GLOBAL EXISTENCE OF SMOOTH SOLUTIONS FOR A NON-CONSERVATIVE BITEMPERATURE EULER MODEL*

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Abstract. The bitemperature Euler model describes a crucial step of Inertial Confinement Fusion (ICF) when the plasma is quasineutral while ionic and electronic temperatures remain distinct. The model is written as a first-order hyperbolic system in non-conservative form with partially dissipative source terms. We consider the polytropic case for both ions and electrons with different γ-law pressures. The system does not fulfill the Shizuta-Kawashima condition and the physical entropy, which is a strictly convex function, does not provide a symmetrizer of the system. In this paper we exhibit a symmetrizer to apply the result on the local existence of smooth solutions in several space dimensions. In the one-dimensional case we establish energy and dissipation estimates leading to global existence for small perturbations of equilibrium states.

Key words. non-conservative hyperbolic system, partial dissipation, symmetrization, energy estimates, Euler type model for plasmas

AMS subject classifications. 35L60, 35F55, 35Q31, 76N10, 76W05

1. Introduction.

This paper is devoted to the study of the global existence of smooth solutions near constant equilibrium states for a bitemperature Euler system. This fluid model describes the interaction of a mixture of one species of ions and one species of electrons in thermal nonequilibrium, with applications in the field of Inertial Confinement Fusion (ICF). It was derived from a kinetic model by using a hydrodynamic limit and the Boltzmann entropy. For this kinetic model, a Discrete Velocity Model (DVM) method with an asymptotic preserving discretization toward Euler equations was obtained. The kinetic approach also allows to design numerical schemes for the bitemperature Euler equations. See [1, 5].

We denote by ρ_e and ρ_i the electronic and ionic densities, $\rho = \rho_e + \rho_i$ the total density, m_e and m_i the related masses, c_e and c_i the mass fractions. These variables satisfy

$$\rho_e = m_e n_e = c_e \rho, \quad \rho_i = m_i n_i = c_i \rho, \quad m_e > 0, \quad m_i > 0, \quad c_e + c_i = 1. \tag{1.1}$$

Quasineutrality is assumed, so that the ionization ratio $Z = n_e/n_i$ is a constant. This implies that the electronic and ionic mass fractions are constant and given by

$$c_e = \frac{Z m_e}{m_i + Z m_e}, \quad c_i = \frac{m_i}{m_i + Z m_e}. \tag{1.2}$$

We suppose that the ionic and electronic velocities are equal: $u_e = u_i = u$, and the pressure of each species satisfies a gamma-law with its own γ exponent :

$$p_e = (\gamma_e - 1)\rho_e \varepsilon_e = n_e k_B T_e, \quad p_i = (\gamma_i - 1)\rho_i \varepsilon_i = n_i k_B T_i, \quad \gamma_e > 1, \quad \gamma_i > 1, \tag{1.3}$$

where k_B is the Boltzmann constant ($k_B > 0$), ε_α and T_α represent respectively the internal specific energy and the temperature of species α for $\alpha = e, i$.

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Denoting by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$, the total energies for the particles are defined by

$$\mathcal{E}_\alpha = \rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha |u|^2 = c_\alpha\Big(\rho\varepsilon_\alpha + \frac{1}{2}\rho|u|^2\Big), \quad \alpha = e, i. \tag{1.4}$$

We denote by $\nu \geq 0$ the interaction coefficient between the electronic and ionic temperatures. Physically this coefficient is a complicated function of the electronic and ionic temperatures and of ρ, see the NRL plasma formulary [11]. A rigorous derivation of ν is obtained *via* a kinetic underlying formulation [1]. It gives $\nu(\rho) = K\rho$ where K is a positive constant. This expression of ν implies that more dense is the plasma, faster it reaches the thermal equilibrium. In order to simplify the notation, we assume that ν is a sufficiently smooth function of ρ, denoted by $\nu = \nu(\rho)$, and satisfies $\nu(\rho) > 0$ for $\rho > 0$. In particular, it suffices to assume that $\nu(1) > 0$ in the study of the global existence of smooth solutions for ρ near 1. From the proof of the main theorem, we will see easily that global existence still holds when ν is a smooth function of (T_e, T_i, ρ) and remains positive at an equilibrium point $(T_e, T_i, \rho) = (\bar{T}, \bar{T}, 1)$ for a positive constant $\bar{T}$.

The model consists of two conservative equations for mass and momentum and two non-conservative equations for each energy:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla(p_e + p_i) = 0, \\ \partial_t \mathcal{E}_e + \operatorname{div}\big(u(\mathcal{E}_e + p_e)\big) - u\cdot(c_i \nabla p_e - c_e \nabla p_i) = \rho\nu(T_i - T_e), \\ \partial_t \mathcal{E}_i + \operatorname{div}\big(u(\mathcal{E}_i + p_i)\big) + u\cdot(c_i \nabla p_e - c_e \nabla p_i) = -\rho\nu(T_i - T_e), \end{cases} \tag{1.5}$$

where "$\cdot$" stands for the inner product in $\mathbb{R}^d$. This is a non-conservative hyperbolic system which can be written in the synthetic form

$$\partial_t \mathcal{W} + \sum_{j=1}^{d} \mathcal{C}_j(\mathcal{W}) \partial_{x_j} \mathcal{W} = F(\mathcal{W}). \tag{1.6}$$

Now we introduce

$$\begin{cases} \eta(\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i) = -\displaystyle\sum_{\alpha=e,i} \frac{\rho_\alpha}{b_\alpha} \ln\Big(\frac{(\gamma_\alpha - 1)\rho_\alpha \varepsilon_\alpha}{\rho_\alpha^{\gamma_\alpha}}\Big), \\ \phi(\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i) = \eta(\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i) u, \end{cases} \tag{1.7}$$

where

$$b_\alpha = \frac{(\gamma_\alpha - 1) m_\alpha}{k_B} > 0, \ \ \alpha = e, i. \tag{1.8}$$

It was proved in [1] (see Theorem 2.9) that the functions (η, ϕ) defined in (1.7) are a pair of entropy-entropy flux of (1.5), and η is strictly convex in the set of state space Ω given by

$$\Omega = \big\{(\rho, u, \varepsilon_e, \varepsilon_i) \in \mathbb{R}^{d+3} \,\big|\, \rho > 0,\, \varepsilon_e > 0,\, \varepsilon_i > 0\big\}.$$

Moreover, any smooth solution of the system satisfies the entropy equality

$$\partial_t \eta(\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i) + \operatorname{div}\phi(\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i) = -\frac{\nu\rho}{T_e T_i}(T_e - T_i)^2, \tag{1.9}$$

which is a partially dissipative condition of the system. It is known that the second-order derivative of a strictly convex entropy provides a symmetrizer of a hyperbolic system in conservative form (see [9, 3]). Unfortunately, the equations for $\mathcal{E}_e$ and $\mathcal{E}_i$ in (1.5) are not in conservative form. As already noticed in [2], $\eta''(\mathcal{W})$ is not a symmetrizer of system (1.5). For the sake of completeness we prove this result in the Appendix of the present article.

According to the theory on the symmetrizable hyperbolic system [14, 12, 15], the existence of a symmetrizer is very important to study smooth solutions in Sobolev spaces. Such a symmetrizer for (1.5) is constructed in Section 2 in any space dimension. It implies the local existence of smooth solutions. See $\mathcal{B}_0(\mathcal{V})$ defined in (2.10) and Proposition 2.1.

In order to study global existence, we may introduce the total energy $\mathcal{E}$ and the total pressure p defined by

$$\mathcal{E} = \mathcal{E}_e + \mathcal{E}_i, \quad p = p_e + p_i.$$

From (1.3) and (1.5), we have

$$\mathcal{E} = \frac{p_e}{\gamma_e - 1} + \frac{p_i}{\gamma_i - 1} + \frac{1}{2}\rho|u|^2, \quad p = \rho\big[(\gamma_e - 1)c_e\varepsilon_e + (\gamma_i - 1)c_i\varepsilon_i\big]$$

and

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ \partial_t\mathcal{E} + \operatorname{div}\big(u(\mathcal{E} + p)\big) = 0, \quad t > 0, \ \ x \in \mathbb{R}^d. \end{cases} \tag{1.10}$$

The last equation in (1.10) shows that the total energy is a conservative variable. If $\gamma_e = \gamma_i$, we introduce a total internal specific energy ε by $\varepsilon = c_e\varepsilon_e + c_i\varepsilon_i$. Then

$$\mathcal{E} = \rho\varepsilon + \frac{1}{2}\rho|u|^2, \quad p = (\gamma_e - 1)\rho\varepsilon.$$

Therefore, (1.10) becomes the gas dynamics equations. In this case, system (1.5) is decoupled and contains (1.10). It is known that smooth solutions to the gas dynamics equations blow up in finite time [13, 23]. Hence, global existence is not expected. In physically realistic situations, one can define a weak solution containing shocks. Existence and uniqueness of weak entropy solutions is rather well understood for one-dimensional strictly hyperbolic systems of conservation laws, see [4] and references therein. For systems with non-conservative products, the authors of [8] gave a definition of shocks, but to our knowledge there is no result on the existence of such solutions for (1.5).

In what follows, we consider the Cauchy problem for (1.5) near constant equilibrium states in case $\gamma_e \neq \gamma_i$. Let us introduce

$$\mathcal{V} = \big(\rho, u^T, \varepsilon_e, \varepsilon_i\big)^T.$$

An equilibrium state $\bar{\mathcal{V}}$ is a constant solution of (1.5). We consider in particular an equilibrium state with zero velocity. Let

$$\bar{\mathcal{V}} = (1, 0, \bar{\varepsilon}_e, \bar{\varepsilon}_i)^T,$$

be such an equilibrium state with $\bar{\varepsilon}_e > 0$ and $\bar{\varepsilon}_i > 0$.

