, using the expressions gives the admissibility conditions (4).

The relaxation model (52) enables the computation of a numerical scheme [9,19,49] for the model (51). However, one notices the complexity of the intermediate states $\mathcal{U}^{L*,R*}$ and an extension including the collisional term $-2\alpha_{ei}f_1/\zeta^3$ is very challenging. Different relaxation models were tested in order to include the collisional source term, but, because of their complexity, they lead to configurations where a Riemann invariant is missing and the problem remains unclosed. In a recent work [30], the same issue is encountered and an additional relation is arbitrarily imposed. In the present situation, this strategy leads to particularly inconvenient solutions and the admissibility conditions are lost.

2.2. The numerical method

The numerical approach presented in the case without electric field is now considered. Contrarily to the relaxation-type procedure, this method enables to include the source term $-\frac{E}{\zeta}(f_0 - f_2)$ naturally.

Integrating the second equation of (49) by $\int_{-a_\zeta \Delta t}^{a_\zeta \Delta t} \int_0^{\Delta t}$ and multiplying by $\frac{1}{2a_\zeta \Delta t}$ gives the following expression

$$f_1^* = \frac{2a_\zeta \zeta^3}{2a_\zeta \zeta^3 + 2\sigma\Delta\zeta}\Big[\frac{f_1^L + f_1^R}{2} - \frac{1}{2a_\zeta}(Ef_2^R - Ef_2^L) + \frac{\Delta\zeta}{2a_\zeta}S_{L,R}\Big], \tag{55}$$

with

$$S_{L,R} = \frac{1}{2}\left[\frac{E}{\zeta_R}(f_0^R - f_2^R) + \frac{E}{\zeta_L}(f_0^L - f_2^L)\right]. \tag{56}$$

The unknown f_0^{L*}, f_0^{R*}, f_0^{n+1} and f_1^{n+1} are computed following the same approach as in the first part

$$\begin{cases} f_{0i}^{n+1} = \frac{a_\zeta \Delta t}{\Delta \zeta} f_{0i-1/2}^{R*} + (1 - \frac{2a_\zeta \Delta t}{\Delta \zeta})f_{0i}^n + \frac{a_\zeta \Delta t}{\Delta \zeta} f_{0i+1/2}^{L*}, \\ f_{1i}^{n+1} = \frac{a_\zeta \Delta t}{\Delta \zeta} f_{1i-1/2}^* + (1 - \frac{2a_\zeta \Delta t}{\Delta \zeta})f_{1i}^n + \frac{a_\zeta \Delta t}{\Delta \zeta} f_{1i+1/2}^*, \end{cases} \tag{57}$$

where the unknowns f_0^{R*} and f_0^{L*} are given by

$$\begin{cases} f_0^{L*} = \tilde{f}_0 - \Gamma\theta, \\ f_0^{R*} = \tilde{f}_0 + \Gamma\theta, \end{cases} \tag{58}$$

with

$$\Gamma = \frac{1}{2}[f_0^R - f_0^L - \frac{\zeta}{a_\zeta}(f_1^L - 2f_1^* + f_1^R)] \tag{59}$$

and

$$\tilde{f}_0 = \frac{f_0^L + f_0^R}{2} - \frac{1}{2a_\zeta}[\zeta f_1^R - \zeta f_1^L]. \tag{60}$$

Using, the same arguments as in the case without electric field, we set $a_\zeta = |E|$

2.3. Properties

In this part, we are interested in the equilibrium solution of system (49). It is shown that the scheme (55)-(57)-(58) preserves this solution. Then, the asymptotic-preserving feature of the scheme is exhibited.

A stationnary solution of system (49) satifies

$$\begin{cases} E\frac{\partial f_1}{\partial \zeta} = 0, \\ E\frac{\partial f_2}{\partial \zeta} - \frac{E}{\zeta}(f_0 - f_2) = -\frac{2\sigma f_1}{\zeta^3}. \end{cases} \tag{61}$$

The first equation of (61) implies that f_1 is independant of ζ. Using the definitions of the angular moments (2) and the definition (3), it follows that $f_1 = 0$ and $f_2 = f_0/3$. Indeed the definitions (2) implies $f_1 = 0$ in $\zeta = 0$. The second equation of the previous system is solved and gives the equibrium solution of the model (49)

$$\begin{cases} f_0 = K\zeta^2, \\ f_1 = 0, \end{cases} \tag{62}$$

where K is a scalar constant.

