Bi-species kinetic model for a cylindrical Langmuir probe.

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Joint work with **Mehdi Badsi** and **Anaïs Crestetto** (Nantes Université, LMJL).

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- It grabs the electrons of the plasma and registers electric current.
- This permits to determine the density, the temperature and the potential of the plasma.

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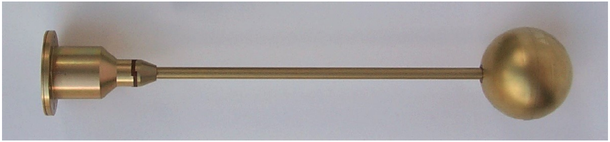
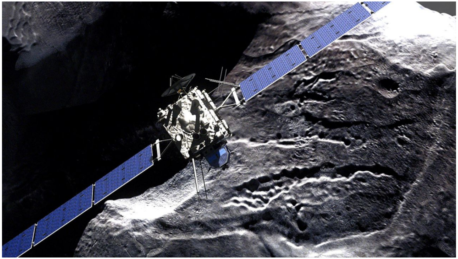


One of two Langmuir probes from the Swedish Institute of Space Physics in Uppsala on board ESA's space vehicle Rosetta (in Titanium).



Rosetta in orbit around the 67P/C–G comet (artist view)

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- Invariance by rotations (Radial Poisson equation).
- Invariance by axial symmetry (No orthoradial macroscopic current).

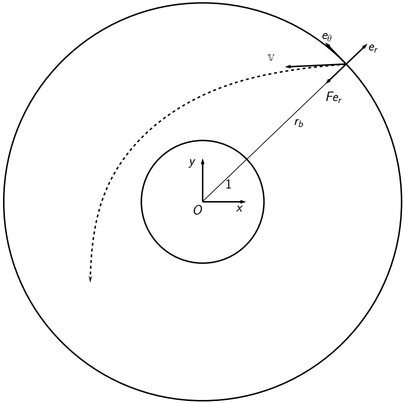

Figure: Sketch of a trajectory of a particle into a radial force field coming from the outer ionizing source (plasma core) into the neigborhood of the probe (at $r = r_b$) with a velocity $\mathbb{V}$.

- Vlasov equation for the ionic density $f_i(r, v_r, v_\theta)$:

$$v_r\,\partial_r f_i - \frac{v_r\,v_\theta}{r}\partial_{v_\theta} f_i + \left(\frac{v_\theta^2}{r} - \partial_r \phi\right)\,\partial_{v_r} f_i = 0.$$

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$$-\frac{1}{r} \frac{d}{dr} \left(r \, \frac{d\phi}{dr} \right)(r) = n_i(r) - n_e(r),$$

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where n_i and n_e are the ions and electrons macroscopic charge densities:

$$n_i(r) = q\int_{\mathbb{R}^2} f_i(r, v_r, v_\theta)\,dv_r\,dv_\theta, \quad n_e(r) = q\int_{\mathbb{R}^2} f_e(r, v_r, v_\theta)\,dv_r\,dv_\theta.$$

- Incoming particles from the plasma core:

$$\forall\ v_r \leq 0, \qquad f_i(r_b, v_r, v_\theta) = f_i^b(v_r, v_\theta), \quad f_e(r_b, v_r, v_\theta) = f_e^b(v_r, v_\theta),$$

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$$\forall\ v_r \geq 0, \qquad f_i(1, v_r, v_\theta) = 0, \quad f_e(1, v_r, v_\theta) = 0.$$

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- Boundary datum for the Poison equation:

$$\phi(1) = \phi_p \in \mathbb{R}, \qquad \phi(r_b) = 0.$$

Theorem (M. Badsi, L.G-C. 2022)

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Sufficient condition for the integrability conditions:

$$\forall\, (v_r, v_\theta) \in \mathbb{R}^2, \qquad |f(v_r, v_\theta)| \leq \frac{1}{|v_r| + |v_\theta|^2 + 1}.$$

Condition satisfied by the Maxwellian distributions.

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- The characteristics for *ionic* Vlasov equation are given by:

$$\begin{cases} \dfrac{d}{dt} r(t) = v_r(t), \\ \dfrac{d}{dt} v_r(t) = \dfrac{v_\theta(t)^2}{r(t)} - \dfrac{d\phi}{dr}(r(t)), \\ \dfrac{d}{dt} v_\theta(t) = -\dfrac{v_r(t)\, v_\theta(t)}{r(t)}. \end{cases}$$

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- Constants of motion: the total energy and the angular momentum.

$$\frac{d}{dt}\left(\frac{v_r^2(t) + v_\theta^2(t)}{2} + \phi(r(t)) \right) = 0,$$

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- The characteristics are contained in the level sets defined for $L \in \mathbb{R}$ and $e \in \mathbb{R}$ by

$$\mathcal{C}_{L,e} := \left\{ (r, v_r, v_\theta) \in (1, r_b) \times \mathbb{R}^2 \ : \ r v_\theta = L \quad \text{and} \quad \frac{v_r^2 + v_\theta^2}{2} + \phi(r) = e \right\}.$$