System (1.5) is supplemented by an initial condition

$$t = 0: \quad \mathcal{V} = \mathcal{V}_0(x) \stackrel{def}{=} \left(\rho_0(x), u_0^T(x), \varepsilon_{e0}(x), \varepsilon_{i0}(x)\right)^T, \quad x \in \mathbb{R}^d. \tag{1.11}$$

For a positive integer m we denote by $H^m(\mathbb{R}^d)$ the usual Sobolev space equipped with the norm $\|\cdot\|_m$. The result of the global existence of solutions holds in one space dimension and can be stated as follows.

THEOREM 1.1. *Let $d = 1$ and $m \geq 2$. Assume $\mathcal{V}_0 - \bar{\mathcal{V}} \in H^m(\mathbb{R})$ and $\gamma_e \neq \gamma_i$. There are two positive constants c and κ_0 such that if $\|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m \leq \kappa_0$, then the Cauchy problem (1.5) and (1.11) admits a unique global solution $\mathcal{V}$ satisfying $\mathcal{V} - \bar{\mathcal{V}} \in C(\mathbb{R}^+; H^m(\mathbb{R})) \cap C^1(\mathbb{R}^+; H^{m-1}(\mathbb{R}))$. Moreover,*

$$\sup_{t \in \mathbb{R}^+} \|\mathcal{V}(t, \cdot) - \bar{\mathcal{V}}\|_m \leq c \|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m. \tag{1.12}$$

For conservative hyperbolic systems with source terms, the global existence of smooth solutions near constant equilibrium states was proved in [10, 26] in a general framework under two main conditions. A typical example in this framework can be seen in [24, 7] for the gas dynamics equations with damping. The first condition required in [10, 26] is an entropy dissipation near an equilibrium state. It implies in particular an L^2 energy estimate of solutions. The second one is the classical Shizuta-Kawashima condition (SK) at the equilibrium state [22]. Unfortunately, these two conditions are not satisfied by system (1.5). The first condition obviously fails because (1.5) is not a conservative system. However, it is known that (SK) is not a necessary condition for the global existence of smooth solutions. There do exist conservative systems for which global existence holds without this condition. We refer the reader to [27, 6, 19, 17] for examples in which different techniques are employed to avoid condition (SK).

Thus, it is important to establish a global existence result for a class of systems including at least one of these examples. In [16] the authors studied energy estimates of smooth solutions near non-constant equilibrium states for conservative systems. In one space dimension, they obtained global existence for systems violating condition (SK) but admitting a very special structure. This allows them to give a proof of global existence by using only a partially dissipative condition via an entropy dissipation. This situation is different from that of the present paper. In the proof of Theorem 1.1, we not only need a partially dissipative condition but also a dissipation estimate for other variables (see Lemma 3.5). In [21] the authors tried to explore a link between the linear degeneracy of characteristic fields and condition (SK) for conservative systems. Under restrictive conditions, they obtained time-decay estimates of solutions which imply global existence. One can check that the conditions in [16] and [21] are not fulfilled by (1.5) and the systems in [27, 6, 19, 17].

Up to our knowledge, Theorem 1.1 provides a first result on the global existence of smooth solutions for a non-conservative partially dissipative hyperbolic system with source terms without condition (SK). The proof of this theorem is based on the local existence of solutions and uniform energy estimates with respect to time through Lagrangian coordinates. It consists of three steps. The first step concerns an L^2 energy estimate. For this purpose, the entropy equality (1.9) is not sufficient because the system is not in conservative form. We need further to prove equilibrium conditions between the system and the entropy η given in (1.7) at the equilibrium state. The verification of these conditions is very complicated and tedious for (1.5). To avoid this, we turn to consider the Cauchy problem in Lagrangian coordinates where these conditions can be easily checked (see Lemma 3.1). The second step is to establish higher-order energy estimates with a dissipation estimate for $T_e - T_i$. This is a classical step which is done by choosing an appropriate symmetrizer of the system (see Lemma 3.4). In the last step, we prove a dissipation estimate for $(\nabla u, \nabla p)$ (see Lemma 3.5). In view of special structures of the system, these estimates are sufficient to obtain the global existence of solutions in Lagrangian coordinates. Then Theorem 1.1 follows from the equivalence result for the solutions between Eulerian and Lagrangian coordinates. Remark that in the proof of Theorem 1.1, we need to use different independent unknown variables in different energy estimates. The difficulty on the lack of condition (SK) for system (1.5) is overcome by choosing appropriate variables connected by C^∞-diffeomorphisms.

Finally, we point out that there exists a result on the global existence of solutions for partially dissipative hyperbolic systems in non-conservative form which satisfy condition (SK). However, the space dimension is required to be bigger than 3 [20] (see Theorem 2.4). System (1.5) is not included in this framework since it does not satisfy condition (SK). So far, global existence in several space dimensions is an open problem for (1.5).

This paper is organized as follows. In the next section, we first exhibit a symmetrizer to apply the result on the local existence of smooth solutions in several space dimensions. Then we study the structure of the system in one space dimension in Eulerian and Lagrangian coordinates. In particular, we show that system (1.5) does not satisfy condition (SK). We also state a result on the global existence of solutions for the system in Lagrangian coordinates (see Theorem 2.3). Section 3 is devoted to the proof of the energy estimates in the three steps mentioned above. In the last section, we give the proof of Theorem 2.3 and then the proof of Theorem 1.1 by using a result on the equivalence of solutions for the Cauchy problem in Eulerian and Lagrangian coordinates.

2. Study of the bitemperature Euler model.

2.1. Symmetrization of the system. System (1.5) can be written in the form

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ \partial_t \mathcal{E}_e + \operatorname{div}\big(u(\mathcal{E}_e + p_e)\big) - u \cdot (c_i \nabla p_e - c_e \nabla p_i) = \nu(\rho) v, \\ \partial_t \mathcal{E}_i + \operatorname{div}\big(u(\mathcal{E}_i + p_i)\big) + u \cdot (c_i \nabla p_e - c_e \nabla p_i) = -\nu(\rho) v, \quad t > 0, \ \ x \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

with relations (1.1)–(1.4) and (1.8) and

$$v = \rho(T_i - T_e), \quad T_\alpha = b_\alpha \varepsilon_\alpha, \quad \alpha = e, i.$$

Now we write the system with variables $(\rho, u, \varepsilon_e, \varepsilon_i)$. We first remark that

$$\operatorname{div}(\rho u \otimes u) = \rho(u \cdot \nabla) u + u \operatorname{div}(\rho u), \quad p = p_e + p_i. \tag{2.2}$$

Then, for $\rho > 0$, the first two equations in (2.1) give

$$\partial_t u + (u \cdot \nabla) u + \rho^{-1} \nabla p = 0. \tag{2.3}$$

By the definition of $\mathcal{E}_\alpha$ and the first two equations in (2.1) together with (2.3), we have

$$\frac{1}{c_\alpha}\big[\partial_t \mathcal{E}_\alpha + \operatorname{div}(u\mathcal{E}_\alpha)\big] = \frac{1}{2}\rho u \cdot \partial_t u + \frac{1}{2} u \cdot \partial_t(\rho u) + \rho \partial_t \varepsilon_\alpha + \varepsilon_\alpha \partial_t \rho + \operatorname{div}\Big(\frac{1}{2}\rho|u|^2 u + \rho u \varepsilon_\alpha\Big)$$
$$= -\frac{1}{2}\rho u \cdot \big[(u\cdot\nabla)u + \rho^{-1}\nabla p\big] - \frac{1}{2}u \cdot \big[\operatorname{div}(\rho u \otimes u) + \nabla p\big] + \rho\partial_t \varepsilon_\alpha$$
$$- \varepsilon_\alpha \operatorname{div}(\rho u) + \operatorname{div}\Big(\frac{1}{2}\rho|u|^2 u + \rho u \varepsilon_\alpha\Big).$$

Since

$$-\rho u \cdot \big[(u\cdot\nabla)u\big] = -\frac{1}{2}\rho u \cdot \nabla(|u|^2),$$
$$\operatorname{div}\Big(\frac{1}{2}\rho|u|^2 u\Big) = \frac{1}{2}|u|^2 \operatorname{div}(\rho u) + \frac{1}{2}\rho u \cdot \nabla(|u|^2),$$

using (2.2), we obtain

$$-\frac{1}{2}\rho u \cdot \big[(u\cdot\nabla)u + \rho^{-1}\nabla p\big] - \frac{1}{2}u \cdot \big[\operatorname{div}(\rho u \otimes u) + \nabla p\big] + \operatorname{div}\Big(\frac{1}{2}\rho|u|^2 u\Big)$$
$$= -\frac{1}{2}\rho u \cdot \nabla(|u|^2) - u\cdot\nabla p - \frac{1}{2}|u|^2 \operatorname{div}(\rho u) + \operatorname{div}\Big(\frac{1}{2}\rho|u|^2 u)\Big)$$
$$= -u\cdot\nabla p.$$

We also have

$$-\varepsilon_\alpha \operatorname{div}(\rho u) + \operatorname{div}(\rho u \varepsilon_\alpha) = \rho u \cdot \nabla \varepsilon_\alpha.$$

These equalities imply that

$$\frac{1}{c_\alpha}\big[\partial_t \mathcal{E}_\alpha + \operatorname{div}(u\mathcal{E}_\alpha)\big] = \rho\partial_t\varepsilon_\alpha + \rho u \cdot \nabla\varepsilon_\alpha - u\cdot\nabla p.$$

Moreover,

$$\begin{cases} \operatorname{div}(up_e) - u\cdot(c_i\nabla p_e - c_e\nabla p_i) = p_e \operatorname{div} u + c_e u \cdot \nabla p, \\ \operatorname{div}(up_i) + u\cdot(c_i\nabla p_e - c_e\nabla p_i) = p_i \operatorname{div} u + c_i u \cdot \nabla p. \end{cases} \tag{2.4}$$

