A DISCRETE VELOCITY NUMERICAL SCHEME FOR THE 2D BITEMPERATURE EULER SYSTEM*

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Abstract. This paper is devoted to the numerical approximation of the bidimensional bitemperature Euler system. This model is a nonconservative hyperbolic system describing an out of equilibrium plasma in a quasi-neutral regime, with applications in Inertial Confinment Fusion (ICF). One main difficulty here is to handle shock solutions involving the product of the velocity by pressure gradients. We develop a second order numerical scheme by using a discrete BGK relaxation model. The second order extension is based on a subdivision of each cartesian cell into four triangles to perform affine reconstructions of the solution. Such ideas have been developed in the litterature for systems of conservation laws. We show here how they can be used in our nonconservative setting. The numerical method is implemented and tested in the last part of the paper.

Key words. nonconservative hyperbolic system, Euler type model for plasmas, discrete BGK approximation, second order

AMS subject classifications. 65M08, 35L60, 76X05, 35Q31

1. Introduction.

This paper is devoted to the numerical resolution of the two dimensional bitemperature Euler system by using a relaxation model under the form of a discrete BGK type approximation.

The bitemperature Euler system is a nonconservative hyperbolic system with a source term. It describes a mixture of electrons and ions in a quasi-neutral regime and in a thermal nonequilibrium. This system is constituted by two conservative equations for mass and momentum and two nonconservative equations on electronic and ionic energies. The non-conservativity is due to source-terms but also to the presence of products of the velocity by pressure gradients. Those products make delicate the definition of weak solutions. Dal Maso, Le Floc'h and Murat developed a general theory to define shocks in such a context, by using families of paths ([18]). This point of view has been considered in a numerical framework ([24]). However even if the path can be theoretically computed, finding the path numerically remains difficult ([1]). In [17], the model is supposed to be isentropic on the electrons and the system is transformed into a conservative form. The same viewpoint is adopted in [20]. In [30], the authors introduce a small parameter representing the mass ratio between electrons and ions. They obtain an hyperbolic system on ions and a parabolic regularisation on electrons.

In the present paper, we generalize a discrete BGK scheme presented in [8]. In this article, the bitemperature Euler system was derived as a fluid limit starting from a Vlasov-BGK model coupled with Ampère and Poisson equations in a quasi-neutral regime when the inter species collisions are dominant. In particular, the nonconservative terms were recovered from the generalized Ohm's law giving the electric field. Entropy dissipation properties were proved. Several numerical schemes were proposed and compared. The approach of the present article was previously validated in one space dimension and first order by comparison with the numerical results of the underlying Vlasov-Maxwell system discretized at the fluid level ([8]) and

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then at the kinetic level by a DVM method ([13]). Then in [2], a Chapman-Engskog expansion was performed where diffusive terms are computed and are shown to be compatible with the entropy of the bitemperature Euler system. The resulting model is a generalization of the system considered in [15]. This underlying Vlasov-BGK model has been extended in order to take into account transverse magnetic fields in [12].

Discrete BGK models have been introduced in a conservative setting in [23] for the approximation of scalar conservation laws. The method was next generalized for systems in [6], (see also [7]) in the degenerate parabolic case. Entropy properties are studied in [10]. In [8], those models are generalized in order to handle the nonconservative terms of the 1D bitemperature Euler system. In particular, the electric force is integrated in the discrete BGK model. Those terms also make difficult the extension to second order. The ideas of [25], [26] necessitate some adaptation to preserve the properties of the first order scheme.

This paper is organised as follows. In section 2, the bitemperature model is introduced with the discrete BGK model that is associated. In section 3, a first order scheme is presented. It is a generalization of the numerical method of [8]. In section 4, the numerical scheme is extended to second order. Finally the last part is dedicated to numerical tests.

2. Underlying discrete BGK model for a nonconservative Euler system.

2.1. The bitemperature Euler system. Superscripts e and i respectively denote electronic and ionic quantities. We denote by ρ^e and ρ^i the electronic and ionic densities, $\rho = \rho^e + \rho^i$ the total density, m^e and m^i the related masses, c^e and c^i the mass fractions. These variables satisfy

$$\rho^e = m^e n^e = c^e \rho, \quad \rho^i = m^i n^i = c^i \rho, \quad m^e > 0, \quad m^i > 0, \quad c^e + c^i = 1. \tag{2.1}$$