Theorem 2.2. *The numerical scheme given by (55)-(57)-(58) is well-balanced in the sense that the stationary states (62) are exactly preserved by the scheme.*

Proof. Using the stationary states (62) into the definition (55) leads to

$$f_1^* = \frac{2a_\zeta \zeta^3}{a_\zeta \zeta^3 + 2\sigma \Delta \zeta} \Big[-\frac{1}{3a_\zeta}(EK\zeta_R^2 - EK\zeta_L^2) + \frac{\Delta \zeta EK}{3a_\zeta}(\zeta_R + \zeta_L) \Big]. \tag{63}$$

Since $(\zeta_R^2 - \zeta_L^2) = (\zeta_R + \zeta_L)(\zeta_R - \zeta_L) = (\zeta_R + \zeta_L)\Delta \zeta$, the calculation of the previous equation gives

$$f_1^* = 0. \tag{64}$$

Using the second equation of (57) leads to

$$f_1^{n+1} = 0. \tag{65}$$

With the definition (32) it follows that

$$\begin{cases} f_0^{R*} = \frac{1}{2}[f_{0L} - \theta f_{0L} + f_{0R} + \theta f_{0R}], \\ f_0^{L*} = \frac{1}{2}[f_{0R} - \theta f_{0R} + f_{0L} + \theta f_{0L}]. \end{cases} \tag{66}$$

The initial conditions (62) implies $\theta = 1$ and inserting (66) into the first equation of (57) gives

$$f_{0i}^{n+1} = \frac{a\Delta t}{\Delta \zeta}K\zeta_i^2 + (1 - \frac{2a\Delta t}{\Delta \zeta})K\zeta_i^2 + \frac{a\Delta t}{\Delta \zeta}K\zeta_i^2. \tag{67}$$

Finally, the previous equation simplifies to give

$$f_{0i}^{n+1} = K\zeta_i^2. \tag{68}$$

The stationary solution (62) is then preserved by the scheme.

$\square$

Using the ideas introduced in the first section, we obtain that the scheme (55)-(57)-(58) is consistent with the limit diffusion equation (50) in the diffusive limit.

Theorem 2.3. *When ε tends to zero, the unknown f_0^{n+1} given by the numerical scheme (55)-(57)-(58) satifies the following discrete equation*

$$\frac{f_{0i}^{n+1,0} - f_{0i}^{n,0}}{\Delta t} - \frac{E}{\Delta \zeta}\Big[\frac{\zeta_{i+1/2}^3}{6\sigma \Delta \zeta}\Big[(Ef_{0i+1}^{n,0} - Ef_{0i}^{n,0})\Big] - \frac{\zeta_{i-1/2}^3}{6\sigma \Delta \zeta}\Big[(Ef_{0i}^{n,0} - Ef_{0i-1}^{n,0})\Big] + \frac{\zeta_{i+1/2}^3 S_{i+1/2}^{n,0}}{2\sigma} - \frac{\zeta_{i-1/2}^3 S_{i-1/2}^{n,0}}{2\sigma}\Big] = 0,$$

with

$$S_{i+1/2}^{n,0} = \frac{E}{3}\Big[\frac{f_{0i+1}^{n,0}}{\zeta_{i+1}} + \frac{f_{0i}^{n,0}}{\zeta_i}\Big].$$

Proof. The proof is the same as in the case without electric field. $\square$

3. Numerical examples

Depending on the regime, we compare the asymptotic-preserving scheme to the standard HLL scheme [43] or to the explicit discretisation of the diffusion equation.

3.1. **Free transport without electric field**

We first consider the system equation (5), without collisions, to validate the numerical scheme proposed in (27)-(32)-(35) on a simple advection of an initial profile. The solution is compared with the exact solution. Consider the initial conditions

$$
\begin{cases}
f_0(x,0) = \sqrt{\dfrac{2}{\pi}} \exp(-\dfrac{x^2}{2}), \\[3mm]
f_1(x,0) = \sqrt{\dfrac{2}{\pi}} \exp(-\dfrac{x^2}{2}),
\end{cases}
\tag{69}
$$

with periodical boundary conditions. In this case we have fixed $\zeta = 5$ and the space step Δx to 5.10^{-3}. In Figure 2, we compare the numerical solution obtained with the scheme (27)-(32)-(35) displayed in red with the exact solution in dashed blue at time t=6. The numerical and exact solutions match perfectly.
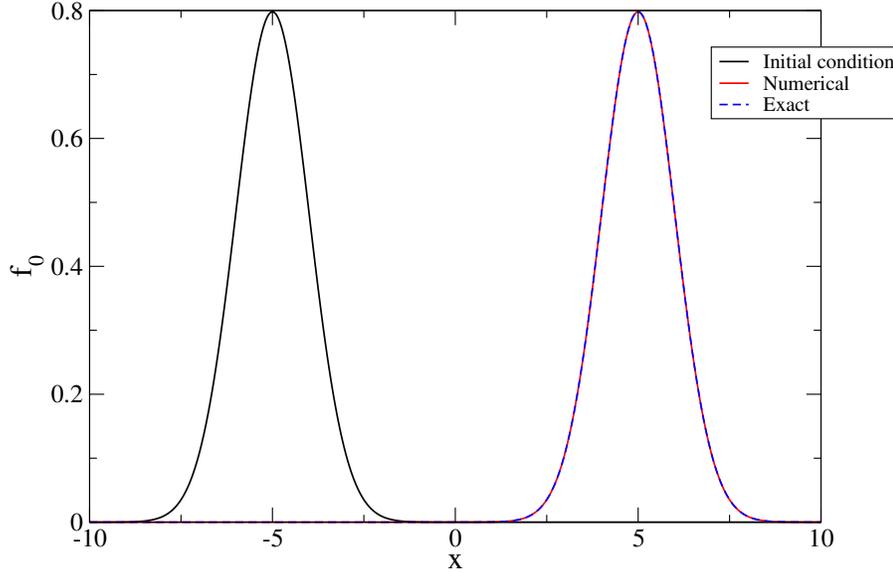
FIGURE 2. Free transport: comparison of the numerical solution (red) and the exact solution (dashed blue) at time t=6.