It follows that

$$\frac{1}{c_e}\big[\partial_t \mathcal{E}_e + \operatorname{div}\big(u(\mathcal{E}_e + p_e)\big) - u\cdot(c_i\nabla p_e - c_e\nabla p_i)\big] = \rho\partial_t\varepsilon_e + \rho u\cdot\nabla\varepsilon_e + \frac{1}{c_e}p_e \operatorname{div} u,$$

$$\frac{1}{c_i}\big[\partial_t \mathcal{E}_i + \operatorname{div}\big(u(\mathcal{E}_i + p_i)\big) + u\cdot(c_i\nabla p_e - c_e\nabla p_i)\big] = \rho\partial_t\varepsilon_i + \rho u\cdot\nabla\varepsilon_i + \frac{1}{c_i}p_i \operatorname{div} u.$$

Finally, by the expression of p_α and the last two equations in (2.1), we obtain

$$\partial_t\varepsilon_e + u\cdot\nabla\varepsilon_e + (\gamma_e - 1)\varepsilon_e \operatorname{div} u = \nu(\rho)(c_e\rho)^{-1}v,$$
$$\partial_t\varepsilon_i + u\cdot\nabla\varepsilon_i + (\gamma_i - 1)\varepsilon_i \operatorname{div} u = -\nu(\rho)(c_i\rho)^{-1}v,$$

which are the equations for ε_e and ε_i. Thus, system (2.1) is equivalent to

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + (u\cdot\nabla)u + \rho^{-1}\nabla p = 0, \\ \partial_t\varepsilon_e + u\cdot\nabla\varepsilon_e + (\gamma_e - 1)\varepsilon_e \operatorname{div} u = \nu(\rho)(c_e\rho)^{-1}v, \\ \partial_t\varepsilon_i + u\cdot\nabla\varepsilon_i + (\gamma_i - 1)\varepsilon_i \operatorname{div} u = -\nu(\rho)(c_i\rho)^{-1}v, \quad t > 0, \ \ x \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

where

$$\begin{cases} p = \rho[c_e(\gamma_e - 1)\varepsilon_e + c_i(\gamma_i - 1)\varepsilon_i], \\ v = \rho(b_i\varepsilon_i - b_e\varepsilon_e). \end{cases} \tag{2.6}$$

Let

$$\mathcal{V} = (\rho, u^T, \varepsilon_e, \varepsilon_i)^T, \quad \varepsilon_1 = c_e(\gamma_e - 1)\varepsilon_e + c_i(\gamma_i - 1)\varepsilon_i,$$

where the superscript T denotes the transpose of a vector. Since $p = \rho\varepsilon_1$ and

$$\rho^{-1}\nabla p = \rho^{-1}\varepsilon_1\nabla\rho + c_e(\gamma_e - 1)\nabla\varepsilon_e + c_i(\gamma_i - 1)\nabla\varepsilon_i,$$

system (2.5) is written in the form

$$\partial_t\mathcal{V} + \sum_{j=1}^{d}\mathcal{B}_j(\mathcal{V})\partial_{x_j}\mathcal{V} = \mathcal{H}(\mathcal{V}), \quad t > 0, \ \ x = (x_1, \cdots, x_d) \in \mathbb{R}^d, \tag{2.7}$$

where

$$\mathcal{B}_j(\mathcal{V}) = \begin{pmatrix} u_j & \rho e_j^T & 0 & 0 \\ \rho^{-1}\varepsilon_1 e_j & u_j I_d & c_e(\gamma_e - 1)e_j & c_i(\gamma_i - 1)e_j \\ 0 & (\gamma_e - 1)\varepsilon_e e_j^T & u_j & 0 \\ 0 & (\gamma_i - 1)\varepsilon_i e_j^T & 0 & u_j \end{pmatrix}, \tag{2.8}$$

and

$$\mathcal{H}(\mathcal{V}) = \begin{pmatrix} 0 \\ 0 \\ \nu(\rho)(c_e\rho)^{-1}v \\ -\nu(\rho)(c_i\rho)^{-1}v \end{pmatrix}, \tag{2.9}$$

with $u = (u_1, \cdots, u_d)^T$, I_d being the unit matrix and $(e_1, \cdots, e_d)$ being the standard basis of $\mathbb{R}^d$.

By a symmetrizer $\mathcal{B}_0(\mathcal{V})$ for system (2.7) we mean that $\mathcal{B}_0(\mathcal{V})$ is a symmetric positive definite matrix such that $\mathcal{B}_0(\mathcal{V})\mathcal{B}_j(\mathcal{W})$ is symmetric for all $j \in \{1, 2, \cdots, d\}$ (see [15]). Now we introduce a diagonal matrix

$$\mathcal{B}_0(\mathcal{V}) = \mathrm{diag}\big(\varepsilon_1\varepsilon_e\varepsilon_i, \rho^2\varepsilon_e\varepsilon_i I_d, c_e\rho^2\varepsilon_i, c_i\rho^2\varepsilon_e\big). \tag{2.10}$$

Obviously, $\mathcal{B}_0(\mathcal{V})$ is symmetric positive definite in Ω. Moreover,

$$\begin{aligned} &\mathcal{B}_0(\mathcal{V})\mathcal{B}_j(\mathcal{V}) \\ &= \begin{pmatrix} u_j\varepsilon_1\varepsilon_e\varepsilon_i & \rho\varepsilon_1\varepsilon_e\varepsilon_i e_j^T & 0 & 0 \\ \rho\varepsilon_1\varepsilon_e\varepsilon_i e_j & \rho^2 u_j\varepsilon_e\varepsilon_i I_d & c_e(\gamma_e - 1)\rho^2\varepsilon_e\varepsilon_i e_j & c_i(\gamma_i - 1)\rho^2\varepsilon_e\varepsilon_i e_j \\ 0 & c_e(\gamma_e - 1)\rho^2\varepsilon_e\varepsilon_i e_j^T & c_e\rho^2 u_j\varepsilon_i & 0 \\ 0 & c_i(\gamma_i - 1)\rho^2\varepsilon_e\varepsilon_i e_j^T & 0 & c_i\rho^2 u_j\varepsilon_e \end{pmatrix} \end{aligned}$$

which is a symmetric matrix. Therefore, $\mathcal{B}_0(\mathcal{V})$ is a symmetrizer and system (2.7) is symmetrizable hyperbolic in the sense of Friedrichs. According to Lax [14] or Kato [12] (see also Majda [15]), for smooth initial data, the Cauchy problem for (2.1) admits a unique smooth solution, locally in time. This result is stated as follows and it holds in any space dimension.

PROPOSITION 2.1. *Let $m > d/2 + 1$ be an integer and $\bar{\varepsilon}_e > 0$ and $\bar{\varepsilon}_i > 0$ be two constants. We suppose that $\mathcal{V}_0 - \bar{\mathcal{V}} \in H^m(\mathbb{R}^d)$ and*

$$\inf_{x\in\mathbb{R}^d} \rho_0(x) > 0, \quad \inf_{x\in\mathbb{R}^d} \varepsilon_{e0}(x) > 0, \quad \inf_{x\in\mathbb{R}^d} \varepsilon_{i0}(x) > 0. \tag{2.11}$$

Then, there exist $T > 0$ and a unique smooth solution $\mathcal{V}$ to the Cauchy problem (1.5) and (1.11). This solution satisfies $\mathcal{V} - \bar{\mathcal{V}} \in C([0,T]; H^m(\mathbb{R}^d)) \cap C^1([0,T]; H^{m-1}(\mathbb{R}^d))$ and

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^d} \rho(t,x) > 0, \quad \inf_{(t,x)\in[0,T]\times\mathbb{R}^d} \varepsilon_e(t,x) > 0, \quad \inf_{(t,x)\in[0,T]\times\mathbb{R}^d} \varepsilon_i(t,x) > 0.$$

Remark 2.2.

Condition $\|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m \le \kappa_0$ in Theorem 1.1 with κ_0 being sufficiently small implies (2.11).

2.2. The system in one space dimension. In one space dimension, systems (2.1) and (2.5) are written as :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t \mathcal{E}_e + \partial_x\big(u(\mathcal{E}_e + p_e)\big) - u(c_i \partial_x p_e - c_e \partial_x p_i) = \nu(\rho) v, \\ \partial_t \mathcal{E}_i + \partial_x\big(u(\mathcal{E}_i + p_i)\big) + u(c_i \partial_x p_e - c_e \partial_x p_i) = -\nu(\rho) v, \quad t > 0, \ \ x \in \mathbb{R} \end{cases} \tag{2.12}$$

and

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t u + u \partial_x u + \rho^{-1} \partial_x p = 0, \\ \partial_t \varepsilon_e + u \partial_x \varepsilon_e + (\gamma_e - 1)\varepsilon_e \partial_x u = \nu(\rho)(c_e \rho)^{-1} v, \\ \partial_t \varepsilon_i + u \partial_x \varepsilon_i + (\gamma_i - 1)\varepsilon_i \partial_x u = -\nu(\rho)(c_i \rho)^{-1} v, \quad t > 0, \ \ x \in \mathbb{R} \end{cases} \tag{2.13}$$

respectively. From (2.6) and (2.13), we further obtain

$$\begin{cases} \partial_t(\rho^2 \varepsilon_e) + \partial_x(\rho^2 \varepsilon_e u) + \gamma_e \rho^2 \varepsilon_e \partial_x u = \dfrac{1}{c_e} \nu(\rho) \rho v, \\ \partial_t(\rho^2 \varepsilon_i) + \partial_x(\rho^2 \varepsilon_i u) + \gamma_i \rho^2 \varepsilon_i \partial_x u = -\dfrac{1}{c_i} \nu(\rho) \rho v, \end{cases}$$

which imply that

$$\begin{cases} \partial_t(\rho p) + \partial_x(\rho p u) + \rho \mu_1 \partial_x u = (\gamma_e - \gamma_i)\nu(\rho)\rho v, \\ \partial_t(\rho v) + \partial_x(\rho v u) + \rho \mu_2 \partial_x u = -\Big(\dfrac{b_i}{c_i} + \dfrac{b_e}{c_e}\Big)\nu(\rho)\rho v, \end{cases} \tag{2.14}$$

where

$$\begin{cases} \mu_1 = \rho\big[c_e \gamma_e(\gamma_e - 1)\varepsilon_e + c_i \gamma_i(\gamma_i - 1)\varepsilon_i\big], \\ \mu_2 = \rho\big(b_i \gamma_i \varepsilon_i - b_e \gamma_e \varepsilon_e\big). \end{cases} \tag{2.15}$$

