Fredholm Property of the Linearized Boltzmann Operator for a Polyatomic Single Gas Model

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Abstract

In the following work, we consider the Boltzmann equation that models a polyatomic gas by representing the microscopic internal energy by a continuous variable I. Under some convenient assumptions on the transition function $\mathcal{B}$, we prove that the linearized Boltzmann operator $\mathcal{L}$ of this model is a Fredholm operator. For this, we write $\mathcal{L}$ as a perturbation of the collision frequency multiplication operator, and we prove that the perturbation operator $\mathcal{K}$ is compact. The result is established after inspecting the kernel form of $\mathcal{K}$ and proving it to be L^2 integrable over its domain using elementary arguments.

1   Introduction

This work is devoted to the study of the Fredholm property of the linearized Boltzmann operator $\mathcal{L}$ for polyatomic gases. In fact, the Fredholm property of $\mathcal{L}$ is essential in the Chapman-Enskog asymptotics and has been assumed for polyatomic gases in the literature, [2, 25] for instance. In contrast to monatomic molecules, the energy in a polyatomic molecule is not totally kinetic, but also partially internal coming from the rotation and vibration of the molecule. This microscopic internal energy is mathematically integrated into the models by being assumed to take discrete values as in [6, 18, 23, 25], or by being represented by a continuous parameter. This continuous representation was introduced in [13] for modeling the Boltzmann equation describing polyatomic gases using the Borgnakke-Larsen procedure [10]. Many formal results were achieved for the model in [13]. Using [20], the Chapman-Enskog method was recently developed in [2], and many macroscopic models of extended thermodynamics were derived [33]. In accordance with [32], two hierarchies of transfer equations for moments were obtained in [31], in contrast to one hierarchy in the case of monatomic gases. In agreement with the macroscopic approach [33], the procedure of maximum entropy principle for the closure of the moments equations was applied to polyatomic gases[7, 32].

In the context of simplifying the polyatomic Boltzmann equation, the simplified ES-BGK model has been developed [1, 14], where the return to equilibrium of the solutions in the homogeneous case has been studied as well in [14]. In [34], an existence result of the ES-BGK model was achieved in the case where the solution lies close to equilibrium.

In order to construct a quantitative theory and to obtain explicit convergence rates to the equilibrium, explicit estimates for the spectral gap of the linearized Boltzmann operator should be obtained. In the monatomic single gas setting, Grad [28] proved that the linearized Boltzmann operator is a Fredholm operator for hard potentials, by writing it as a compact perturbation of a coercive multiplication operator (see also [17, 22]). Bobylev was the first to find explicit estimates for the spectral gap for Maxwell molecules by implementing Fourier methods [8, 9]. Using the spectral gap of Maxwell molecules, coercivity estimates for hard sphere gases were recently established in [3], and for monatomic gases without angular cut-off in [29]. Under certain assumptions on the collision cross-section, the following results were achieved. For monatomic mixtures, $\mathcal{L}$ was proved to be Fredholm in [4, 12]. Coercivity estimates on the spectral gap of the linearized Boltzmann operator were obtained [19]. For a single diatomic gas, the operator $\mathcal{L}$ was proved

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to be a Fredholm operator, in which $\mathcal{K}$ was proved to be Hilbert-Schmidt in [15] under some assumption on the transition function. In [5], $\mathcal{L}$ was proved to be Fredholm for a single polyatomic gas with continuous internal energy using a different approach, while in [4], discrete internal energy was considered. In [11], the compactness of $\mathcal{K}$ was proved for polyatomic gases undergoing resonance. In this current work, we aim to generalize the work [15] for a single polyatomic gas using elementary arguments. For this, we write $\mathcal{L}$ as a compact perturbation of the collision frequency multiplication operator, and we prove as well that the collision frequency ν is coercive. This implies that $\mathcal{L}$ is a Fredholm operator.

The plan of the paper is as follows: In Section 2, we give a brief recall on the collision model [13], which describes the microscopic state of polyatomic gases, and give an equivalent formulation of the collision operator. In Section 3, we define the linearized operator $\mathcal{L}$, which is obtained by approximating the distribution function f around the Maxwellian M. The main aim of this paper is to prove that the linearized Boltzmann operator is a Fredholm operator, which is achieved in Section 4. In particular, we write $\mathcal{L}$ as $\mathcal{L} = \mathcal{K} - \nu\,\mathrm{Id}$ and we prove that $\mathcal{K}$ is compact, and ν is coercive. As a result, $\mathcal{L}$ is viewed as a compact perturbation of the multiplication operator $\nu\,\mathrm{Id}$. To prove $\mathcal{K}$ is compact, we write $\mathcal{K}$ as $\mathcal{K}_3 + \mathcal{K}_2 - \mathcal{K}_1$, and we prove each $\mathcal{K}_i$, with $i = 1, 2, 3$, to be a Hilbert-Schmidt operator. In Section 5, we prove the coercivity and we give the monotonicity property of the collision frequency, which helps to locate the essential spectrum of $\mathcal{L}$.

2 The Classical model

We present first the model in [13] on which our work is mainly based. We start with physical conservation equations and proceed by parameterizing them.

2.1 Boltzmann Equation Without loss of generality, we first assume that the molecule mass equals unity, and we denote as usual by (v, v_*), (I, I_*) and (v', v'_*), (I', I'_*) the pre-collisional and post-collisional velocity and internal energy pairs respectively. In this model, the internal energies are assumed to be continuous. The following conservation of momentum and total energy equations hold:

$$v + v_* = v' + v'_* \tag{1}$$

$$\frac{1}{2}|v|^2 + \frac{1}{2}|v_*|^2 + I + I_* = \frac{1}{2}|v'|^2 + \frac{1}{2}|v'_*|^2 + I' + I'_*. \tag{2}$$

From the above equations, we can deduce the following equation representing the conservation of total energy in the center of mass reference frame:

$$\frac{1}{4}|v - v_*|^2 + I + I_* = \frac{1}{4}|v' - v'_*|^2 + I' + I'_* = E,$$

with E denoting the total energy. We introduce in addition the parameter $R \in [0, 1]$ which represents the portion allocated to the post-kinetic energy out of the total energy, and the parameter $r \in [0, 1]$ which represents the distribution of the post internal energy among the two interacting molecules. Namely,

$$\frac{1}{4}|v' - v'_*|^2 = RE$$
$$I' + I'_* = (1 - R)E,$$

and

$$I' = r(1 - R)E$$
$$I'_* = (1 - r)(1 - R)E.$$

Using the above equations, we can express the post-collisional velocities in terms of the other quantities by the following

$$v' \equiv v'(v, v_*, I, I_*, \sigma, R) = \frac{v + v_*}{2} + \sqrt{RE}\,\sigma$$
$$v'_* \equiv v'_*(v, v_*, I, I_*, \sigma, R) = \frac{v + v_*}{2} - \sqrt{RE}\,\sigma,$$

where $\sigma = \frac{v'-v'_*}{|v'-v'_*|} \in S^2$ is regarded as a parameter. In addition, we define the parameters $r' \in [0,1]$ and $R' \in [0,1]$ for the pre-collisional terms in the same manner as r and R. In particular

$$\frac{1}{4}|v-v_*|^2 = R'E$$
$$I + I_* = (1-R')E,$$

and

$$I = r'(1-R')E$$
$$I_* = (1-r')(1-R')E.$$

Finally, for $(r', R') \in (0,1)^2$, the post-collisional energies can be given in terms of the pre-collisional energies by the following relation

$$I' = \frac{r(1-R)}{r'(1-R')}I$$
$$I'_* = \frac{(1-r)(1-R)}{(1-r')(1-R')}I_*.$$

The Boltzmann equation for an interacting single polyatomic gas reads

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{3}$$

where $f = f(t,x,v,I) \geq 0$ is the distribution function, with $t \geq 0, x \in \mathbb{R}^3, v \in \mathbb{R}^3$, and $I \geq 0$. The operator $Q(f,f)$ is the quadratic Boltzmann operator [13] given as

$$Q(f,f)(v,I) = \int_{(0,1)^2 \times S^2 \times \mathbb{R}_+ \times \mathbb{R}^3} \left(\frac{f'f'_*}{(I'I'_*)^\alpha} - \frac{ff_*}{(II_*)^\alpha}\right) \times \mathcal{B} \times (r(1-r))^\alpha (1-R)^{2\alpha} \times$$
$$I^\alpha I_*^\alpha (1-R)R^{1/2} \, \mathrm{d}R\mathrm{d}r\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*, \tag{4}$$

where $\alpha \geq 0$, and we use the standard notations $f_* = f(v_*, I_*), f' = f(v', I')$, and $f'_* = f(v'_*, I'_*)$. Choosing the power α in the measure of the above integral is essential for the operator $\mathcal{K}$ (23) to be a Hilbert-Schmidt operator (see the proof of compactness of $\mathcal{K}_2$ on page 9).