3.2. **Hot wall with collisions without electric field**

We now consider the system equation (5) with collisions to validate the numerical scheme (27)-(32)-(35) taking into account the collisional part. The solution obtained with the scheme presented in this paper is compared with the classical HLL scheme.

Consider the initial conditions

$$
\begin{cases}
f_0(x,\zeta,0) = \sqrt{\dfrac{2}{\pi}} \dfrac{\zeta^2}{T_{ini}(x)^{3/2}} \exp\left(-\dfrac{\zeta^2}{2T_{ini}(x)}\right), \\[3mm]
f_1(x,\zeta,0) = 0,
\end{cases}
$$

with $T_{ini}(x) = 0.1$ and the left boundary condition

$$\begin{cases} f_0(0, \zeta, t) = \sqrt{\dfrac{2}{\pi}} \dfrac{\zeta^2}{T_{ext}(x)^{3/2}} \exp\left(-\dfrac{\zeta^2}{2T_{ext}(x)}\right), \\ f_1(0, \zeta, t) = 0, \end{cases}$$

with $T_{ext}(x) = 1$ and $\sigma = 1$. The energy range chosen is $[0, 6]$ with an energy step $\Delta\zeta = 0.1$ and the space range is $[0, 2]$ with a space step $\Delta x = 10^{-3}$. In Figure 3, we compare the numerical solution obtained with the AP scheme (27)-(32)-(35) (AP) displayed in continuous lines with the solution given by HLL scheme in dashed lines at time 0.1, 0.25 and 0.5. The (AP) numerical scheme and the HLL scheme gives comparable results. However, because of the large discontinuity between the initial temperature profile and the temperature at the left boundary in addition to the important numerical viscosity of the HLL scheme, one can notice the slight differences of behavior between the two numerical schemes.
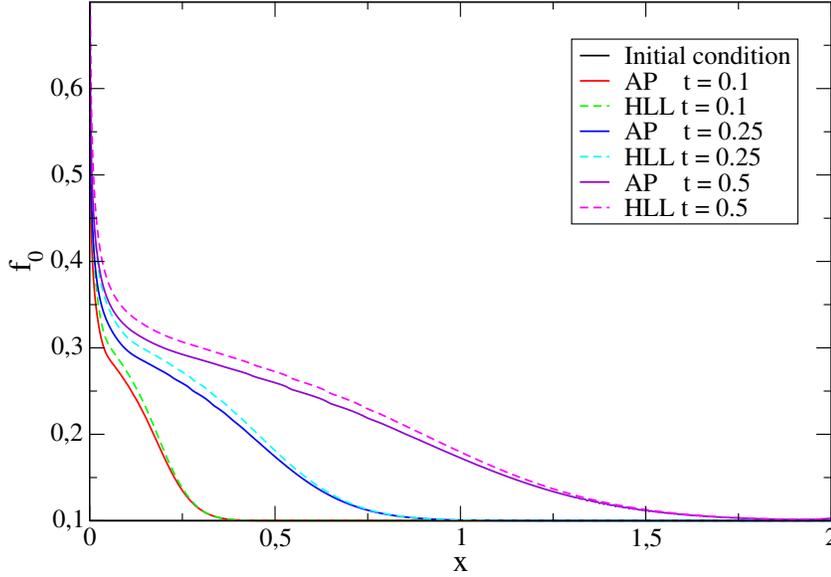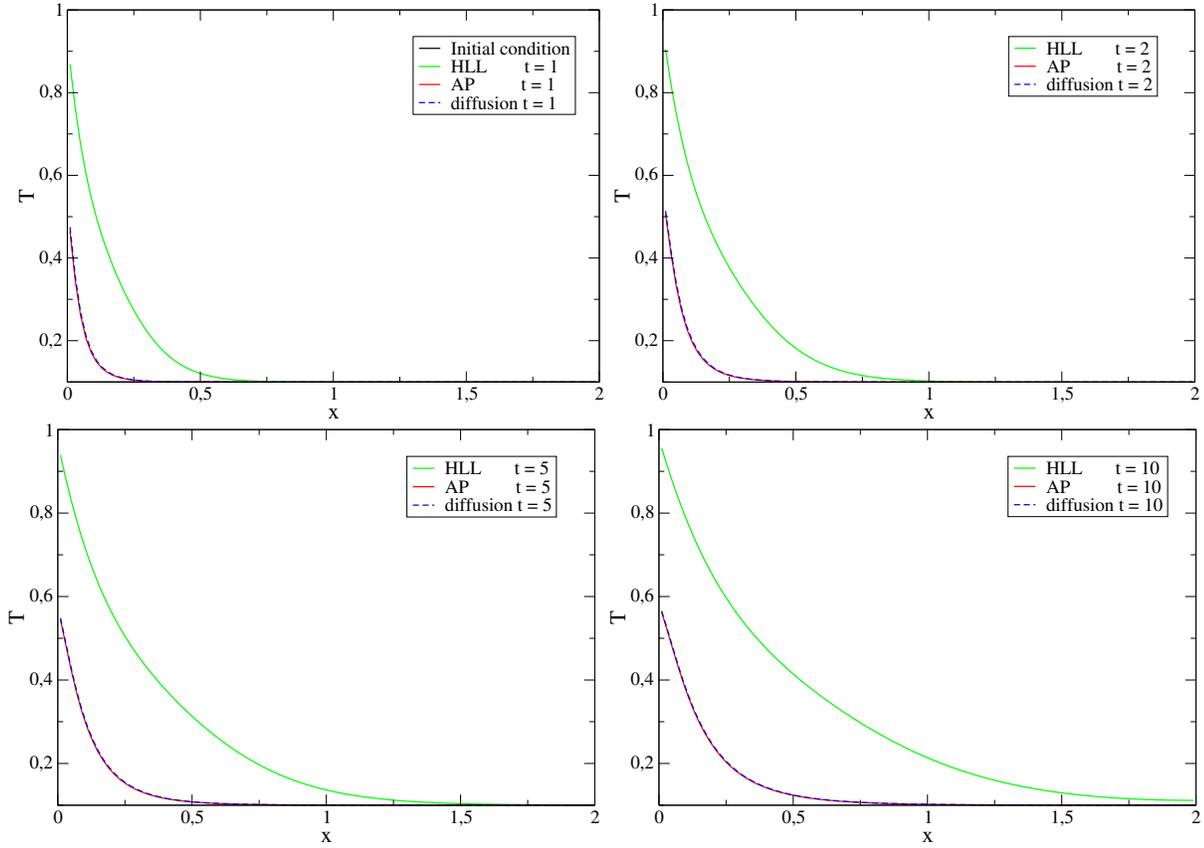