By (2.6) and the expression of μ_2 above, we see that μ_1 and μ_2 can further be expressed as linear functions of p and v as

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = M \begin{pmatrix} p \\ v \end{pmatrix}, \tag{2.16}$$

where M is a constant invertible matrix given by

$$M = \frac{1}{(m_e c_i + m_i c_e) k_B} \begin{pmatrix} k_B(m_e c_i \gamma_i + m_i c_e \gamma_e) & c_e c_i k_B^2 (\gamma_i - \gamma_e) \\ m_e m_i (\gamma_i - \gamma_e) & k_B(m_e c_i \gamma_e + m_i c_e \gamma_i) \end{pmatrix}. \tag{2.17}$$

By the expression of $\mathcal{B}_j$ given in (2.8), we can calculate the eigenvalues λ_i and the eigenvectors r_i of (2.13). They are given by

$$\lambda_1(\mathcal{V}) = u - a, \quad \lambda_2(\mathcal{V}) = \lambda_3(\mathcal{V}) = u, \quad \lambda_4(\mathcal{V}) = u + a,$$

$$r_1(\mathcal{V}) = \begin{pmatrix} \rho \\ -a \\ (\gamma_e - 1)\varepsilon_e \\ (\gamma_i - 1)\varepsilon_i \end{pmatrix}, \quad r_2(\mathcal{V}) = \begin{pmatrix} 0 \\ 0 \\ -(\gamma_i - 1)c_i \\ (\gamma_e - 1)c_e \end{pmatrix}$$

$$r_3(\mathcal{V}) = \begin{pmatrix} -\rho \\ 0 \\ \varepsilon_e \\ \varepsilon_i \end{pmatrix}, \quad r_4(\mathcal{V}) = \begin{pmatrix} \rho \\ a \\ (\gamma_e - 1)\varepsilon_e \\ (\gamma_i - 1)\varepsilon_i \end{pmatrix},$$

where

$$a(\varepsilon_e, \varepsilon_i) = \sqrt{c_e \gamma_e (\gamma_e - 1)\varepsilon_e + c_i \gamma_i (\gamma_i - 1)\varepsilon_i}.$$

Moreover, by (2.9), we have

$$\mathcal{H}'(\bar{\mathcal{V}}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{b_e}{c_e}\nu(1) & \frac{b_i}{c_e}\nu(1) \\ 0 & 0 & \frac{b_e}{c_i}\nu(1) & -\frac{b_i}{c_i}\nu(1) \end{pmatrix}.$$

It is known that condition (SK) is invariant under a change of unknown variables by a C^1-diffeomorphism [10]. This condition shows a coupling property between the eigenvectors and the source terms of the system. At a given equilibrium state $\bar{\mathcal{V}}$, it means that $\mathcal{H}'(\bar{\mathcal{V}}) r_i(\bar{\mathcal{V}}) \neq 0$ for all $i = 1, 2, 3, 4$. From (3.3), we see easily that $\mathcal{H}'(\bar{\mathcal{V}}) r_3(\bar{\mathcal{V}}) = 0$. This shows that condition (SK) is not satisfied for system (2.13).

2.3. The system in Lagrangian coordinates. Let $(\rho, u) \in C^1(\mathbb{R}^+ \times \mathbb{R})$ satisfying $\rho \geq \text{const} > 0$ in $\mathbb{R}^+ \times \mathbb{R}$ and

$$\partial_t \rho + \partial_x (\rho u) = 0. \tag{2.18}$$

The Euler-Lagrange change of variables from (t, x) to (t', y) is defined by

$$t' = t, \quad dy = \rho dx - \rho u dt,$$

or equivalently for y :

$$y = \int_{X_1(t)}^{x} \rho(t, \xi) d\xi, \quad \text{with } X_1'(t) = u(t, X_1(t)).$$

It is clear that this change of variables is a diffeomorphism from $\mathbb{R}^+ \times \mathbb{R}$ to itself. For simplicity, we use the same notation for unknown variables in Eulerian coordinates (t, x) and in Lagrangian coordinates (t, y).

Consider smooth solutions for (2.12). Let

$$\tau = \rho^{-1}, \quad E_\alpha = \tau \mathcal{E}_\alpha = \frac{1}{2} c_\alpha u^2 + c_\alpha \varepsilon_\alpha, \quad \alpha = e, i. \tag{2.19}$$

Given a first-order partial differential equation

$$\partial_t w + \partial_x z_1 + b \partial_x z_2 = f. \tag{2.20}$$

By (2.18), in Lagrangian coordinates this equation is written equivalently as

$$\partial_t (\tau w) + \partial_y (z_1 - w u) + b \partial_y z_2 = \tau f. \tag{2.21}$$

Applying this to (2.12), we obtain

$$\begin{cases} \partial_t \tau - \partial_y u = 0, \\ \partial_t u + \partial_y p = 0, \\ \partial_t E_e + p_e \partial_y u + c_e u \partial_y p = \nu \tau v, \\ \partial_t E_i + p_i \partial_y u + c_i u \partial_y p = -\nu \tau v. \end{cases} \tag{2.22}$$

Similarly to (2.14), by (2.4), we obtain

$$\begin{cases} \partial_t p + \tau^{-1} \mu_1 \partial_y u = (\gamma_e - \gamma_i) \nu v, \\ \partial_t v + \tau^{-1} \mu_2 \partial_y u = -\Big(\dfrac{b_i}{c_i} + \dfrac{b_e}{c_e} \Big) \nu v, \end{cases} \tag{2.23}$$

where μ_1 and μ_2 are given in (2.16).

Regarding p_α and p as functions of (τ, u, E_e, E_i), we have

$$p_\alpha = \frac{1}{\tau} (\gamma_\alpha - 1) \Big(E_\alpha - \frac{1}{2} c_\alpha u^2 \Big), \quad \alpha = e, i, \quad p = p_e + p_i.$$

Hence, system (2.22) can be written as

$$\partial_t \mathcal{U} + \mathcal{A}(\mathcal{U}) \partial_y \mathcal{U} = \mathcal{G}(\mathcal{U}), \quad t > 0, \ \ y \in \mathbb{R}, \quad \mathcal{U} = (\tau, u, E_e, E_i)^T, \tag{2.24}$$

which is supplemented by an initial condition

$$t = 0 : \quad \mathcal{U} = \mathcal{U}_0(y) \overset{def}{=} \big(\tau_0(y), u_0(y), E_{e0}(y), E_{i0}(y) \big), \quad y \in \mathbb{R}. \tag{2.25}$$

Here,

$$\mathcal{A}(\mathcal{U}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \partial_\tau p & \partial_u p & \partial_{E_e} p & \partial_{E_i} p \\ c_e u \partial_\tau p & p_e + c_e u \partial_u p & c_e u \partial_{E_e} p & c_e u \partial_{E_i} p \\ c_i u \partial_\tau p & p_i + c_i u \partial_u p & c_i u \partial_{E_e} p & c_i u \partial_{E_i} p \end{pmatrix}, \quad \mathcal{G}(\mathcal{U}) = \begin{pmatrix} 0 \\ 0 \\ \nu \tau v \\ -\nu \tau v \end{pmatrix},$$

with

$$\partial_\tau p = -\frac{p}{\tau}, \quad \partial_u p = -\frac{[c_e(\gamma_e - 1) + c_i(\gamma_i - 1)] u}{\tau}, \quad \partial_{E_e} p = \frac{\gamma_e - 1}{\tau}, \quad \partial_{E_i} p = \frac{\gamma_i - 1}{\tau}.$$

Let

$$\bar{\mathcal{U}} = (1, 0, \bar{E}_e, \bar{E}_i)^T,$$

which is an equilibrium state of (2.24) with $\bar{E}_e > 0$ and $\bar{E}_i > 0$. The result of global existence of solutions to (2.24) and (2.25) can be stated as follows.

THEOREM 2.3. *Let $m \geq 2$ and $\mathcal{U}_0 - \bar{\mathcal{U}} \in H^m(\mathbb{R})$. Assume $\gamma_e \neq \gamma_i$. There are two positive constants c and κ_1 such that if $\|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m \leq \kappa_1$, then the Cauchy problem (2.24) and (2.25) admits a unique global solution $\mathcal{U}$ satisfying $\mathcal{U} - \bar{\mathcal{U}} \in C(\mathbb{R}^+; H^m(\mathbb{R})) \cap C^1(\mathbb{R}^+; H^{m-1}(\mathbb{R}))$. Moreover,*

$$\sup_{t\in\mathbb{R}^+} \|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|_m^2 + \int_0^{+\infty} \big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\big)\,dt' \leq c\|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m^2. \tag{2.26}$$

3. Energy estimates in Lagrangian coordinates. We study energy estimates for the Cauchy problem (2.24) and (2.25). Let $m \geq 2$ be an integer and $T > 0$ such that the local smooth solution $\mathcal{U}$ is defined on time interval $[0,T]$. We denote by $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|\cdot\|_l$ the usual norms of $L^2(\mathbb{R})$, $L^\infty(\mathbb{R})$ and $H^l(\mathbb{R})$ for $l \in \mathbb{N}$, respectively. We also denote

$$\mathcal{U}_T = \max_{t\in[0,T]} \|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|_m.$$

We consider a smooth solution $\mathcal{U}$ near $\bar{\mathcal{U}}$, namely, $\mathcal{U}_T$ is small. In the proof below, we denote by $C > 0$ and $c_0 > 0$ generic constants independent of t and T.

The global existence of smooth solutions to (2.24) and (2.25) will be proved in the three steps shown in Introduction.

