Particle Deceleration in Random Media

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Analysis, modeling and numerical method for kinetic and related models
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Motivation

•In ICF (Inertial Confinement Fusion) pellets, the reaction

$$D + D \to T + p$$

produces suprathermal T ions.

•These T ions decelerate in the plasma, stop, and react with the background D ions

$$D + T \to \alpha + n + 17.6 MeV$$

This reaction has a much higher cross-section than the previous one and is the main source of energy in this process.

•In practice, the D plasma is polluted by outer shell debris due to hydrodynamic instabilities. A statistical description of this process is therefore important to evaluate the number of $D + T$ reactions.

ICF Pellet Schematics

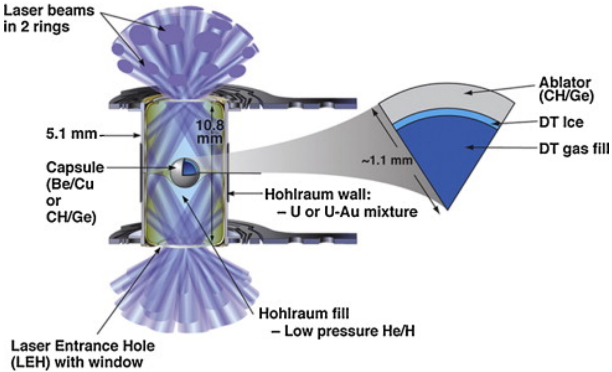

Figure: An ICF pellet in a hohlraum (indirect drive).

ICF: Hydrodynamic Instablities

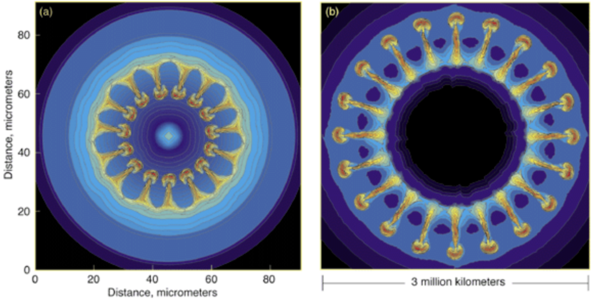

Figure: The Richtmyer-Meshkov instability in an ICF pellet, & in a core-collapse implosion of a supernova. Left: Sakagami-Nishihara, Phys. Fluids B **2**, 2715 (1990); right: Hachisu & al., Astrophysics Journal **368**, L27 (1991)

Build-up of the Instability

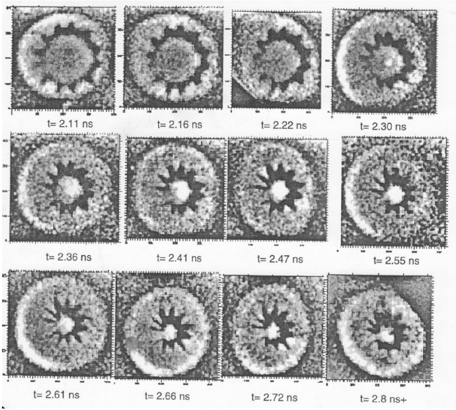

Figure: Compression of a foam filled cylindrical target with machined perturbations by the Nova Laser (1995); growth of the hydrodynamic instabilities.

Some References

(1) C. D. Levermore, G. B. Zimmerman, J. Math. Phys. 34, 4725–4729, (1993)

Based on the Levermore-Pomraning-Sanzo-Wong 1986 analysis of random media which are Markovian along particle paths.

(2) J.-F. Clouet, FG, M. Puel, R. Sentis, KRM 1, 387–404, (2008)

Boolean random medium: inhomogeneities are balls centered at points distributed according to a Poisson point process with intensity proportional to the Lebesgue measure in the space of positions

Unknown Distribution function $f \equiv f(t,x,v) \geq 0$ satisfying

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) - \kappa(x) S(|v|) v \cdot \nabla_v f(t,x,v) = 0$$

Data

(a) Universal function $S(|v|)$ — typically of the form

$$S(w) = s_0 \Big(\underbrace{\frac{w}{\beta^3}}_{electrons} + \underbrace{\frac{1}{w^2}}_{ions} \Big)$$

Then β depends in general on the plasma temperature — will be considered a constant here.