The function $\mathcal{B}$ is the transition function; a function of $(v, v_*, I, I_*, r, R, \sigma)$. we give some assumptions on $\mathcal{B}$ inspired from [24], which are extended from Grad's assumption for collision kernels of monatomic gases. In general, $\mathcal{B}$ is assumed to be an almost everywhere positive function satisfying the following microreversibility conditions

$$\mathcal{B}(v, v_*, I, I_*, r, R, \sigma) = \mathcal{B}(v_*, v, I_*, I, 1-r, R, -\sigma),$$
$$\mathcal{B}(v, v_*, I, I_*, r, R, \sigma) = \mathcal{B}(v', v'_*, I', I'_*, r', R', \sigma'), \tag{5}$$

where $\sigma' = \frac{v-v_*}{|v-v_*|}$.

Remark 1. *The expression of the collision operator Q in (4), which was introduced in [13] is equivalent to the expression given in (54), in which the integration is carried out with a measure preserving E and G. The equivalence of these expressions is shown in Appendix A.*

2.2 Main Assumptions on the Transition Function $\mathcal{B}$ Throughout this paper, $c > 0$ will
denote a generic constant. Together with the above assumption (5), we assume the following boundedness assumptions on the transition function $\mathcal{B}$. In particular, for a given $\gamma \geq 0$ we assume

$$\gamma \geq 0, \quad \Phi_\gamma(r, R)\left(|v-v_*|^\gamma + I^{\gamma/2} + I_*^{\gamma/2}\right) \leq \mathcal{B}(v, v_*, I, I_*, r, R, \sigma), \tag{6}$$

and

$$\gamma \geq 0, \quad \mathcal{B}(v, v_*, I, I_*, r, R, \sigma) \leq \Psi_\gamma(r, R)\left(|v-v_*|^\gamma + I^{\gamma/2} + I_*^{\gamma/2}\right). \tag{7}$$

For $-2 - 2\alpha < \gamma < 0$, we prove that $\mathcal{K}$ remains compact under the following upper bound assumption on $\mathcal{B}$

$$-2 - 2\alpha < \gamma < 0, \quad \mathcal{B}(v, v_*, I, I_*, r, R, \sigma) \le \Psi_\gamma(r, R)\, E^{\gamma/2}. \tag{8}$$

We assume that Φ_γ (for $\gamma \ge 0$) and Ψ_γ (for $\gamma \ge 0$ or $-2 - 2\alpha < \gamma < 0$), are positive functions such that

$$\Phi_\gamma \le \Psi_\gamma,$$

and

$$\Phi_\gamma(r, R) = \Phi_\gamma(1 - r, R), \quad \Psi_\gamma(r, R) = \Psi_\gamma(1 - r, R). \tag{9}$$

In addition, Ψ_γ satisfies the following

$$\Psi_\gamma^2(r, R)(1 - r)^{2\alpha - 1 - \gamma} r^{\alpha - 1} R(1 - R)^{3\alpha - \gamma} \quad \in L^1((0, 1)^2). \tag{10}$$

In fact, though assumption (10) seems to be strict, yet it covers several physical models. In addition, one may notice that for bigger values of α or smaller values of γ, condition (10) covers a wider class of functions Ψ_γ.

Remark 2. *We remark that physically, it is possible that γ may only belong to $(-3, 2]$ (as in the monatomic case). However, we prove in this paper that from a mathematical point of view, the compactness of $\mathcal{K}$ is valid even under assumption (8), where $-2 - 2\alpha < \gamma < 0$. Yet in this case, ν is not coercive, which implies that $\mathcal{L}$ does not satisfy the Fredholm property.*

In order to give some models of $\mathcal{B}$ that satisfy condition (10), we first present the relation between the number of atoms in a molecule and the value of α, see [30]. Let D be the number of degrees of freedom in a molecule of N atoms, then α is given in terms of D by the formula:

$$\alpha = \frac{D - 5}{2}. \tag{11}$$

To relate D with N, we consider the following cases of molecules:

1. **Non-vibrating molecules** In this case, we distinguish between linear and non-linear molecules. Regarding the fact that vibrations are not occurring, a linear (respectively non-linear) molecule will always remain linear (respectively non-linear) even after collisions. The number of degrees of freedom D in this case will be the sum of the rotational and translational degrees of freedom in $\mathbb{R}^3$, and will not depend on the number of atoms in the gas molecule.

   - Linear molecules:
     - translational degrees of freedom: 3
     - rotational degrees of freedom: 2

     and therefore $D = 5$ and $\alpha = 0$.
   - Non-linear molecules:
     - translational degrees of freedom: 3
     - rotational degrees of freedom: 3

     and therefore $D = 6$ and $\alpha = 1/2$.

2. **Vibrating molecules** In this case, the total number of degrees of freedom depends on N. Consider a molecule of N atoms in $\mathbb{R}^3$, then as long as the shape of the molecule is deformable due to vibrations, the position of each atom will be determined freely by 3 degrees of freedom. Thus, the total number of degrees of freedom of the molecule of N atoms will equal to $3N$. Hence, $D = 3N$ and

$$\alpha = \frac{3N - 5}{2}. \tag{12}$$

   We notice that in this case, the fact that the molecule is linear or non-linear doesn't have an impact on D and α.

We now give some models that satisfy assumptions (10), which have been recently studied in the literature (see [21, 24]).

Examples

In [24], for $\gamma \in (0, 2]$, and an angular function $b(\cos(\theta)) \in L^1([0, \pi])$ the following transition function model was suggested

$$\mathcal{B}\left(v, v_*, I, I_*, r, R, \sigma\right) = b\left(\sigma \cdot \frac{v - v_*}{|v - v_*|}\right)\left(|v - v_*|^\gamma + I^{\gamma/2} + I_*^{\gamma/2}\right), \tag{13}$$

which is equivalent to the model

$$\mathcal{B}\left(v, v_*, I, I_*, r, R, \sigma\right) = b\left(\sigma \cdot \frac{v - v_*}{|v - v_*|}\right) E^{\gamma/2}. \tag{14}$$

In addition, the following models were suggested

$$\mathcal{B}\left(v, v_*, I, I_*, r, R, \sigma\right) = b\left(\sigma \cdot \frac{v - v_*}{|v - v_*|}\right)\left(R^{\gamma/2}|v - v_*|^\gamma + (1 - R)^{\gamma/2}\left(I + I_*\right)^{\gamma/2}\right) \tag{15}$$

$$\mathcal{B}\left(v, v_*, I, I_*, r, R, \sigma\right) = b\left(\sigma \cdot \frac{v - v_*}{|v - v_*|}\right)\left(R^{\gamma/2}|v - v_*|^\gamma + \left(r(1 - R)I\right)^{\gamma/2} + \left((1 - r)(1 - R)I_*\right)^{\gamma/2}\right). \tag{16}$$

The above models satisfy assumptions (6),(7), (9), and (10) for a constant angular function b, for $\alpha > 0$, and $\gamma < 2\alpha$. Under these constraints, the parameters α and γ range according to the molecule state as described in the following table.

Molecule state	Range of α	γ
vibrating	$\alpha \geq 1/2$	$\gamma < 2\alpha$
not vibrating, non-linear	$\alpha = 1/2$	$\gamma < 1$

Table 1: Range of α and possible range of γ based on the degrees of freedom

Models (13), (15), and (16) are obtained by taking for model (13)

$$\Phi_\gamma(r, R) = \Psi_\gamma(r, R) = 1,$$

for model (15)

$$\Phi_\gamma(r, R) = \min\{R, (1 - R)\}^{\gamma/2}, \text{ and } \Psi_\gamma(r, R) = \max\{R, 1 - R\}^{\gamma/2},$$

and for model (16)

$$\Phi_\gamma(r, R) = \min\{R, (1 - R)\}^{\gamma/2}\min\{r, (1 - r)\}^{\gamma/2}, \text{ and } \Psi_\gamma(r, R) = \max\{R, 1 - R\}^{\gamma/2}.$$

In [21], the authors considered the class of transition functions having the expression

$$\mathcal{B}\left(v, v_*, I, I_*, r, R, \sigma\right) = b\left(\sigma \cdot \frac{v - v_*}{|v - v_*|}\right)\left(R^{\gamma/2}|v - v_*|^\gamma + \left(r(1 - R)I\right)^{\gamma/2} + \left((1 - r)(1 - R)I_*\right)^{\gamma/2}\right), \tag{17}$$

where $b(\cos\theta)$ was assumed to be L^1 integrable while establishing the first six fields equations, whereas b was assumed constant for the fourteen moments model.