3.1. An L^2 estimate. We first look at the entropy equality (1.9) in Lagrangian coordinates. From (2.6), we have

$$\begin{cases} \varepsilon_e = \dfrac{(m_i p - c_i k_B v)\tau}{(\gamma_e - 1)(m_e c_i + m_i c_e)}, \\ \varepsilon_i = \dfrac{(m_e p + c_e k_B v)\tau}{(\gamma_i - 1)(m_e c_i + m_i c_e)}, \end{cases}$$

which are strictly positive in a neighborhood of $v = 0$ when $\tau > 0$ and $p > 0$. It follows from the definition of b_α, T_α and v that

$$-\frac{\nu\rho}{T_e T_i}(T_i - T_e)^2 = -\nu_1 \rho v^2,$$

where $\nu_1 = \nu_1(\tau, p, v)$ given by

$$\nu_1 = \frac{k_B^2 (m_e c_i + m_i c_e)^2 \nu}{m_e m_i (m_i p - c_i k_B v)(m_e p + c_e k_B v)}. \tag{3.1}$$

It is clear that, for all $\tau > 0$ and $p > 0$, $\nu_1 > 0$ in a neighborhood of $v = 0$. We introduce a new variable

$$s = \tau\eta = -\sum_{\alpha=e,i} \frac{c_\alpha}{b_\alpha} \ln\Big[\Big(\frac{(\gamma_\alpha - 1)\tau^{\gamma_\alpha - 1}}{c_\alpha^{\gamma_\alpha}}\Big)\Big(E_\alpha - \frac{c_\alpha}{2}u^2\Big)\Big],$$

which is a function of variable $\mathcal{U}$. According to the equivalence of equations (2.20) and (2.21) in two coordinates, the entropy equality (1.9) in variables (t,y) becomes

$$\partial_t s = -\nu_1 v^2, \tag{3.2}$$

which means that s is an entropy of system (2.24) with 0 as its entropy-flux.

Recall that an equilibrium state with zero velocity is of the from

$$\bar{\mathcal{V}} = (1, 0, \bar{\varepsilon}_e, \bar{\varepsilon}_i)^T.$$

By definition, $\bar{\mathcal{V}}$ is an equilibrium state for (2.7) if $\mathcal{H}(\bar{\mathcal{V}}) = 0$. Since $\nu > 0$, by the definition of $\mathcal{H}$ and v, we have

$$b_e\bar{\varepsilon}_e = b_i\bar{\varepsilon}_i. \tag{3.3}$$

Combining this with (2.19) yields

$$\frac{b_e\bar{E}_e}{c_e} = \frac{b_i\bar{E}_i}{c_i} \stackrel{def}{=} \bar{E}_* > 0.$$

LEMMA 3.1. *For all $\mathcal{U}$ in the domain under consideration, it holds*

$$\nabla s(\bar{\mathcal{U}})\mathcal{G}(\mathcal{U}) = 0 \tag{3.4}$$

and

$$\nabla s(\bar{\mathcal{U}})\mathcal{A}(\mathcal{U}) = \nabla \mathcal{F}(\mathcal{U}), \tag{3.5}$$

where

$$\mathcal{F}(\mathcal{U}) = \frac{1}{\bar{E}_*}up - k_B\Big(\frac{c_e}{m_e} + \frac{c_i}{m_i}\Big)u.$$

Proof. A straightforward calculation gives

$$-\partial_\tau s(\mathcal{U}) = \frac{k_B}{\tau}\Big(\frac{c_e}{m_e} + \frac{c_i}{m_i}\Big),$$
$$-\partial_u s(\mathcal{U}) = -\Big(\frac{c_e}{b_e\varepsilon_e} + \frac{c_i}{b_i\varepsilon_i}\Big)u,$$
$$-\partial_{E_e} s(\mathcal{U}) = \frac{1}{b_e\varepsilon_e}, \quad -\partial_{E_i} s(\mathcal{U}) = \frac{1}{b_i\varepsilon_i},$$

where

$$\varepsilon_\alpha = \frac{1}{c_\alpha}E_\alpha - \frac{1}{2}u^2, \quad \alpha = e, i.$$

Therefore,

$$-\partial_\tau s(\bar{\mathcal{U}}) = k_B\Big(\frac{c_e}{m_e} + \frac{c_i}{m_i}\Big),$$
$$-\partial_u s(\bar{\mathcal{U}}) = 0,$$
$$-\partial_{E_e} s(\bar{\mathcal{U}}) = -\partial_{E_i} s(\bar{\mathcal{U}}) = \frac{1}{\bar{E}_*}.$$

Hence, it is easy to check that (3.4) and (3.5) are satisfied. ∎

LEMMA 3.2. *In a neighborhood of $\bar{\mathcal{U}}$, it holds*

$$\|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|^2 + \int_0^t \|v(t',\cdot)\|^2 dt' \le C\|\mathcal{U}_0 - \bar{\mathcal{U}}\|^2, \quad \forall\, t \in [0, T]. \tag{3.6}$$

Proof. We introduce

$$S(\mathcal{U}) = s(\mathcal{U}) - s(\bar{\mathcal{U}}) + \nabla s(\bar{\mathcal{U}})(\mathcal{U} - \bar{\mathcal{U}}).$$

Since η is a strictly convex entropy for (2.1), by a result in [25], s is a strictly convex entropy for (2.24). Hence, by Taylor formula, in a neighborhood of $\bar{\mathcal{U}}$, these exist two constants $c_2 \geq c_1 > 0$ such that

$$c_1 |\mathcal{U} - \bar{\mathcal{U}}|^2 \leq S(\mathcal{U}) \leq c_2 |\mathcal{U} - \bar{\mathcal{U}}|^2.$$

Using (2.24) and (3.2), we have

$$\partial_t S - \nabla s(\bar{\mathcal{U}}) \mathcal{A}(\mathcal{U}) \partial_y \mathcal{U} = -\nu_1 v^2 - \nabla s(\bar{\mathcal{U}}) \mathcal{G}(\mathcal{U}).$$

It follows from Lemma 3.1 that

$$\partial_t S - \partial_y \mathcal{F}(\mathcal{U}) = -\nu_1 v^2.$$

In a neighborhood of $\bar{\mathcal{U}}$, there is a constant $\bar{\nu}_1 > 0$ such that $\nu_1 \geq \bar{\nu}_1$. Thus, integrating this equality over $[0, t] \times \mathbb{R}$ with $t \in [0, T]$, we obtain (3.6). ∎

3.2. Higher-order energy estimates. Let $U = (u, p, v, s)^T$. We use variable U in higher-order energy estimates. From (2.22), (2.23), and (3.2), we have

$$\begin{cases} \partial_t u + \partial_y p = 0, \\ \partial_t p + \tau^{-1} \mu_1 \partial_y u = -(\gamma_i - \gamma_e) \nu v, \\ \partial_t v + \tau^{-1} \mu_2 \partial_y u = -b \nu v, \\ \partial_t s = -\nu_1 v^2, \qquad t > 0, \ \ x \in \mathbb{R}, \end{cases} \tag{3.7}$$

where ν_1 is defined in (3.1), μ_1 and μ_2 are defined in (2.16) and (2.17), and

$$b = \frac{b_i}{c_i} + \frac{b_e}{c_e} > 0.$$

In particular, μ_1 and μ_2 are linear functions of p and v. This system can be written as

$$\partial_t U + A(U) \partial_y U = G(U), \quad t > 0, \ \ x \in \mathbb{R}, \tag{3.8}$$

where

$$A(U) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \tau^{-1} \mu_1 & 0 & 0 & 0 \\ \tau^{-1} \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G(U) = \begin{pmatrix} 0 \\ -(\gamma_i - \gamma_e) \nu v \\ -b \nu v \\ -\nu_1 v^2 \end{pmatrix},$$

and τ is regarded as a function of U. By the definition in (2.6), the equilibrium state for U is $\bar{U} = (0, \bar{p}, \bar{v}, \bar{s})$ with

$$\bar{p} = (\gamma_e - 1) \bar{E}_e + (\gamma_i - 1) \bar{E}_i > 0, \quad \bar{v} = 0, \quad \bar{s} = s(\bar{\mathcal{U}}).$$

We first prove the following useful property.

LEMMA 3.3. *Let $\delta(p,v)$ be defined by*

$$\delta(p,v) = b\mu_1(p,v) - (\gamma_i - \gamma_e)\mu_2(p,v). \tag{3.9}$$

There is a constant $\bar{\delta} > 0$ such that $\delta(p,v) \geq \bar{\delta}$ in a neighborhood of $(\bar{p}, 0)$.

Proof. By continuity, it is sufficient to prove that $\delta(\bar{p},0) > 0$. From (1.2), we have

$$\frac{c_e}{c_i} = \frac{Zm_e}{m_i}.$$

It follows from the definition of b_α in (1.8) that

$$\frac{b_i c_e(\gamma_e - 1)}{c_i} = Zb_e(\gamma_i - 1), \quad \frac{b_e c_i(\gamma_i - 1)}{c_e} = \frac{b_i}{Z}(\gamma_e - 1).$$

Since $\bar{\rho} = 1$, from (2.15) we obtain

$$\delta(\bar{p},0) = \left(\gamma_i - 1 + \frac{\gamma_e - 1}{Z}\right)(Z\gamma_e b_e \bar{\varepsilon}_e + \gamma_i b_i \bar{\varepsilon}_i) - (\gamma_i - \gamma_e)(\gamma_i b_i \bar{\varepsilon}_i - \gamma_e b_e \bar{\varepsilon}_e).$$

Using the fact that $b_e \bar{\varepsilon}_e = b_i \bar{\varepsilon}_i > 0$ (see (3.3), $\delta(\bar{p},0) > 0$ if and only if

$$\left(\gamma_i - 1 + \frac{\gamma_e - 1}{Z}\right)(Z\gamma_e + \gamma_i) > (\gamma_i - \gamma_e)^2,$$

or equivalently,

$$\gamma_e Z(\gamma_i - 1) + \gamma_i(\gamma_e - 1) > 0.$$

Lemma 3.3 is proved since $Z > 0$, $\gamma_i > 1$ and $\gamma_e > 1$. ◻

LEMMA 3.4. *Let the conditions of Theorem 2.3 hold. If $\|\mathcal{U} - \bar{\mathcal{U}}\|_m$ is sufficiently small, for all $t \in [0,T]$, we have*

$$\begin{aligned} &\|\mathcal{U}(t,\cdot) - \bar{\mathcal{U}}\|_m^2 + \int_0^t \|v(t',\cdot)\|_m^2 dt' \\ &\leq C\|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m^2 + C\int_0^t \left(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\right)\|\mathcal{U} - \bar{\mathcal{U}}\|_m dt'. \end{aligned} \tag{3.10}$$

