Stability theory for the linear symmetric hyperbolic system with general relaxation

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Analysis, modeling and numerical method
for kinetic and related models

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Key Words: Stability Condition, Regularity-loss structure

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1. Introduction

Symmetric hyperbolic systems with relaxation:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \qquad \text{(SHS)}$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$.

Assume that

(a) A^0 is symmetric and positive definite,

(b) A^j is symmetric for each j,

(c) L is symmetric and non-negative definite.

Applying the Fourier transform, we obtain

$$A^0 \hat{u}_t + i|\xi| A(\omega)\hat{u} + L\hat{u} = 0,$$

where $A(\omega) = \sum_{j=1}^{n} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^{n-1}$.

Eigenvalue problem

Eigenvalue problem:

$$\lambda A^0\varphi + (i|\xi|A(\omega) + L)\varphi = 0.$$

$\lambda = \lambda(|\xi|, \omega)$: Eigenvalue, $\varphi = \varphi(|\xi|, \omega)$: Eigenvector.

Jin-Xin model:

$$\rho_t + v_x = 0,$$

$$v_t + \rho_x + v = 0.$$

We rewrite Jin-Xin model that

$$u_t + Au_x + Lu = 0,$$

where

$$u = \begin{pmatrix} \rho \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Stability conditions

Condition(SC): Shizuta & Kawashima (1985)

For any $(\mu,\omega) \in \mathbb{R} \times S^{n-1}$, $\mathrm{Ker}(\mu I + A(\omega)) \cap \mathrm{Ker}(L) = \{0\}$.

Condition(R): Kalman,Ho & Narendra (1963), Beauchard & Zuazua (2011)

For any $\omega \in S^{n-1}$,

$$\mathrm{rank}\begin{pmatrix} L \\ L(A^0)^{-1}A(\omega) \\ \vdots \\ L((A^0)^{-1}A(\omega))^{m-1} \end{pmatrix} = m.$$

Condition(K): Umeda, Kawashima & Shizuta (1984)

There exists $K(\omega)$ with the following properties:

(i) $K(-\omega) = -K(\omega)$. (ii) $K(\omega)A^0$ is skew-symmetric.

(iii) $L + (K(\omega)A(\omega))^\sharp$ is positive definite.

Here $X^\sharp$ is the symmetric part of X.

Characterization and decay estimate

Theorem 1.1 (Characterization for the dissipative structure)

The following conditions are equivalent.

(i) $\mathrm{Condition(SC)}$. (ii) $\mathrm{Condition(R)}$. (iii) $\mathrm{Condition(K)}$.

(iv) $\mathrm{Re}\lambda(|\xi|,\omega) < 0$ *for* $|\xi| \neq 0$. (v) $\mathrm{Re}\lambda(|\xi|,\omega) \leq -c|\xi|^2/(1+|\xi|^2)$.

Theorem 1.2 (Decay estimate)

Under $\mathrm{Condition(K)}$, *the solutions to* (SHS) *satisfy the pointwise estimate*

$$|\hat{u}(\xi,t)| \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|,$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. *Namely, we obtain*

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}, \quad k \geq 0.$$

Q: Can we extend these conditions for (SHS) with non-symmetric L??

Examples

Dissipative Timoshenko system(linear):

$$\phi_{tt} - (\phi_x - \psi)_x = 0,$$
$$\psi_{tt} - \psi_{xx} - (\phi_x - \psi) + \psi_t = 0.$$

Putting $\rho = \phi_x - \psi$, $v = \phi_t$, $z = \psi_x$, $y = \psi_t$, we obtain the symmetric hyperbolic system

$$u_t + Au_x + Lu = 0,$$

where $u = (\rho, v, z, y)^\top$ and

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

2. General stability condition

Symmetric hyperbolic systems with relaxation:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0. \qquad \text{(SHS)}$$

Assume that

(a) A^0 is symmetric and positive definite,

(b) A^j is symmetric for each j,

(c) $L^\sharp$ is non-negative definite (not necessary symmetric).

$$X^\sharp := \frac{1}{2}(X + X^*), \qquad X^\flat := \frac{1}{2}(X - X^*), \qquad X^* := \bar{X}^\top$$

ODE in Fourier space:

$$A^0 \hat{u}_t + i|\xi| A(\omega)\hat{u} + L\hat{u} = 0.$$

$A(\omega) = \sum_{j=1}^{n} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^{n-1}.$

Eigenvalue problem:

$$\lambda A^0 \varphi + (irA(\omega) + L)\varphi = 0. \qquad \text{(EP)}$$

$\lambda = \lambda(r,\omega)$: Eigenvalue, $\varphi = \varphi(r,\omega)$: Eigenvector.

Remark 2.1

$\mathrm{Re}\lambda(r,\omega) \le 0$ for $r \ge 0$, $\omega \in S^{n-1}$

Indeed, taking a $\mathbb{C}^m$ inner product (EP) with φ, and taking a real part for the resultant equation, we obtain

$$\mathrm{Re}\lambda\langle A^0\varphi, \varphi\rangle + \langle L^\sharp\varphi, \varphi\rangle = 0.$$

Here, we used the symmetric property for $A(\omega)$. Therefore, since A^0 is positive definite, and $L^\sharp$ is non-negative definite, we arrive at Remark 2.1.

New stability conditions

Define $\mathcal{A}(\nu,\omega) := (A^0)^{-1}(\nu A(\omega) - iL^\flat)$.

Here, $\nu A(\omega) - iL^\flat$ is a complex valued Hermitan matrix.

Stability Condition(GSC):

For any $(\mu,\nu,\omega) \in \mathbb{R} \times \mathbb{R}_+ \times S^{n-1}$,

$$\mathrm{Ker}(\mu I + \mathcal{A}(\nu,\omega)) \cap \mathrm{Ker}(L^\sharp) = \{0\}$$

Kalman Rank Condition(GR):

For any $(\nu,\omega) \in \mathbb{R}_+ \times S^{n-1}$,

$$\mathrm{rank}\begin{pmatrix} L^\sharp \\ L^\sharp \mathcal{A}(\nu,\omega) \\ \vdots \\ L^\sharp \mathcal{A}(\nu,\omega)^{m-1} \end{pmatrix} = m.$$

Characterization for the strict dissipativity

Craftsmanship Condition(GK):

There exists $\mathcal{K}(\nu,\omega) \in C(\mathbb{R}_+ \times S^{n-1})$ with the following properties:

(i) $\bar{\mathcal{K}}(\nu,-\omega) = -\mathcal{K}(\nu,\omega)$. (ii) $\mathcal{K}(\nu,\omega)^* = -\mathcal{K}(\nu,\omega)$.