Remark 3. *In comparison to the result of [5], the compactness of $\mathcal{K}$ was proved to be valid in [5] without such restrictions on γ, and under an assumption on $\mathcal{B}$ that governs the above models suggested in [24] for $\gamma \in (0, 2]$. Yet for proving the Hilbert-Schmidt property of $\mathcal{K}$, the restriction $\alpha > 1/4$ is needed. In our approach, the assumption $\alpha > \gamma/2$ is rather needed.*

3   The Linearized Boltzmann Operator

We state first the H-theorem for polyatomic gases which was initially established in [13]. In particular, the entropy production functional satisfies

$$D(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} Q(f,f) \log f \ \mathrm{d}I \mathrm{d}v \le 0,$$

and the following are equivalent

1. The collision operator $Q(f,f)$ vanishes, i.e. $Q(f,f)(v,I) = 0$ for every $v \in \mathbb{R}^3$ and $I \ge 0$.

2. The entropy production vanishes, i.e. $D(f) = 0$.

3. There exists $T > 0$, $n > 0$, and $u \in \mathbb{R}^3$ such that

$$f(v,I) = M_{n,u,T}(v,I) = \frac{n}{(2\pi)^{3/2} \Gamma(\alpha+1)(\kappa T)^{\alpha+5/2}} I^\alpha e^{-(1/kT)(\frac{1}{2}|v-u|^2+I)}, \tag{18}$$

where κ in (18) is the Boltzmann constant. The linearization of the Boltzmann equation of polyatomic gases could be taken around the local Maxwellian function $M_{n,u,T}$, which represents the equilibrium state of a gas, where n, u, and T are the number of molecules per unit volume, the hydrodynamic velocity, and the temperature respectively. In particular,

$$n = \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} f \mathrm{d}I \mathrm{d}v, \quad nu = \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} vf \mathrm{d}I \mathrm{d}v, \quad \left(\alpha + \frac{5}{2}\right) n\kappa T = \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \left(\frac{|v-u|^2}{2} + I\right) f \mathrm{d}I \mathrm{d}v.$$

Without loss of generality, we will consider in the sequel a normalized version $M_{1,0,1}$, by assuming $\kappa T = n = 1$ and $u = 0$. For the sake of simplicity, the index will be dropped. In particular,

$$M(v,I) = M_{1,0,1}(v,I) = \frac{1}{(2\pi)^{3/2} \Gamma(\alpha+1)} I^\alpha e^{-\frac{1}{2}|v|^2-I}. \tag{19}$$

We look for a solution f around M defined in (19) having the form

$$f(v,I) = M(v,I) + M^{1/2}(v,I)g(v,I).$$

The linearization of the Boltzmann operator (4) around M leads to introduce the linearized Boltzmann operator $\mathcal{L}$ given as

$$\mathcal{L}g = M^{-1/2}[Q(M, M^{1/2}g) + Q(M^{1/2}g, M)].$$

More explicitly, $\mathcal{L}$ writes

$$\mathcal{L}g = M^{-1/2} \int_\Delta \left[\frac{M'(M'_*)^{1/2}}{\sqrt{(I'_*)^\alpha}(I')^\alpha} \frac{g'_*}{\sqrt{(I'_*)^\alpha}} - \frac{MM_*^{1/2}}{I^\alpha \sqrt{I_*^\alpha}} \frac{g_*}{\sqrt{I_*^\alpha}} + \frac{M'_*(M')^{1/2}}{(I'_*)^\alpha \sqrt{(I')^\alpha}} \frac{g'}{\sqrt{(I')^\alpha}} - \frac{M_*M^{1/2}}{I_*^\alpha \sqrt{I^\alpha}} \frac{g}{\sqrt{I^\alpha}} \right] \times$$
$$(r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \ \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*. \tag{20}$$

Thanks to the conservation of total energy (2) we have $\frac{M}{I^\alpha} \frac{M_*}{I_*^\alpha} = \frac{M'}{(I')^\alpha} \frac{M'_*}{(I'_*)^\alpha}$, and so $\mathcal{L}$ has the following form:

$$\mathcal{L}(g) = - I^{-\alpha/2} \int_\Delta \frac{g_*}{I_*^{\alpha/2}} \frac{M^{1/2}}{I^{\alpha/2}} \frac{M_*^{1/2}}{I_*^{\alpha/2}} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*$$

$$- I^{-\alpha} \int_\Delta g \frac{M_*}{I_*^\alpha} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*$$

$$+ I^{-\alpha/2} \int_\Delta \frac{g'_*}{(I'_*)^{\alpha/2}} \frac{M_*^{1/2}}{I_*^{\alpha/2}} \frac{(M')^{1/2}}{(I')^{\alpha/2}} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*$$

$$+ I^{-\alpha/2} \int_\Delta \frac{g'}{(I')^{\alpha/2}} \frac{M_*^{1/2}}{I_*^{\alpha/2}} \frac{(M'_*)^{1/2}}{(I'_*)^{\alpha/2}} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*.$$

Here, Δ refers to the open set $(0,1)^2 \times S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$. The operator $\mathcal{L}$ can be written in the form

$$\mathcal{L} = \mathcal{K} - \nu \, \mathrm{Id},$$

where

$$\nu(v, I) = I^{-\alpha} \int_\Delta \frac{M_*}{I_*^\alpha} \left(r(1-r)\right)^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*, \tag{21}$$

represents the collision frequency. We write also $\mathcal{K}$ as $\mathcal{K} = \mathcal{K}_3 + \mathcal{K}_2 - \mathcal{K}_1$ with

$$\mathcal{K}_1 g = I^{-\alpha/2} \int_\Delta \frac{g_*}{I_*^{\alpha/2}} \frac{M^{1/2}}{I^{\alpha/2}} \frac{M_*^{1/2}}{I_*^{\alpha/2}} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*, \tag{22}$$

$$\mathcal{K}_2 g = I^{-\alpha/2} \int_\Delta \frac{g'_*}{(I'_*)^{\alpha/2}} \frac{M_*^{1/2}}{I_*^{\alpha/2}} \frac{(M')^{1/2}}{(I')^{\alpha/2}} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*, \tag{23}$$

and

$$\mathcal{K}_3 g = I^{-\alpha/2} \int_\Delta \frac{g'}{(I')^{\alpha/2}} \frac{M_*^{1/2}}{I_*^{\alpha/2}} \frac{(M'_*)^{1/2}}{(I'_*)^{\alpha/2}} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^\alpha I_*^\alpha \mathcal{B} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*. \tag{24}$$

The linearized operator $\mathcal{L}$ is a symmetric operator, with kernel

$$\ker \mathcal{L} = M^{1/2} \, \mathrm{span}\{1, v_1, v_2, v_3, \tfrac{1}{2}|v|^2 + I\}. \tag{25}$$

Since $\mathcal{L}$ is symmetric and $\nu \, \mathrm{Id}$ is self-adjoint on

$$Dom(\nu \, \mathrm{Id}) = \{g \in L^2(\mathbb{R}^3 \times \mathbb{R}_+) : \nu g \in L^2(\mathbb{R}^3 \times \mathbb{R}_+)\},$$

then $\mathcal{K}$ is symmetric. In the following section, we prove that $\mathcal{K}$ is a bounded compact operator on $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$. Hence, $\mathcal{L}$ is a self-adjoint operator on Dom $(\mathcal{L}) = \mathrm{Dom}(\nu \, \mathrm{Id})$. In Section 5 we prove that ν is coercive, and therefore, the coercivity of ν and the compactness of $\mathcal{K}$, imply that $\mathcal{L}$ is a Fredholm operator on $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$.

4 Main Result

We state the following theorem, which is the main result of the paper.

Theorem 1. *1. For $\gamma \geq 0$, and under assumptions (7),(9), and (10), the operator $\mathcal{K}$ defined by (22)-(24) for a single polyatomic gas is a compact operator from $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$, and by assumption (6) the multiplication operator by ν is coercive. As a result, under assumptions (6),(7), (9), and (10) the linearized Boltzmann operator $\mathcal{L}$ is an unbounded self-adjoint Fredholm operator from $Dom (\mathcal{L}) = Dom (\nu \, Id) \subset L^2(\mathbb{R}^3 \times \mathbb{R}_+)$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$.*

2. For $-2 - 2\alpha < \gamma < 0$, under assumptions (8),(9), and (10), $\mathcal{K}$ is a compact operator from $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)$.

Proof. We give the proof of compactness of $\mathcal{K}$ for both cases of γ ($\gamma \geq 0$ and $-2 - 2\alpha < \gamma < 0$) right after the following corollary. In addition, we prove that ν is coercive for $\gamma \geq 0$ in Section 5. As a result, by Theorem 4.3 in [26], $\mathcal{L}$ is a Fredholm operator for $\gamma \geq 0$ and under assumptions (6),(7), (9), and (10). $\qquad \square$

For $\gamma \geq 0$, where under assumptions (7),(9), and (10) $\mathcal{K}$ is compact, and under assumption (6) ν is coercive, we can deduce the following corollary (see [27] in the monatomic case and [5] in the polyatomic case).