Proof. Let

$$A_0(U) = \begin{pmatrix} \mu_0 & 0 & 0 & 0 \\ 0 & \frac{b\mu_2}{\gamma_i - \gamma_e} & -\mu_2 & 0 \\ 0 & -\mu_2 & \mu_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\mu_0(\tau,p,v) = \frac{1}{(\gamma_i - \gamma_e)\tau}\mu_2(p,v)\delta(p,v),$$

and $\delta(p,v)$ is defined in (3.9). By Lemma 3.3, in a neighborhood of $\bar{\mathcal{U}}$, there are positive constants $\bar{\mu}_1$, $\bar{\mu}_2$ and $\bar{\mu}_0$ such that

$$\mu_1(p,v) \geq \bar{\mu}_1, \quad (\gamma_i - \gamma_e)\mu_2(p,v) \geq \bar{\mu}_2, \quad \mu_0(\tau,p,v) \geq \bar{\mu}_0.$$

Then it is easy to check that, in a neighborhood of $\bar{\mathcal{U}}$, $A_0(U)$ is a symmetrizer of system (3.8), namely, $A_0(U)$ is symmetric positive definite and $A_0(U)A(U)$ is symmetric. In particular,

$$A_0(U)A(U) = \begin{pmatrix} 0 & \mu_0 & 0 & 0 \\ \mu_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_0(U)G(U) = \begin{pmatrix} 0 \\ 0 \\ -\delta\nu v \\ -\nu_1 v^2 \end{pmatrix}.$$

Let $1 \le k \le m$ be an integer. We denote $U_k = \partial_y^k U$. From (3.8), we have

$$\partial_t U_k + A(U)\partial_y U_k = \partial_y^k G(U) + J_k, \tag{3.11}$$

where

$$J_k = A(U)\partial_y U_k - \partial_y^k (A(U)\partial_y U).$$

Taking the inner product of (3.11) with $A_0(U)U_k$ in $L^2(\mathbb{R})$, we obtain the Friedrichs energy equality

$$\begin{aligned} \frac{d}{dt}\big\langle A_0(U)U_k, U_k\big\rangle = 2\big\langle A_0(U)\partial_y^k G(U), U_k\big\rangle + 2\big\langle A_0(U)J_k, U_k\big\rangle \\ + \big\langle \operatorname{div}\vec{A}(U)U_k, U_k\big\rangle, \end{aligned} \tag{3.12}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\mathbb{R})$ and

$$\operatorname{div}\vec{A}(U) = \partial_t A_0(U) + \partial_y \tilde{A}(U), \quad \tilde{A} = A_0 A.$$

By the definition of A_0 and $\tilde{A}$, we have

$$\operatorname{div}\vec{A}(U) = \begin{pmatrix} \partial_t\mu_0 & \partial_y\mu_0 & 0 & 0 \\ \partial_y\mu_0 & \frac{b}{\gamma_i-\gamma_e}\partial_t\mu_2 & -\partial_t\mu_2 & 0 \\ 0 & -\partial_t\mu_2 & \partial_t\mu_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned} \partial_y\mu_0(U) &= \mu_0'(U)\partial_y U, \\ \partial_t\mu_i(U) &= \mu_i'(U)\partial_t U = \mu_i'(U)\big(G(U) - A(U)\partial_y U\big), \quad i = 0, 1, 2. \end{aligned}$$

Since $G(U) = O(v)$ and the imbedding from $H^m(\mathbb{R})$ to $W^{1,\infty}(\mathbb{R})$ is continuous, we obtain

$$\big\langle \operatorname{div}\vec{A}(U)U_k, U_k\big\rangle \le C\big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|\partial_y v\|_{m-1}^2\big)\|U - \bar{U}\|_m. \tag{3.13}$$

Next, a direct calculation yields

$$A_0(U)J_k = \begin{pmatrix} 0 \\ \Big[\tau^{-1}\big(\frac{b}{\gamma_i-\gamma_e}\mu_1 - \mu_2\big)\partial_y^{k+1}u - \partial_y^k\big(\tau^{-1}\big(\frac{b}{\gamma_i-\gamma_e}\mu_1 - \mu_2\big)\partial_y u\big)\Big]\mu_2 \\ \mu_2\partial_y^k(\tau^{-1}\mu_1\partial_y u) - \mu_1\partial_y^k(\tau^{-1}\mu_2\partial_y u) \\ 0 \end{pmatrix}.$$

By the Moser-type inequalities [15], we have

$$2\big\langle A_0(U)J_k, U_k\big\rangle \le C\big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|\partial_y v\|_{m-1}^2\big)\|U-\bar U\|_m. \tag{3.14}$$

Moreover,

$$\partial_y^k G(U) = \begin{pmatrix} 0 \\ -(\gamma_i-\gamma_e)\nu\partial_y^k v \\ -b\nu\partial_y^k v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (\gamma_i-\gamma_e)\big(\nu\partial_y^k v - \partial_y^k(\nu v)\big) \\ b\nu\partial_y^k v - b\partial_y^k(\nu v) \\ -\partial_y^k(\nu_1 v^2) \end{pmatrix} \stackrel{def}{=} G_1 + G_2,$$

with

$$A_0(U)G_1 = \begin{pmatrix} 0 \\ 0 \\ -\delta\nu\partial_y^k v \\ 0 \end{pmatrix}$$

and

$$A_0(U)G_2 = \begin{pmatrix} 0 \\ a_{22}(\gamma_i-\gamma_e)\big(\nu\partial_y^k v - \partial_y^k(\nu v)\big) - \mu_2\big[b\nu\partial_y^k v - b\partial_y^k(\nu v)\big] \\ \mu_1\big[b\nu\partial_y^k v - b\partial_y^k(\nu v)\big] - \mu_2(\gamma_i-\gamma_e)\big(\nu\partial_y^k v - \partial_y^k(\nu v)\big) \\ -\partial_y^k(\nu_1 v^2) \end{pmatrix},$$

where

$$a_{22} = \frac{b\mu_2}{\gamma_i-\gamma_e}.$$

These equalities imply that

$$A_0(U)G_1\cdot U_k = -\delta\nu|\partial_y^k v|^2$$

and

$$\begin{aligned} A_0(U)G_2\cdot U_k =& \Big(a_{22}(\gamma_i-\gamma_e)\big(\nu\partial_y^k v - \partial_y^k(\nu v)\big) - \mu_2\big[b\nu\partial_y^k v - b\partial_y^k(\nu v)\big]\Big)\partial_y^k u \\ &+ \Big(\mu_1\big[b\nu\partial_y^k v - b\partial_y^k(\nu v)\big] - \mu_2(\gamma_i-\gamma_e)\big(\nu\partial_y^k v - \partial_y^k(\nu v)\big)\Big)\partial_y^k p \\ &- \partial_y^k(\nu_1 v^2)\partial_y^k s. \end{aligned} \tag{3.15}$$

Observe that each of three terms on the right-hand side of (3.15) is quadratic in variables (u,p,v) with coefficients depending on derivatives of $U-\bar U$ up to order m. Moreover, using Lemma 3.3, we have $\delta\nu \ge \bar\delta\bar\nu$ in a neighborhood of $\bar{\mathcal U}$, where $\bar\nu > 0$ is a constant. Thus, the Moser-type inequalities imply that

$$\begin{aligned} &\big\langle A_0(U)\partial_y^k G(U), U_k\big\rangle + \bar\delta\bar\nu\|\partial_y^k v\|^2 \\ &\le C\big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\big)\|U-\bar U\|_m. \end{aligned} \tag{3.16}$$

Since $A_0(U)$ is positive definite, $\big\langle A_0(U)U_k, U_k\big\rangle$ is equivalent to $\|U_k\|^2$. Combining (3.12)-(3.16) and integrating (3.12) over $[0,t]$ with $t\in[0,T]$, we have

$$\|U_k\|^2 + \int_0^t \|\partial_y^k v(t',\cdot)\|^2 dt'$$

$$\le C\|U_0-\bar U\|_m^2 + C\int_0^t \big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\big)\|U-\bar U\|_m dt',$$

where U_0 is the initial data of U. Finally, the change of variables $U \longmapsto \mathcal{U}$ is a C^∞-diffeomorphism in a neighborhood of $\bar{U}$. Then, $\|U-\bar{U}\|_l$ is equivalent to $\|\mathcal{U}-\bar{\mathcal{U}}\|_l$ for all $l \in \mathbb{N}$. Summing up this inequality for all $k = 1, 2, \cdots, m$, and using Lemma 3.2, we obtain (3.10). ∎

3.3. Dissipation estimates.

LEMMA 3.5. *Let the conditions of Theorem 2.3 hold. If $\|\mathcal{U}-\bar{\mathcal{U}}\|_m$ is sufficiently small, for all $t \in [0,T]$, we have*

$$\begin{aligned}&\int_0^t \big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2\big)dt' \\ &\le C\|\mathcal{U}_0-\bar{\mathcal{U}}\|_m^2 + C\int_0^t \big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\big)\|\mathcal{U}-\bar{\mathcal{U}}\|_m dt'.\end{aligned} \tag{3.17}$$