(iii) $\exists C_K$ s.t. $\|\mathcal{K}(\nu,\omega)\| \le C_K$ for $(\nu,\omega) \in \mathbb{R}_+ \times S^{n-1}$.

(iv) $\exists c_K$ s.t.

$$\langle (L^\sharp + (\mathcal{K}(\nu,\omega)\mathcal{A}(\nu,\omega))^\sharp)\sigma,\sigma\rangle > \frac{c_K\nu^2}{1+\nu^2}|\mathcal{K}(\nu,\omega)\sigma|^2$$

for $(\nu,\omega,\sigma) \in \mathbb{R}_+ \times S^{n-1} \times \mathbb{S}^{m-1}$, where $\mathbb{S}^{m-1} := \{\sigma \in \mathbb{C}^m; |\sigma| = 1\}$.

Theorem 2.2 (Characterization for the strict dissipativity,U(2018))

The following conditions are equivalent.

(i) Condition(GSC). (ii) Condition(GR). (iii) Condition(GK).

(iv) $\mathrm{Re}\lambda(r,\omega) < 0$ *for* $r > 0$, $\omega \in S^{n-1}$ (called Strictly dissipative).

Characterization for the strict dissipativity

Theorem 2.3 (Characterization for the strict dissipativity, U(2021))

Let $n = 1$. The following condition is equivalent to (i)–(iv)*.*

(v) $\mathrm{Re}\lambda(r, \omega) \le -cr^{2(m-1)}/(1+r^2)^{2(m-1)}$ *for* $\omega \in S^{n-1}$ (called Uniformly dissipative).

Let $n \ge 2$. Suppose that $\mathrm{Ker}\big(L^\sharp G\big(((A^0)^{-1}A(\omega))^\ell, (-i(A^0)^{-1}L^\flat)^{k-\ell}\big)\big)$ *does not depend on* $\omega \in S^{n-1}$*. The conditions* (i)–(v) *are equivalent.*

For matrices X and Y, we define $(X+Y)^k = \sum_{\ell=0}^k G(X^\ell, Y^{k-\ell})$, where $G(X^\ell, Y^{k-\ell})$ denotes a polynomial of X and Y, which degrees of X and Y are ℓ and $k-\ell$, respectively. Then $\mathcal{A}(r,\omega)^k$ is represented by

$$\begin{aligned}\mathcal{A}(r,\omega)^k &= \big((A^0)^{-1}(rA(\omega) - iL^\flat)\big)^k \\ &= \sum_{\ell=0}^k r^\ell G\big(((A^0)^{-1}A(\omega))^\ell, (-i(A^0)^{-1}L^\flat)^{k-\ell}\big).\end{aligned}$$

Decay estimate

Corollary 2.4 (Decay estimate, U(2021))

Under the condition (v) in Theorem 2.3, the solutions to (SHS) satisfy the pointwise estimate

$$|\hat{u}(\xi,t)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \qquad \rho(\xi) = \frac{|\xi|^{2(m-1)}}{(1+|\xi|^2)^{2(m-1)}}.$$

Namely, we obtain

$$\|u(t)\|_{L^2} \le C(1+t)^{-\frac{n}{4(m-1)}}\|u_0\|_{L^1} + C(1+t)^{-\frac{\ell}{2(m-1)}}\|\partial_x^\ell u_0\|_{L^2}, \quad \ell \ge 0.$$

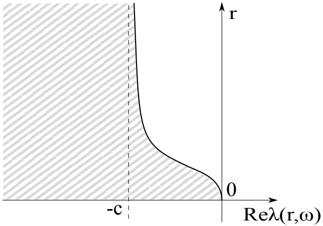


Figure: Standard type

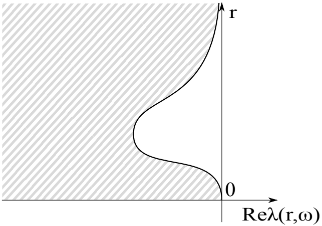


Figure: Regularity-loss type

Remarks

Remark 2.5

The pointwise estimate in Corollary 4.4 might not be optimal.
(e.g. Timoshenko system is $\mathrm{Re}\lambda(r,\omega) \le -cr^2/(1+r^2)^2$ but $m = 4$.

Outline of the proof of Theorems 2.2 and 2.3:

♣ $\mathrm{Re}\lambda(r,\omega) < 0 \Longleftrightarrow$ Condition(GSC) $\Longleftrightarrow$ Condition(GR)

($\because$ Ccontradiction argument and the Celey-Hamilton theorem.)

♣ Condition(GR) $\Longrightarrow$ Condition(GK) $\Longrightarrow \mathrm{Re}\lambda(r,\omega) < 0$

($\because$ Use the energy method.)

♣ Condition(GR) $\Longrightarrow \mathrm{Re}\lambda(r,\omega) \le -c\dfrac{r^{2(m-1)}}{(1+r^2)^{2(m-1)}}$

$$\Longrightarrow \mathrm{Re}\lambda(r,\omega) < 0$$

($\because$ Construct the Lyapunov function.)

Lyapunov function

Let κ be a small positive number. Then we chose κ_k such that

$$0 = \kappa_0 < \kappa_1 < \cdots < \kappa_m, \qquad \kappa_k - \frac{1}{2}(\kappa_{k-1} + \kappa_{k+1}) \geq \kappa > 0.$$

Lemma 2.6 (Lyapunov function)

Define

$$\mathcal{E}(\hat{u}) := \langle A^0 \hat{u}, \hat{u} \rangle + \delta h(|\xi|, \omega) \sum_{k=1}^{m-1} \varepsilon^{\kappa_k} \frac{\mathrm{Im}\langle L^\sharp \mathcal{A}(|\xi|, \omega)^{k-1} \hat{u}, L^\sharp \mathcal{A}(|\xi|, \omega)^k \hat{u} \rangle}{\|\mathcal{A}(|\xi|, \omega)\|^{2k}}$$

for $\delta > 0$ *and* $\varepsilon > 0$*, where*

$$h(|\xi|, \omega) := \frac{\|\mathcal{A}(|\xi|, \omega)\|^2}{(\|\mathcal{A}(|\xi|, \omega)\| + \|(A^0)^{-1}\| \|L^\sharp\|)^2}.$$

Then there exist δ_0 *and* ε_0 *such that*

$$\frac{\partial}{\partial t} \mathcal{E}(\hat{u}) + c_0 |L^\sharp \hat{u}|^2 + c_1 h(|\xi|, \omega) \sum_{k=1}^{m-1} \varepsilon^{\kappa_k} \frac{|L^\sharp \mathcal{A}(|\xi|, \omega)^k \hat{u}|^2}{\|\mathcal{A}(|\xi|, \omega)\|^{2k}} \leq 0,$$

and $c_* |\hat{u}|^2 \leq \mathcal{E}(\hat{u}) \leq C_* |\hat{u}|^2$ *for* $\delta = \delta_0$ *and* $0 < \varepsilon < \varepsilon_0$.