Corollary 1.1. *For $\gamma \geq 0$, there exists $C > 0$ such that, for each $\phi \in L^2(\nu dv dI)$, the following coercivity estimate holds*

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} \phi \mathcal{L}(\phi) dv dI \geq C \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\phi - \mathbf{P}\phi)^2 \nu(v, I) dv dI, \tag{26}$$

where $\mathbf{P}$ is the orthogonal projection on $\ker \mathcal{L}$ given in (25).

The proof of the corollary is similar to that in the monatomic case [27]. Therefore, we only give the proof of Theorem 1. We carry out the proof of the coercivity of ν Id in Section 5, and we dedicate the rest of this section to the proof of the compactness of $\mathcal{K}$.

Proof of compactness of $\mathcal{K}$ We will prove the compactness of each $\mathcal{K}_i$, with $i = 1, 2, 3$, separately.

Compactness of $\mathcal{K}_1$. The compactness of $\mathcal{K}_1$ is straightforward as $\mathcal{K}_1$ already possesses a kernel form. Thus, we can inspect the operator kernel of $\mathcal{K}_1$ (22) to be

$$k_1(v, I, v_*, I_*) = \frac{1}{\Gamma(\alpha + 1)(2\pi)^{3/2}} \int_{(0,1)^2 \times S^2} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^{\alpha/2} I_*^{\alpha/2} \mathcal{B} e^{-\frac{1}{4}|v_*|^2 - \frac{1}{4}|v|^2 - \frac{1}{2}I_* - \frac{1}{2}I} \, drdRd\sigma,$$

and therefore

$$\mathcal{K}_1 g(v, I) = \int_{\mathbb{R}^3 \times \mathbb{R}_+} g(v_*, I_*) k_1(v, I, v_*, I_*) \, dv_* dI_* \quad \forall (v, I) \in \mathbb{R}^3 \times \mathbb{R}_+.$$

We give the following lemma that yields to the compactness of $\mathcal{K}_1$.

Lemma 2. *Using assumptions (7),(8), and (10) on $\mathcal{B}$, the function $k_1 \in L^2(\mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+)$.*

Proof. Applying Cauchy-Schwarz inequality we get

$$||k_1||_{L^2}^2 \leq \frac{1}{\Gamma(\alpha + 1)(2\pi)^{3/2}} \int_{\mathbb{R}^3 \times \mathbb{R}_+} \int_\Delta I^\alpha I_*^\alpha (r(1-r))^{2\alpha} (1-R)^{4\alpha+2} R \mathcal{B}^2 e^{-\frac{1}{2}|v_*|^2 - \frac{1}{2}|v|^2 - I_* - I} drdRd\sigma dI dv dI_* dv_*.$$

For $\gamma \geq 0$ we use assumptions (7) and (10) to get

$$||k_1||_{L^2}^2 \leq c \int_{(\mathbb{R}_+ \times \mathbb{R}^3)^2} I^\alpha I_*^\alpha (I^\gamma + I_*^\gamma + |v - v_*|^{2\gamma}) e^{-\frac{1}{2}|v_*|^2 - \frac{1}{2}|v|^2 - I_* - I} dI dv dI_* dv_*$$

$$\leq c_0 + c \int_{\mathbb{R}^3} e^{-\frac{1}{2}|v_*|^2} \left[\int_{|v-v_*| \leq 1} e^{-\frac{1}{2}|v|^2} dv + \int_{|v-v_*| \geq 1} |v - v_*|^{\lceil 2\gamma \rceil} e^{-\frac{1}{2}|v|^2} dv \right] dv_*$$

$$\leq c_0 + c \int_{\mathbb{R}^3} e^{-\frac{1}{2}|v_*|^2} \left[\int_{|v-v_*| \geq 1} \sum_{k=0}^{\lceil 2\gamma \rceil} |v|^k |v_*|^{\lceil 2\gamma \rceil - k} e^{-\frac{1}{2}|v|^2} dv \right] dv_*$$

$$\leq c_0 + c \sum_{k=0}^{\lceil 2\gamma \rceil} \int_{\mathbb{R}^3} |v_*|^{\lceil 2\gamma \rceil - k} e^{-\frac{1}{2}|v_*|^2} \left[\int_{\mathbb{R}^3} |v|^k e^{-\frac{1}{2}|v|^2} dv \right] dv_* < \infty,$$

where $\lceil 2\gamma \rceil$ is the ceiling of 2γ, and c_0 is such that

$$c \int_{(\mathbb{R}_+ \times \mathbb{R}^3)^2} I^\alpha I_*^\alpha (I^\gamma + I_*^\gamma) e^{-\frac{1}{2}|v_*|^2 - \frac{1}{2}|v|^2 - I_* - I} dI dv dI_* dv_* \leq c_0. \tag{27}$$

For $-2 - 2\alpha < \gamma < 0$, we use assumptions (8) and (10) to obtain

$$||k_1||_{L^2}^2 \leq c \int_{(\mathbb{R}_+ \times \mathbb{R}^3)^2} I^\alpha I_*^\alpha (I + I_* + |v - v_*|)^\gamma e^{-\frac{1}{2}|v_*|^2 - \frac{1}{2}|v|^2 - I_* - I} dI dv dI_* dv_*.$$

As $\gamma < 0$, the inequality

$$(I + I_* + |v - v_*|)^\gamma \leq I^{\gamma/2} I_*^{\gamma/2}$$

holds. Therefore we get

$$||k_1||_{L^2}^2 \le c \int_{(\mathbb{R}_+)^2} I^{\alpha+\gamma/2} I_*^{\alpha+\gamma/2} e^{-\frac{1}{2}|v_*|^2 - \frac{1}{2}|v|^2 - I_* - I} \mathrm{d}I \mathrm{d}I_* < \infty.$$

$\square$

This implies that $\mathcal{K}_1$ is a Hilbert-Schmidt operator, and thus compact. We now prove the compactness of $\mathcal{K}_2$ similarly, by proving it to be a Hilbert-Schmidt operator.

Compactness of $\mathcal{K}_2$. Additional work is required to inspect the kernel form of $\mathcal{K}_2$, since the kernel is not obvious as $\mathcal{K}_2$ is given explicitly as

$$\mathcal{K}_2 g(v, I) = \int_\Delta (I_*')^{-\alpha/2} e^{-\frac{I_*}{2} - \frac{1}{2}r(1-R)\left(\frac{|v-v_*|^2}{4} + I + I_*\right) - \frac{1}{4}|v_*|^2 - \frac{1}{4}\left(\frac{v+v_*}{2} + \sqrt{R(\frac{1}{4}|v-v_*|^2 + I + I_*)}\sigma\right)^2} \times$$
$$g\left(\frac{v+v_*}{2} - \sqrt{R(\frac{1}{4}|v-v_*|^2 + I + I_*)}\sigma, (1-R)(1-r)\left[\frac{1}{4}|v-v_*|^2 + I + I_*\right]\right) \times \qquad (28)$$
$$\frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}}(r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^{\alpha/2} I_*^\alpha \mathcal{B} \ \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*.$$

We seek then to write $\mathcal{K}_2$ in its kernel form. For this, we define $h_{v,I,r,R,\sigma}$; where for simplicity the index will be omitted; as

$$h : \mathbb{R}^3 \times \mathbb{R}_+ \longmapsto h(\mathbb{R}^3 \times \mathbb{R}_+) \subset \mathbb{R}^3 \times \mathbb{R}_+$$
$$(v_*, I_*) \longmapsto (x, y) = \left(\frac{v+v_*}{2} - \sqrt{R(\frac{1}{4}|v-v_*|^2 + I + I_*)}\sigma,\right.$$
$$\left.(1-R)(1-r)\left[\frac{1}{4}|v-v_*|^2 + I + I_*\right]\right),$$

for fixed $v, I, r, R,$ and σ. The function h is invertible, and (v_*, I_*, v', I') can be expressed in terms of (x, y) as

$$v_* = 2x + 2\sqrt{Ray}\sigma - v, \quad I_* = ay - I - (x - v + \sqrt{Ray}\sigma)^2,$$

and

$$v' = x + 2\sqrt{Ray}\sigma, \quad I' = \frac{r}{1-r}y,$$

where $a = \frac{1}{(1-r)(1-R)}$. The Jacobian of h^{-1} is computed to be

$$J = \left|\frac{\partial v_* \partial I_*}{\partial x \partial y}\right| = \frac{8}{(1-r)(1-R)}, \qquad (29)$$

and the positivity of I_* restricts the variation of the variables (x, y) in integral (28) to the space

$$H_{R,r,\sigma}^{v,I} = h(\mathbb{R}^3 \times \mathbb{R}_+) = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}_+ : ay - I - (x - v + \sqrt{Ray}\sigma)^2 > 0\}. \qquad (30)$$

