Fredholm Property of the Linearized Boltzmann Operator

Mixture of Polyatomic Gases

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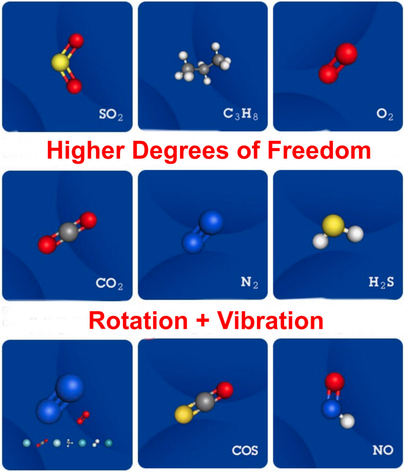
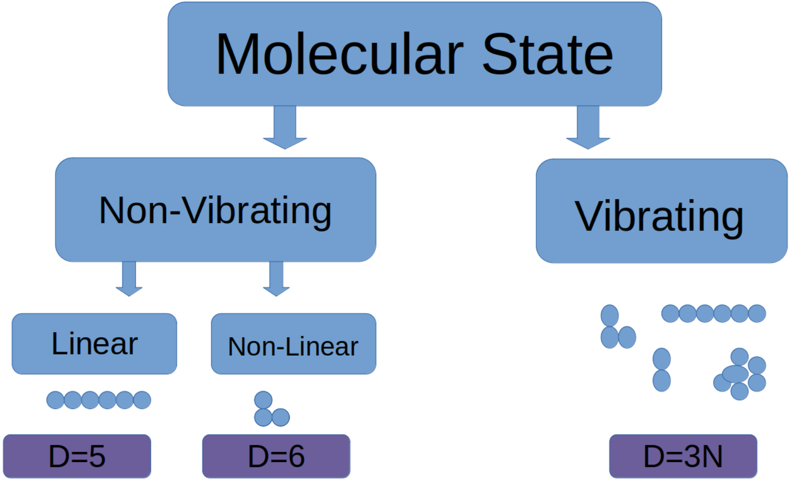
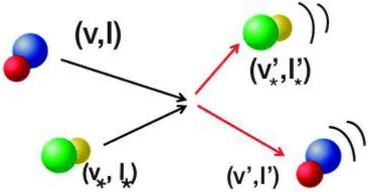
Polyatomic Gases



Degrees of Freedom



Post-Collisonal Variables



$$m_i v + m_j v_* = m_i v' + m_j v'_*$$

$$\frac{m_i}{2} v^2 + \frac{m_j}{2} {v_*}^2 + I + I_* = \frac{m_i}{2} v'^2 + \frac{m_j}{2} {v'_*}^2 + I'_* + I'$$

$$\Downarrow$$

$$m_i v + m_j v_* = m_i v' + m_j v'_*$$

$$\frac{\mu_{ij}}{2}(v - v_*)^2 + I_* + I = E = \frac{\mu_{ij}}{2}(v' - v'_*)^2 + I'_* + I'$$

The Borgnakke-Larsen Procedure[1]

- **Equivalent Formulation of Conservation Equations**

$$m_i v + m_j v_* = m_i v' + m_j v'_*$$

$$\frac{\mu_{ij}}{2}(v - v_*)^2 + I + I_* = \frac{\mu_{ij}}{2}(v' - v'_*)^2 + I'_* + I' = E$$

- **Partition of total energy by the variable** $R \in [0, 1]$

$$\frac{\mu_{ij}}{2}(v' - v'_*)^2 = RE$$

$$I' + I'_* = (1 - R)E$$

- **Partition of internal energy by the variable** $r \in [0, 1]$

$$I' = r(1 - R)E$$

$$I'_* = (1 - r)(1 - R)E$$

[1] Borgnakke, Larsen (1975)

Borgnakke-Larsen Procedure

- **Post-Collisional Velocities**

$$v' = \frac{m_i v + m_j v_*}{m_i + m_j} + \frac{m_j}{m_i + m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma$$

$$v'_* = \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma,$$

with $\sigma \in S^2$

- **Post-Collisional Internal Energies**

$$I' = r(1-R)E$$

$$I'_* = (1-r)(1-R)E$$

Internal Energy

References

- Continuous internal energy
  1. Borgnakke, Larsen (1975).
  2. Bourgat, Desvillettes, Le Tallec, Perthame (1994)
  3. Desvillettes, Monaco, Salvarani (2005)
  4. Baranger, Bisi, Brull, Desvillettes (2018)
  5. Park, Yun (2019)
  6. Gamba, Pavić-Čolić (2020)
  7. Duan, Li (2023)