(b) Slowing rate $\kappa(x)$ depending on whether x is in the light medium, or in the dense inhomogeneties

Poisson Random Medium

Assume that

• The **slowing rate** takes 2 values

$$\kappa(x) = 0 \text{ in the light medium,} \quad \kappa(x) = \kappa/\epsilon \text{ in the dense medium}$$

• The dense inhomogeneities consist of balls of radius ϵ

• The centers of the inhomogeneities are realizations of a Poisson point process in $\mathbf{R}^d$ with intensity $\lambda_\epsilon |\cdot|$, where $|\cdot|$ is the Lebesgue measure on $\mathbf{R}^d$ and $\lambda_\epsilon = \lambda/\epsilon^{d-1}$:

for each countable set $C = \{c_j : j \geq 1\}$ of points in $\mathbf{R}^d$ and each measurable $A \subset \mathbf{R}^d$

$$\mathbf{P}\{C \subset (\mathbf{R}^d)^{\mathbf{N}^*} \text{ s.t. } \#(C \cap A) = n\} = \frac{\lambda_\epsilon^n |A|^n}{n!} e^{-\lambda_\epsilon |A|}$$

For each configuration of inhomogeneities C (countable $C \subset \mathbf{R}^d$)

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f_\epsilon(t,x,v;C) = \underbrace{\frac{\kappa}{\epsilon}\mathbf{1}_{\mathrm{dist}(x,C)<\epsilon}}_{\kappa_\epsilon(x;C)} \mathrm{div}_v(f_\epsilon(t,x,v;C)S(|v|)v) \\ f_\epsilon\big|_{t=0} = f^{in} \end{cases}$$

The associated characteristic flow is $t \mapsto (X_\epsilon, V_\epsilon)(t,x,v;C)$ s.t.

$$\begin{cases} \dot{X}_\epsilon = V_\epsilon\,, & X_\epsilon\big|_{t=0} = x \\ \dot{V}_\epsilon = -\frac{\kappa}{\epsilon}\mathbf{1}_{\mathrm{dist}(X_\epsilon,C)<\epsilon} S(|V_\epsilon|)V_\epsilon\,, & V_\epsilon\big|_{t=0} = v \end{cases}$$

Its Jacobian $J_\epsilon(t,x,v;C) = \det \frac{\partial(X_\epsilon,V_\epsilon)}{\partial(x,v)}(t,x,v;C)$ satisfies

$$\begin{cases} \dot{J}_\epsilon = -\frac{\kappa}{\epsilon}\mathbf{1}_{\mathrm{dist}(X_\epsilon,C)<\epsilon}(dS(|V_\epsilon|) + S'(|V_\epsilon|)|V_\epsilon|)J_\epsilon \\ J_\epsilon\big|_{t=0} = 1 \end{cases}$$

Solving the transport equation above by the method of characteristics

$$f_\epsilon(t,x,v;C) = \frac{f^{in}((X_\epsilon,V_\epsilon)(-t,x,v;C))}{J_\epsilon(t,(X_\epsilon,V_\epsilon)(-t,x,v;C);C)}$$

Equivalently

$$f_\epsilon(t,\cdot,\cdot;C)dxdv = (X_\epsilon,V_\epsilon)(t,\cdot,\cdot;C)\# f^{in}dxdv$$

meaning that

$$\int_{\mathbf{R}^{2d}} f_\epsilon(t,x,v;C)\phi(x,v)dxdv = \int_{\mathbf{R}^{2d}} f^{in}(x,v)\underbrace{\phi((X_\epsilon,V_\epsilon)(t,x,v;C))}_{=:\Phi_\epsilon(t,x,v;C)}dxdv$$

where Φ_ϵ is the solution of the **adjoint problem**

$$\begin{cases}\partial_t\Phi_\epsilon(t,x,v;C) = (v\cdot\nabla_x - \frac{\kappa}{\epsilon}\mathbf{1}_{\mathrm{dist}(x,C)<\epsilon}S(|v|)v\cdot\nabla_v)\Phi_\epsilon(t,x,v;C)\\ \Phi_\epsilon\big|_{t=0} = \phi\end{cases}$$

THM Let $0 \leq f^{in} \in L^\infty(\mathbf{R}^2)$ have support in $\overline{B(0,R)} \times \overline{B(0,R)}$. Then $\mathbf{E}f_\epsilon(t,\cdot) \to F(t,\cdot)$ narrowly on $\mathbf{R}^{2d}$ as $\epsilon \to 0^+$, where F is the **unique** solution of