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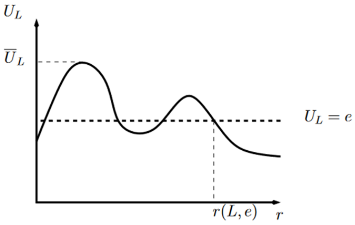
- Its maximum value is denoted :

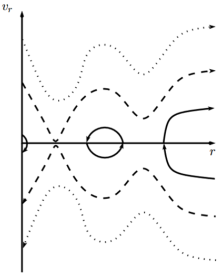
$$\overline{U_L} := \max_{r \in [1, r_b]} U_L(r).$$

The maximal value $\overline{U_L}$ defines a global potential barrier that separates trajectories that collapse with the Langmuir probe with others.

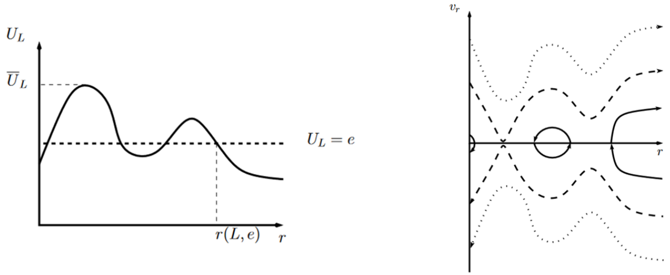
Explicit resolution of the Vlasov equation

ionic phase diagram and characteristics





Effective potential, potential barrier position and phase diagram for the particles dynamics.



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- Position of the potential barrier when $e \leq \overline{U_L}$:

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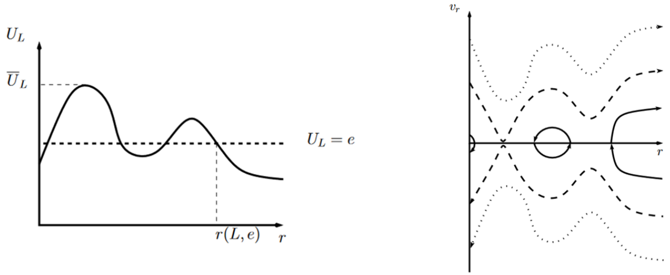

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The quantities $\overline{U_L}$ and $r_i(L,e)$ are non-local with respect to ϕ.

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For every point $(r, v_r) \in \mathcal{D}_i^b$ there exists a unique characteristics curves that passes through (r, v_r) and originates from $r = r_b$ with a negative velocity.

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$$f_i(r, v_r, v_\theta) := \begin{cases} f_i^b\left(-\sqrt{v_r^2 + 2(U_L(r) - U_L(r_b))}; \frac{rv_\theta}{r_b}\right) \text{ if } (r, v_r) \in \mathcal{D}_i^b(L) \text{ with } L = rv_\theta. \\ 0 \text{ otherwise.} \end{cases}$$

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Proposition

The function f_i is a weak solution of the ionic Vlasov equation with macroscopic electric potential ϕ and with incoming fluxes $f_i^b \in L^1_{loc}$ at radius $r = r_b$.

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Proposition

The function f_i is a weak solution of the ionic Vlasov equation with macroscopic electric potential ϕ and with incoming fluxes $f_i^b \in L^1_{loc}$ at radius $r = r_b$.

- For the electrons: replace ϕ by $-\phi$ (define $V_L := L^2/2r^2 - \phi$).

- Define:

$$\widetilde{\rho}[\psi](e) := \inf\left\{a \in [1, r_b] \ : \ \text{ for a.e } \ s \in [a, r_b], \ \psi(s) \leq e\right\}.$$

Explicit resolution of the Vlasov equation

computation of the macroscopic densities

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- Analogous formula for electrons with $-\phi$.

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- Recall that:

$$U_L(r) = \frac{L^2}{2r^2} + \phi(r), \qquad V_L(r) = \frac{L^2}{2r^2} - \phi(r), \qquad \overline{\psi} = \max_{r \in [1, r_b]} \psi(r),$$

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$$\mathfrak{R}_i^{n+1}(v,L) := \widetilde{\rho}\Big[\phi_n + \frac{L^2}{2\,\cdot^{\,2}}\Big]\left(\frac{v^2}{2} + \frac{L^2}{2\,r_b^2}\right),$$

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$$\mathfrak{U}_L^{n+1} := \max_{r\in[1,r_b]} \phi_n(r) + \frac{L^2}{2\,r^2}, \qquad \mathfrak{V}_L^{n+1} := \max_{r\in[1,r_b]} -\phi_n(r) + \frac{L^2}{2\,r^2}.$$

Iterated potentials:

Define $\phi_{n+1} : [1, r_b] \to \mathbb{R}$ as being a solution of the studied non-linear elliptic equation, **BUT** the non-local terms are replaced by $\mathfrak{R}_i^{n+1}(v,L)$, $\mathfrak{R}_e^{n+1}(v,L)$, $\mathfrak{U}_L^{n+1}$, and $\mathfrak{V}_L^{n+1}$.