FIGURE 3. Hot wall: comparison of the temperature profile for the numerical solution (AP) and for the HLL scheme (HLL) at time 0.1, 0.25 and 0.5.

3.3. Hot wall in the diffusive regime without electric field

In this example, the numerical scheme (27)-(32)-(35) is validated in the diffusive regime. The results are compared with the diffusion solution and with the one obtained with the HLL scheme.

Consider the initial conditions

$$\begin{cases} f_0(x, \zeta, 0) = \sqrt{\dfrac{2}{\pi}} \dfrac{\zeta^2}{T_{ini}(x)^{3/2}} \exp\left(-\dfrac{\zeta^2}{2T_{ini}(x)}\right), \\ f_1(x, \zeta, 0) = 0, \end{cases} \tag{70}$$

with $T_{ini}(x) = 0.1$ and the left boundary condition

$$\begin{cases} f_0(0, \zeta, t) = \sqrt{\dfrac{2}{\pi}} \dfrac{\zeta^2}{T_{ext}(x)^{3/2}} \exp\Big(-\dfrac{\zeta^2}{2T_{ext}(x)}\Big), \\ f_1(0, \zeta, t) = 0, \end{cases} \tag{71}$$

with $T_{ext}(x) = 1$ and $\sigma = 10^4$. The energy range chosen is $[0, 6]$ with an energy step $\Delta\zeta = 0.1$ and the space range is $[0, 2]$ with a space step $\Delta x = 10^{-2}$. In Figure 4, we compare the numerical solution obtained with the scheme (27)-(32)-(35) (AP) displayed in continuous red lines with the solution given by HLL scheme in continuous green lines and the diffusion solution in dashed blue lines at time t=1, t=2, 5 and 10. The (AP) numerical scheme and the diffusion solution match perfectly while the HLL scheme gives very inaccurate results.

FIGURE 4. Hot wall in the diffusive limit: comparison of the temperature profile for the numerical solution (AP), for the HLL scheme (HLL) and the diffusion solution at time t=1, 2, 5 and 10.

3.4. **Discontinous initial condition in the diffusive regime without electric field**

In this case, a discontinous initial condition in the diffusive regime without electric field is considered. The results are compared with the diffusion equation solution and the HLL scheme. The energy range chosen is $[0, 6]$ with an energy step $\Delta\zeta = 0.1$ and the space range $L=[-10, 10]$ with a space step $\Delta x = 5.10^{-2}$. Consider the initial conditions

$$\begin{cases} f_0(x, \zeta, 0) = \begin{cases} 1 & \text{if } x \leq L/3, \\ 0 & \text{if } L/3 \leq x \leq 2L/3, \\ 1 & \text{if } L/3 \leq x, \end{cases} \\ f_1(x, \zeta, 0) = 0, \end{cases}$$

with periodical boundary conditions and $\sigma = 10^4$. In Figure 5, we compare the numerical solution obtained with the asymptotic-preserving scheme displayed in red with the diffusion solution in dashed blue and the HLL scheme in green at time t=200. The AP and diffusion solutions match prefectly while the HLL scheme is very inaccurate.

FIGURE 5. discontinuous initial condition: comparison of the f_0 profile for the asymptotic-preserving scheme (AP), for the HLL scheme (HLL) and the diffusion solution at time t=200.