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \sigma|v|)F(t,z) = \sigma\mathcal{K}^*(|\cdot|F(t,x,\cdot))(v) + \lambda(t,x)\delta_{v=0} \\ \partial_t \int_{\mathbf{R}^d} F(t,z)dv + \mathrm{div}_x \int_{\mathbf{R}^d} vF(t,z)dv = 0 \\ F\big|_{t=0} = f^{in} \end{cases}$$

where

$$\mathcal{K}^*(\psi)(w) := \int_{|\nu|\leq 1} \psi(\mathcal{V}^{-1}(|\nu|,|w|)\tfrac{w}{|w|})\frac{\mathcal{V}^{-1}(|\nu|,|w|)^{d-1}}{|w|^{d-1}}\frac{S(\mathcal{V}^{-1}(|\nu|,|w|))}{S(|w|)}\frac{d\nu}{|\mathbf{B}^d|}$$

$$\mathcal{V}^{-1}(h,v) := \Sigma'^{-1}(\Sigma'(v)+2\kappa\sqrt{1-h^2}), \qquad \Sigma(v) := \int_0^v (v-u)\frac{du}{S(u)}$$

$\lambda(t,x) =$ Lagrange multiplier needed for local conservation of mass

Notations for Particle Paths

For all $x, v \in \mathbf{R}^d$ and all $t > 0$, let

$$\mathcal{A}_\epsilon(t,x,v) := \left\{ C \;:\; \left[\begin{array}{l}(x+[0,t]v)\cap B(c,\epsilon)\neq\varnothing\\(x+[0,t]v)\cap B(c',\epsilon)\neq\varnothing\end{array}\right] \Rightarrow |c-c'|>2\epsilon \right\}$$

For each configuration C of ball centers, set

$$\mathcal{X}_\epsilon[C] := \mathbf{R}^d \setminus \bigcup_{c\in C} \overline{B(c,\epsilon)}$$

and, for all $x \in \mathcal{X}_\epsilon[C]$ and all $v \neq 0$

$$\Theta_\epsilon(x,v;C) := \{t>0 \text{ s.t. } X_\epsilon(t,x,v;C) \notin \mathcal{X}_\epsilon[C]\}$$
$$= [\tau_1^\epsilon(x,v;C), \tau_2^\epsilon(x,v;C)] \cup [\tau_3^\epsilon(x,v;C), \tau_4^\epsilon(x,v;C)] \cup \ldots$$

For each $t > 0$, set

$$\mathcal{N}_\epsilon(t,x,v;C) := \inf\{n \geq 0 \text{ s.t. } \tau_n^\epsilon(x,v;C) \leq t\}$$

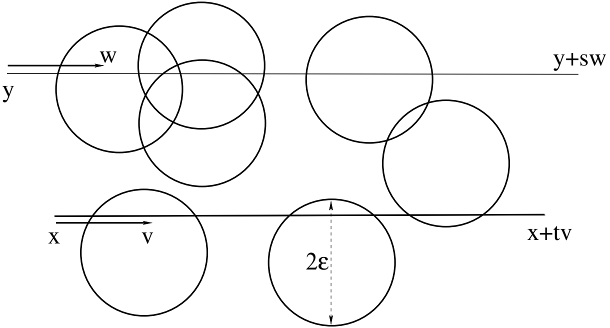

Figure: Example of a configuration C which belongs to $\mathcal{A}_\epsilon(t, x, v)$ but not to $\mathcal{A}_\epsilon(s, y, w)$.

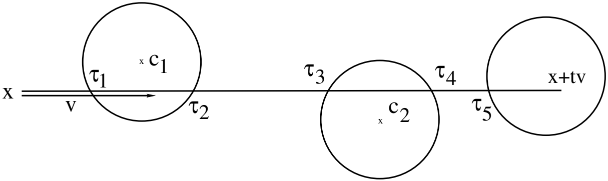

Figure: The times $\tau_j^\epsilon(x, v; C)$ for $j = 1, 2, 3, 4, 5$, in a case where $\mathcal{N}_\epsilon(t, x, v; C) = 5$.

Let $\phi \in C_c^1(\mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}))$. Then

$$\begin{aligned}
\mathbf{E}\Phi_\epsilon(t,z) =& \mathbf{E}(\phi(X_\epsilon, V_\epsilon)(t,z;\cdot)) \\
=& \underbrace{\mathbf{E}\left(\phi(X_\epsilon, V_\epsilon)(t,z;\cdot)\mathbf{1}_{C\in\mathcal{A}_\epsilon(t,z)}\mathbf{1}_{x\in\mathcal{X}_\epsilon[C]}\right)}_{=(1)} \\
&+ \underbrace{\mathbf{E}\left(\phi(X_\epsilon, V_\epsilon)(t,z;\cdot)\mathbf{1}_{C\in\mathcal{A}_\epsilon(t,z)}\mathbf{1}_{x\in\mathcal{X}^c_\epsilon[C]}\right)}_{=(2)} \\
&+ \underbrace{\mathbf{E}\left(\phi(X_\epsilon, V_\epsilon)(t,z;\cdot)\mathbf{1}_{C\in\mathcal{A}^c_\epsilon(t,z)}\right)}_{=(3)}
\end{aligned}$$