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This gives the continuity property required to pass to the limit in the right-hand side.

How to pass to the limit in the quantity:

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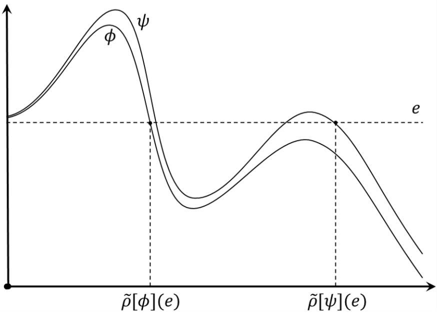

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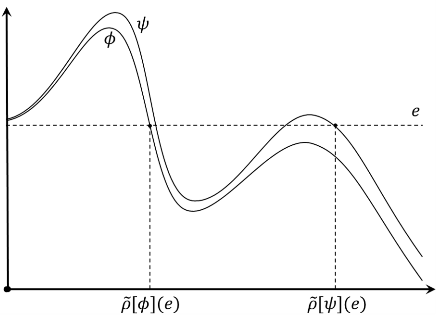

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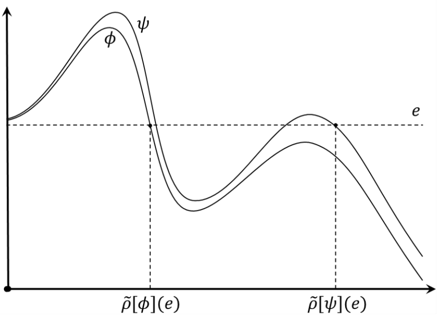


Lemma (Convergence property for $\widetilde{\rho}$)

Let (ψ_n) be a sequence of continuous functions converging towards ψ in L^∞. Then for almost every $e \in \mathbb{R}$,

$$\widetilde{\rho}[\psi_n](e) \longrightarrow \widetilde{\rho}[\psi](e).$$

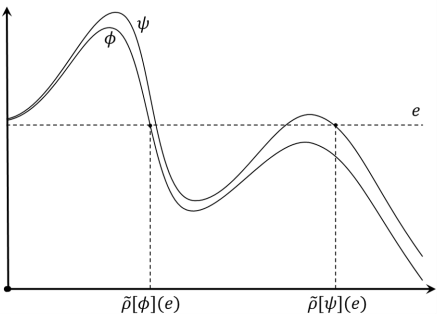
Existence of solution for the Poisson equation

Continuity of the barrier parameters

How to pass to the limit in the quantity:

$$\widetilde{\rho}[\psi](e) := \inf\left\{a \in [1, r_b] \;:\; \forall\, s \geq a, \psi(s) \leq e\right\}$$

It is in general not continuous with respect to ψ for the L^∞ or the $W^{s,p}$ topologies.



Lemma (Convergence property for $\widetilde{\rho}$)

Let (ψ_n) be a sequence of continuous functions converging towards ψ in L^∞. Then for almost every $e \in \mathbb{R}$,

$$\widetilde{\rho}[\psi_n](e) \longrightarrow \widetilde{\rho}[\psi](e).$$

Enough to conclude since $\widetilde{\rho}$ only appears under an integral.

Technical proofs

Technical proofs

- Proof of the bound on the right-hand side.
- Proof of the Hölder regularity for the right-hand side.
- Proof of the Convergence property for $\widetilde{\rho}$.
- Existence for the non-linear local elliptic equation.
- Conclusion of the proof

Let $p \in [1,2)$ and let $h \in (0,1/2]$. let $L, \nu \in \mathbb{R}$ and let $r \in [1, r_b]$. We define the set

$$\mathcal{O}_{h,r}^{L,\nu} := \left\{ v \in \mathbb{R} : \left| v^2 - L^2 \left(\frac{1}{r^2} - \frac{1}{r_b^2} \right) - 2\nu \right| \le h\, v^2 \right\}.$$

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By definition of $\mathcal{O}_{b,r}^{L,\nu}$,

$$\int_{\mathbb{R}\setminus\mathcal{O}_{b,r}^{L,\nu}} \frac{|v|^p}{\left|v^2 - L^2\left(\frac{1}{r^2} - \frac{1}{r_b^2}\right) - 2\nu\right|^{\frac{p}{2}}} \left|f(v,L)\right| dv \leq \frac{1}{h^{\frac{p}{2}}} \int_{-\infty}^{+\infty} \left|f(v,L)\right| dv.$$

Technical proofs

Proof of the bound on the right-hand side.