Proof. Let k be an integer with $0 \le k \le m-1$. Applying ∂_y^k to the first three equations in (3.7) yields

$$\begin{cases}\partial_t\partial_y^k u + \partial_y^{k+1} p = 0,\\ \partial_t\partial_y^k p + \tau^{-1}\mu_1\partial_y^{k+1}u = \tau^{-1}\mu_1\partial_y^{k+1}u - \partial_y^k\big(\tau^{-1}\mu_1\partial_y u\big) - (\gamma_i-\gamma_e)\partial_y^k(\nu v),\\ \partial_t\partial_y^k v + \tau^{-1}\mu_2\partial_y^{k+1}u = \tau^{-1}\mu_2\partial_y^{k+1}u - \partial_y^k\big(\tau^{-1}\mu_2\partial_y u\big) - b\partial_y^k(\nu v).\end{cases} \tag{3.18}$$

We multiply the third equation in (3.18) by $(\gamma_i-\gamma_e)$ and take the inner product with $\partial_y^{k+1}u$ in $L^2(\mathbb{R})$. Using $(\gamma_i-\gamma_e)\tau^{-1}\mu_2 \ge 3c_0$ it yields

$$\begin{aligned}3c_0\|\partial_y^{k+1}u\|^2 \le &-(\gamma_i-\gamma_e)\big\langle \partial_t\partial_y^k v, \partial_y^{k+1}u\big\rangle \\ &+ (\gamma_i-\gamma_e)\big\langle \tau^{-1}\mu_2\partial_y^{k+1}u - \partial_y^k\big(\tau^{-1}\mu_2\partial_y u\big) - b\partial_y^k(\nu v), \partial_y^{k+1}u\big\rangle.\end{aligned}$$

By the Young inequality and the Moser-type inequalities, the last term above is bounded by

$$c_0\|\partial_y^{k+1}u\|^2 + C\|v\|_m^2 + C\|\partial_y u\|_{m-1}^2\|\mathcal{U}-\bar{\mathcal{U}}\|_m.$$

Moreover, by the first equation in (3.18) and an integration by parts, we have

$$\begin{aligned}-(\gamma_i-\gamma_e)\big\langle \partial_t\partial_y^k v, \partial_y^{k+1}u\big\rangle &= -(\gamma_i-\gamma_e)\frac{d}{dt}\big\langle \partial_y^k v, \partial_y^{k+1}u\big\rangle + (\gamma_i-\gamma_e)\big\langle \partial_y^{k+1}v, \partial_y^{k+1}p\big\rangle \\ &\le -(\gamma_i-\gamma_e)\frac{d}{dt}\big\langle \partial_y^k v, \partial_y^{k+1}u\big\rangle + \beta\|\partial_y^{k+1}p\|^2 + C\|v\|_m^2,\end{aligned}$$

where $\beta > 0$ is a small constant to be chosen. This implies that

$$\begin{aligned}2c_0\|\partial_y^{k+1}u\|^2 \le &-(\gamma_i-\gamma_e)\frac{d}{dt}\big\langle \partial_y^k v, \partial_y^{k+1}u\big\rangle \\ &+ \beta\|\partial_y^{k+1}p\|^2 + C\|v\|_m^2 + C\|\partial_y u\|_{m-1}^2\|\mathcal{U}-\bar{\mathcal{U}}\|_m.\end{aligned} \tag{3.19}$$

Similarly, taking the inner product of the first equation in (3.18) with $\partial_y^{k+1}p$ in $L^2(\mathbb{R})$ and using an integration by parts, we have

$$\|\partial_y^{k+1}p\|^2 = -\frac{d}{dt}\big\langle \partial_y^k u, \partial_y^{k+1}p\big\rangle - \big\langle \partial_y^{k+1}u, \partial_y^k\partial_t p\big\rangle.$$

By the second equation in (3.18), we obtain as above

$$\begin{aligned}-\big\langle \partial_y^{k+1}u, \partial_y^k\partial_t p\big\rangle &= \big\langle \partial_y^k(\tau^{-1}\mu_1\partial_y u) - \tau^{-1}\mu_1\partial_y^{k+1}u + (\gamma_i-\gamma_e)\partial_y^k(\nu v), \partial_y^{k+1}u\big\rangle \\ &\quad + \big\langle \tau^{-1}\mu_1\partial_y^{k+1}u, \partial_y^{k+1}u\big\rangle \\ &\le C\|\partial_y^{k+1}u\|^2 + C\|v\|_m^2 + C\|\partial_y u\|_{m-1}^2\|\mathcal{U}-\bar{\mathcal{U}}\|_m.\end{aligned}$$

Hence,

$$\|\partial_y^{k+1}p\|^2 \le -\frac{d}{dt}\langle \partial_y^k u, \partial_y^{k+1}p\rangle + C\|\partial_y^{k+1}u\|^2 + C\|v\|_m^2 + C\|\partial_y u\|_{m-1}^2\|\mathcal{U}-\bar{\mathcal{U}}\|_m. \tag{3.20}$$

Combining (3.19) and (3.20), and choosing $\beta > 0$ to be sufficiently small, it yields

$$\begin{aligned} c_0\|\partial_y^{k+1}u\|^2 + \beta\|\partial_y^{k+1}p\|^2 \le& -\frac{d}{dt}\big[(\gamma_i-\gamma_e)\langle \partial_y^k v, \partial_y^{k+1}u\rangle + 2\beta\langle \partial_y^k u, \partial_y^{k+1}p\rangle\big] \\ &+ C\|v\|_m^2 + C\|\partial_y u\|_{m-1}^2\|\mathcal{U}-\bar{\mathcal{U}}\|_m. \end{aligned} \tag{3.21}$$

Finally, since $0 \le k \le m-1$, we have

$$\big|\langle \partial_y^k v, \partial_y^{k+1}u\rangle\big| + \big|\langle \partial_y^k u, \partial_y^{k+1}p\rangle\big| \le C\|\mathcal{U}-\bar{\mathcal{U}}\|_m^2.$$

Integrating (3.21) over $[0,t]$ with $t \in [0,T]$, we obtain

$$\begin{aligned} \int_0^t \big(\|\partial_y^{k+1}u\|^2 + \|\partial_y^{k+1}p\|^2\big)dt' \le& C\|\mathcal{U}-\bar{\mathcal{U}}\|_m^2 + C\|\mathcal{U}_0-\bar{\mathcal{U}}\|_m^2 \\ &+ C\int_0^t \big(\|v\|_m^2 + \|\partial_y u\|_{m-1}^2\|\mathcal{U}-\bar{\mathcal{U}}\|_m\big)dt'. \end{aligned}$$

Summing this inequality for all $k = 0, 1, \cdots, m-1$ and using Lemma 3.4, we obtain (3.17). ∎

3.4. Proof of Theorem 2.3. From (3.10) and (3.17), we have

$$\begin{aligned} &\|\mathcal{U}(t,\cdot)-\bar{\mathcal{U}}\|_m^2 + \int_0^t \big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\big)dt' \\ &\le C\|\mathcal{U}_0-\bar{\mathcal{U}}\|_m^2 + C\mathcal{U}_T\int_0^t \big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\big)dt', \quad \forall\, t \in [0,T]. \end{aligned}$$

Since $\mathcal{U}_T$ is sufficiently small, we further obtain

$$\|\mathcal{U}(t,\cdot)-\bar{\mathcal{U}}\|_m^2 + \int_0^t \big(\|\partial_y u\|_{m-1}^2 + \|\partial_y p\|_{m-1}^2 + \|v\|_m^2\big)dt' \le C\|\mathcal{U}_0-\bar{\mathcal{U}}\|_m^2, \quad \forall\, t \in [0,T].$$

This estimate together with a bootstrap argument implies (2.26) and the global existence of a solution $\mathcal{U}$ to (2.24) and (2.25), provided that $\|\mathcal{U}_0-\bar{\mathcal{U}}\|_m$ is sufficiently small. □

4. Proof of Theorem 1.1. For the Cauchy problem for (2.12) with initial data given in (1.11), we first define

$$Y_0(x) = \int_0^x \rho_0(\xi)d\xi.$$

Then $Y_0' = \rho_0$. By the condition in Theorem 1.1, we have $\inf_{x\in\mathbb{R}} \rho_0(x) > 0$ and $\rho_0 - 1 \in H^m(\mathbb{R})$. Therefore, the continuous imbedding from $H^m(\mathbb{R})$ to $C^{m-1}(\mathbb{R})$ implies that Y_0 is a C^m-diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. We denote by X_0 the inverse C^m-diffeomorphism of Y_0 and define

$$\mathcal{U}_0(y) = \Big(\frac{1}{\rho_0}, u_0, \frac{1}{2}c_e u_0^2 + c_e\varepsilon_{e0}, \frac{1}{2}c_i u_0^2 + c_i\varepsilon_{i0}\Big)(X_0(y)).$$

Then condition $\mathcal{V}_0 - \bar{\mathcal{V}} \in H^m(\mathbb{R})$ implies that $\mathcal{U}_0 - \bar{\mathcal{U}} \in H^m(\mathbb{R})$ and condition $\|\mathcal{V}_0 - \bar{\mathcal{V}}\|_m \le \kappa_0$ with κ_0 being sufficiently small implies that $\|\mathcal{U}_0 - \bar{\mathcal{U}}\|_m$ is sufficiently small. According to Theorem 2.3, there exists a global smooth solution $\mathcal{U}(t,y) = (\tau(t,y), u(t,y), E_e(t,y), E_i(t,y))^T$ to the Cauchy problem (2.24) and (2.25). Then, we define

$$\rho(t,y) = \frac{1}{\tau(t,y)}, \quad \varepsilon_\alpha(t,y) = \frac{1}{c_\alpha} E_\alpha(t,y) - \frac{1}{2} u^2(t,y), \quad \alpha = e, i.$$

On the other hand, the result in Theorem 2.3 also implies that $\mathcal{U} \in C^1(\mathbb{R}^+ \times \mathbb{R})$ and $\mathcal{U}$ is globally Lipschitzian on $\mathbb{R}$ with respect to y (in particular for τ and u). Then the Cauchy problem to the following ordinary differential equation

$$Y_1'(t) = u(t, Y_1(t)), \quad Y_1(0) = 0,$$

admits a unique global solution $Y_1 \in C^2(\mathbb{R}^+)$. Let us further define a function X by

$$X(t,y) = \int_{Y_1(t)}^{y} \tau(t,\eta) d\eta.$$

Then, $X \in C^1(\mathbb{R}^+ \times \mathbb{R})$. Similarly to Y_0, for all $t \in \mathbb{R}^+$, $X(t,\cdot)$ is a C^m-diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Let us denote by $Y(t,\cdot)$ the inverse C^m-diffeomorphism of $X(t,\cdot)$. It is easy to see that

$$X(0,y) = X_0(y), \quad Y(0,x) = Y_0(x).$$

Finally, we define

$$\mathcal{V}(t,x) = (\rho, u, \varepsilon_e, \varepsilon_i)^T \big(t, Y(t,x)\big).$$

It is proved in [18] (see also [25]) that entropy solutions of the Cauchy problem for a hyperbolic system of conservation laws are equivalent in Eulerian and Lagrangian coordinates. Moreover, there are explicit formulations of the solutions between two coordinates. Since the solutions studied here are smooth, it is obvious that this equivalence result holds for non-conservative systems. Applying this result, we see that $\mathcal{V}$ is a smooth solution to the Cauchy problem (2.13) and (1.11). Estimate (1.12) follows from (2.26) together with Moser-type inequalities. □

Appendix A. Strictly convex entropy and symmetrizer. There is a well-known result showing that the second-order derivative of a strictly convex entropy is a symmetrizer for the hyperbolic system of conservation laws [9, 3]. In general, this result does not hold for a non-conservative system. In this Appendix, we want to show that the bitemperature Euler model, which is a non-conservative system, provides a good example on this topic.