- Discrete internal energy
  - Giovangigli, Multi-component Flow Modeling (1999)

- Undifferentiated
  - Bisi, Borsoni, Groppi (2022)

Polyatomic Boltzmann Equation (Monospecies)

- Distribution function: $f(t, x, v, I)$
- Boltzmann equation:

$$\boxed{\partial_t f + v.\nabla_x f = Q(f, f)}$$

- Boltzmann Collision Operator[2]:

$$Q(f,f)(v,I) = \int \left(\frac{f' f'_*}{(I' I'_*)^\alpha} - \frac{f f_*}{(I I_*)^\alpha} \right) (I I_*)^\alpha \mathcal{B} \\ (r(1-r))^\alpha (1-R)^{2\alpha} (1-R) R^{1/2} \mathrm{d}R \mathrm{d}r \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*,$$

- $\alpha = \frac{D-5}{2}$
- D : degrees of freedom

[2] Bourgat, Desvillettes, Le Tallec, Perthame (1994)

Polyatomic Boltzmann Equation (Mixtures)

- Boltzmann Equation:

$$\partial_t f_i + v.\nabla_x f_i = \sum_{j=1}^{n} Q_{ij}(f_i, f_j), \quad 1 \le i \le n,$$

- Boltzmann Collision Operator:

$$Q_{ij}(f_i, f_j)(v, I) = \int_{\mathbb{R}^3\times\mathbb{R}_+\times S^2\times(0,1)^2} \left(\frac{f_i' f_{j*}'}{I'^{\alpha_i} I_*'^{\alpha_j}} - \frac{f_i f_{j*}}{I^{\alpha_i} I_*^{\alpha_j}} \right) \times \mathcal{B}_{ij} \times r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j} I^{\alpha_i} I_*^{\alpha_j}(1-R) R^{1/2} \, \mathrm{d}R \mathrm{d}r \mathrm{d}\sigma \mathrm{d}I_* \, \mathrm{d}v_*,$$

- $\alpha_k = \frac{D_k - 5}{2}, \quad k = 1, \cdots n$
- $f_{j*} = f_j(v_*, I_*)$, $f_i' = f_i(v', I')$, and $f_{j*}' = f_j(v_*', I_*')$

The Collision Cross-Section

- Micro-reversibility conditions:

$$\mathcal{B}_{ij}(v, v_*, I, I_*, r, R, \sigma) = \mathcal{B}_{ji}(v_*, v, I_*, I, 1-r, R, \sigma)$$
$$\mathcal{B}_{ij}(v, v_*, I, I_*, r, R, \sigma) = \mathcal{B}_{ij}(v', v'_*, I', I'_*, r', R', \sigma'),$$

- Bounds
  - $\gamma_{ij} \geq 0$ (hard potential/ Maxwell like)[3]

$$\boxed{\Phi_{ij}(r, R)\ E^{\gamma_{ij}/2} \leq \mathcal{B}_{ij}(v, v_*, I, I_*, r, R, \sigma) \leq \Psi_{ij}(r, R) E^{\gamma_{ij}/2}}$$

  - $-1 < \gamma_{ij} < 0$ (soft like potential)

$$\boxed{\mathcal{B}_{ij}(v, v_*, I, I_*, r, R, \sigma) \leq \Psi_{ij}(r, R)\ E^{\gamma_{ij}/2}}$$

[3] Gamba, Pavić-Čolić: On the Cauchy problem for Boltzmann equation modelling a polyatomic gas

Properties of the Cross-section

- Symmetry
  - $\Phi_{ij}(r, R) = \Phi_{ij}(1-r, R)$,
  - $\Psi_{ij}(r, R) = \Psi_{ij}(1-r, R)$
- Boundedness
  - $\Psi_{ij}^2(r, R) r^{\alpha_i+\alpha_j-1-\gamma_{ij}}(1-r)^{\alpha_j-1}(1-R)^{\alpha_i+2\alpha_j-\gamma_{ij}} R \in L^1((0,1)^2)$
  - $\Psi_{ij}^2(r, R) r^{\alpha_i-1}(1-r)^{2\alpha_j-\gamma_{ij}-1}(1-R)^{\alpha_i+2\alpha_j-\gamma_{ij}} R \in L^1((0,1)^2)$