Let $p \in [1,2)$ and let $h \in (0,1/2]$. let $L, \nu \in \mathbb{R}$ and let $r \in [1, r_b]$. We define the set

$$\mathcal{O}_{h,r}^{L,\nu} := \left\{ v \in \mathbb{R} : \left| v^2 - L^2 \left(\frac{1}{r^2} - \frac{1}{r_b^2} \right) - 2\nu \right| \le h\, v^2 \right\}.$$

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On the other hand, since $h \le 1/2$,

$$\begin{aligned} v \in \mathcal{O}_{h,r}^{L,\nu} \quad &\Longleftrightarrow \quad (h-1)v^2 \le L^2\Big(\frac{1}{r^2} - \frac{1}{r_b^2}\Big) + 2\nu \le (h+1)v^2 \\ &\Longleftrightarrow \quad \frac{L^2\left(\frac{1}{r^2} - \frac{1}{r_b^2}\right) + 2\nu}{1+h} \le v^2 \le \frac{L^2\left(\frac{1}{r^2} - \frac{1}{r_b^2}\right) + 2\nu}{1-h}. \end{aligned}$$

The set $\mathcal{O}_{h,r}^{L,\nu}$ is non-empty if and only if $L^2(1/r^2 - 1/r_b^2) + 2\nu \ge 0$. In this case we can define $\lambda \ge 0$ such that $\lambda^2 = L^2(1/r^2 - 1/r_b^2) + 2\nu$.

Technical proofs

Proof of the bound on the right-hand side.

For λ a positive number, direct computation gives

$$\int_{\frac{\lambda}{\sqrt{1+h}}}^{\lambda} \frac{|v|^{p-1}dv}{\left|v^2-\lambda^2\right|^{\frac{p}{2}}} \leq \frac{1}{1-\frac{p}{2}} \quad \text{and} \quad \int_{\lambda}^{\frac{\lambda}{\sqrt{1-h}}} \frac{|v|^{p-1}dv}{\left|v^2-\lambda^2\right|^{\frac{p}{2}}} \leq \frac{1}{1-\frac{p}{2}}.$$

Technical proofs

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Then,

$$\begin{aligned}
&\int_{\mathcal{O}_{h,r}^{L,\nu}} \frac{|v|^p}{\left|v^2 - L^2\left(\frac{1}{r^2} - \frac{1}{r_b^2}\right) - 2\nu\right|^{\frac{p}{2}}} \left|f(v,L)\right| dv \\
\leq\ & \left(\sup_v |v| \left|f(v,L)\right|\right) \int_{\mathcal{O}_{h,r}^{L,\nu}} \frac{|v|^{p-1}}{\left|v^2 - L^2\left(\frac{1}{r^2} - \frac{1}{r_b^2}\right) - 2\nu\right|^{\frac{p}{2}}} dv \\
\leq\ & \frac{2}{1 - \frac{p}{2}} \left(\sup_v |v| \left|f(v,L)\right|\right).
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&\leq \left(\sup_v |v| \left|f(v,L)\right|\right) \int_{\mathcal{O}_{h,r}^{L,\nu}} \frac{|v|^{p-1}}{\left|v^2-L^2\left(\frac{1}{r^2}-\frac{1}{r_b^2}\right)-2\nu\right|^{\frac{p}{2}}} dv \\
&\leq \frac{2}{1-\frac{p}{2}} \left(\sup_v |v| \left|f(v,L)\right|\right).
\end{aligned}$$

Integrate now for the variable L and gather the two obtained estimates:

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} &\frac{|v|^p}{\left|v^2-L^2\left(\frac{1}{r^2}-\frac{1}{r_b^2}\right)-2\nu\right|^{\frac{p}{2}}} \left|f(v,L)\right| dv\, dL \\
&\leq 2^{\frac{p}{2}} \|f\|_{L^1} + \frac{2}{1-\frac{p}{2}} \int_{-\infty}^{+\infty} \left(\sup_v |v|\left|f(v,L)\right|\right) dL.
\end{aligned}$$

This concludes the proof. □

Let $\nu' < \nu \in \mathbb{R}$ such that $\nu - \nu' \leq 1$ and let $r \in [1, r_b)$. We consider the number $1 < p < 2$ such that $\gamma = (p-1)/(3-p)$. We define

$$\mathcal{P}^{r,p}_{\nu,\nu'} := \left\{ (v, L) \in \mathbb{R}^2 \ : \ \left| v^2 - L^2 \left(\frac{1}{r^2} - \frac{1}{r_b^2} \right) - 2\nu' \right| \geq \frac{\nu - \nu'}{|v|^{2\frac{p-1}{3-p}}} \right\}.$$

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Step 1: Regularity property on $\mathcal{P}^{r,p}_{\nu,\nu'}$. By convexity inequality, $\dfrac{1}{\sqrt{a}} - \dfrac{1}{\sqrt{a+h}} \leq \dfrac{h}{2\sqrt{a}^3}$. Thus,