In fact, $H_{R,r,\sigma}^{v,I}$ can be explicitly expressed as

$$H_{R,r,\sigma}^{v,I} = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}_+ : x \in B_{v-\sqrt{Ray}\sigma}(\sqrt{ay - I}) \text{ and } y \in ((1-r)(1-R)I, +\infty)\}.$$

Therefore, equation (28) becomes

$$\mathcal{K}_2 g = \frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}} \int_{(0,1)^2 \times S^2} \int_{H_{R,r,\sigma}^{v,I}} y^{-\alpha/2}(r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} I^{\alpha/2} I_*^\alpha \mathcal{B} J \times$$
$$g(x, y) e^{-\frac{ay - I - (x - v + \sqrt{Ray}\sigma)^2}{2} - \frac{r}{2(1-r)}y - \frac{1}{4}(2x + 2\sqrt{Ray}\sigma - v)^2 - \frac{1}{4}(x + 2\sqrt{Ray}\sigma)^2} \mathrm{d}y \mathrm{d}x \mathrm{d}\sigma \mathrm{d}r \mathrm{d}R. \qquad (31)$$

Now we point out the kernel form of $\mathcal{K}_2$ and prove after by the help of assumption (7) that the kernel of $\mathcal{K}_2$ is in $L^2(\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+)$. Indeed, we recall the definition of Δ, with $\Delta := (0,1)^2 \times S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$, and we define $H^{v,I}$ to be

$$H^{v,I} := \{(R,r,\sigma,x,y) \in \Delta \ : \ R \in (0,1), \ r \in (0,1), \ \sigma \in S^2, \ x \in B_{v-\sqrt{Ray}\sigma}(\sqrt{ay-I}),$$
$$\text{and } y \in ((1-r)(1-R)I,+\infty)\}.$$

We remark that $H^{v,I}_{R,r,\sigma}$ is a slice of $H^{v,I}$, and we define the slice $H^{v,I}_{x,y} \subset (0,1) \times (0,1) \times S^2$ such that $H^{v,I} = H^{v,I}_{x,y} \times \mathbb{R}^3 \times \mathbb{R}_+$. In particular,

$$H^{v,I}_{x,y} = \{(r,R,\sigma) \in (0,1) \times (0,1) \times S^2 : (y,x,\sigma,r,R) \in H^{v,I}\}. \tag{32}$$

Then by Fubini's theorem, it holds that

$$
\begin{aligned}
\mathcal{K}_2 g(v,I) &= \frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}} \int_{H^{v,I}} y^{-\alpha/2}(r(1-r))^\alpha(1-R)^{2\alpha+1}R^{1/2}I^{\alpha/2}I_*^\alpha \mathcal{B}Jg(x,y)\times \\
&\quad e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2} - \frac{r}{2(1-r)}y - \frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2 - \frac{1}{4}(x+2\sqrt{Ray}\sigma)^2} \,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}} \int_{\mathbb{R}^3\times\mathbb{R}_+} \int_{H^{v,I}_{x,y}} y^{-\alpha/2}(r(1-r))^\alpha(1-R)^{2\alpha+1}R^{1/2}I^{\alpha/2}I_*^\alpha \mathcal{B}J\times \\
&\quad g(x,y)e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2} - \frac{r}{2(1-r)}y - \frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2 - \frac{1}{4}(x+2\sqrt{Ray}\sigma)^2} \,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}y\mathrm{d}x.
\end{aligned}
\tag{33}
$$

The kernel of $\mathcal{K}_2$ is thus inspected and written explicitly in the following lemma.

Lemma 3. *Using assumptions (7),(8), and (10) on $\mathcal{B}$, the kernel of $\mathcal{K}_2$ given by*

$$
\begin{aligned}
k_2(v,I,x,y) &= \frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}} \int_{H^{v,I}_{x,y}} y^{-\alpha/2}(r(1-r))^\alpha(1-R)^{2\alpha+1}R^{1/2}I^{\alpha/2}I_*^\alpha \mathcal{B}J\times \\
&\quad e^{-\frac{ay-I-(x-v+\sqrt{Ray}\sigma)^2}{2} - \frac{r}{2(1-r)}y - \frac{1}{4}(2x+2\sqrt{Ray}\sigma-v)^2 - \frac{1}{4}(x+2\sqrt{Ray}\sigma)^2} \,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma
\end{aligned}
$$

is in $L^2(\ \mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+)$.

Proof. Applying Cauchy-Schwarz inequality we get

$$
\begin{aligned}
\|k_2\|^2_{L^2} &\le c \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{H^{v,I}_{x,y}} y^{-\alpha}(r(1-r))^{2\alpha}(1-R)^{4\alpha+2}RI^\alpha I_*^{2\alpha}J^2\mathcal{B}^2\times \\
&\quad e^{-[ay-I-(x-v+\sqrt{Ray}\sigma)^2] - \frac{r}{(1-r)}y - \frac{1}{2}(2x+2\sqrt{Ray}\sigma-v)^2 - \frac{1}{2}(x+2\sqrt{Ray}\sigma)^2} \,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v.
\end{aligned}
$$

By means of h^{-1} we have then

$$
\begin{aligned}
\|k_2\|^2_{L^2} &\le c \int_{(\mathbb{R}_+\times\mathbb{R}^3)^2} \int_{(0,1)^2\times S^2} E^{-\alpha}e^{-I_* - \frac{1}{2}v_*^2 - r(1-R)\left(\frac{|v-v_*|^2}{4}+I+I_*\right) - \frac{1}{2}\left(\frac{v+v_*}{2}+\sqrt{R(\frac{1}{4}|v-v_*|^2+I+I_*)}\sigma\right)^2}\times \\
&\quad r^{2\alpha}(1-r)^\alpha(1-R)^{3\alpha+2}RI^\alpha I_*^{2\alpha}J\mathcal{B}^2\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}I\mathrm{d}v.
\end{aligned}
\tag{34}
$$

Using the inequality

$$I^\alpha \le \left(\frac{1}{4}|v-v_*|^2+I+I_*\right)^\alpha = E^\alpha, \tag{35}$$

we eliminate $E^{-\alpha}$ and I^α from the above integral. Furthermore, if $\gamma \ge 0$, we use assumption (7) on $\mathcal{B}$ together with the inequality

$$|v-v_*|^{2\gamma}+I^\gamma + I_*^\gamma \le cE^\gamma, \tag{36}$$

and if $-2 - 2\alpha < \gamma < 0$, we use assumption (8). In both cases we get

$$\|k_2\|_{L^2}^2 \leq c \int_{(0,1)^2 \times S^2} \int_{(\mathbb{R}_+ \times \mathbb{R}^3)^2} e^{-I_* - \frac{1}{2}|v_*|^2 - r(1-R)\left(\frac{|v-v_*|^2}{4} + I + I_*\right) - \frac{1}{2}\left(\frac{v+v_*}{2} + \sqrt{R(\frac{1}{4}|v-v_*|^2 + I + I_*)}\sigma\right)^2} \times$$

$$\Psi_\gamma^2(r,R)E^\gamma r^{2\alpha}(1-r)^\alpha(1-R)^{3\alpha+2}JRI_*^{2\alpha}\mathrm{d}I\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma. \tag{37}$$

We remark that choosing α to be the power of the measure of integral (4), is essential for eliminating I^α from (34), which is not integrable. This elimination is possible thanks to (35). Perform now the change of variable $I \longmapsto E = I + I_* + \frac{1}{4}|v - v_*|^2$, then as $\mathrm{d}I = \mathrm{d}E$, (37) becomes

$$\|k_2\|_{L^2}^2 \leq c \int_{(0,1)^2 \times S^2} \int_{(\mathbb{R}_+ \times \mathbb{R}^3)^2} e^{-I_* - \frac{1}{2}|v_*|^2 - r(1-R)E - \frac{1}{2}\left(\frac{v+v_*}{2} + \sqrt{RE}\sigma\right)^2} \times$$

$$\Psi_\gamma^2(r,R)r^{2\alpha}(1-r)^{\alpha-1}(1-R)^{3\alpha+1}RI_*^{2\alpha}E^\gamma\mathrm{d}E\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma$$

$$= c \int_{(0,1)^2} \int_{\mathbb{R}^3} \int_{(\mathbb{R}_+)^2} e^{-I_* - \frac{1}{2}|v_*|^2 - r(1-R)E} \left[\int_{S^2} \int_{\mathbb{R}^3} e^{-\frac{1}{2}\left(\frac{v+v_*}{2} + \sqrt{RE}\sigma\right)^2}\mathrm{d}v\mathrm{d}\sigma\right] \times$$

$$\Psi_\gamma^2(r,R)r^{2\alpha}(1-r)^{\alpha-1}(1-R)^{3\alpha+1}RI_*^{2\alpha}E^\gamma\mathrm{d}E\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R. \tag{38}$$