The terms (2) and (3) are negligible as $\epsilon \to 0$, and will be disposed of by using elementary bounds involving the intensity of the Poisson process, which is $\lambda\epsilon^{1-d}$ times the Lebesgue measure on $\mathbf{R}^d$

•The term (2) is mastered by

$$|(2)|/\|\phi\|_{L^\infty} \leq \mathbf{E}\left(\mathbf{1}_{x\in\mathcal{X}_\epsilon^c[C]}\right) = \mathbf{E}\left(\mathbf{1}_{C\cap\overline{B(x,\epsilon)}\neq\varnothing}\right)$$

$$= \sum_{n\geq 1} \mathbf{P}\{\#(\overline{B(x,\epsilon)}\cap C) = n\} = \sum_{n\geq 1} \frac{(\lambda\epsilon|\mathbf{B}^d|)^n}{n!} e^{-\lambda\epsilon|\mathbf{B}^d|} \leq \lambda\epsilon|\mathbf{B}^d|$$

•Assume that $\operatorname{supp}(\phi) \subset B(0,R)\times(B(0,R)\setminus B(0,r))$, so that

$$\operatorname{supp}(z\mapsto\phi(X_\epsilon,V_\epsilon)(t,z;C)) \subset B(0,R(1+t))\times(B(0,R)\setminus B(0,r))$$

Thus the term (3) satisfies

$$|(3)| = \left|\mathbf{E}\left(\phi(X_\epsilon,V_\epsilon)(t,z;\cdot)\mathbf{1}_{|x|<R(1+t)}\mathbf{1}_{r<|v|<R}\mathbf{1}_{C\in\mathcal{A}_\epsilon^c(t,z)}\right)\right|$$

$$\leq \|\phi\|_{L^\infty} \sup_{\substack{|x|<R(1+t)\\ r<|v|<R}} \mathbf{E}\left(\mathbf{1}_{C\in\mathcal{A}_\epsilon^c(t,z)}\right)$$

Now, whenever $|x| < R(1+t)$ and $r < |v| < R$

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_{C\in\mathcal{A}_\epsilon^c(t,z)}\right)\\
&=\mathbf{E}\left(\mathbf{1}_{\exists(c,c')\in C^2\,:\mathbf{1}_{\max(\mathrm{dist}(c,x+[0,t]v),\mathrm{dist}(c',x+[0,t]v))\le\epsilon}\mathbf{1}_{0<|c-c'|\le 2\epsilon}=1}\right)\\
&\le\mathbf{P}\left(\left\{C:\sum_{(c,c')\in C^2}\mathbf{1}_{\mathrm{dist}(c,x+[0,t]v)\le\epsilon}\mathbf{1}_{0<|c-c'|\le 2\epsilon}\ge 1\right\}\right)\\
&\le\mathbf{E}\left(\sum_{(c,c')\in C^2}\mathbf{1}_{\mathrm{dist}(c,x+[0,t]v)\le\epsilon}\mathbf{1}_{0<|c-c'|\le 2\epsilon}\right)
\end{aligned}$$

where the 1st inequality follows from $\mathbf{1}_{\max(a,b)\le\epsilon}\le\mathbf{1}_{a\le\epsilon}$, while the 2nd is the Markov inequality.

By Thm 4.4 in [Last-Penrose, Poisson Process, Cambridge 2018] — i.e. the bivariate Campbell formula

$$\begin{aligned}
&\mathbf{E}\left(\sum_{(c,c')\in C^2}\mathbf{1}_{\mathrm{dist}(c,x+[0,t]v)\le\epsilon}\mathbf{1}_{0<|c-c'|\le 2\epsilon}\right)\\
&=\iint_{\mathbf{R}^d\times\mathbf{R}^d}\mathbf{1}_{\mathrm{dist}(z,x+[0,t]v)\le\epsilon}\mathbf{1}_{0<|z-z'|\le 2\epsilon}\lambda_\epsilon^2\,dz\,dz'\\
&\le\lambda_\epsilon^2|B(x,\epsilon)+[0,t]v||B(0,\epsilon)|\\
&=\lambda|\mathbf{B}^{d-1}|(tR+2\epsilon)\cdot\lambda|\mathbf{B}^d|\epsilon
\end{aligned}$$

Hence

$$\sup_{\substack{|x|<R(1+t)\\ r<|v|<R}}|(3)|/\|\phi\|_{L^\infty}\le\lambda^2|\mathbf{B}^d||\mathbf{B}^{d-1}|(tR+2\epsilon)\epsilon$$

so that $|(2)|+|(3)|=O(\epsilon)$ and we can focus on (1).