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$$I^{r,p}_{\nu,\nu'} := \int_{\mathcal{P}^{r,p}_{\nu,\nu'}} \left| \Gamma(\nu', r, v, L) - \Gamma(\nu, r, v, L) \right| \left| f_i^b \left(v, \frac{L}{r_b} \right) \right| \left(1 + \mathbb{1}_{v^2 + \frac{L^2}{r_b^2} < 2\mathfrak{U}_L} \right) \mathbb{1}_{r \geq \mathfrak{R}_i(v,L)} \, dv \, dL$$

$$\leq \int_{\mathcal{P}^{r,p}_{\nu,\nu'}} \frac{|v| \ (\nu - \nu')}{\left| v^2 - L^2 \left(\frac{1}{r^2} - \frac{1}{r_b^2} \right) - 2\nu' \right|^{\frac{3}{2}}} \left| f_i^b \left(v, \frac{L}{r_b} \right) \right| dv \, dL$$

$$\leq \int_{\mathbb{R}^2} \frac{|v|^p \ (\nu - \nu')^{\frac{p-1}{2}}}{\left| v^2 - L^2 \left(\frac{1}{r^2} - \frac{1}{r_b^2} \right) - 2\nu' \right|^{\frac{p}{2}}} \left| f_i^b \left(v, \frac{L}{r_b} \right) \right| dv \, dL,$$

where for the last inequality we used the definition of $\mathcal{P}^{r,p}_{\nu,\nu'}$.

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The bound given by the previous lemma gives the conclusion.

Technical proofs

Technical proofs

Step 2: Regularity property on $\mathbb{R}^2 \setminus \mathcal{P}^{r,p}_{\nu,\nu'}$. This case reduces to study

$$J^r_{\nu,\nu'} := \int_{\mathbb{R}^2 \setminus \mathcal{P}^{r,p}_{\nu,\nu'}} \frac{|v|}{\left| v^2 - L^2\left(\frac{1}{r^2} - \frac{1}{r_b^2}\right) - 2\nu' \right|^{\frac{1}{2}}} \left| f_i^b\left(v, \frac{L}{r_b}\right) \right| dv\, dL.$$

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By the Hölder inequality and the bound given by the previous lemma,

$$J^r_{\nu,\nu'} \leq \frac{C}{2-q'} \left(K^{r,p}_{\nu,\nu'}\right)^{\frac{1}{q}}, \qquad \text{where} \qquad K^{r,p}_{\nu,\nu'} := \int_{\mathbb{R}^2 \setminus \mathcal{P}^{r,p}_{\nu,\nu'}} \left| f_i^b\left(v, \frac{L}{r_b}\right)\right| \, dv\, dL.$$

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We now observe that

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The Fubini theorem then gives $K^{r,p}_{\nu,\nu'} = 2\int_{-\infty}^{+\infty}\int_{(M^1_{v,\nu,\nu'})_+^{1/2}}^{(M^2_{v,\nu,\nu'})_+^{1/2}} \left| f_i^b\left(v, \frac{L}{r_b}\right)\right| dL\, dv$, where

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This eventually gives, with p such that $\gamma = (p-1)/(3-p)$,

$$K^{r,p}_{\nu,\nu'} \leq C(r)\sqrt{\nu-\nu'}\int_{-\infty}^{+\infty} \sup_{L\in\mathbb{R}} \left| f_i^b(v, L)\right| \frac{dv}{|v|^{\gamma}}. \qquad \square$$

For $\phi : [1, r_b] \to \mathbb{R}$ be a continuous function, we define $\phi^\dagger(r) := \max_{r' \in [r, r_b]} \phi(r')$.

Technical proofs

Proof of the Convergence property for $\widetilde{\rho}$.

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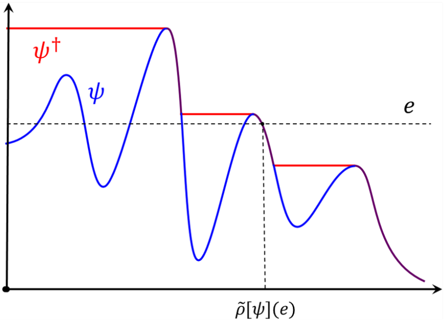

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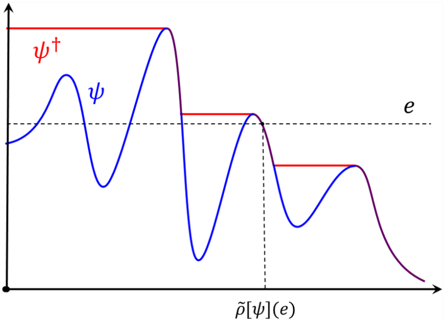

Lemma (Application † is Lipschitz)

Let ϕ and ψ be two continuous functions on $[1, r_b]$. Then: $\|\phi^\dagger - \psi^\dagger\|_{L^\infty} \leq \|\phi - \psi\|_{L^\infty}$.