3.5. **Relaxation of a Gaussian profile, in the homogeneous case in the diffusive regime with electric field**

We consider the system equation (49) with collisions and the source term $\frac{E}{\zeta}(f_0 - f_2)$ to validate the numerical scheme (55)-(57)-(58) in the diffusive limit. The energy range chosen is $[0, 20]$ with an energy step $\Delta\zeta = 10^{-2}$. We consider the initial conditions

$$\begin{cases} f_0(\zeta, 0) = \sqrt{\frac{2}{\pi}} \exp(-\frac{\zeta^2}{2}), \\ f_1(\zeta, 0) = 0. \end{cases} \tag{72}$$

In Figure 6, we compare the numerical solution obtained with the scheme (55)-(57)-(58) displayed in red with the diffusion solution in dashed blue and the HLL scheme at time t=20. The asymptotic-preserving and diffusion solutions match prefectly while the HLL scheme is very diffusive.
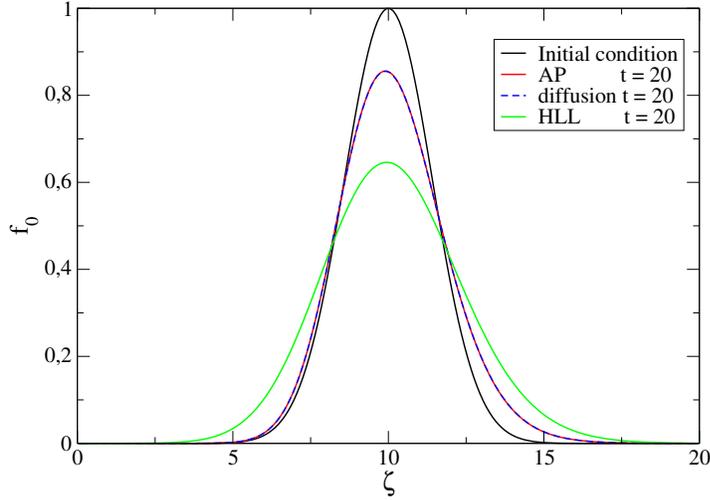

FIGURE 6. Relaxation of a Gaussian profile: comparison of the f_0 profile for the asymptotic-preserving scheme (AP), for the HLL scheme (HLL) and the diffusion solution at time t=20.

3.6. Relaxation of a Gaussian profile in the diffusive regime without electric field in the case of a non-constant collisional parameter

In this example, the numerical scheme (27)-(32)-(35) is validated in the diffusive regime without electric field in a inhomogeneous collisional plasma. In this case the coefficient σ is not constant and follows the linear profile

$$\sigma(x) = (5x/8 + 15/2).10^{-3} \tag{73}$$

Then $\sigma(-4) = 5.10^{-3}$ and $\sigma(4) = 10^{-4}$. The energy range chosen is $[0, 8]$ with an energy step $\Delta\zeta = 0.1$ and the space range $[-4, 4]$ with a space step $\Delta x = 5.10^{-2}$. The initial conditions are the following

$$\begin{cases} f_0(x, \zeta, 0) = \zeta^2 \exp(-\dfrac{x^2}{2}), \\ f_1(x, \zeta, 0) = 0. \end{cases} \tag{74}$$

In Figure 7, we compare the numerical solution obtained with the asymptotic-preserving scheme displayed in red with the diffusion solution in dashed blue at time t=150. In this case, the asymptotic-preserving and diffusion solutions also match prefectly. The HLL scheme results are not given in Figure 7, since the final time t=150 is important the HLL results are completely wrong.

4. CONCLUSION

In this work, we have proposed a numerical scheme for the electronic M_1 model in the case without electric field and in the homogeneous case. We have exhibited an approximate Riemann solver that satisfies the ad-missibility conditions. Contrarily to the HLL scheme, the proposed numerical scheme is asymptotic-preserving
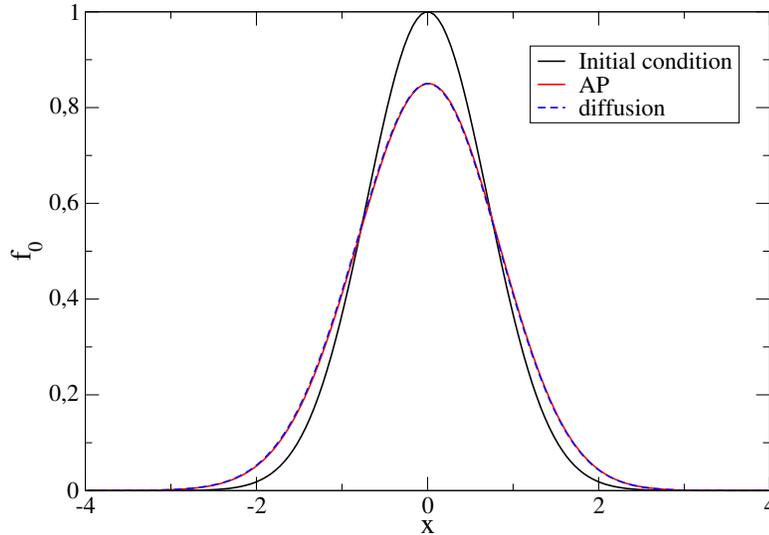
FIGURE 7. Relaxation of a Gaussian profile in the case of a linear collisionnal parameter: comparison of the f_0 profile for the asymptotic-preserving scheme (AP) and the diffusion solution at time t=150.