Computing (1): Impact Parameter

Definition If a particle enters $B(c,\epsilon)$ at the point $x_- \in \partial B(c,\epsilon)$ with velocity $V_- \neq 0$, its **impact parameter** is

$$h := \sqrt{1 - \left(\frac{(x_- - c)\cdot V_-}{|x_- - c||V_-|}\right)^2}$$

and the straight line $x + \mathbf{R}V_-$ crosses $B(c,\epsilon)$ on a chord of length

$$\ell := 2\epsilon\sqrt{1-h^2}$$

Impact Parameter/Chord Length

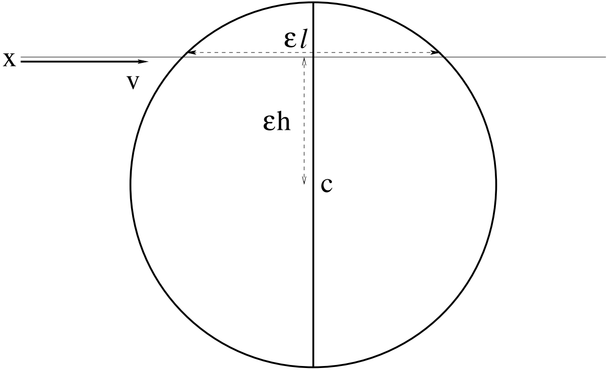

Figure: The impact parameter h and the chord length ℓ

Lemma 1

If a particle enters $B(c,\epsilon)$ at speed v_- with impact parameter h,
(i) either it stops in the ball, and we set $\mathcal{V}(h,v_-)=0$ if

$$\Sigma'(v_-)<2\kappa\sqrt{1-h^2}\,,\quad \Sigma(v):=\int_0^v(v-u)\frac{du}{S(u)}$$

(ii) or exits at speed v_+ with

$$\begin{aligned}\Sigma'(v_+)&=\Sigma'(v_-)-2\kappa\sqrt{1-h^2}\\ v_+=(\Sigma^*)'\left(\Sigma'(v_-)-2\kappa\sqrt{1-h^2}\right)&=:\mathcal{V}(h,v_-)\end{aligned}$$

where Σ^* is the Legendre transform of Σ defined by

$$\Sigma^*(w)=\sup_{v\geq 0}(wv-\Sigma(v))$$

Lemma 2

Let $z = (x, v)$ with $v \neq 0$, and let $C \in \mathcal{A}_\epsilon(t, z)$ s.t. $C \subset \overline{B(x,\epsilon)}^c$. Let $\tau_j^\epsilon := \tau_j^\epsilon(z; C)$ and $\mathcal{N}_\epsilon := \mathcal{N}_\epsilon(t, z; C)$. Then

$$V_\epsilon(t, z; C) = \begin{cases} v_0 = v & 0 < t < \tau_1^\epsilon \\ v_1 = \mathcal{V}(h_1, |v|)\frac{v}{|v|} & \tau_2^\epsilon < t < \tau_3^\epsilon \\ v_2 = \mathcal{V}(h_2, |v_1|)\frac{v}{|v|} & \tau_4^\epsilon < t < \tau_5^\epsilon \\ \dots & \dots \\ v_{[\frac{\mathcal{N}_\epsilon}{2}]} = \mathcal{V}(h_{[\frac{\mathcal{N}_\epsilon}{2}]}, |v_{[\frac{\mathcal{N}_\epsilon}{2}]-1}|)\frac{v}{|v|} & \tau^\epsilon_{2[\frac{\mathcal{N}_\epsilon}{2}]} < t < \tau^\epsilon_{2[\frac{\mathcal{N}_\epsilon}{2}]+1} \end{cases}$$

where h_1 is the impact parameter on the first ball, h_2 the impact parameter on the second ball, and so on.