Lemma (Convergence property for $\widetilde{\rho}$)

Let (ϕ_n) be a sequence of continuous functions that is uniformly converging towards ϕ. Then for almost every $e \in \mathbb{R}$,

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$$\begin{aligned} 0 = \int_{\mathbb{R}^d} 0\, dx &= \int_{\mathbb{R}^d} \text{meas}\,\big\{y \in \mathbb{R} : f(x) = y\big\}\, dx \\ &= \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{1}_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}\,:\, f(x)=y\}}\, dx\, dy \\ &= \int_{\mathbb{R}} \text{meas}\,\big\{x \in \mathbb{R}^d : f(x) = y\big\}\, dy. \end{aligned}$$

□

Technical proofs

Existence for the non-linear local elliptic equation.

We study

$$-\Delta\phi(x) = g\left(\phi(x), x\right), \qquad \phi(x) = 0 \text{ on } \partial\Omega.$$

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$$\mathcal{J}(\psi) := \int_\Omega \left(\frac{1}{2}\left|\nabla\psi(x)\right|^2 - G\left(\psi(x),\, x\right)\right) dx,$$

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The function $\mathcal{J}$ satisfy the following inequality: $\displaystyle \frac{1}{2}\int_0^1 \left|\nabla\psi(x)\right|^2 dx \;\leq C_\Omega \left(\mathcal{J}(\psi) + \|g\|_{L^\infty}^2\right).$

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We now bound $\mathcal{J}$ from below using the Young inequality and the Poincaré inequality :

$$\begin{aligned}
\mathcal{J}(\psi) &\geq \frac{1}{2}\int_\Omega \left|\nabla\psi(x)\right|^2 dx - C_\Omega \|g\|_{L^\infty} \|\psi\|_{L^2} \\
&\geq \frac{1}{2}\int_\Omega \left|\nabla\psi(x)(x)\right|^2 dx - \frac{C_\Omega}{4\varepsilon}\|g\|_{L^\infty}^2 - \varepsilon\, C_\Omega \|\psi\|_{L^2}^2 \\
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These two facts together imply, since ψ_n is a minimizing sequence for $\mathcal{J}$,

$$\mathcal{J}(\psi^\star) \le \inf_{\psi\in H_0^1([0,1])} \mathcal{J}(\psi),$$

which eventually gives the existence of a minimizer for $\mathcal{J}$. □

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- The mass of the electron is small in comparison with the mass of ions. Is it possible to do an asymptotic expansion ?
- Describing the obtained solution with calculus of variations techniques (*partial results*).

Radial solutions

Analysis of radial solutions

- We are interested in solving the following 1D elliptic equation:

$$-\frac{\lambda^2}{r}\frac{d}{dr}\left(r\,\frac{d\phi}{dr}\right)(r) = (n_i - n_e)(r), \qquad \phi(1) = \phi_p \quad \phi(r_b) = 0,$$

with

$$r\,n_i(r) = \int_{\mathbb{R}^2} \Gamma\big(\phi(r), r, v, L\big)\, f_i^b\left(v, \frac{L}{r_b}\right)\left(1 + \mathbb{1}_{v^2 + \frac{L^2}{r_b^2} < 2\overline{U_L}}\right)\mathbb{1}_{r \geq \widetilde{\rho}[U_L]\left(\frac{v^2}{2} + \frac{L^2}{2r_b^2}\right)}\, dv\, dL.$$

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- **We focus on the radial case :** $\boldsymbol{f^b(v, L) = g^b(v)\,\delta_{L=0}}$**.**

Theorem (M. Badsi, L.G-C., 2022)

Let $g_i^b : \mathbb{R}_-^* \longrightarrow \mathbb{R}_+$ *and* $g_e^b : \mathbb{R}_-^* \longrightarrow \mathbb{R}_+$ *two incoming distribution functions having the property:*

$$\sup_{v\in\mathbb{R}} \left| v\, g^b(v) \right| < +\infty, \qquad \text{and} \qquad \int_{\mathbb{R}} g^b(v)\, dv < +\infty.$$

Then there exists a solution $\phi \in \mathcal{C}^2[1, r_b]$ *to the non-linear Poisson problem, and* (g_i, g_e, ϕ) *a measure valued solution to the original Vlasov-Poisson equation.*

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YES : under physical and technical assumptions.

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- *Electric neutrality of the plasma core.*
- *A Bohm condition:* $\displaystyle\int_{-\infty}^{0} g_i^b(v)\frac{dv}{v^2} < \frac{g_e^b\left(-\sqrt{-2\phi_p}\right)}{\sqrt{-2\phi_p}} + \int_{-\infty}^{\sqrt{-\phi_p}} (g_e^b)'(v)\frac{dv}{|v|}.$

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Then, *the Poisson problem admits a solution* $\phi_\lambda \in \mathcal{C}^2$ *s.t.*

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Important argument : reduction of the problem to a **local** equation.