Energy estimate

Corollary 2.7

$$\frac{d}{dt}\mathcal{E}(\hat{u}) + c\mathcal{D}(|\xi|,\omega,\hat{u}) \le 0,$$

where

$$\mathcal{D}(|\xi|,\omega,\hat{u}) := \begin{cases} \displaystyle\sum_{k=0}^{m-1} \frac{1}{(1+|\xi|^2)^k} |L^\sharp \mathcal{A}(|\xi|,\omega)^k \hat{u}|^2 & \text{if } A(\omega) \neq O, L^\flat \neq O, \\ \displaystyle\frac{|\xi|^2}{1+|\xi|^2} \sum_{k=0}^{m-1} |L^\sharp A(\omega)^k \hat{u}|^2 & \text{if } A(\omega) \neq O, L^\flat = O, \\ \displaystyle\sum_{k=0}^{m-1} |L^\sharp (L^\flat)^k \hat{u}|^2 & \text{if } A(\omega) = O, L^\flat \neq O. \end{cases}$$

Remark 2.8

When we prove Lemma 2.6, we do not use the conditions in Theorem 2.2.

3. Optimality for the pointwise estimates in Fourier space

Analysis for the Low frequency part

In this section, we assume $A^0 = I$ and consider

$$\hat{u}_t + \mathfrak{A}(\xi)\hat{u} = 0, \quad \mathfrak{A}(\xi) := i|\xi|A(\omega) + L. \quad (\mathfrak{A}(\xi) = i\mathcal{A}(|\xi|, \omega) + L^\sharp)$$

Definition 3.1 (Orthogonal projections for the Low frequency part)

(1) $\mathbb{P}_{s_0} : \mathbb{C}^m \to \mathrm{Ker}\,(is_0 I + L)$

(2) $\mathbb{P}_{s_1,s_0,\omega} : \mathbb{P}_{s_0}\mathbb{C}^m \to \mathrm{Ker}\,\big((s_1 I + \mathbb{P}_{s_0}A(\omega))|_{\mathbb{P}_{s_0}\mathbb{C}^m}\big)$

(3) $\mathbb{P}_{s_2,s_1,s_0,\omega} : \mathbb{P}_{s_1,s_0,\omega}\mathbb{C}^m \to \mathrm{Ker}\Big(\big(is_2 I + \mathbb{P}_{s_1,s_0,\omega}L^{(1)}_{low}(s_0,\omega)\big)|_{\mathbb{P}_{s_1,s_0,\omega}\mathbb{C}^m}\Big)$,

where $L^{(1)}_{low}(s_0,\omega) = \mathbb{P}_{s_0}A(\omega)\big(is_0 I + L(\omega)\big)|^{-1}_{\mathbb{P}^\perp_{s_0}\mathbb{C}^m}\mathbb{P}^\perp_{s_0}A(\omega)|_{\mathbb{P}_{s_0}\mathbb{C}^m}$

Here the notation $\mathbb{F} : Y \to Z$ denotes that $\mathbb{F}$ is the orthogonal projection from the subspace Y of $\mathbb{C}^m$ to the subspace Z of Y. We also denote by $\mathbb{F}^\perp$ the orthogonal projection $I|_Y - \mathbb{F}$, where $I|_Y$ is the identity map on Y. Each s_j is a given real number and $\omega \in S^{n-1}$.

Definition 3.2 (Singular sets for the low frequency part)

$$\mathcal{S}_{low,0} = \{s_0 \in \mathbb{R} \,|\, \mathrm{Ran}\,(\mathbb{P}_{s_0}) \neq \{0\}\},$$
$$\mathcal{S}_{low,1} = \{(s_2, s_1, s_0, \omega) \in \mathbb{R}^3 \times S^{n-1} \,|\, \mathrm{Ran}\,(\mathbb{P}_{s_2,s_1,s_0,\omega}) \neq \{0\}\},$$
$$\tilde{\mathcal{S}}_{low,1} = \{(s_1, s_0, \omega) \in \mathbb{R}^2 \times S^{n-1} \,|\, \tilde{V}^{low,1}(s_1, s_0, \omega) \neq \{0\}\},$$

where

$$\tilde{V}^{low,1}(s_1, s_0, \omega) = \mathrm{Ran}\,(\mathbb{P}_{s_1,s_0,\omega}) \cap \mathrm{Ker}\,\left(L^\sharp(is_0 I + L)|^{-1}_{\mathbb{P}^\perp_{s_0}\mathbb{C}^m}\mathbb{P}^\perp_{s_0} A(\omega)\right)$$

Remark 3.3

$$\mathrm{Ran}\,(\mathbb{P}_{s_2,s_1,s_0,\omega}) \subset \tilde{V}^{low,1}(s_1, s_0, \omega) \subset \mathrm{Ran}\,(\mathbb{P}_{s_0})$$

Theorem 3.4 (Maekawa-U(2021))

Let $n = 1$ *and* $\alpha \in \{0, 1\}$. *Assume* (GSC) *holds.*

$\mathcal{S}_{low,\alpha} = \emptyset \iff \{e^{-t\mathfrak{A}(r\omega)}\}_{t \geq 0}$ *satisfies* (LowEst).

$$\|e^{-t\mathfrak{A}(r\omega)}\| \leq Ce^{-cr^{2\alpha}t}, \quad t \geq 0,\ 0 < r \leq 1,\ \omega \in S^{n-1}. \qquad \text{(LowEst)}$$

Theorem 3.5 (Maekawa-U(2021))

Assume (GSC) *holds.*

$\mathcal{S}_{low,0} = \emptyset \Longrightarrow \{e^{-t\mathfrak{A}(r\omega)}\}_{t\geq 0}$ *satisfies* (LowEst) *with* $\alpha = 0$.

Theorem 3.6 (Maekawa-U(2021))

Assume (GSC) *holds.*

(i) *or* $(\mathrm{ii}) \Longrightarrow \{e^{-t\mathfrak{A}(r\omega)}\}_{t\geq 0}$ *satisfies* (LowEst) *with* $\alpha = 1$.

(i) $\tilde{\mathcal{S}}_{low,1} = \emptyset$.