In particular

$$\tau_{2j}^\epsilon(z; C) - \tau_{2j-1}^\epsilon(z; C) \leq 2\epsilon/|v_j|$$

Lemma 3

Let $z = (x, v)$ with $v \neq 0$, and let $C \in \mathcal{A}_\epsilon(t, z)$ s.t. $C \subset \overline{B(x,\epsilon)}^c$.
Then, with the same notations as in Lemma 2

$$X_\epsilon(t,z;C) = x + \tau_1^\epsilon v + \ell_1 + (\tau_3^\epsilon - \tau_2^\epsilon)v_1 + \ell_2 + (\tau_5^\epsilon - \tau_4^\epsilon)v_2 + \ldots + (t \wedge \tau^\epsilon_{2[\frac{\mathcal{N}_\epsilon}{2}]+1} - \tau^\epsilon_{2[\frac{\mathcal{N}_\epsilon}{2}]})_+ v_{[\frac{\mathcal{N}_\epsilon}{2}]} + O(\epsilon)$$

where

$$\ell_j = 2\epsilon\sqrt{1-h_j^2} \le 2\epsilon\,, \qquad \text{for } j = 1, 2, \ldots, [\tfrac{N_\epsilon}{2}]$$

Definition Interaction cylinder

$$\mathcal{T}_\epsilon(t,z;C) := \bigcup_{0\le s\le t} \overline{B(X_\epsilon(t,z;C),\epsilon)}$$

Remark The particle path $[0,t] \mapsto Z_\epsilon(s,z;C) = (X_\epsilon, V_\epsilon)(s,z;C)$ depends only on $C \cap \mathcal{T}_\epsilon(t,z;C)$

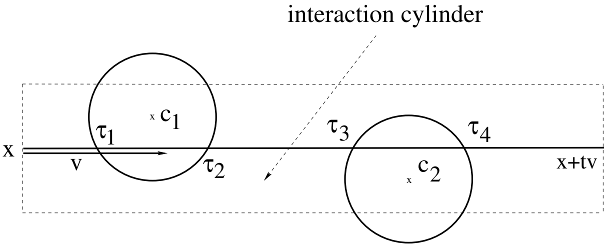

Figure: The interaction cylinder with 2 balls, with the times $\tau_j^\epsilon(x, v; C)$ for $j = 1, 2, 3, 4$ and with $\mathcal{N}_\epsilon(t, x, v; C) = 4$.

Denoting $Z_\epsilon = (X_\epsilon, V_\epsilon)$, one has

$$(1) = \sum_{N\ge 0} \int \left(\phi(Z_\epsilon(t,z;C))\mathbf{1}_{\#(C\cap\mathcal{T}_\epsilon(t,z;C))=N}\mathbf{1}_{C\in\mathcal{A}_\epsilon(t,z)}\mathbf{1}_{x\in\mathcal{X}_\epsilon[C]}\right)\mathbf{P}(dC)$$

$$= \sum_{N\ge 0}\frac{\lambda_\epsilon^N}{N!}\int e^{-\lambda_\epsilon|\mathcal{T}_\epsilon(t,z;\{c_1,\dots,c_N\})|}\phi(Z_\epsilon(t,z;\{c_1,\dots,c_N\}))$$

$$\times\prod_{j=1}^N \mathbf{1}_{|c_j-x|>\epsilon}\prod_{1\le j<k\le N}\mathbf{1}_{|c_j-c_k|>2\epsilon}dc_1\dots dc_N$$

Define $n_1,\dots,n_{2N}\in\mathbf{S}^{d-1}$ by the prescription

$$X_\epsilon(\tau^\epsilon_{2k-1}(z;C),z;C) = c_k+\epsilon n_{2k-1}\,,\quad X_\epsilon(\tau^\epsilon_{2k}(z;C),z;C) = c_k+\epsilon n_{2k}$$

and set

$$\nu_k := n_{2k-1} - \underbrace{\left(\frac{v}{|v|}\cdot n_{2k-1}\right)}_{=h_k}\frac{v}{|v|} = n_{2k} - \underbrace{\left(\frac{v}{|v|}\cdot n_{2k}\right)}_{=h_k}\frac{v}{|v|}$$

The New Variables for the kth Ball

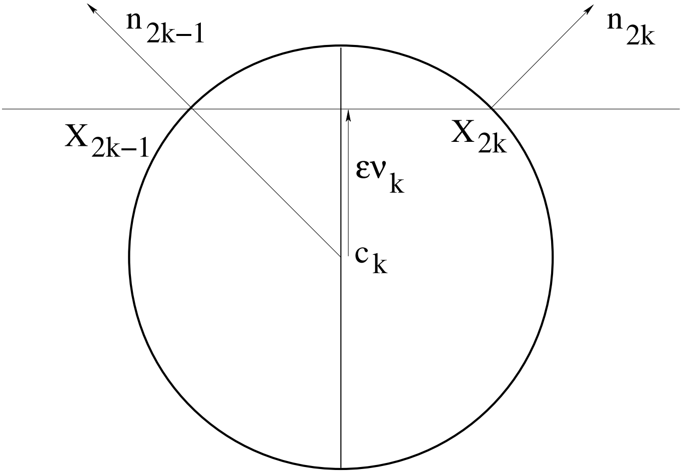

Figure: The incoming and outgoing points $X_\epsilon(\tau^\epsilon_{2k-1}(z;C),z;C)$ and $X_\epsilon(\tau^\epsilon_{2k}(z;C),z;C)$, the unit vectors n_{2k-1} and n_{2k}, and the vector ν_k.