Numerical investigations

The numerical method

Define the Energy associated to this Dirichlet Problem:

$$E[\psi](\phi) := \frac{1}{2} \int_1^{r_b} \left| \frac{d}{dr}\phi(r) \right|^2 r\, dr \; + \; N[\psi](\phi(r), r).$$

The function ψ is the parameters for the non-local terms.The natural function space is:

$$X := \left\{ \phi : [1, r_b] \to \mathbb{R} \; : \; \frac{d\phi}{dr} \in L^2, \;\; \phi(1) = \phi_p, \;\; \phi(r_b) = 0 \right\}.$$

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We use a gradient descent approach:

$$\phi_{n+1} := \phi_n - \rho\, \nabla E[\phi_n](\phi_n).$$

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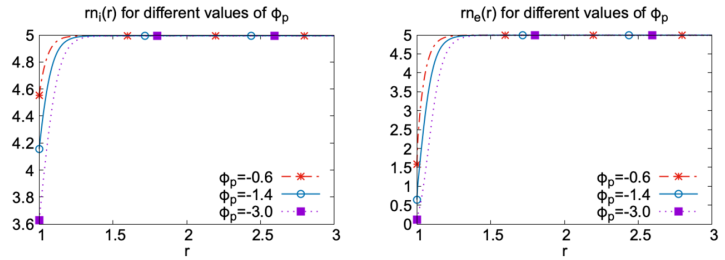
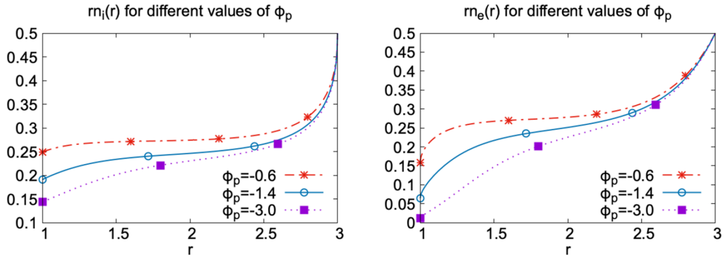
We are able to converge as long as $\varepsilon > 0$ and $k \in \mathbb{N}$ are not too large.

Numerical results



Radial case with satisfied Bohm condition: ionic density $rn_i(r)$ (left) and electronic density $rn_e(r)$ (right) for ϕ_p varying from -0.6 to -3.



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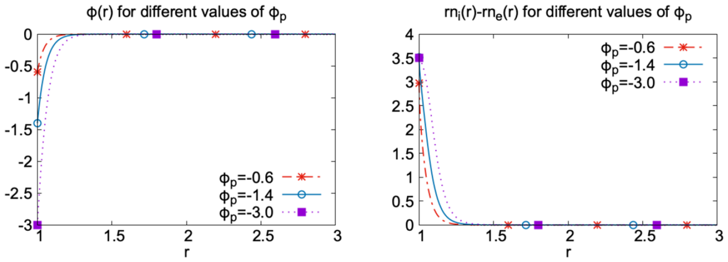
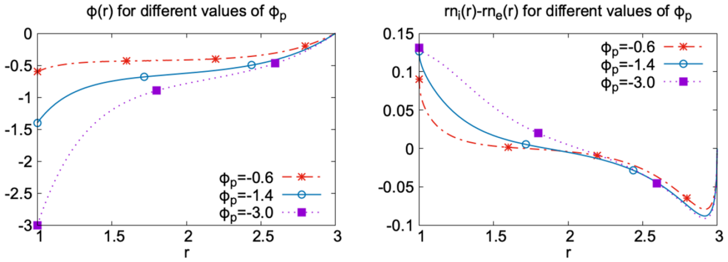


Radial case with satisfied Bohm condition: potential $\phi(r)$ (left) and density difference $rn_i(r) - rn_e(r)$ (right) for ϕ_p varying from -0.6 to -3.



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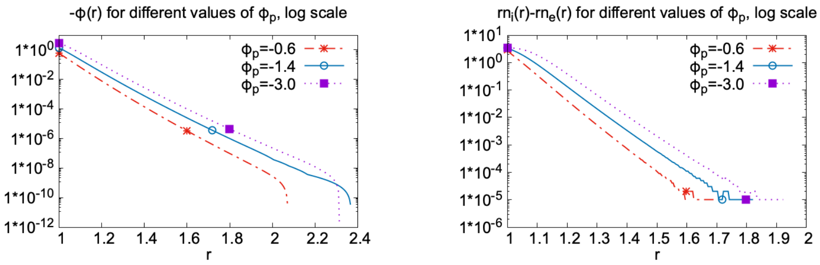
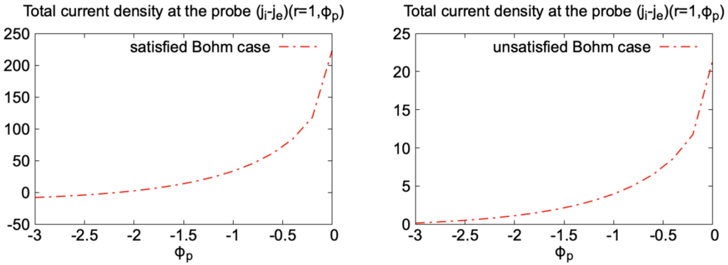
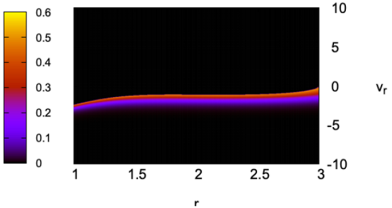


Radial case with satisfied Bohm condition: potential $\phi(r)$ (left) and density difference $rn_i(r) - rn_e(r)$ (right) for ϕ_p varying from -0.6 to -3. y-log scale.