Quasineutrality is assumed, so that the ionization ratio $Z = n^e/n^i$ is a constant. This implies that the electronic and ionic mass fractions are constant and given by

$$c^e = \frac{Zm^e}{m^i + Zm^e}, \quad c^i = \frac{m^i}{m^i + Zm^e}. \tag{2.2}$$

Electronic and ionic velocities u^e, u^i are assumed to be in thermodynamic equilibrium in the model. Hence, $u^e = u^i = u$, where u denotes mixture velocity. The pressure of each species satisfies a gamma-law with its own γ exponent :

$$p^e = (\gamma^e - 1)\rho^e \varepsilon^e = n^e k_B T^e, \quad p^i = (\gamma^i - 1)\rho^i \varepsilon^i = n^i k_B T^i, \quad \gamma^e > 1, \quad \gamma^i > 1, \tag{2.3}$$

where k_B is the Boltzmann constant ($k_B > 0$), ε^α and T^α represent respectively the internal specific energy and the temperature of species α for $\alpha = e, i$.

Denoting by $|\cdot|$ the euclidean norm in $\mathbb{R}^D$, the total energies for the particles are defined by

$$\mathcal{E}^\alpha = \rho^\alpha \varepsilon^\alpha + \frac{1}{2}\rho^\alpha |u|^2 = c^\alpha\Big(\rho\varepsilon^\alpha + \frac{1}{2}\rho|u|^2\Big), \quad \alpha = e, i. \tag{2.4}$$

We denote by $\nu^{ei} \geq 0$ the interaction coefficient between the electronic and ionic temperatures. The model consists of two conservative equations for mass and momentum and two nonconservative equations for each energy:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + (p^e + p^i)\mathrm{I}) = 0, \\ \partial_t \mathcal{E}^e + \operatorname{div}(u(\mathcal{E}^e + p^e)) - u \cdot \nabla \left(c^i p^e - c^e p^i\right) = \nu^{ei}(T^i - T^e), \\ \partial_t \mathcal{E}^i + \operatorname{div}(u(\mathcal{E}^i + p^i)) + u \cdot \nabla \left(c^i p^e - c^e p^i\right) = -\nu^{ei}(T^i - T^e), \end{cases} \tag{2.5}$$

where I represents the identity matrix in $\mathbb{R}^D$. In the following we denote

$$\mathcal{U} = (\rho, \rho u, \mathcal{E}^e, \mathcal{E}^i), \quad U^\alpha = (c^\alpha \rho, c^\alpha \rho u, \mathcal{E}^\alpha). \tag{2.6}$$

The system (2.5) is hyperbolic, diagonalisable and owns 3 eigenvalues λ_-, λ_0 (with multiplicity $D+1$ where D is the space dimension), λ_+:

$$\lambda_- = u \cdot \omega - a, \qquad \lambda_0 = u \cdot \omega, \qquad \lambda_+ = u \cdot \omega + a$$

where

$$a = \sqrt{\sum_{\alpha=e,i} \frac{\gamma^\alpha p^\alpha}{\rho}} \tag{2.7}$$

is the sound velocity. The fields related to $\lambda_\pm$ are genuinely nonlinear, while the field related to λ_0 is linearly degenerate.

Defining the total energy $\mathcal{E} = \mathcal{E}^e + \mathcal{E}^i$ and the total pressure $p = p^e + p^i$, one can note that if $\mathcal{U}$ is a solution of system (2.5) then $(\rho, \rho u, \mathcal{E})$ satisfies the following conservative system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p\mathrm{I}) = 0, \\ \partial_t \mathcal{E} + \operatorname{div}(u(\mathcal{E} + p)) = 0. \end{cases} \tag{2.8}$$

If $\gamma^e = \gamma^i$ this is the wellknown monotemperature Euler system. But even in this case, one has to deal with one more equation to determine electronic and ionic temperatures. If $\gamma^e \neq \gamma^i$ system (2.8) is not closed. We want to underline the fact that in both cases, the solutions of system (2.5) are to be defined in the context of nonconservative equations were the product of a possibly discontinuous function with a Dirac measure appears. To give a sense to such solutions, one has to bring more physical information. In [8] we obtain solutions of (2.5) as hydrodynamic limits of solutions of an underlying, physically realistic BGK model. The entropy-entropy flux of species α being defined as

$$\eta^\alpha(U^\alpha) = -\frac{\rho^\alpha}{m^\alpha(\gamma^\alpha - 1)} \left[\ln\left(\frac{(\gamma^\alpha - 1)\rho^\alpha \varepsilon^\alpha}{(\rho^\alpha)^{\gamma^\alpha}}\right) + C\right], \quad Q^\alpha(U^\alpha) = \eta^\alpha(U^\alpha) u, \tag{2.9}$$

the total entropy-entropy flux pair for (2.5) is

$$\eta(\mathcal{U}) = \eta^e(U^e) + \eta^i(U^i), \qquad Q(\mathcal{U}) = \eta(\mathcal{U}) u \tag{2.10}$$

and we proved the following entropy inequality for these hydrodynamic limits:

$$\partial_t \eta(\mathcal{U}) + \operatorname{div} Q(\mathcal{U}) \leq -\frac{\nu^{ei}}{k_B T^i T^e}(T^i - T^e)^2. \tag{2.11}$$

We then defined an admissible solution of (2.5) as a solution satisfying this inequality.