and recovers the correct diffusion equation in the diffusive limit. It has been shown, in the homogeneous case, that the method presented, enables to include the source term $-E(f_0 - f_2)/\zeta$, while a relaxation type method seems inconvenient. In addition, the scheme is well-balanced, capturing the steady state considered. Several numerical tests have been performed, it has been shown that the presented scheme behaves correctly in the classical regime and in the diffusive limit. Indeed, while, the HLL scheme is very inaccurate in the diffusive regime, the asymptotic-preserving scheme matches perfectly with the expected diffusion solution. Also, the method correctely handles the case where the collisionnal parameter is not constant. The present study can be extended to the general electronic M_1 model (1). However, the correct treatment of the mixed-derivatives, arising in the diffusive limit when considering the entire model is a challenging issue. This problem will be investigated in a forthcoming paper.

REFERENCES

[1] A.Bermudez and M. E. Vazquez. Upwind methods for hyperbolic conservation laws with source terms. Computers and Fluids 23 (1994), no. 8, 10491071.

[2] G.W. Alldredge, C.D. Hauck, and A.L. Tits. High-order entropy-based closures for linear transport in slab geometry II: A computational study of the optimization problem. SIAM Journal on Scientific Computing Vol. 34-4 (2012), pp. B361-B391.

[3] E. Audit, P. Charrier, J.-P. Chièze, and B. Dubroca. A radiation hydrodynamics scheme valid from the transport to the diffusion limit. preprint astro-ph 0206281. (2002).

[4] R. Balescu. Transport Processes in Plasma, Elsevier, (Amsterdam, 1988), Vol. 1.

[5] M. Bennoune, M. Lemou, and L. Mieussens. Uniformly stable numerical schemes for the Boltzmann equation preserving the compressible NavierStokes asymptotics. J. Comput. Phys. 227 (8) (2008) 37813803.

[6] C. Berthon, P. Charrier, and B. Dubroca. An asymptotic preserving relaxation scheme for a moment model of radiative transfer. C. R. Acad. Sci. Paris, Ser. I 344 (2007).

[7] C. Berthon, P. Charrier, and B. Dubroca. An HLLC Scheme to Solve The M1 Model of Radiative Transfer in Two Space Dimensions. Journal of Scientific Computing, Vol. 31, No. 3, 2007.

[8] C. Berthon and R.Turpault. Asymptotic preserving HLL schemes. Numerical Methods for Partial Differential Equations, 27 (6) (2011) 1396-1422.

[9] F. Bouchut. Nonlinear Stability of Finite Volume Methods for Hyperbolic Conservation Laws, and Well-Balanced Schemes for sources. Frontiers in Mathematics series, Birkhauser. (2004).

[10] F. Bouchut and T. Morales. A subsonic-well-balanced reconstruction scheme for shallow water flows. SIAM Journal on Numerical Analysis 48 (2010), no. 5, 17331758.

[11] S.I. Braginskii. Reviews of Plasma Physics. M.A Leontovich, Ed., Consultants Bureau (New York, 1965), Vol. 1, p.205.

[12] A.V. Brantov, V.Yu. Bychenkov, O.V. Batishchev, and W.Rozmus. Nonlocal heat wave propagation due to skin layer plasma heating by short laser pulses. Computer Physics communications 164 67, 2004.

[13] C. Buet and S. Cordier. Asymptotic Preserving Scheme and Numerical Methods for Radiative Hydrodynamic Models. 951956 C. R. Acad. Sci. Paris, Tome 338, Srie I (2004).

[14] C. Buet, S. Cordier, B. Lucquin-Desreux, and S. Mancini. Diffusion limit of the Lorentz model: asymptotic preserving schemes. M2AN Math. Model. Numer. Anal. 36 (4) (2002) 631655.

[15] C. Buet and B. Després. Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics. J. Quant. Spectrosc. Radiat. Transfer, 85, 3-4, 385418 (2004).

[16] C. Buet and B. Després. Asymptotic preserving and positive schemes for radiation hydrodynamics. J. Compt. Phy., 215, 717740 (2006).

[17] P. Cargo and A.-Y. Le Roux. Un schéma équilibre adapté au modèle d'atmosphère avec termes de gravité. Comptes rendus de l'Académie des sciences. Série 1, Mathématique 318 (1994), no. 1, 7376.

[18] J.A. Carrillo, T. Goudon, P. Lafitte, and F. Vecil. Numerical schemes of diffusion asymptotics and moment closures for kinetic equations. J. Sci. Comput. 36 (1) (2008) 113149.

[19] C.Berthon. Numerical approximations of the 10-moment Gaussian closure. Math. Comp. 75 (2006) 1809-1831.

[20] C. Chalons, F. Coquel, E. Godlewski, P.-A. Raviart, and N. Seguin. Godunov-type schemes for hyperbolic systems with parameter-dependent source. The case of Euler system with friction. Math. Models Methods Appl. Sci. 20 (2010), no. 11, 21092166. MR 2740716 (2011m:65179).