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[4] Gamba, Pavić-Čolić: On the Cauchy problem for Boltzmann equation modelling a polyatomic gas

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- Symmetry
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  - $\Psi_{ij}^2(r,R)r^{\alpha_i-1}(1-r)^{2\alpha_j-\gamma_{ij}-1}(1-R)^{\alpha_i+2\alpha_j-\gamma_{ij}}R \in L^1((0,1)^2)$

MODEL[4]: $\gamma_{ij} < \alpha_i + \alpha_j, \alpha_i > 0,$ and $\alpha_j > 0$

$$\boxed{\mathcal{B}_{ij} = \frac{\mu_{ij}}{2}|v - v_*|^{\gamma_{ij}} + I^{\gamma_{ij}/2} + I_*^{\gamma_{ij}/2}}$$

[4] Gamba, Pavić-Čolić: On the Cauchy problem for Boltzmann equation modelling a polyatomic gas

Linearization

- Global Maxwellian function:

$$M_i(v,I) = \frac{(m_i)^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}\Gamma(\alpha_i+1)} I^{\alpha_i} e^{-\frac{m_i}{2}v^2 - I}$$

- Perturbation: $f_i(t,x,v,I) = M_i(v,I) + M_i^{1/2}(v,I) g_i(t,x,v,I)$

- Linearized Boltzmann operator:

$$[\mathcal{L}g]_i = \sum_{j=1}^{n} M_i^{-\frac{1}{2}} [Q_{ij}(M_i, M_j^{\frac{1}{2}} g_j) + Q_{ij}(M_i^{1/2} g_i, M_j)]$$

- $g = (g_1, \cdots, g_n)$

- Linearized Boltzmann Operator:

$$
\left.
\begin{aligned}
[\mathcal{L}g]_i = \; & \sum_{j=1}^{n} \int M_*^{\frac{1}{2}} M'^{\frac{1}{2}} \left(\frac{I}{I'}\right)^{\frac{\alpha_i}{2}} \left(\frac{I_*}{I'_*}\right)^{\frac{\alpha_j}{2}} g_j(v'_*, I'_*) \tilde{\mathcal{B}}_{ij} \,\mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_* \\
& + \sum_{j=1}^{n} \int M_{j*} M'^{\frac{1}{2}}_{j*} \left(\frac{I}{I'}\right)^{\frac{\alpha_i}{2}} \left(\frac{I_*}{I'_*}\right)^{\frac{\alpha_j}{2}} g_i(v', I') \tilde{\mathcal{B}}_{ij} \,\mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_* \\
& - \sum_{j=1}^{n} \int M^{\frac{1}{2}} M_*^{\frac{1}{2}} g_j(v_*, I_*) \tilde{\mathcal{B}}_{ij} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*
\end{aligned}
\right\} = [\mathcal{K}]_i
$$

$$
- \sum_{j=1}^{n} \int M_{j*} g_i(v, I) \tilde{\mathcal{B}}_{ij} \,\mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_* \;\Big\} = \nu_i
$$

- Write

$$
\mathcal{L}g(v, I) = \underbrace{\mathcal{K}}_{\text{Perturbation operator}} g(v, I) - \underbrace{\nu(v, I)}_{\text{Collision Frequency}} g(v, I)
$$

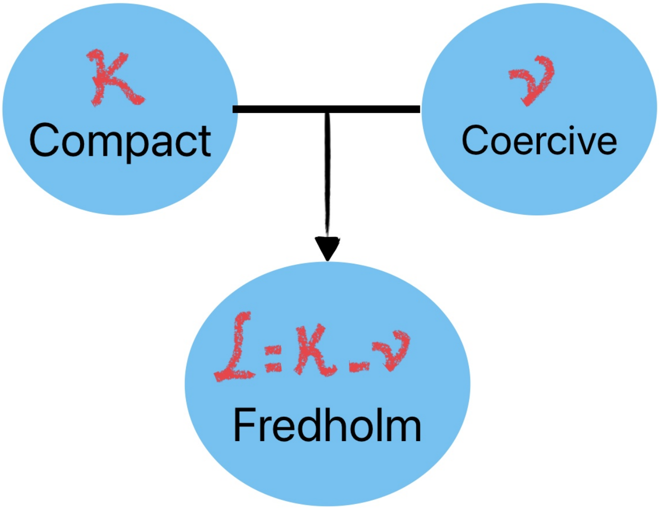
Part I: Fredholm Property of the Linearized Operator

S. Brull, M. Shahine, P. Thieullen (2022). Fredholm Property of the Linearized Boltzmann Operator for a Mixture of Polyatomic Gases. *preprint*.