(ii) *If* $\tilde{\mathcal{S}}_{low,1} \neq \emptyset$ *then both* (a) *and* (b) *hold for any* $(s_1, s_0, \omega) \in \tilde{\mathcal{S}}_{low,1}$:

(a) $\{(\mathbb{P}_{s_0} A(\omega')|_{\mathbb{P}_{s_0}\mathbb{C}^m}, \mathbb{P}_{s_0}\mathbb{C}^m)\}_{\omega' \in S^{n-1}}$ *has no-splitting real eigenvalues.*

(b) $\mathrm{Ran}\,(\mathbb{P}_{s_2,s_1,s_0,\omega}) = \{0\}$ *for any* $s_2 \in \mathbb{R}$, *that is,* $\mathcal{S}_{low,1} = \emptyset$.

Remark 3.7

(GSC) and $\mathrm{Ker}\,(L) = \mathrm{Ker}\,(L^\sharp) \implies \tilde{\mathcal{S}}_{low,1} = \emptyset$ (Stability Cond.)

Theorem 3.8 (Maekawa-U(2021))

Let $\alpha \in \{0, 1\}$. *Assume* (GSC) *holds.*

$\mathcal{S}_{low,\alpha} \neq \emptyset \Longrightarrow \{e^{-t\mathfrak{A}(r\omega)}\}_{t\geq 0}$ *does not satisfy* (LowEst).

Definition 3.9 (no-splitting condition)

Let $\{Z_\omega\}_{\omega \in S^{n-1}}$ be a family of the subspaces of $\mathbb{C}^m$, and $\{M_\omega\}_{\omega \in S^{n-1}}$, $M_\omega : Z_\omega \to Z_\omega$, be a family of linear operators.

It is called that $\{(M_\omega, Z_\omega)\}_{\omega \in S^{n-1}}$ has no-splitting real eigenvalues if the following two conditions are satisfied.

(i) The map $\omega \mapsto (M_\omega \mathcal{P}_{Z_\omega}, \mathcal{P}_{Z_\omega}) \in (\mathbb{C}^{m\times m})^2$ is continuous, where $\mathcal{P}_{Z_\omega}$ is the orthogonal projection from $\mathbb{C}^m$ to Z_ω.

(ii) The numbers $\#\sigma(M_\omega)$ and $\#\big(\sigma(M_\omega) \cap \mathbb{R}\big)$ are independent of $\omega \in S^{n-1}$, where $\sigma(M_\omega)$ is the set of the eigenvalues of M_ω.

Definition 3.10 (Orthogonal projections for the high frequency part)

(4) $\mathbb{Q}_{s_0,\omega} : \mathbb{C}^m \to \mathrm{Ker}\,\big(s_0 I + A(\omega)\big)$

(5) $\mathbb{Q}_{s_1,s_0,\omega} : \mathbb{Q}_{s_0,\omega}\mathbb{C}^m \to \mathrm{Ker}\,\big((is_1 I + \mathbb{Q}_{s_0,\omega}L)|_{\mathbb{Q}_{s_0,\omega}\mathbb{C}^m}\big)$

(6) $\mathbb{Q}_{s_2,s_1,s_0,\omega} : \mathbb{Q}_{s_1,s_0,\omega}\mathbb{C}^m \to \mathrm{Ker}\,\big(\big(s_2 I + \mathbb{Q}_{s_1,s_0,\omega}A^{(1)}(s_0,\omega)\big)|_{\mathbb{Q}_{s_1,s_0,\omega}\mathbb{C}^m}\big),$

where

$$A^{(1)}(s_0,\omega) := \mathbb{Q}_{s_0,\omega}LK(s_0,\omega)L^*|_{\mathbb{Q}_{s_0,\omega}\mathbb{C}^m},$$
$$K(s_0,\omega) := -\big(s_0 I + A(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega}\mathbb{C}^m}\mathbb{Q}^\perp_{s_0,\omega}$$

(7) $\mathbb{Q}_{s_3,s_2,s_1,s_0,\omega} : \mathbb{Q}_{s_2,s_1,s_0,\omega}\mathbb{C}^m \to$
$$\mathrm{Ker}\,\big(\big(is_3 I + \mathbb{Q}_{s_2,s_1,s_0,\omega}L^{(1)}_{high}(s_1,s_0,\omega)\big)|_{\mathbb{Q}_{s_2,s_1,s_0,\omega}\mathbb{C}^m}\big),$$

where

$$L^{(1)}_{high}(s_1,s_0,\omega) := is_1\mathbb{Q}_{s_1,s_0,\omega}LK(s_0,\omega)^2L^*|_{\mathbb{Q}_{s_1,s_0,\omega}\mathbb{C}^m} + \mathbb{Q}_{s_1,s_0,\omega}G(s_1,s_0,\omega)|_{\mathbb{Q}_{s_1,s_0,\omega}\mathbb{C}^m},$$

$$G(s_1,s_0,\omega) := A^{(1)}(s_0,\omega)\big(is_1 I + \mathbb{Q}_{s_0,\omega}L\big)|^{-1}_{\mathbb{Q}^\perp_{s_1,s_0,\omega}X_\omega}\mathbb{Q}^\perp_{s_1,s_0,\omega}A^{(1)}(s_0,\omega) - LK(s_0,\omega)L^*K(s_0,\omega)L^*.$$

Definition 3.11 (Singular sets for the high frequency part)

$$\mathcal{S}_{high,0} = \big\{(s_1, s_0, \omega) \in \mathbb{R}^2 \times S^{n-1} \,|\, \mathrm{Ran}\,(\mathbb{Q}_{s_1,s_0,\omega}) \neq \{0\}\big\},$$

$$\mathcal{S}_{high,1} = \big\{(s_3, s_2, s_1, s_0, \omega) \in \mathbb{R}^4 \times S^{n-1} \,|\, \mathrm{Ran}\,(\mathbb{Q}_{s_3,s_2,s_1,s_0,\omega}) \neq \{0\}\big\},$$

$$\tilde{\mathcal{S}}_{high,0} = \big\{(s_0, \omega) \in \mathbb{R} \times S^{n-1} \,|\, \tilde{V}^{high,0}(s_0, \omega) \neq \{0\}\big\},$$

$$\mathcal{S}^{(1)}_{high,1} = \big\{(s_1, s_0, \omega) \in \mathbb{R}^2 \times S^{n-1} \,|\, V^{high,1,(1)}(s_1, s_0, \omega) \neq \{0\}\big\},$$

$$\mathcal{S}^{(2)}_{high,1} = \big\{(s_2, s_1, s_0, \omega) \in \mathbb{R}^3 \times S^{n-1} \,|\, V^{high,1,(2)}(s_2, s_1, s_0, \omega) \neq \{0\}\big\},$$

where

$$\tilde{V}^{high,0}(s_0, \omega) = \mathrm{Ran}\,(\mathbb{Q}_{s_0,\omega}) \cap \mathrm{Ker}\,(L^\sharp),$$

$$V^{high,1,(1)}(s_1, s_0, \omega) = \mathrm{Ran}\,(\mathbb{Q}_{s_1,s_0,\omega}) \cap \mathrm{Ker}\,\big(L^\sharp K(s_0, \omega) L^*\big),$$

$$V^{high,1,(2)}(s_2, s_1, s_0, \omega) = \mathrm{Ran}\,(\mathbb{Q}_{s_2,s_1,s_0,\omega}) \cap$$

$$\cap\, \mathrm{Ker}\,\big(L^\sharp\{(is_1 I + \mathbb{Q}_{s_0,\omega} L)|^{-1}_{\mathbb{Q}^\perp_{s_1,s_0,\omega}\mathbb{C}^m} \mathbb{Q}^\perp_{s_1,s_0,\omega} A^{(1)}(s_0, \omega) - K(s_0, \omega) L^*\}\big).$$