[21] C. Chalons, F. Coquel, and C. Marmignon. Well-balanced time implicit formulation of relaxation schemes for the euler equations. SIAM Journal on Scientific Computing 30 (2008), no. 1, 394415.

[22] S. Chapman and T. G. Cowling. The Mathematical Theory of Non-Uniform Gases. Cambridge University Press, Cambridge, England, 1995.

[23] P. Charrier, B. Dubroca, G. Duffa, and R. Turpault. Multigroup model for radiating flows during atmospheric hypersonic re-entry. Proceedings of International Workshop on Radiation of High Temperature Gases in Atmospheric Entry, pp. 103110. Lisbonne, Portugal. (2003).

[24] F. Coron and B. Perthame. Numerical passage from kinetic to fluid equations. SIAM J. Numer. Anal. 28 (1991) pp. 26-42.

[25] P. Crispel, P. Degond, and M.-H. Vignal. Quasi-neutral fluid models for current-carrying plasmas. Journal Comp. Phys. 205 408-438, 2005.

[26] P. Crispel, P. Degond, and M.-H. Vignal. An asymptotic preserving scheme for the two-fluid Euler-Poisson model in the quasi-neutral limit. J. Comput. Phys. 223 (2007) 208-234, 2007.

[27] P. Degond, F. Deluzet, L. Navoret, A. Sun, and M. Vignal. Asymptotic-Preserving Particle-In-Cell method for the Vlasov-Poisson system near quasineutrality. J. Comput. Phys, 229 no 16, 5630-5652, 2010.

[28] P. Degond, H. Liu, D. Savelief, and M-H. Vignal. Numerical approximation of the Euler-Poisson-Boltzmann model in the quasineutral limit. C. R. Acad. Sci. Paris, Ser. I 341 323328, 2005.

[29] P. Degond, D. Savelief, and F. Deluzet. Numerical approximation of the Euler-Maxwell model in the quasineutral limit. arXiv:1104.1870v1.

[30] V. Desveaux, M. Zenk, C. Berthon, and C. Klingenberg. Well-balanced schemes to capture non-explicit steady states. Part 1: Ripa model. Submitted.

[31] J.F. Drake, P.K. Kaw, Y.C. Lee, G. Schmidt, C.S. Liu, and M.N. Rosenbluth. Parametric instabilities of electromagnetic waves in plasmas. Phys. Fluids 17, 778, 1974.

[32] B. Dubroca, J.-L. Feugeas, and M. Frank. Angular moment model for the Fokker-Planck equation. European Phys. Journal D, 60, 301, 2010.

[33] B. Dubroca and J.L. Feugeas. étude théorique et numérique d'une hiérarchie de modèles aux moments pour le transfert radiatif. C. R. Acad. Sci. Paris, t. 329, SCrie I, p. 915-920, 1999.

[34] B. Dubroca and J.L. Feugeas. Entropic moment closure hierarchy for the radiative transfert equation. C. R. Acad. Sci. Paris Ser. I, 329 915, 1999.

[35] E. Epperlein and R. Short. Phys. Fluids B 4, 2211 1992.

[36] G. Gallice. Positive and entropy stable Godunov-type schemes for gas dynamics and MHD equations in Lagrangian or Eulerian coordinates. Numer. Math. 94 (2003), no. 4, 673713. MR 1990589 (2004e:65094).

[37] L. Gosse and G. Toscani. An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations. C. R. Math. Acad. Sci. Paris 334 (4) (2002) 337342.

[38] L. Gosse and G. Toscani. Space localization and well-balanced schemes for discrete kinetic models in diffusive regimes. SIAM J. Numer. Anal. 41 (2) (2003) 641658.

[39] H. Grad. On the kinetic theory of rarefied gases. Commun. Pure Appl. Math. 2, 331-407 (1949).

[40] J. M. Greenberg and A. Y. Leroux. A well-balanced scheme for the numerical processing of source terms in hyperbolic equations. SIAM J. Numer. Anal., 33, 116 (1996).

[41] C.P.T. Groth and J.G. McDonald. Towards physically-realizable and hyperbolic moment closures for kinetic theory. Continuum Mech. Thermodyn. 21, 467-493 (2009).

[42] S. Guisset, S. Brull, B. Dubroca, E. d'Humières, S. Karpov, and I. Potapenko. Asymptotic-preserving scheme for the Fokker-Planck-Landau-Maxwell system in the quasi-neutral regime. Submitted.

[43] A. Harten, P.D. Lax, and B. Van Leer. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. SIAM Review 25 (1983), 35-61.

[44] P. Lafitte J.A. Carrillo, T. Goudon. Simulation of fluid and particles flows: asymptotic preserving schemes for bubbling and flowing regimes. J. Comput. Phys. 227 (16) (2008) 79297951.

[45] S. Jin and C.D. Levermore. Fully discrete numerical transfer in diffusive regimes. Transport Theory Statist. Phys. 22 (6) 739791. (1993).

[46] S. Jin and C.D. Levermore. The discrete-ordinate method in diffusive regimes. Transport Theory Statist. Phys. 20 (56) 413439. (1991).