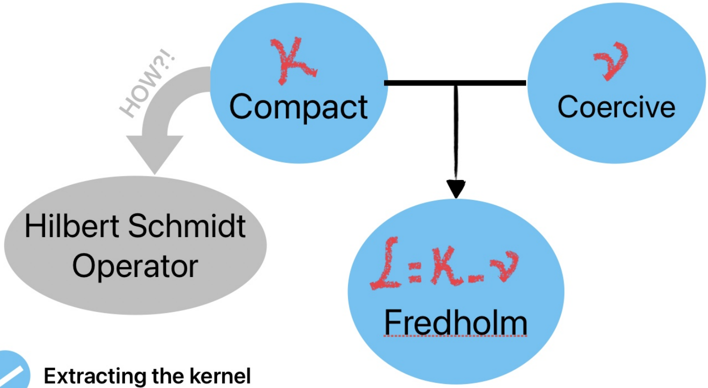
S. Brull, M. Shahine, P. Thieullen (2022). Fredholm Property of the Linearized Boltzmann Operator for a Single Polyatomic Gas. *to appear in Kinetic and Related Models*.

S. Brull, M. Shahine, P. Thieullen (2022). Compactness Property of the Linearized Boltzmann Operator for a Single Diatomic Gas. *Networks and Heterogeneous Media, 17, 847-861*.

Plan



Plan



- ✓ **Extracting the kernel**
- ✓ **Proving square integrability**

Kernel of $\mathcal{K}_1$

- Integral form of $\mathcal{K}_1$

$$[\mathcal{K}_1 g]_i(v, I) = \sum_{j=1}^{n} \int_{\mathbb{R}^3 \times \mathbb{R}_+} g_j\,(v_*, I_*)\, k_1^{ij}\,(v, I, v_*, I_*) \mathrm{d}I_* \mathrm{d}v_*$$

- Kernel of $\mathcal{K}_1$

$$k_1^{ij} = \int_{(0,1)^2 \times S^2} M^{\frac{1}{2}} M_*^{\frac{1}{2}} \mathcal{B}_{ij} r^{\alpha_i} (1-r)^{\alpha_j} (1-R)^{\alpha_i+\alpha_j+1} R^{1/2} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma$$

- Assumption on $\mathcal{B}_{ij} \rightsquigarrow k_1$ is L^2 integrable

Kernel of $\mathcal{K}_2$

- Explicit expression of $\mathcal{K}_2$

$$[\mathcal{K}_2 g]_i = \sum_{j=1}^{n} \int M_{j*}^{\frac{1}{2}} M_i'^{\frac{1}{2}} \left(\frac{I}{I'}\right)^{\frac{\alpha_i}{2}} \left(\frac{I_*}{I_*'}\right)^{\frac{\alpha_j}{2}} \tilde{\mathcal{B}}_{ij}$$
$$g_j\left(\frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2 + I + I_*\right)}\,\sigma,\right.$$
$$\left.(1-R)(1-r)\left[\frac{\mu_{ij}}{2}(v-v_*)^2 + I + I_*\right]\right)$$
$$\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}I_*$$

- Could we express $\mathcal{K}_2$ as follows ?

$$\mathcal{K}_2 g = \int_H g(x,y) k_2(v, I, x, y) J \mathrm{d}y\mathrm{d}x$$

Kernel of $\mathcal{K}_2$

- Define the change of variable:

$$\mathbf{h} : \mathbb{R}^3 \times \mathbb{R}_+ \longmapsto \mathbf{h}(\mathbb{R}^3 \times \mathbb{R}_+) \subset \mathbb{R}^3 \times \mathbb{R}_+$$

$$(v_*, I_*) \longmapsto \begin{cases} x = & \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{(m_i+m_j)} \sqrt{\frac{2R}{\mu_{ij}} \left(\frac{\mu_{ij}}{2} (v - v_*)^2 + I + I_* \right)} \, \sigma \\ y = & (1-R)(1-r)\left[\frac{\mu_{ij}}{2}(v - v_*)^2 + I + I_* \right] \end{cases}$$