Remark 3.12

(i) $V^{high,1,(1)}(s_1, s_0, \omega) \subset \mathrm{Ran}\,(\mathbb{Q}_{s_1,s_0,\omega}) \subset \tilde{V}^{high,0}(s_0, \omega)$.

(ii) $\mathrm{Ran}\,(\mathbb{Q}_{s_3,s_2,s_1,s_0,\omega}) \subset V^{high,1,(2)}(s_2, s_1, s_0, \omega) \subset \mathrm{Ran}\,(\mathbb{Q}_{s_1,s_0,\omega})$.

(iii) If $L^\sharp K(s_0,\omega) L^\flat \mathbb{Q}_{s_0,\omega}\mathbb{C}^m \subset \mathbb{Q}^\perp_{s_0,\omega}\mathbb{C}^m$ holds for any $s_0 \in \mathbb{R}$, $\omega \in S^{n-1}$ then $V^{high,1,(2)}(s_2, s_1, s_0, \omega) \subset V^{high,1,(1)}(s_1, s_0, \omega)$.

Theorem 3.13 (Maekawa-U(2021))

Let $n = 1$ *and* $\beta \in \{0, 1\}$. *Assume* (GSC) *holds.*

$\mathcal{S}_{high,\beta} = \emptyset \Longleftrightarrow \{e^{-t\mathfrak{A}(r\omega)}\}_{t\ge 0}$ *satisfies* (HighEst).

$$\|e^{-t\mathfrak{A}(r\omega)}\| \le Ce^{-cr^{-2\beta}t}, \quad t \ge 0,\ r \ge 1,\ \omega \in S^{n-1}. \qquad \text{(HighEst)}$$

Theorem 3.14 (Maekawa-U(2021))

Let $\beta \in \{0, 1\}$. *Assume* (GSC) *holds.*

$\mathcal{S}_{high,\beta} \neq \emptyset \Longrightarrow \{e^{-t\mathfrak{A}(r\omega)}\}_{t\ge 0}$ *does not satisfy* (HighEst).

Theorem 3.15 (Maekawa-U(2021))

Assume (GSC) *holds.*

(i) *or* (ii) $\Longrightarrow \{e^{-t\mathfrak{A}(r\omega)}\}_{t\geq 0}$ *satisfies* (HighEst) *with* $\beta = 0$.

(i) $\tilde{\mathcal{S}}_{high,0} = \emptyset$.

(ii) *If* $\tilde{\mathcal{S}}_{high,0} \neq \emptyset$ *then both* (a) *and* (b) *hold for any* $(s_0, \omega) \in \tilde{\mathcal{S}}_{high,0}$:

(a) $\{(A(\omega'), \mathbb{C}^m)\}_{\omega' \in S^{n-1}}$ *has no-splitting real eigenvalues.*

(b) $\mathrm{Ran}\,(\mathbb{Q}_{s_1,s_0,\omega}) = \{0\}$ *for any* $s_1 \in \mathbb{R}$, *that is,* $\mathcal{S}_{high,0} = \emptyset$.

Remark 3.16

(GSC) and $\mathrm{Ker}\,(L) = \mathrm{Ker}\,(L^\sharp) \implies \tilde{\mathcal{S}}_{high,0} = \emptyset$ (Stability Cond.)

Theorem 3.17 (Maekawa-U(2021))

Assume (GSC) *holds. Also assume* $\{(A(\tilde\omega),\mathbb{C}^m)\}_{\tilde\omega\in S^{n-1}}$ *has no-splitting real eigenvalues, and* $\mathcal{S}_{high,0}\neq\emptyset$.

(i) *or* (ii) $\Longrightarrow \{e^{-t\mathfrak{A}(r\omega)}\}_{t\geq 0}$ *satisfies* (HighEst) *with* $\beta=1$.

(i) *Both* (i-a) *and* (i-b) *hold for any* $(s_1,s_0,\omega)\in\mathcal{S}_{high,0}$:

(i-a) $L^\sharp K(s_0(\tilde\omega),\tilde\omega)L^\flat\mathbb{Q}_{s_0(\tilde\omega),\tilde\omega}\mathbb{C}^m\subset\mathbb{Q}^\perp_{s_0(\tilde\omega),\tilde\omega}\mathbb{C}^m$, $\quad\tilde\omega\in S^{n-1}$.

(i-b) $V^{high,1,(1)}(s_1,s_0,\omega)=\emptyset$ *for any* $s_1\in\mathbb{C}$, *that is,* $\mathcal{S}^{(1)}_{high,1}=\emptyset$.

(ii) *Both* (ii-a) *and* (ii-b) *hold for any* $(s_1,s_0,\omega)\in\mathcal{S}_{high,0}$:

(ii-a) $\{(\mathbb{Q}_{s_0(\tilde\omega),\tilde\omega}iL|_{\mathbb{Q}_{s_0(\tilde\omega),\tilde\omega}\mathbb{C}^m},\mathbb{Q}_{s_0(\tilde\omega),\tilde\omega}\mathbb{C}^m)\}_{\tilde\omega\in S^{n-1}}$ *has no-splitting real e.v.*

(ii-b) *If* $\mathcal{S}^{(2)}_{high,1}\neq\emptyset$ *then* (iii-b1), (iii-b2) *hold for* $(s_2,s_1,s_0,\omega)\in\mathcal{S}^{(2)}_{high,1}$:

(ii-b1) $\{(\mathbb{Q}_{s_1(\tilde\omega),s_0(\tilde\omega),\tilde\omega}A^{(1)}(s_0(\tilde\omega),\tilde\omega)|_{\mathbb{Q}_{s_1(\tilde\omega),s_0(\tilde\omega),\tilde\omega}\mathbb{C}^m},\mathbb{Q}_{s_1(\tilde\omega),s_0(\tilde\omega),\tilde\omega}\mathbb{C}^m)\}_{\tilde\omega\in S^{n-1}}$ *has no-splitting real eigenvalues.*

(ii-b2) $\mathrm{Ran}\,(\mathbb{Q}_{s_3,s_2,s_1,s_0,\omega})=\{0\}$ *for any* $s_3\in\mathbb{R}$, *that is,* $\mathcal{S}_{high,1}=\emptyset$.