(a) In the r.h.s. of (1), integrate by substitution with

$$(c_1,\ldots,c_N)\mapsto(\tau_1^\epsilon(z;\{c_1\}),\nu_1,\ldots,\nu_N,\tau_N^\epsilon(z;\{c_1,\ldots,c_N\}))$$

As above, $\tau_{2j-1}^\epsilon(z;C)$ and $\tau_{2j}^\epsilon(z;C)$ depend only on $\{c_1,\ldots,c_j\}$, so that the Jacobian determinant of the transformation above is triangular. It is the **product of the determinants** of $x_k\mapsto(\tau_{2k-1}^\epsilon,\nu_k)$.

(b) Next one has

$$c_k=X_\epsilon(\tau_{2k-1}^\epsilon,z,\{c_1,\ldots,c_{k-1}\})+\epsilon\sqrt{1-|\nu_k|^2}\frac{v}{|v|}-\epsilon\nu_k$$

so that, in the decomposition $\mathbf{R}^d=\mathbf{R}v\oplus(\mathbf{R}v)^\perp$, the Jacobian determinant of the transformation above is

$$\det\left(\frac{\partial c_k}{\partial\tau_{2k-1}^\epsilon\partial\nu_k}\right)=\begin{vmatrix}|V_\epsilon(\tau_{2k-1}^\epsilon,z;\{c_1,\ldots,c_{k-1}\})| & 0\\ -\epsilon\frac{\frac{v}{|v|}\otimes\nu_k}{\sqrt{1-|\nu_k|^2}} & -\epsilon I_{(\mathbf{R}v)^\perp}\end{vmatrix}$$
$$=(-\epsilon)^{d-1}|V_\epsilon(\tau_{2k-1}^\epsilon,z;\{c_1,\ldots,c_{k-1}\})|$$

We have **ordered** $c_1, \dots, c_N$, thereby **cancelling** the $\frac{1}{N!}$ factor, so that

$$(1) \simeq \sum_{N\geq 0} \int \mathbf{1}_{0<\tau_1<\tau_3<\ldots<\tau_{2N-1}<t} d\tau_1 d\tau_3 \ldots d\tau_{2N-1}$$

$$\times \int_{(\mathbf{B}^{d-1})^N} d\nu_1 \ldots d\nu_N \underbrace{\lambda_\epsilon^N \epsilon^{(d-1)N}}_{=\lambda^N} |v||v_1| \ldots |v_{N-1}|$$

$$\times e^{-\lambda_\epsilon \epsilon^{d-1}|\mathbf{B}^{d-1}|(\tau_1|v|+(\tau_3-\tau_2)|v_1|+\ldots+(t-\tau_{2N})_+|v_N|+O(\epsilon))}$$

$$\times \phi(x+\tau_1|v|+(\tau_3-\tau_2)|v_1|+\ldots+ \underbrace{(t-\tau_{2N})_+}_{=(t-\tau_{2N-1})+O(\epsilon)} |v_N| + O(\epsilon),$$

$$\mathcal{W}(t-\tau_{2N-1}, |\nu_n|, v_{N-1})\tfrac{v}{|v|})$$

where, for all $t > \tau_{2N-1}$

$$\int_{\mathcal{W}(t-\tau_{2N-1},h_n,v_{N-1})}^{v_{N-1}} \frac{du}{S(u)u} = \frac{\kappa}{\epsilon}(t \wedge \tau_{2N} - \tau_{2N-1})$$

and

$$|v_k| = \mathcal{V}(|\nu_k|, \cdot) \circ \ldots \circ \mathcal{V}(|\nu_1|, \cdot)(|v|)$$

Summarizing

$$\Phi(t,x,v)=\lim_{\epsilon\to0^+}\mathbf{E}\Phi_\epsilon(t,x,v;\cdot)$$

$$=\sum_{N\ge0}\int_0^t d\tau_{2N-1}\int_0^{\tau_{2N-1}}d\tau_{2N-3}\dots\int_0^{\tau_3}d\tau_1\int_{(\mathbf{B}^{d-1})^N}d\nu_1\dots d\nu_N$$

$$\times\lambda^N|v||v_1|\dots|v_{N-1}|e^{-\lambda|\mathbf{B}^{d-1}|(\tau_1|v|+(\tau_3-\tau_2)|v_1|+\dots+(t-\tau_{2N-1})|v_N|)}$$

$$\times\phi(x+(\tau_1|v|+(\tau_3-\tau_2)|v_1|+\dots+(t-\tau_{2N-1})|v_N|)\tfrac{v}{|v|},|v_N|\tfrac{v}{|v|})$$