- Jacobian:

$$J = \left| \frac{\partial v_* \partial I_*}{\partial x \partial y} \right| = \left(\frac{m_i + m_j}{m_j} \right)^3 \frac{1}{(1-r)(1-R)}$$

Proof

$$(v_*, I_*) \rightsquigarrow \begin{cases} x = \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{(m_i+m_j)} \sqrt{\frac{2R}{\mu_{ij}} \left[\frac{\mu_{ij}}{2}(v - v_*)^2 + I + I_* \right]}\, \sigma \\ y = (1-R)(1-r)\left[\frac{\mu_{ij}}{2}(v - v_*)^2 + I + I_* \right] \end{cases}$$

$$\Bigg\Downarrow dI_* = dE \qquad \boxed{E = \frac{\mu_{ij}}{2}(v - v_*)^2 + I + I_*}$$

$$(v_*, E) \rightsquigarrow \begin{cases} x = \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{(m_i+m_j)} \sqrt{\frac{2R}{\mu_{ij}} E}\, \sigma \\ y = (1-R)(1-r)E \end{cases}$$

Kernel of $\mathcal{K}_2$

- Integral form of $\mathcal{K}_2$

$$[\mathcal{K}_2 g]_i(v,I) = \sum_{j=1}^{n} \int_{\mathbb{R}^3\times\mathbb{R}_+} g_j\,(x,y)\, k_2^{ij}\,(v,I,x,y)\mathrm{d}y\mathrm{d}x$$

- Kernel

$$k_2^{ij}(v,I,x,y) = \int \left[M_*^{\frac{1}{2}} M'^{\frac{1}{2}} \left(\frac{I}{I'}\right)^{\frac{\alpha_i}{2}} \left(\frac{I_*}{I'_*}\right)^{\frac{\alpha_j}{2}} \tilde{\mathcal{B}}_{ij} \right] \circ h^{-1}(x,y)\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma$$

- Kernel of $\mathcal{K}_2$ in $L^2(\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+)$?

L^2 integrability of kernel

- L^2 norm:

$$\left\|k_2^{ij}\right\|_{L^2}^2 = \int_{R^6\times R_+^2} k_2^{ij\,2}(x,y,v,I)\,\mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v$$

$$\leq \int \left[\mathbf{J} M_{j*}^{\frac{1}{2}} M_i'^{\frac{1}{2}} \left(\frac{I}{I'}\right)^{\frac{\alpha_i}{2}} \left(\frac{I_*}{I_*'}\right)^{\frac{\alpha_j}{2}} \tilde{\mathcal{B}}_{ij}\right] \circ h^{-1}(x,y)^2 drdRd\sigma\mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v$$

- Move backwards to (v_*, I_*) (C.O.V: $(x,y)\mapsto(v_*,I_*)$)
- C.O.V: $(I,v)\mapsto(E,v')$:

$$\rightsquigarrow \begin{cases} E = \frac{\mu_{ij}}{2}(v-v_*)^2 + I + I_* \\ v' = \frac{m_i v + m_j v_*}{m_i+m_j} - \frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}E}\,\sigma \end{cases}$$

Work on $\mathcal{K}_3$

$$[\mathcal{K}_3 g]_i(v,I) = \sum_{j=1}^{n} \int_{\mathbb{R}^3\times\mathbb{R}_+} g_j\,(x,y)\; k_3^{ij}\,(v,I,x,y)\mathrm{d}y\mathrm{d}x$$

- Change of variables :

$$\begin{aligned} \mathbb{R}^3 \times \mathbb{R}_+ &\longmapsto \mathbb{R}^3 \times \mathbb{R}_+ \\ (v_*, I_*) &\longmapsto (v', I') \end{aligned}$$

Jacobian

$$\tilde{J} = \left| \frac{\partial v_* \partial I_*}{\partial v' \partial I'} \right| = \left(\frac{m_i + m_j}{m_j} \right)^3 \frac{1}{r(1-R)}$$

- Final requirement:

$$\int_{(0,1)^2} \Psi_{ij}^2(r,R) r^{\alpha_i - 1} (1-r)^{2\alpha_j - \gamma_{ij} - 1} (1-R)^{\alpha_j + 2\alpha_i - \gamma_{ij}} R \mathrm{d}r \mathrm{d}R < \infty$$

Properties

Collision frequency ν

- ν Id is **Coercive** with bounds

$$\nu(v, I) \geq c(|v|^{\gamma} + I^{\frac{\gamma}{2}} + 1)$$

- Collision cross section monotonic $\rightsquigarrow$ Collision frequency monotonic.