Here $s_j(\cdot):S^{n-1}\to\mathbb{R}$ *is the conti. map s.t.* $s_j(\omega)=s_j$ *and that each* $s_j(\omega)$ *is an eigenvalue of* $-A(\omega)$ *if* $j=0$, *or* $\mathbb{Q}_{s_0(\omega),\omega}iL|_{\mathbb{Q}_{s_0(\omega),\omega}\mathbb{C}^m}$ *if* $j=1$.

Gearhart-Prüss type theorem

X : Hilbert space.

$A : D(A) \to X$, densely defined closed operator in X with $D(A) \subset X$.

The operator A is called m-accretive if the left open half-plane is contained in the resolvent set $\rho(-A)$ with $\|(\lambda I + A)^{-1}\|_{X\to X} \le 1/\mathrm{Re}\lambda$ for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$. We denote by $\Psi(A)$ the pseudospectral bound of A:

$$\Psi(A) = \left(\sup_{\lambda\in\mathbb{R}} \|(i\lambda I + A)^{-1}\|_{X\to X} \right)^{-1}.$$

Theorem 3.18 (Wei(2021))

Let A be an m-accretive operator in a Hilbert space X. Then

$$\|e^{-tA}\|_{X\to X} \le e^{-t\Psi(A)+\pi/2}, \qquad t > 0.$$

♣ Since (GSC), $\mathfrak{A}(\xi)$ is an m-accretive operator.

Resolvent analysis

We introduce $\Phi(r,\omega) := \sup_{\lambda\in\mathbb{R}} \|\big(i\lambda I + irA(\omega) + L(\omega)\big)^{-1}\|$.

$$\big(\Psi(r,\omega) := \Phi(r,\omega)^{-1} \text{ is a pseudospectral bound.}\big)$$

Proposition 3.19

Suppose the same assumption as in Theorem 3.5. Then there exists $C > 0$ such that

$$\sup_{\omega\in S^{n-1}} \Phi(r,\omega) \le C, \qquad 0 < r \le 1.$$

Proposition 3.20

Suppose the same assumption as in Theorem 3.6. Then there exists $C > 0$ such that

$$\sup_{\omega\in S^{n-1}} \Phi(r,\omega) \le Cr^{-2}, \qquad 0 < r \le 1. \tag{$*$}$$

♣ Propositions 3.19 and 3.20 $\Longrightarrow$ Theorems 3.5 and 3.6. ($\because$Theorem3.18.)

Outline of the proof of Proposition 3.20

The proof is based on the reduction argument (T.Kato)
and the contradiction argument.

Step 1: Set $M = \sup_{\omega \in S^{n-1}}(1 + \|A(\omega)\| + \|L\|)$. It is easy to see from the Neumann series argument that

$$\sup_{0<r\le 1, \omega\in S^{n-1}, |\lambda|\ge M} \|(i\lambda I + irA(\omega) + L)^{-1}\| \le C.$$

So it suffices to consider the case $|\lambda| \le M$.

Step 2: Suppose that $(*)$ does not hold. Then there exist a sequence $\{r_N, \lambda_N, \omega_N, u_N\}$ with $r_N \in (0,1]$, $\omega_N \in S^{n-1}$, $\lambda_N \in \mathbb{R}$ with $|\lambda_N| \le M$, and $u_N \in \mathbb{C}^m$ such that $|u_N| = 1$ and

$$\lim_{N\to\infty} r_N^{-2}(i\lambda_N I + ir_N A(\omega_N) + L)u_N = 0\,.$$

By taking a suitable subsequence if necessary, we may also assume

$$\lim_{N\to\infty}(r_N, \lambda_N, \omega_N, u_N) = (r_*, \lambda_*, \omega_*, u_*)$$

for some $r_* \in [0,1]$, $|\lambda_*| \le M$, $\omega_* \in S^{n-1}$, $u_* \in \mathbb{C}^m$ with $|u_*| = 1$.

Then the limit $u_* \in \mathbb{C}^m \setminus \{0\}$ satisfies $(i\lambda_* I + ir_* A(\omega_*) + L)u_* = 0$.

If $r_* > 0 \implies u_* = 0$ ($\because$ (GSC)). This is a contradiction.
If $r_* = 0 \implies u_* \in \mathrm{Ker}\,(i\lambda_* I + L) = \mathrm{Ran}\,(\mathbb{P}_{\lambda_*})$.

Step 3: Suppose the assumption in Theorem 3.6 holds.
Set $f_N = (i\lambda_N I + ir_N A(\omega_N) + L)u_N$, and this gives $f_N = o(r_N^2)$.
Let us decompose $u_N = w_N + w_N^\perp$ with $w_N := \mathbb{P}_{\lambda_*} u_N$ and $w_N^\perp := \mathbb{P}_{\lambda_*}^\perp u_N$.
Then f_N is also decomposed by $f_N = \mathbb{P}_{\lambda_*} f_N + \mathbb{P}_{\lambda_*}^\perp f_N$ that

$$\mathbb{P}_{\lambda_*} f_N = i(\lambda_N - \lambda_*) w_N + ir_N \mathbb{P}_{\lambda_*} A(\omega_N) u_N = o(r_N^2)\,,$$

$$\mathbb{P}_{\lambda_*}^\perp f_N = (i\lambda_N I + L)|_{\mathbb{P}_{\lambda_*}^\perp \mathbb{C}^m} w_N^\perp + ir_N \mathbb{P}_{\lambda_*}^\perp A(\omega_N) u_N = o(r_N^2)\,.$$

Here we have

$$w_N^\perp = -ir_N \big(i\lambda_N I + L + ir_N \mathbb{P}_{\lambda_*}^\perp A(\omega_N)\big)|_{\mathbb{P}_{\lambda_*}^\perp \mathbb{C}^m}^{-1} \mathbb{P}_{\lambda_*}^\perp A(\omega_N) w_N + o(r_N^2),$$

$\Big(\because (i\lambda_N I + L + ir_N \mathbb{P}_{\lambda_*}^\perp A(\omega_N))|_{\mathbb{P}_{\lambda_*}^\perp \mathbb{C}^m}$ is invertible for large N. $\Big)$