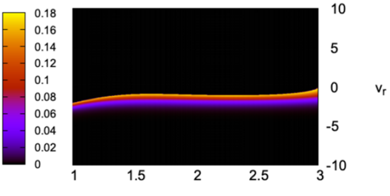


Radial case: total current density at the probe $(j_i - j_e)(r = 1, \phi_p)$ as a function of the probe potential, the Bohm condition being satisfied (left) or unsatisfied (right).

ionic distribution $f_i(r,v_r)$ for v_θ=0, T=1.10^{-1}



ionic distribution $f_i(r,v_r)$ for v_θ≈0.45, T=1.10^{-1}



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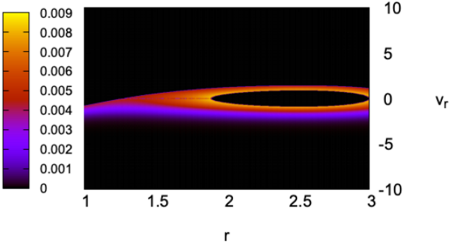

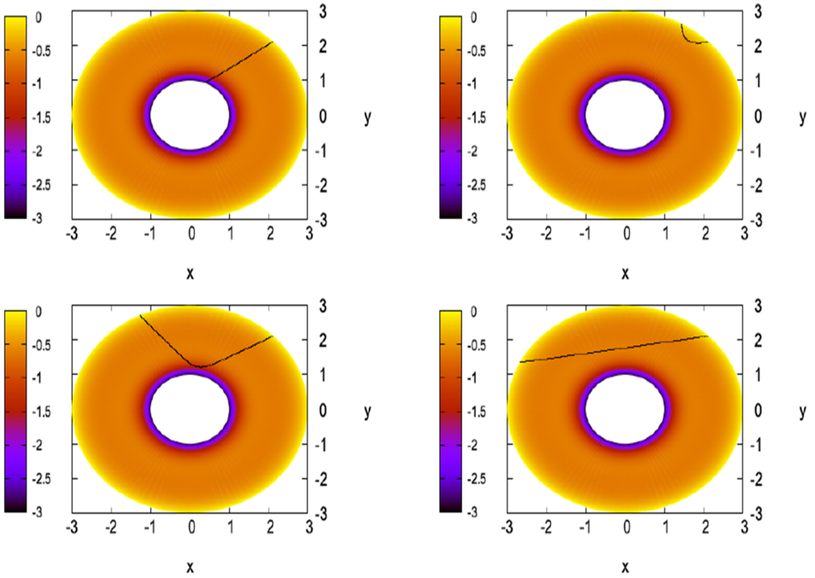
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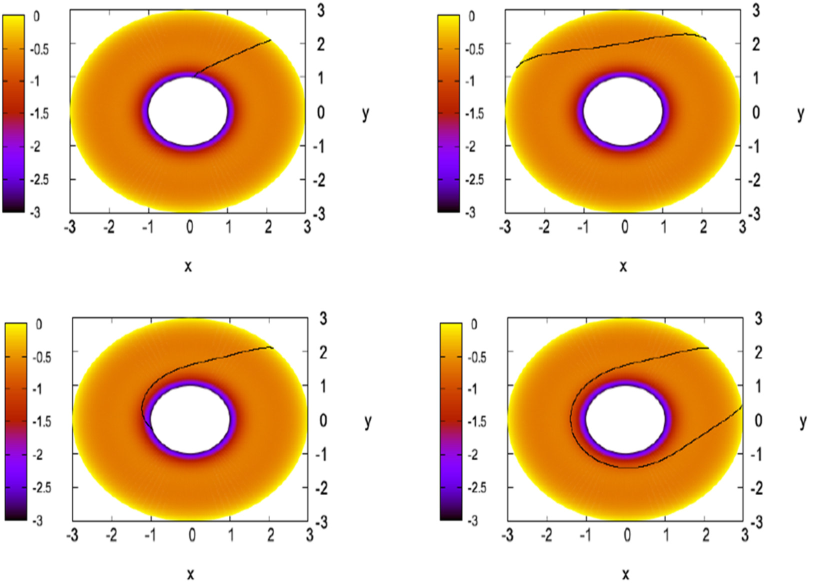
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Thank-you for your attention !