Properties

Collision frequency ν

- ν Id is **Coercive** with bounds

$$\nu(v, I) \geq c(|v|^{\gamma} + I^{\frac{\gamma}{2}} + 1)$$

- Collision cross section monotonic $\rightsquigarrow$ Collision frequency monotonic.

Linearized operator $\mathcal{L}$

- Compactness of $\mathcal{K}$ & Coercivity of ν $\rightsquigarrow$ Fredholm alternative of $\mathcal{L}$
- $\mathcal{L}$ is a self-adjoint unbounded operator

$$Dom(\mathcal{L}) = Dom(\nu \mathrm{Id}) = \{f \in L^2(\mathbb{R}^3 \times \mathbb{R}_+) \ : \ \nu f \in L^2\}$$

Monatomic Gases?

$$[\mathcal{K}_2 g]_i = \sum_{j=1}^{n} \iint_{\mathbb{R}^3 \times S^2} {M'_i}^{1/2} {M_{j*}}^{1/2} g_j\left(v'_*\right) \mathcal{B}_{ij}\left(\sigma, v, v_*\right) \mathrm{d}\sigma \mathrm{d}v_*$$

- Change of variable

$$h : \mathbb{R}^3 \mapsto \mathbb{R}^3$$
$$v_* \mapsto x = \frac{m_i v + m_j v_*}{m_i + m_j} - \frac{m_i}{m_i + m_j} \frac{|v - v_*|}{2} \sigma$$

- Jacobian:

$$\mu_j^2 \left| 1 - \frac{\mu_i}{2} \left|\sigma - \sigma'\right|^2 \right|$$

- $\sigma' = \frac{v - v_*}{|v - v_*|}, \quad \sigma = \frac{v' - v'_*}{|v' - v'_*|}$

State of Art

- Single monoatomic gas
  - Grad (1963): Hard Potentials
  - Drange (1975): Soft Potentials
- Mixture of monoatomic gases
  - Boudin, Grec, Pavic, Salvarani (2014)
  - Bernhoff (2022)
- Single polyatomic gas
  - Brull, S., Thieullen (2022): diatomic gases
  - Brull, S., Thieullen (to appear): polyatomic gases
  - Bernhoff (2022): polyatomic gases
  - Borsoni, Boudin, Salvarani (2022): resonant model
- Mixture of polyatomic gases
  - Brull, S., Thieullen (2022): continuous Internal Energy
  - Bernhoff (2022): discrete Internal Energy

Part II: Application

Macroscopic Equations derived from the Boltzmann Equation

Hydrodynamic Limits

Scaled Boltzmann equation

$$\varepsilon\partial_t f_\varepsilon + v.\nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon)$$

$$\Downarrow$$

Incompressible Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla_x) u + \nabla_x p = \nu \Delta u,$$
$$\partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta \theta,$$
$$\nabla_x \cdot u = 0, \quad \nabla_x(\rho + \theta) = 0.$$

Navier stokes equations

- Scaled Boltzmann equation

$$\boxed{\varepsilon \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon)}$$

- Perturbation near equilibrium:

$$\boxed{f_\varepsilon(v, I) = M(v, I) + \varepsilon M(v, I) g_\varepsilon(v, I)}$$

- Linearized equation:

$$\boxed{\varepsilon \partial_t g_\varepsilon + v.\nabla_x g_\varepsilon = \frac{1}{\varepsilon} \mathcal{L}(g_\varepsilon) + \Gamma(g_\varepsilon, g_\varepsilon)}$$

- Kernel:

$$\boxed{\ker \mathcal{L} = M^{1/2} \ \text{span}\{1, v_1, v_2, v_3, \frac{1}{2}|v|^2 + I\}}$$

Main result

Theorem

Assume g_ε converges a.e. to a function g as $\varepsilon \to 0$. Then

$$g = \rho + u.v + \theta(\frac{1}{2}v^2 + I - \alpha - \frac{5}{2}),$$

such that

$$\nabla_x.u = 0, \quad \nabla_x(\rho + \theta) = 0,$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x P = \nu \Delta u,$$
$$\partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta \theta,$$

where ν : viscosity, κ : thermal conductivity.