Since $\lim_{N\to\infty} w_N = u_* \neq 0$ and $\lim_{N\to\infty} \lambda_N = \lambda_*$, we find that $|\lambda_N - \lambda_*| \leq C r_N$ and $|w_N^\perp| \leq C r_N$ are satisfied for large N. Then we set

$$\tilde{\lambda}_N = \frac{\lambda_N - \lambda_*}{r_N}, \qquad \tilde{w}_N^\perp = \frac{w_N^\perp}{r_N},$$

which are bounded uniformly in N. Thus by taking a subsequence if necessary we may assume that $\lim_{N\to\infty} \tilde{\lambda}_N = \tilde{\lambda}_*$ and $\lim_{N\to\infty} \tilde{w}_N^\perp = \tilde{w}_*^\perp$. Since $\tilde{w}_N^\perp \in \mathrm{Ran}\,(\mathbb{P}_{\lambda_*}^\perp)$ we have $\tilde{w}_*^\perp \in \mathrm{Ran}\,(\mathbb{P}_{\lambda_*}^\perp)$, and we obtain

$$u_* \in \mathrm{Ker}\left((\tilde{\lambda}_* I + \mathbb{P}_{\lambda_*} A(\omega_*))|_{\mathbb{P}_{\lambda_*}\mathbb{C}^m}\right),$$

$$\tilde{w}_*^\perp = -i(i\lambda_* I + L)|_{\mathbb{P}_{\lambda_*}^\perp \mathbb{C}^m}^{-1} \mathbb{P}_{\lambda_*}^\perp A(\omega_*) u_*.$$

Furthermore, using $L^\sharp \tilde{w}_*^\perp = 0$, we get

$$u_* \in \mathrm{Ker}\,(L^\sharp (i\lambda_* I + L)|_{\mathbb{P}_{\lambda_*}^\perp \mathbb{C}^m}^{-1} \mathbb{P}_{\lambda_*}^\perp A(\omega_*)).$$

As a summary, we arrive at $u_* \neq 0$ and $u_* \in \tilde{V}^{low,1}(\tilde{\lambda}_*, \lambda_*, \omega_*)$.

Thus, this is a contradiction because of the condition (i) in Theorem 3.6, that is, $\tilde{\mathcal{S}}_{low,1} = \emptyset$.

In the case for $\tilde{\mathcal{S}}_{low,1} \neq \emptyset$, we start from

$$\begin{aligned}&\tilde{\lambda}_N w_N + \mathbb{P}_{\lambda_*} A(\omega_N) w_N \\ &- i r_N \mathbb{P}_{\lambda_*} A(\omega_N) \big(i\lambda_* I + L\big)|^{-1}_{\mathbb{P}^{\perp}_{\lambda_*}\mathbb{C}^m} \mathbb{P}^{\perp}_{\lambda_*} A(\omega_N) w_N = o(r_N).\end{aligned}$$

In virtue of (ii-a), there exists a continuous curve $s_1(\cdot) : S^{n-1} \to \mathbb{R}$ such that $s_1(\omega_*) = \tilde{\lambda}_*$ and each $s_1(\omega)$ is the eigenvalue of $\mathbb{P}_{\lambda_*} A(\omega)|_{\mathbb{P}_{\lambda_*}\mathbb{C}^m}$.

Then we have

$$\begin{aligned}&(\tilde{\lambda}_N - s_1(\omega_N))\mathbb{P}_{s_1(\omega_N),\lambda_*,\omega_N} w_N \\ &- i r_N \mathbb{P}_{s_1(\omega_N),\lambda_*,\omega_N} A(\omega_N)(i\lambda_* I + L)|^{-1}_{\mathbb{P}^{\perp}_{\lambda_*}\mathbb{C}^m} \mathbb{P}^{\perp}_{\lambda_*} A(\omega_N) w_N = o(r_N)\,.\end{aligned}$$

Since $|\mathbb{P}_{s_1(\omega_N),\lambda_*,\omega_N} w_N|$ must be positive uniformly for large N, $\tilde{\lambda}'_N = (\tilde{\lambda}_N - s_1(\omega_N))/r_N$ is uniformly bounded in N. Then we may assume that $\tilde{\lambda}'_N$ converges to $\tilde{\lambda}'_*$ by taking a subsequence if necessary.

Thus we obtain

$$i\tilde{\lambda}'_* u_* + \mathbb{P}_{\tilde{\lambda}_*,\lambda_*,\omega_*} A(\omega_*)(i\lambda_* I + L)|^{-1}_{\mathbb{P}^{\perp}_{\lambda_*}\mathbb{C}^m} \mathbb{P}^{\perp}_{\lambda_*} A(\omega_*) u_* = 0,$$

and hence $u_* \in \mathrm{Ran}\,(\mathbb{P}_{\tilde{\lambda}'_*,\tilde{\lambda}_*,\lambda_*,\omega_*})$. Therfore, $u_* = 0$ by the condition (ii-b) in Theorem 3.6, which is a contradiction. The proof is complete. □

6. Application and weak dissipative structure

Cond.(GSC): $\mu A^0\varphi + (\nu A(\omega) - iL^\flat)\varphi = 0, \quad \varphi \in \mathrm{Ker}(L^\sharp) \implies \varphi = 0$

Dissipative Timoshenko system:

$$A^0 = I, \quad A(\omega) = -\omega \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Corollary 4.1

The Timoshenko system satisfies $\mathrm{Condition(GSC)}$. Namely, this system is strictly dissipative.

Proof: For $(\mu, \nu, \omega) \in \mathbb{R} \times \mathbb{R}_+ \times \{-1, 1\}$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^\top \in \mathbb{C}^4$,

$$\begin{cases} \mu\varphi_1 - \nu\omega\varphi_2 - i\varphi_4 = 0, \\ \mu\varphi_2 - \nu\omega\varphi_1 = 0, \\ \mu\varphi_3 - \nu\omega\varphi_4 = 0, \\ \mu\varphi_4 - \nu\omega\varphi_3 + i\varphi_1 = 0, \end{cases} \quad \text{and} \quad \varphi_4 = 0. \quad \implies \quad \varphi = 0.$$

□

Corollary 4.2 (Low frequency part)

The solution operator to the Timoshenko system satisfies

$$\|e^{-t\mathfrak{A}(\xi)}\| \le Ce^{-c|\xi|^2 t}, \qquad |\xi| \le 1,$$

Proof: We have

$$\mathbb{P}_0 y = \begin{pmatrix} 0 \\ y_2 \\ y_3 \\ 0 \end{pmatrix}, \qquad \mathbb{P}_{s_0} y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for } s_0 \neq 0,$$

where $y = (y_1, y_2, y_3, y_4)^\top \in \mathbb{C}^4$. Hence we have $\mathcal{S}_{low,0} = \{0\} \neq \emptyset$. This means that the solution operator does not satisfy the exponential decay estimate ($\because$ Theorem 3.5).