[47] S. Jin and L. Pareschi. Discretization of the multiscale semiconductor Boltzmann equation by diffusive relaxation scheme. J. Comput. Phys. 161 (2000) 312-330.

[48] S. Jin, L. Pareschi, and G. Toscani. Uniformly accurate diffusive relaxation schemes for multiscale transport equations. SIAM J. Numer. Anal. 38 (3) (2000) 913936.

[49] S. Jin and Z. Xin. The relaxation scheme for systems of conservation laws in arbitrary space dimension. Comm. Pure Appl. Math. 45, (1995) 235276.

[50] A. Klar. An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit. SIAM J. Numer. Anal. 35 (3) (1998) 10731094.

[51] A. Klar. An asymptotic preserving numerical scheme for kinetic equations in the low Mach number limit. SIAM J. Numer. Anal. 36 (1999) 15071527.

[52] A. Klar and C. Schmeiser. Numerical passage from radiative heat transfer to nonlinear diffusion models. Math. Models Methods Appl. Sci. 11 (5) (2001) 749767.

[53] L. Landau. On the vibration of the electronic plasma. J. Phys. USSR 10 (1946).

[54] A.W. Larsen and J.E. Morel. Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes II. J. Comput. Phys. 83 (1) (1989) 212236.

[55] A.W. Larsen, J.E. Morel, and W.F. Miller Jr. Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. J. Comput. Phys. 69 (2) (1987) 283324.

[56] M. Lemou and L. Mieussens. A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit. SIAM J. Sci. Comput. 31 (1) (2008) 334368.

[57] C.D. Levermore. Moment closure hierarchies for kinetic theories. J. Stat. Phys. 83, 1021-1065 (1996).

[58] J. Mallet, S. Brull, and B. Dubroca. An entropic scheme for an angular moment model for the classical Fokker-Planck-Landau equation of electrons. Comm. Comput. Phys., 422, 2013.

[59] J. Mallet, S. Brull, and B. Dubroca. General moment system for plasma physics based on minimum entropy principle. Submitted.

[60] A. Marocchino, M. Tzoufras, S. Atzeni, A. Schiavi, Ph. D. Nicolaï, J. Mallet, V. Tikhonchuk, and J.-L. Feugeas. Nonlocal heat wave propagation due to skin layer plasma heating by short laser pulses. Phys. Plasmas 20, 022702, 2013.

[61] J.G. McDonald and C.P.T. Groth. Towards realizable hyperbolic moment closures for viscous heat-conducting gas flows based on a maximum-entropy distribution. Continuum Mech. Thermodyn. 25, 573-603 (2012).

[62] N. Meezan, L. Divol, M. Marinak, G. Kerbel, L. Suter, R. Stevenson, G. Slark, and K. Oades. Phys. Plasmas 11, 5573 2004.

[63] G.N. Minerbo. Maximum entropy Eddigton Factors. J. Quant. Spectrosc. Radiat. Transfer, 20, 541, 1978.

[64] I. Muller and T. Ruggeri. Rational Extended Thermodynamics. Springer, New York (1998).

[65] Ph. Nicolaï, M. Vandenboomgaerde, B. Canaud, and F. Chaigneau. 7, 4250 2000.

[66] S. Jin R. Caflish and G. Russo. Uniformly accurate schemes for hyperbolic systems with relaxation. SIAM J. Numer. Ana. 34 (1997) 246281.

[67] J.-F. Ripoll. An averaged formulation of the M1 radiation model with presumed probability density function for turbulent flows. J. Quant. Spectrosc. Radiat. Trans. 83 (34), 493517. (2004).

[68] J.-F. Ripoll, B. Dubroca, and E. Audit. A factored operator method for solving coupled radiation-hydrodynamics models. Trans. Theory. Stat. Phys. 31, 531557. (2002).

[69] W. Rozmus, V. T. Tikhonchuk, and R. Cauble. A model of ultrashort laser pulse absorption in solid targets. Phys. Plasmas 3, 360 (1996).

[70] K. Shigemori, H. Azechi, M. Nakai, M. Honda, K. Meguro, N. Miyanaga, H. Takabe, and K. Mima. Phys. Rev. Lett. 78, 250 1997.

[71] I.P. Shkarofsky, T.W. Johnston, and The Particle Kinetics of Plasmas M.P. Bachynski. Addison-Wesley (Reading, Massachusetts, 1966).

[72] L. Spitzer and R. Härm. Phys. Rev. 89 (1953) 977.

[73] H. Struchtrup. Macroscopic Transport Equations for Rarefied Gas Flows. Springer, Berlin (2005).

[74] E.F. Toro. *Riemann Solvers and Numerical Methods for Fluids dynamics*. Springer, Berlin, 1999.

[75] R. Turpault. A consistent multigroup model for radiative transfer and its underlying mean opacity. J. Quant. Spectrosc. Radiat. Transfer 94, 357371 (2005).

[76] R. Turpault, M. Frank, B. Dubroca, and A. Klar. Multigroup half space moment appproximations to the radiative heat transfer equations. J. Comput. Phys. 198 363 (2004).

[77] A. Velikovich, J. Dahlburg, J. Gardner, and R. Taylor. Phys. Plasmas 5, 1491 1998.