Setting $\sigma:=\lambda|\mathbf{B}^{d-1}|$, this is the Duhamel series for the solution of

$$\begin{cases}(\partial_t-v\cdot\nabla_x)\Phi+\sigma|v|\Phi=\underbrace{\frac{\sigma|v|}{|\mathbf{B}^{d-1}|}\int_{|\nu|\le1}\Phi(t,x,\mathcal{V}(|\nu|,|v|)\tfrac{v}{|v|})d\nu}_{=:\sigma|v|\mathcal{K}\Phi(t,z)}\\[2ex]\Phi(0,x,v)=\phi(x,v)\end{cases}$$

Thus $\mathbf{E}f_\epsilon(t,\cdot;\cdot)\to f(t,\cdot)$ vaguely in $\mathbf{R}^d\times(\mathbf{R}^d\setminus\{0\})$ as $\epsilon\to 0$, where

$$\int_{\mathbf{R}^{2d}}\phi(z)f(t,z)dxdv=\int_{\mathbf{R}^{2d}}f^{in}(z)\Phi(t,z)dz$$

for each $\phi\in C^1_c(\mathbf{R}^d\times\mathbf{R}^d\setminus\{0\})$. Deriving both sides in t at $t=0$, we arrive at the expression of the generator of the dynamics of the limiting averaged distribution function f restricted to $v\neq 0$

$$(\partial_t+v\cdot\nabla_x+\sigma|v|)f(t,z)=\sigma\mathcal{K}^*(|\cdot|f(t,x,\cdot))(v)$$

in $\mathcal{D}'((0,+\infty)\times\mathbf{R}^d\times(\mathbf{R}^d\setminus\{0\}))$ where, for $|w|>0$

$$\mathcal{K}^*(\psi)(w)=\int_{|\nu|\le 1}\psi(\mathcal{V}^{-1}(|\nu|,|w|)\tfrac{w}{|w|})\frac{\mathcal{V}^{-1}(|\nu|,|w|)^{d-1}}{|w|^{d-1}}\frac{S(\mathcal{V}^{-1}(|\nu|,|w|))}{S(|w|)}\frac{d\nu}{|\mathbf{B}^d|}$$

On the other hand, for each $\epsilon > 0$

$$\begin{aligned} \operatorname{supp}(f^{in}) \subset \overline{B(0,R)} \times \overline{B(0,R)} \\ \implies \operatorname{supp}((x,v) \mapsto f_\epsilon(t,x,v;C)) \subset \overline{B(0,R(1+t))} \times \overline{B(0,R)} \end{aligned}$$

Since

$$\begin{aligned} \partial_t \int_{\mathbf{R}^d} f_\epsilon(t,z;C)dv + \operatorname{div}_x \int_{\mathbf{R}^d} v f_\epsilon(t,z;C)dv = 0 \\ \frac{d}{dt} \iint_{\mathbf{R}^{2d}} f_\epsilon(t,z;C)dz = 0 \end{aligned}$$

and since $\mathbf{E}f_\epsilon$ is **compactly supported** in $\mathbf{R}^{2d}$, hence **tight**, it is relatively compact for the topology of narrow convergence on $\mathbf{R}^{2d}$, hence

$$\mathbf{E}f_{\epsilon_j} \to F \text{ narrowly} \quad \implies \frac{d}{dt} \iint_{\mathbf{R}^{2d}} \mathbf{E}F(t,z)dz = 0$$

$$\partial_t \int_{\mathbf{R}^d} F(t,z)dv + \operatorname{div}_x \int_{\mathbf{R}^d} vF(t,z)dv = 0$$

Conclusions

•We have studied the process of deceleration of particles in a random media consisting of identical balls centered at the realizations of Poisson point process.

•The radius of the inhomogeneities is small and the slowing rate in each ball is such that the loss of kinetic energy is of order 1 when crossing a ball.

•The average distribution function of the decelerating particles satisfies a linear kinetic equation in the space of positions and kinetic energies.

•The collision integral **does not** satisfy the conservation of mass.

•Instead, the local conservation of mass is an additional constraint imposed on solutions and preserved by the evolution by means of a Lagrange multiplier times a Dirac measure at $v = 0$.