Furthermore we have

$$\tilde{V}^{low,1}(s_1, 0, \omega) = \mathrm{Ker}\,\big(L^\sharp L|_{\mathbb{P}_0^\perp \mathbb{C}^4}^{-1} \mathbb{P}_0^\perp A(\omega)\big) \cap \mathrm{Ker}\,\big((s_1 I + \mathbb{P}_0 A(\omega))|_{\mathbb{P}_0 \mathbb{C}^4}\big),$$

and this gives

$$\tilde{V}^{low,1}(s_1,0,\omega) = \begin{cases} \left\{ \begin{pmatrix} 0 \\ 0 \\ y_3 \\ 0 \end{pmatrix} \,\middle|\, y_3 \in \mathbb{C} \right\} & \text{for} \quad s_1 = 0, \\ \{0\} & \text{for} \quad s_1 \neq 0. \end{cases}$$

Hence we obtain $\tilde{\mathcal{S}}_{low,1} = \{(0,0,\pm 1)\} \neq \emptyset$, it suffices to consider the set $V^{low,1}(s_2,0,0,\pm 1)$. For $y = (0,0,y_3,0)^\top$ and $s_2 \in \mathbb{R}$, we have

$$is_2 y + \mathbb{P}_0 A(\omega) L|^{-1}_{\mathbb{P}_0^\perp \mathbb{C}^4} \mathbb{P}_0^\perp A(\omega) y = \begin{pmatrix} 0 \\ -a y_3 \\ is_2 y_3 \\ 0 \end{pmatrix}.$$

Thus, this gives

$$\mathrm{Ker}\left(\left(is_2 I + \mathbb{P}_0 A(\omega) L|^{-1}_{\mathbb{P}_0^\perp \mathbb{C}^4} \mathbb{P}_0^\perp A(\omega) \right)|_{\mathbb{P}_{0,0,\omega}\mathbb{C}^4} \right) \cap \tilde{V}^{low,1}(0,0,\omega) = \{0\}$$

for all $s_2 \in \mathbb{R}$ and $\omega \in \{\pm 1\}$. This implies $\mathcal{S}_{low,1} = \emptyset$ and therefore the condition (ii-b) in Theorem 3.6 is satisfied. □

Dissipative Bresse system: $(\ell \neq 0)$

$$\begin{aligned} \phi_{tt} - (\phi_x + \psi + \ell w)_x - \ell(w_x - \ell\phi) &= 0, \\ \psi_{tt} - \psi_{xx} + (\phi_x + \psi + \ell w) + \psi_t &= 0, \\ w_{tt} - (w_x - \ell\phi)_x + \ell(\phi_x + \psi + \ell w) &= 0. \end{aligned} \qquad \text{(BS)}$$

♣ If $\ell = 0$, this system is reduced to the dissipative Timoshenko system.

Putting $\rho = \phi_x + \psi + \ell w$, $v = \phi_t$, $z = \psi_x$, $y = \psi_t$, $q = w_x - \ell\phi$, $p = w_t$, we obtain the symmetric hyperbolic system $u_t + Au_x + Lu = 0$, where $u = (\rho, v, z, y, q, p)^T$ and

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -\ell \\ 0 & 0 & 0 & 0 & -\ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \ell & 0 & 0 & 0 & 0 \\ \ell & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

New dissipative structure

Corollary 4.3

The Bresse system does not satisfy $\mathrm{Condition(GSC)}$. Namely, this system is not strictly dissipative.

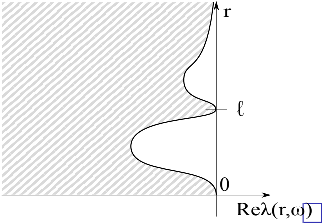
Proof: For $(\mu, \nu, \omega) \in \mathbb{R} \times \mathbb{R}_+ \times \{-1, 1\}$ and $\varphi = (\varphi_1, \cdots, \varphi_6)^\top \in \mathbb{C}^6$,

$$\begin{cases} \mu\varphi_1 - \nu\omega\varphi_2 + i\varphi_4 + i\ell\varphi_6 = 0, \\ \mu\varphi_2 - \nu\omega\varphi_1 + i\ell\varphi_5 = 0, \\ \mu\varphi_3 - \nu\omega\varphi_4 = 0, \\ \mu\varphi_4 - \nu\omega\varphi_3 - i\varphi_1 = 0, \\ \mu\varphi_5 - \nu\omega\varphi_6 - i\ell\varphi_2 = 0, \\ \mu\varphi_6 - \nu\omega\varphi_5 - i\ell\varphi_1 = 0, \end{cases} \quad \text{and} \quad \varphi_4 = 0. \tag{1}$$

Let $(\mu, \nu) = (0, |\ell|)$, then

$$\varphi = (\sigma_1, \sigma_2, -i\frac{1}{|\ell|}\sigma_1, 0, -i\frac{\ell}{|\ell|}\sigma_1, -i\frac{\ell}{|\ell|}\sigma_2)^T$$

satisfy (1) for $(\sigma_1, \sigma_2) \in \mathbb{C}^2$.



Decay estimate

Theorem 4.4 (Decay estimate, U(2022))

The solutions to (BS) *satisfy the pointwise estimate*

$$|\hat{u}(\xi,t)| \le Ce^{-c\eta(\xi)t}|\hat{u}_0(\xi)|, \qquad \eta(\xi) = \frac{\xi^2(\xi-\ell)^2(\xi+\ell)^2}{(1+\xi^2)^8}.$$

Namely, we obtain

$$\|u(t)\|_{L^2} \le \underbrace{C(1+t)^{-\frac{1}{4}}\|u_0\|_{L^1}}_{Low\ \ freq.} + \underbrace{C(1+t)^{-\frac{1}{4}}\|u_0\|_{L^1}}_{Middle\ \ freq.} + \underbrace{C(1+t)^{-\frac{\ell}{2}}\|\partial_x^\ell u_0\|_{L^2}}_{High\ \ freq.}, \qquad \ell \ge 0.$$

Future Works

- Can we derive the relationship between the known results??

  ♣ Duan-Kawashima-U(2012) :

  Condition(S) + (K) $\Longrightarrow \mathrm{Re}\lambda \leq -c\dfrac{|\xi|^2}{(1+|\xi|^2)^2}$

  (e.g. Timoshenko system, Euler-Maxwell system)

  ♣ Duan-Kawashima-U(2017) :

  Craftsmanship Condition $\Longrightarrow \mathrm{Re}\lambda \leq -c\dfrac{|\xi|^4}{(1+|\xi|^2)^3}$

  (e.g. Timoshenko system with memory)

- How about the asymptotic profile of the solution to the dissipative Bresse system??

- Can we apply these results to nonlinear problems??