More precisely, we consider the system in the form (1.5) or equivalently (1.6). Denote $\mathcal{W} = (\rho, \rho u^T, \mathcal{E}_e, \mathcal{E}_i)$. Since η defined in (1.7) is a strictly convex entropy, $\eta''(\mathcal{W})$ is a symmetric positive definite matrix. The result below implies that $\eta''(\mathcal{W})\mathcal{C}_j(\mathcal{W})$ is not symmetric in one space dimension.

Proposition. Consider the one dimensional system (2.12) and denote by $\mathcal{C}_1(\mathcal{W}) = \mathcal{C}(\mathcal{W})$ the related matrix. Then $\eta''(\mathcal{W})\mathcal{C}(\mathcal{W})$ is symmetric if and only if $T_e = T_i$.

Proof. We denote $\Gamma = c_e\gamma_e + c_i\gamma_i$. A straightforward calculation using (1.5) gives

$$\mathcal{C}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2}(\Gamma-3)u^2 & -(\Gamma-3)u & \gamma_e-1 & \gamma_i-1 \\ -\frac{\gamma_e u p_e}{(\gamma_e-1)\rho}+\frac{1}{2}c_e(\Gamma-2)u^3 & \frac{\gamma_e p_e}{(\gamma_e-1)\rho}+c_e(\frac{3}{2}-\Gamma)u^2 & (\gamma_e c_e+c_i)u & c_e(\gamma_i-1)u \\ -\frac{\gamma_i u p_i}{(\gamma_i-1)\rho}+\frac{1}{2}c_i(\Gamma-2)u^3 & \frac{\gamma_i p_i}{(\gamma_i-1)\rho}+c_i(\frac{3}{2}-\Gamma)u^2 & c_i(\gamma_e-1)u & (\gamma_i c_i+c_e)u \end{pmatrix}.$$

Let $q = \rho u$. From (1.3) and (1.4), we may write p_α in variable $\mathcal{W}$ as

$$p_\alpha = (\gamma_\alpha - 1)\Big(\mathcal{E}_\alpha - \frac{c_\alpha q^2}{2\rho}\Big).$$

Then η defined in (1.7) can be expressed as

$$\eta = \eta_e + \eta_i, \quad \eta_\alpha = -\frac{c_\alpha\rho}{b_\alpha}\ln\Big(\frac{p_\alpha}{c_\alpha^{\gamma_\alpha}\rho^{\gamma_\alpha}}\Big), \qquad \alpha = e, i.$$

Obviously,

$$\begin{aligned} \eta_e'(\mathcal{W}) &= \left(\frac{\eta_e}{\rho} - \frac{k_B c_e^2 q^2}{2m_e\rho p_e} + \frac{\gamma_e c_e}{b_e}, \frac{k_B c_e^2 q}{m_e p_e}, -\frac{k_B c_e\rho}{m_e p_e}, 0\right), \\ \eta_i'(\mathcal{W}) &= \left(\frac{\eta_i}{\rho} - \frac{k_B c_i^2 q^2}{2m_i\rho p_i} + \frac{\gamma_i c_i}{b_i}, \frac{k_B c_i^2 q}{m_i p_i}, 0, -\frac{k_B c_i\rho}{m_i p_i}\right), \\ \eta'(\mathcal{W}) &= \eta_e'(\mathcal{W}) + \eta_i'(\mathcal{W}). \end{aligned}$$

Since $\partial^2_{\mathcal{E}_e\mathcal{E}_i}\eta = 0$, the hessian matrix of η is of the following form :

$$\eta''(\mathcal{W}) = \begin{pmatrix} \partial^2_{\rho\rho}(\eta_e+\eta_i) & \partial^2_{\rho q}(\eta_e+\eta_i) & \partial^2_{\rho\mathcal{E}_e}\eta_e & \partial^2_{\rho\mathcal{E}_i}\eta_i \\ \partial^2_{\rho q}(\eta_e+\eta_i) & \partial^2_{qq}(\eta_e+\eta_i) & \partial^2_{q\mathcal{E}_e}\eta_e & \partial^2_{q\mathcal{E}_i}\eta_i \\ \partial^2_{\rho\mathcal{E}_e}\eta_e & \partial^2_{q\mathcal{E}_e}\eta_e & \partial^2_{\mathcal{E}_e\mathcal{E}_e}\eta_e & 0 \\ \partial^2_{\rho\mathcal{E}_i}\eta_i & \partial^2_{q\mathcal{E}_i}\eta_i & 0 & \partial^2_{\mathcal{E}_i\mathcal{E}_i}\eta_i \end{pmatrix},$$

with

$$\partial^2_{\rho\rho}\eta_\alpha = \frac{\gamma_\alpha c_\alpha}{b_\alpha\rho} + \frac{c_\alpha u^4}{4b_\alpha\rho\varepsilon_\alpha^2}, \quad \partial^2_{\rho q}\eta_\alpha = -\frac{c_\alpha u^3}{2b_\alpha\rho\varepsilon_\alpha^2}, \quad \partial^2_{\rho\mathcal{E}_\alpha}\eta_\alpha = -\frac{1}{b_\alpha\rho\varepsilon_\alpha} + \frac{u^2}{2b_\alpha\rho\varepsilon_\alpha^2},$$

$$\partial^2_{qq}\eta_\alpha = \frac{c_\alpha}{b_\alpha\rho\varepsilon_\alpha} + \frac{c_\alpha u^2}{b_\alpha\rho\varepsilon_\alpha^2}, \quad \partial^2_{q\mathcal{E}_\alpha}\eta_\alpha = -\frac{u}{b_\alpha\rho\varepsilon_\alpha^2},$$

$$\partial^2_{\mathcal{E}_\alpha\mathcal{E}_\alpha}\eta_\alpha = \frac{1}{c_\alpha b_\alpha\rho\varepsilon_\alpha^2}, \qquad \alpha = e, i.$$

Hence we obtain

$$\partial^2_{\rho\rho}\eta = \sum_{\alpha=e,i}\left(\frac{\gamma_\alpha c_\alpha}{b_\alpha\rho} + \frac{c_\alpha u^4}{4b_\alpha\rho\varepsilon_\alpha^2}\right), \quad \partial^2_{\rho q}\eta = -\sum_{\alpha=e,i}\frac{c_\alpha u^3}{2b_\alpha\rho\varepsilon_\alpha^2}, \quad \partial^2_{\rho\mathcal{E}_\alpha}\eta = \frac{1}{b_\alpha\rho\varepsilon_\alpha^2}(\frac{u^2}{2}-\varepsilon_\alpha)$$

$$\partial^2_{qq}\eta = \sum_{\alpha=e,i}\frac{c_\alpha}{b_\alpha\rho\varepsilon_\alpha^2}\left(\varepsilon_\alpha + u^2\right), \quad \partial^2_{q\mathcal{E}_\alpha}\eta = -\frac{u}{b_\alpha\rho\varepsilon_\alpha^2},$$

$$\partial^2_{\mathcal{E}_\alpha\mathcal{E}_\alpha}\eta = \frac{1}{b_\alpha c_\alpha\rho\varepsilon_\alpha^2}, \qquad \alpha = e, i.$$

The entry in the 3-th row and 1-th column of $\eta''\mathcal{C}$ is

$$\begin{aligned}(\eta''\mathcal{C})_{31} &= \partial^2_{q\mathcal{E}_e}\eta_e \times \frac{1}{2}(c_e\gamma_e + c_i\gamma_i - 3)u^2 + \partial^2_{\mathcal{E}_e\mathcal{E}_e}\eta_e[\frac{1}{2}c_e(c_e\gamma_e + c_i\gamma_i - 2)u^3 - c_e\gamma_e u\varepsilon_e] \\ &= \frac{u^3}{2b_e\rho\varepsilon_e^2} - \frac{\gamma_e u}{b_e\rho\varepsilon_e}.\end{aligned}$$

Similarly,

$$(\eta''\mathcal{C})_{13} = \frac{u^3}{2b_e\rho\varepsilon_e^2} - \Big(\frac{\gamma_e c_e + c_i}{b_e\varepsilon_e} + \frac{c_i(\gamma_e - 1)}{b_i\varepsilon_i}\Big)\frac{u}{\rho}.$$

Therefore, $(\eta''\mathcal{C})_{31} = (\eta''\mathcal{C})_{13}$ if and only if $b_e\varepsilon_e = b_i\varepsilon_i$, that is to say $T_e = T_i$. This proves that η is not a symmetrizer of the system. In a same way, we can show that $\eta''\mathcal{C}$ is symmetric if and only if $T_i = T_e$. ◻

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