Let $\tilde{V} = \frac{v}{2} + \frac{v_*}{2} + \sqrt{RE}\sigma$. Performing this change of variable in v, we get

$$\|k_2\|_{L^2}^2 \leq c \int_{(0,1)^2} \int_{\mathbb{R}^3} \int_{(\mathbb{R}_+)^2} e^{-I_* - \frac{1}{2}|v_*|^2 - r(1-R)E} \left[\int_{\mathbb{R}^3} \int_{S^2} e^{-\frac{1}{2}|\tilde{V}|^2}\mathrm{d}\tilde{V}\mathrm{d}\sigma\right] \times$$

$$\Psi_\gamma^2(r,R)r^{2\alpha}(1-r)^{\alpha-1}(1-R)^{3\alpha+1}RI_*^{2\alpha}E^\gamma\mathrm{d}E\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R. \tag{39}$$

Therefore the integral in (39) becomes

$$\|k_2\|_{L^2}^2 \leq \int_{(0,1)^2} \left[\int_{\mathbb{R}_+} E^\gamma e^{-r(1-R)E}\mathrm{d}E\right] \Psi_\gamma^2(r,R)r^{2\alpha}(1-r)^{\alpha-1}(1-R)^{3\alpha+1}R\mathrm{d}r\mathrm{d}R$$

$$\leq c \int_{(0,1)^2} \Psi_\gamma^2(r,R)r^{2\alpha-1-\gamma}(1-r)^{\alpha-1}R(1-R)^{3\alpha-\gamma}\mathrm{d}r\mathrm{d}R. \tag{40}$$

By (10), the integral

$$\int_{(0,1)^2} \Psi_\gamma^2(r,R)r^{2\alpha-1-\gamma}(1-r)^{\alpha-1}R(1-R)^{3\alpha-\gamma}\mathrm{d}r\mathrm{d}R < \infty. \tag{41}$$

This implies that $\mathcal{K}_2$ is a Hilbert-Schmidt operator. $\qquad\square$

Compactness of $\mathcal{K}_3$. The proof of the compactness of $\mathcal{K}_3$ (24) is similar to that of $\mathcal{K}_2$. The operator $\mathcal{K}_3$ which has the explicit form

$$\mathcal{K}_3 g(v, I) = \int_\Delta e^{-\frac{I_*}{2} - \frac{1}{2}(1-r)(1-R)\left(\frac{|v-v_*|^2}{4} + I + I_*\right)} e^{-\frac{1}{4}|v_*|^2 - \frac{1}{4}\left(\frac{v+v_*}{2} - \sqrt{R(\frac{1}{4}|v-v_*|^2 + I + I_*)}\sigma\right)^2} \times$$

$$(I')^{-\alpha} g\left(\frac{v+v_*}{2} + \sqrt{R(\frac{1}{4}|v-v_*|^2 + I + I_*)}\sigma, r(1-R)\left[\frac{1}{4}|v-v_*|^2 + I + I_*\right]\right) \times$$

$$\frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}}(r(1-r))^\alpha(1-R)^{2\alpha+1}R^{1/2}I^{\alpha/2}I_*^\alpha\mathcal{B}\ \mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*,$$

inherits the same form as $\mathcal{K}_2$, with a remark that the Jacobian of the needed transformation

$$\tilde{h} : \mathbb{R}^3 \times \mathbb{R}_+ \longmapsto \mathbb{R}^3 \times \mathbb{R}_+$$

$$(v_*, I_*) \longmapsto (x, y) = \left(\frac{v+v_*}{2} + \sqrt{R(\frac{1}{4}|v-v_*|^2 + I + I_*)}\sigma,\right.$$

$$\left. r(1-R)\left[\frac{1}{4}|v-v_*|^2 + I + I_*\right]\right),$$

is calculated to be

$$\tilde{J} = \frac{8}{r(1-R)}.$$

For the kernel of $\mathcal{K}_3$ to be L^2 integrable, the final computations require

$$\Psi_\gamma^2(r,R)(1-r)^{2\alpha-1-\gamma}r^{\alpha-1}R(1-R)^{3\alpha-\gamma} \quad \in L^1((0,1)^2). \tag{42}$$

Applying the change of variable $r \mapsto 1 - r$, and using the symmetry assumption (9) of Ψ_γ, (42) is satisfied by (10).

To this extent, the perturbation operator $\mathcal{K}$ is proved to be Hilbert-Schmidt, and thus $\mathcal{K}$ is a compact operator.

5 Properties of the Collision Frequency

In this section, we give some properties of ν. The first is the coercivity property, which implies that $\mathcal{L}$ is a Fredholm operator, and we prove the monotonicity of ν which depends on the choice of the transition function $\mathcal{B}$. The latter property is usually used in the monatomic case for locating the essential spectrum of $\mathcal{L}$ (see [16] Chapter 4).

Proposition 4 (Coercivity of ν Id). *With the assumption* (6)*, there exists $c > 0$ such that*

$$\nu(v,I) \geq c(|v|^\gamma + I^{\gamma/2} + 1),$$

for any $\gamma \geq 0$. As a result, the multiplication operator ν Id is coercive.

Proof. The collision frequency (21) is

$$\nu(v,I) = \frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}} \int_\Delta \mathcal{B}I_*^\alpha(r(1-r))^\alpha(1-R)^{2\alpha+1}R^{1/2}e^{-I_* - \frac{1}{2}|v_*|^2} \,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*,$$

where by (6) we get

$$\nu(v,I) \geq c \int_{\mathbb{R}^3} \left(|v - v_*|^\gamma + I^{\gamma/2}\right) e^{-\frac{1}{2}|v_*|^2} \,\mathrm{d}v_*$$

$$\geq c\left(I^{\gamma/2} + \int_{\mathbb{R}^3} ||v| - |v_*||^\gamma e^{-\frac{1}{2}|v_*|^2} \,\mathrm{d}v_*\right),$$

where c is a generic constant. We consider the two cases, $|v| \geq 1$ and $|v| \leq 1$. If $|v| \geq 1$ we have

$$\nu(v,I) \geq c\left(I^{\gamma/2} + \int_{|v_*| \leq \frac{1}{2}|v|} (|v| - |v_*|)^\gamma e^{-\frac{1}{2}|v_*|^2} \,\mathrm{d}v_*\right) \geq c\left(I^{\gamma/2} + |v|^\gamma \int_{|v_*| \leq \frac{1}{2}} e^{-\frac{1}{2}|v_*|^2}\mathrm{d}v_*\right) \geq c(|v|^\gamma + I^{\gamma/2} + 1).$$

For $|v| \leq 1$,

$$\nu(v,I) \geq c\left(I^{\gamma/2} + \int_{|v_*| \geq 2} (|v_*| - |v|)^\gamma e^{-\frac{1}{2}|v_*|^2} \,\mathrm{d}v_*\right) \geq c\left(I^{\gamma/2} + \int_{|v_*| \geq 2} e^{-\frac{1}{2}|v_*|^2} \,\mathrm{d}v_*\right) \geq c(1 + I^{\gamma/2} + |v|^\gamma).$$

$\square$

As a result of Theorem 1 and Proposition 4, $\mathcal{L}$ is a Fredholm operator for $\gamma \geq 0$ under assumptions (6),(7), (9), and (10). We now give the following proposition, which is a generalization of the work of Grad [28], in which he proved that the collision frequency of monatomic single gases is monotonic based on the choice of the transition function $\mathcal{B}$.

Proposition 5 (monotonicity of ν). *Under the assumption that*

$$\int_{(0,1)^2 \times S^2} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} \mathcal{B}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \tag{43}$$

is increasing (respectively decreasing) in $|V|$ and I for every I_*, the collision frequency ν is increasing (respectively decreasing), where $|V| = |v - v_*|$.

Remark 4. *If $\mathcal{B}$ is increasing (respectively decreasing) in $|V|$ and I, then (43) is increasing (respectively decreasing) in $|V|$ and I.*

In particular, for Maxwell molecules, where $\mathcal{B}$ is constant in $|V|$ and I, ν is constant. On the other hand, for transition functions of the form

$$\mathcal{B}(v, v_*, I, I_*, r, R, \sigma) = \Phi_\gamma(r, R)\Big(|v - v_*|^\gamma + I^{\gamma/2} + I_*^{\gamma/2}\Big),$$

integral (43) is increasing, and thus ν is increasing, $\gamma \geq 0$, and Φ_γ is a positive function such that

$$\Phi_\gamma(r, R) = \Phi_\gamma(1 - r, R),$$

and

$$\Phi_\gamma(r, R)(r(1-r))^\alpha R^{1/2}(1-R)^{2\alpha+1} \in L^1((0,1)^2).$$

In fact, if Φ_γ for instance satisfies

$$\Phi_\gamma^2(r, R) r^{\alpha-1}(1-r)^{2\alpha-1-\gamma} R(1-R)^{3\alpha-\gamma} \in L^1((0,1)^2)$$

then this transition function satisfies our main assumptions (6)-(10).