Idea of the proof:

$$\boxed{\varepsilon\partial_t g_\varepsilon + v.\nabla_x g_\varepsilon = \frac{1}{\varepsilon}\mathcal{L}(g_\varepsilon) + \Gamma(g_\varepsilon, g_\varepsilon)}$$

- multiply by ε, then $\varepsilon \to 0 \Longrightarrow \boxed{g = \rho + v.u + \left(\frac{1}{2}v^2 + I - \alpha - \frac{5}{2}\right)\theta}$
- multiply by M and Mv and integrate over (v, I)

$$\varepsilon\partial_t \langle g_\varepsilon \rangle + \nabla_x \langle v g_\varepsilon \rangle = 0,$$
$$\varepsilon\partial_t \langle v g_\varepsilon \rangle + \nabla_x \langle v \otimes v g_\varepsilon \rangle = 0$$

$\varepsilon \to 0$

$$\nabla_x \langle v g \rangle = 0, \quad \nabla_x \langle v \otimes v g \rangle = 0$$

expression of g

$$\boxed{\nabla_x u = 0} \qquad \boxed{\nabla_x (\rho + \theta) = 0}$$

Incompressibility *Boussinesq*

Idea of the proof:

- Deriving the limiting momentum equation

$$\varepsilon\partial_t \langle v g_\varepsilon \rangle + \nabla_x \langle v \otimes v g_\varepsilon \rangle = 0$$

$$\varepsilon\partial_t \langle v g_\varepsilon \rangle + \nabla_x \left\langle \underbrace{\left(v \otimes v - \frac{1}{3} v^2 \mathbf{I}\right)}_{A(v)} g_\varepsilon \right\rangle + \nabla_x \underbrace{\left\langle \frac{1}{3} v^2 \mathbf{I} g_\varepsilon \right\rangle}_{P_\varepsilon} = 0$$

$$\boxed{\partial_t \langle v g_\varepsilon \rangle + \nabla_x \frac{1}{\varepsilon} \langle A(v) g_\varepsilon \rangle + \nabla_x P_\varepsilon = 0}$$

$$\downarrow \varepsilon \to 0$$

$$\boxed{\partial_t u + \lim_{\varepsilon \to 0} \nabla_x \frac{1}{\varepsilon} \langle A(v) g_\varepsilon \rangle + \nabla_x P = 0}$$

Idea of the proof:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\langle A(v)g_\varepsilon\rangle \overset{\text{Fredholm}}{\doteq} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\langle \mathcal{L}\tilde{A}(v,I)g_\varepsilon\right\rangle \overset{\text{self-adjoint}}{\doteq} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\langle \tilde{A}(v,I)\mathcal{L}g_\varepsilon\right\rangle,$$
$$= \left\langle \tilde{A}(v,I)v.\nabla_x g\right\rangle - \left\langle \tilde{A}(v,I)\Gamma(g,g)\right\rangle$$

$$\boxed{\text{Galilean Invariance of } \mathcal{L} : \exists\, a \text{ such that } \tilde{A}(v,I) = a(|v|,I)A(v)}$$

- $\left\langle \tilde{A}v.\nabla_x g\right\rangle = \boxed{\frac{1}{15}\left(\int a(|v|,I)v^4 M dv dI\right)}\left(\nabla_x u + \nabla_x^T u - \frac{2}{3}div_x u\mathbf{I}\right)$
- $\left\langle \tilde{A}(v,I)\Gamma(g,g)\right\rangle = \frac{1}{2}\left\langle \tilde{A}(v,I)\mathcal{L}(g^2)\right\rangle = \frac{1}{2}\left\langle A(v)g^2\right\rangle = u\otimes u - \frac{1}{3}|u|^2\mathbf{I}$

Transport Coefficients

- Viscosity

$$\nu = \frac{1}{15}\left(\int a(|v|, I)v^4 M dv dI\right)$$

- Thermal conductivity

$$\kappa = \frac{1}{\left(\alpha + \frac{7}{2}\right)} \int b(|v|, I)|B(v, I)|^2 M(v, I) dv dI$$

- Sonine polynomial

$$B(v, I) = v\left(\frac{1}{2}|v|^2 + I - \frac{7}{2} - \alpha\right)$$

Perspectives

- **Fredholm Property of the Linearized Boltzmann Operator**:
  Improving assumptions+ mono-poly mixture
- **Spectral Gap Estimates**:
  Use the monotony property of ν
- **Incompressible Navier-Stokes Equations:**
  Mixture of polyatomic gases

Thank you for your attention!