Proof. We remark first that ν is a radial function in $|v|$ and I. In fact, we perform the change of variable $V = v - v_*$ in the integral (21), where the expression of ν becomes

$$\nu(|v|, I) = \frac{1}{\Gamma(\alpha+1)(2\pi)^{3/2}} \int_\Delta \mathcal{B}(|V|, I, I_*, r, R, \sigma) I_*^\alpha (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} e^{-\frac{1}{2}|v-V|^2 - I_*} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}V, \tag{44}$$

where $\Delta = \mathbb{R}^3 \times \mathbb{R}_+ \times S^2 \times (0,1)^2$. The integration in V in the above integral (44) is carried in the spherical coordinates of V, with fixing one of the axes of the reference frame along v, and therefore, the above integral will be a function of $|v|$ and I.

The partial derivative of ν in the v_i direction is

$$\frac{\partial \nu}{\partial v_i} = \int_\Delta \frac{I_*^\alpha (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2}}{\Gamma(\alpha+1)(2\pi)^{3/2}} \frac{v_i - v_{*i}}{|v - v_*|} \frac{\partial \mathcal{B}}{\partial |v - v_*|}(|v - v_*|, I, I_*, r, R, \sigma) e^{-\frac{1}{2}|v_*|^2 - I_*} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*. \tag{45}$$

Perform the change of variable $V = v - v_*$ in (45), then

$$\frac{\partial \nu}{\partial v_i} = \int_\Delta \frac{I_*^\alpha (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2}}{\Gamma(\alpha+1)(2\pi)^{3/2}} \frac{V_i}{|V|} \frac{\partial \mathcal{B}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) e^{-\frac{1}{2}|v-V|^2 - I_*} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}V,$$

and thus,

$$\sum_{i=1}^3 v_i \frac{\partial \nu}{\partial v_i} = \int_\Delta \frac{I_*^\alpha (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2}}{\Gamma(\alpha+1)(2\pi)^{3/2}} \frac{v \cdot V}{|V|} \frac{\partial \mathcal{B}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) e^{-\frac{1}{2}|v-V|^2 - I_*} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}V. \tag{46}$$

Applying Fubini's theorem, we write (46) as

$$\sum_{i=1}^{3} v_i \frac{\partial \nu}{\partial v_i} = \int_{\mathbb{R}_+ \times \mathbb{R}^3} \left[\int_{(0,1)^2 \times S^2} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} \frac{\partial \mathcal{B}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \right] \times$$
$$\frac{I_*^\alpha}{\Gamma(\alpha+1)(2\pi)^{3/2}} \frac{v \cdot V}{|V|} e^{-\frac{1}{2}|v-V|^2 - I_*} \mathrm{d}I_* \mathrm{d}V.$$

The partial derivative of ν along I is

$$\frac{\partial \nu}{\partial I} = \int_\Delta \frac{I_*^\alpha (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2}}{\Gamma(\alpha+1)(2\pi)^{3/2}} \frac{\partial \mathcal{B}}{\partial I}(|V|, I, I_*, r, R, \sigma) e^{-\frac{1}{2}|v-V|^2 - I_*} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}V$$
$$= \int_{\mathbb{R}_+ \times \mathbb{R}^3} \frac{I_*^\alpha}{\Gamma(\alpha+1)(2\pi)^{3/2}} \left[\int_{(0,1)^2 \times S^2} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} \frac{\partial \mathcal{B}}{\partial I}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \right] \times \quad (47)$$
$$e^{-\frac{1}{2}|v-V|^2 - I_*} \mathrm{d}I_* \mathrm{d}V.$$

When $v \cdot V > 0$, the exponential in integral (46) is greater than when $v \cdot V < 0$, and so the term $v \cdot V$ doesn't affect the sign of the partial derivatives of ν. Therefore, the sign of the partial derivative of ν along $|v|$ has the same sign as

$$\int_{(0,1)^2 \times S^2} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} \frac{\partial \mathcal{B}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma.$$

It's clear as well that the partial derivative of ν with respect to I (47) has the same sign as

$$\int_{(0,1)^2 \times S^2} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} \frac{\partial \mathcal{B}}{\partial I}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma.$$

As a result, for a transition function $\mathcal{B}$ satisfying the condition that the integral

$$\int_{(0,1)^2 \times S^2} (r(1-r))^\alpha (1-R)^{2\alpha+1} R^{1/2} \mathcal{B}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma$$

is increasing (respectively decreasing) in $|V|$ and I, the collision frequency is increasing (respectively decreasing). $\qquad \square$

6 Conclusion

With what proceeds, we conclude that for $\gamma > 0$, and under assumptions (7),(9), and (10), the operator $\mathcal{K}$ for a single polyatomic gas is a compact operator, and by assumption (6) the multiplication operator by ν is coercive. As a result, the linearized Boltzmann operator $\mathcal{L}$ is an unbounded self-adjoint Fredholm operator. For $-2 - 2\alpha < \gamma < 0$, under assumptions (8),(9), and (10), we conclude that $\mathcal{K}$ remains a compact operator.

In comparison with the monatomic case, if the ω- parameterization is used, one can notice that the change of variable maps $v_* \mapsto v_*'$ and $v_* \mapsto v'$ are not invertible for monatomic molecules, and therefore, it is not possible to respectively apply these changes of variables in the linearized Boltzmann operators K_2 and K_3 in order to extract the kernel. That is why in the literature, when the ω- representation has been used, K_2 and K_3 were written in the Carleman representation in order to extract the kernel of K. However, the extracted kernels are not L^2 integrable. Nevertheless, if the σ− representation is used in the monatomic case, the maps $v_* \mapsto v_*'$ and $v_* \mapsto v'$ are invertible, and the Hilbert-Schmidt property can be proved to be valid by the same approach, yet also under a stricter assumption on the collision cross-section (10).
In the polyatomic case, the change of variable maps $(v_*, I_*) \mapsto (v_*', I_*')$ and $(v_*, I_*) \mapsto (v', I')$ are both invertible with Jacobians $\frac{1}{(1-r)(1-R)}$ and $\frac{1}{r(1-R)}$ respectively (see (29)). In order to prove the L^2 compactness of

the kernel, the singularities coming from these Jacobians should be overcome by imposing further constraints on the transition function $\mathcal{B}$. From the expressions of these Jacobians, the constraints should be obviously set on the generic function $\Psi_\gamma(r, R)$, which is in the upper bound of B. In particular, the assumptions needed are (9) and (10).

A Appendix

We display the equivalence of the collision operator form (4) to the form (54), where in the latter, the preserved quantities E and G appear in the Lebesgue measure. The derivation of this formulation is a result of subsequent changes of variables, see (48). The final result sought is the Jacobian of the following map:

$$\mathcal{T} : \mathbb{R}^6 \times \mathbb{R}_+^2 \times (0,1)^2 \times S^2 \to \mathbb{R}^6 \times \mathbb{R}_+^2 \times \mathbb{R}^3 \times \mathbb{R}_+$$
$$(v, v_*, I, I_*, r, R, \sigma) \mapsto (v, G, E, I, v', I'), \tag{48}$$

where $g = v - v_*$ and $G = \frac{v + v_*}{2}$. For this transformation, the following Jacobians are elementary:

$$J_{(v, v_* I, I_*, r, R, \sigma) \mapsto (g, G, I, I_*, r, R, \sigma)} = 1,$$

and

$$J_{(g, G, I, I_*, r, R, \sigma) \mapsto (g, G, I, E, r, R, \sigma)} = 1. \tag{49}$$

Equation (49) is due to the fact that only E is a function of I_*. What remains in deducing the Jacobian of $\mathcal{T}$ is calculating the Jacobian of the transformation $(g, G, I, E, r, R, \sigma) \mapsto (v, G, I, E, v', I')$. As an intermediate step we define

$$\lambda = \sqrt{RE}, \quad \mu = r(1 - R),$$

which induces the Jacobian

$$J_{(g, G, I, E, r, R, \sigma) \mapsto (g, G, I, E, \lambda, \mu, \sigma)} = \frac{1}{2} \frac{(1 - R)}{\sqrt{R}} \sqrt{E}.$$

Thus the final sub-transformation is $(g, G, I, E, \lambda, \mu, \sigma) \mapsto (v, G, I, E, v', I')$, where specifically,

$$v' = G + \lambda \sigma, \quad \text{and} \quad I' = \mu E. \tag{50}$$

It's clear that

$$J_{(g, G, I, E, \lambda, \mu, \sigma) \mapsto (g, G, I, E, \lambda, I', \sigma)} = E$$

and for v' we have

$$J_{(g, G, I, E, \lambda, I', \sigma) \mapsto (g, G, I, E, v', I')} = \lambda^2 = RE, \tag{51}$$

since (λ, σ) is the spherical representation of $v' - G$. As $v = \frac{1}{2}g + G$, then the Jacobian

$$J_{(g, G, I, E, v', I') \mapsto (v, G, I, E, v', I')} = \frac{1}{8}. \tag{52}$$

Finally, combining the preceding transformations, the Jacobian of $\mathcal{T}$ is

$$J_\mathcal{T} = \frac{1}{16} R^{1/2} (1 - R) E^{5/2}. \tag{53}$$

In other words,

$$\mathrm{d}v \mathrm{d}G \mathrm{d}I \mathrm{d}E \mathrm{d}v' \mathrm{d}I' = \frac{1}{16} R^{1/2} (1 - R) E^{5/2} \mathrm{d}v \mathrm{d}v_* \mathrm{d}I \mathrm{d}I_* \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma.$$

The equivalent model of (4), based on the above computations is therefore

$$Q(f,f)(v,I) = \int_{(\mathbb{R}^3 \times \mathbb{R}_+)^2} \left(\frac{f'f'_*}{(I'I'_*)^\alpha} - \frac{ff_*}{(II_*)^\alpha} \right) W(v,I,v',I',G,E)\, \mathrm{d}G\mathrm{d}E\, \mathrm{d}v'\mathrm{d}I', \tag{54}$$

where

$$W(v,I,v',I',G,E) = 16(I'I'_*II_*)^\alpha E^{-\frac{5}{2}-2\alpha}\mathcal{B}(v,v_*,I,I_*,r,R,\sigma), \tag{55}$$

where $I_* = I_*(v,I,G,E)$, $I'_* = I'_*(v',I',G,E)$, $v'_* = v'_*(G,v')$, $v_* = v_*(G,v)$, $\sigma = \sigma(v',G)$, $R = R(v',E,G)$, and $r = r(I',v',E,G)$.

Moreover, W in (55) is clearly microreversible, and the measure $\mathrm{d}E\mathrm{d}G\mathrm{d}v\mathrm{d}I\mathrm{d}v'\mathrm{d}I'$ is invariant if time is reversed.

References

[1] Pierre Andries, Patrick Le Tallec, Jean-Philippe Perlat, and Benoît Perthame. The Gaussian-BGK model of Boltzmann equation with small Prandtl number. *European Journal of Mechanics - B/Fluids*, 19(6):813–830, 2000.

[2] Céline Baranger, Marzia Bisi, Stéphane Brull, and Laurent Desvillettes. On the Chapman-Enskog asymptotics for a mixture of monoatomic and polyatomic rarefied gases. *Kinetic and Related Models*, 11(4):821–858, 2018.

[3] Céline Baranger and Clément Mouhot. Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials. *Revista Matematica Iberoamericana*, 21:819–841, 2005.

[4] Niclas Bernhoff. Linearized boltzmann collision operator: I. Polyatomic molecules modeled by a discrete internal energy variable and multicomponent mixtures. *Acta Applicandae Mathematicae*, 183(1), 2022.

[5] Niclas Bernhoff. Linearized Boltzmann collision operator: II. Polyatomic molecules modeled by a continuous internal energy variable. *Kinetic and Related Models*, 2023.

[6] Marzia Bisi and María J. Cáceres. A BGK relaxation model for polyatomic gas mixtures. *Communications in Mathematical Sciences*, 14:297–325, 2016.

[7] Marzia Bisi, Tommaso Ruggeri, and Giampiero Spiga. Dynamical pressure in a polyatomic gas: Interplay between kinetic theory and extended thermodynamics. *Kinetic and Related Models*, 11(2):1–25, 2018.

[8] Alexander Bobylev. Fourier transform method in the theory of the Boltzmann equation for Maxwellian molecules. *Akademiia Nauk SSSR Doklady*, 225:1041–1044, 1975.

[9] Alexander Bobylev. The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. *Mathematical Physics Reviews*, 7:111–233, 1988.

[10] Claus Borgnakke and Poul S. Larsen. Statistical collision model for Monte Carlo simulation of polyatomic gas mixture. *Journal of Computational Physics*, 18(4):405–420, 1975.

[11] Thomas Borsoni, Laurent Boudin, and Francesco Salvarani. Compactness property of the linearized Boltzmann operator for a polyatomic gas undergoing resonant collisions. *Journal of Mathematical Analysis and Applications*, 517(1):126579, 2023.

[12] Laurent Boudin, Bérénice Grec, Milana Pavić-Čolić, and Francesco Salvarani. Diffusion asymptotics of a kinetic model for gaseous mixtures. *Kinetic and Related Models*, 6(1):137–157, 2013.

[13] Jean-François Bourgat, Laurent Desvillettes, Patrick Le Tallec, and Benoît Perthame. Microreversible collisions for polyatomic gases and Boltzmann's theorem. *European Journal of Mechanics B/fluids*, 13:237–254, 1994.

[14] Stéphane Brull and Jacques Schneider. On the ellipsoidal statistical model for polyatomic gases. *Continuum Mechanics and Thermodynamics*, 20(8):489–508, 2009.

[15] Stéphane Brull, Marwa Shahine, and Philippe Thieullen. Compactness property of the linearized Boltzmann operator for a diatomic single gas model. *Networks and Heterogeneous Media*, 17(6):847–861, 2022.

[16] Carlo Cercignani. *The Boltzmann Equation and Its Applications*. Springer-Verlag, 1988.

[17] Carlo Cercignani, Reinhard Illner, and Mario Pulvirenti. *The Mathematical Theory of Dilute Gases*. Springer New York, NY, 1st edition, 1994.

[18] Sydney Chapman and Thomas G. Cowling. *The Mathematical Theory of Non-Uniform Gases*. Cambridge University Press, 3rd edition, 1990.

[19] Esther S. Daus, Ansgar Jüngel, Clément Mouhot, and Nicola Zamponi. Hypocoercivity for a linearized multispecies Boltzmann system. *SIAM Journal on Mathematical Analysis*, 48(1):538–568, 2016.

[20] Laurent Desvillettes, Roberto Monaco, and Francesco Salvarani. A kinetic model allowing to obtain the energy law of polytropic gases in the presence of chemical reactions. *European Journal of Mechanics - B/Fluids*, 24(2):219–236, 2005.

[21] Vladimir Djordjić, Milana Pavić-Čolić, and Nikola Spasojević. Polytropic gas modelling at kinetic and macroscopic levels. *Kinetic and Related Models*, 14(3):483–522, 2021.

[22] Hans Birger Drange. The linearized boltzmann collision operator for cut-off potentials. *SIAM Journal on Applied Mathematics*, 29(4):665–676, 1975.

[23] Alexandre Ern and Vincent Giovangigli. The kinetic chemical equilibrium regime. *Physica A: Statistical Mechanics and its Applications*, 260(1):49–72, 1998.

[24] Irene M. Gamba and Milana Pavić-Čolić. On the Cauchy problem for Boltzmann equation modeling a polyatomic gas. *Journal of Mathematical Physics*, 64(1):013303, 2023.

[25] Vincent Giovangigli. *Multicomponent Flow Modeling*. Birkhäuser Boston, MA, 1st edition, 1999.

[26] Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek. *Unbounded Fredholm Operators and Perturbation Theorems*, pages 369–381. Birkhäuser Basel, Basel, 1990.

[27] François Golse. The Boltzmann equation and its hydrodynamic limits. In *Handbook of Differential Equations Evolutionary Equations*, volume 2, pages 159–301. North-Holland, 2005.

[28] Harold Grad. Asymptotic theory of the Boltzmann equation II. *Rarefied Gas Dynamics*, 1:26–29, 1963.

[29] Clément Mouhot and Robert M. Strain. Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff. *Journal de Mathématiques Pures et Appliquées*, 87(5):515–535, 2007.

[30] Raj K. Pathria and Paul D. Beale. *The Theory of Simple Gases*. Statistical Mechanics, 3rd edition, 2011.

[31] Milana Pavić, Tommaso Ruggeri, and Srboljub Simić. Maximum entropy principle for rarefied polyatomic gases. *Physica A: Statistical Mechanics and its Applications*, 392(6):1302–1317, 2013.

[32] Tommaso Ruggeri. Non-linear maximum entropy principle for a polyatomic gas subject to the dynamic pressure. *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, 11:1–22, 2016.

[33] Tommaso Ruggeri and Masaru Sugiyama. *Rational Extended Thermodynamics beyond the Monatomic Gas*. Springer, New York, 2015.

[34] Seok-Bae Yun. Ellipsoidal BGK model for polyatomic molecules near Maxwellians: A dichotomy in the dissipation estimate. *Journal of Differential Equations*, 266(9):5566–5614, 2019.