We now introduce for numerical purpose a relaxing “BGK type” approximation of system (2.5) in the spirit of [6]. It should be noted that this approximation differs from the underlying BGK system mentioned just above, despite a formal resemblance.

2.2. A BGK-type kinetic model for a system of conservation laws. In order for the article to be self-contained we briefly recall the formalism for a system of conservation laws

$$\partial_t U + \sum_{d=1}^{D} \partial_{x_d} F_d(U) = 0, \tag{2.12}$$

where $U(x,t) \in \Omega$, $\Omega \subset \mathbb{R}^K$ convex, and $F = (F_1, \dots, F_D)$ is a smooth function defined on Ω with values in $(\mathbb{R}^K)^D$. In [5], [6] we constructed relaxation approximations of such a system as a set of transport equations with source term:

$$\partial_t f^\varepsilon + \sum_{d=1}^{D} \Lambda_d \partial_{x_d} f^\varepsilon = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right), \tag{2.13}$$

with

$$f^\varepsilon = (f_1^\varepsilon, \dots, f_L^\varepsilon), \quad f^\varepsilon(x,t) \in (\mathbb{R}^K)^L, \quad \Lambda_d = \operatorname{diag}\left(v_{d,1}\mathrm{I}_K, \dots, v_{d,L}\mathrm{I}_K\right), \quad v_{d,l} \in \mathbb{R}, \tag{2.14}$$

$P \in \mathcal{L}\left((\mathbb{R}^K)^L, \mathbb{R}^K\right)$, and $M = (M_1, \dots, M_L)$, a function defined on Ω with values in $(\mathbb{R}^K)^L$. Equivalently we can write

$$\partial_t f_l^\varepsilon + \sum_{d=1}^{D} v_{d,l} \partial_{x_d} f_l^\varepsilon = \frac{1}{\varepsilon}\left(M_l(Pf^\varepsilon) - f_l^\varepsilon\right), \quad 1 \le l \le L. \tag{2.15}$$

The compatibility between systems (2.12) and (2.13) is insured by the following conditions:

$$\forall U \in \Omega, \qquad P(M(U)) = U, \qquad P(\Lambda_d M(U)) = F_d(U), \qquad d = 1, \dots, D. \tag{2.16}$$

By analogy with the gas kinetic theory, we called (2.13) a discrete BGK system, M being the maxwellian function and P being the moment operator. By applying the moment operator P to (2.13) one has

$$\partial_t(Pf^\varepsilon) + \sum_{d=1}^{D} \partial_{x_d} P(\Lambda_d f^\varepsilon) = 0.$$

Moreover, if $f^\varepsilon \to f$ then $f = M(Pf)$. Therefore, formally, $U = Pf$ is a solution of (2.12).

In the present article we use the following model, written for $D = 2$ for the sake of clarity. We set $L = 4$, define P as

$$\forall f \in (\mathbb{R}^K)^4, \quad Pf = \sum_{l=1}^{4} f_l\,. \tag{2.17}$$

Let $\lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^- \in \mathbb{R}$ be such that $\lambda_1^+ > \lambda_1^-$ and $\lambda_2^+ > \lambda_2^-$. We define the discrete velocities $V_l = (v_{1,l}, v_{2,l})$ as

$$V_1 = (\lambda_1^-, 0), \quad V_2 = (0, \lambda_2^-), \quad V_3 = (\lambda_1^+, 0), \quad V_4 = (0, \lambda_2^+) \tag{2.18}$$

and the maxwellians functions

$$M(U) = \begin{pmatrix} \dfrac{1}{\lambda_1^+ - \lambda_1^-}\left(\dfrac{\lambda_1^+}{2}U - F_1(U)\right) \\ \dfrac{1}{\lambda_2^+ - \lambda_2^-}\left(\dfrac{\lambda_2^+}{2}U - F_2(U)\right) \\ \dfrac{1}{\lambda_1^+ - \lambda_1^-}\left(\dfrac{-\lambda_1^-}{2}U + F_1(U)\right) \\ \dfrac{1}{\lambda_2^+ - \lambda_2^-}\left(-\dfrac{\lambda_2^-}{2}U + F_2(U)\right) \end{pmatrix}. \tag{2.19}$$

System (2.13) is a relaxation system for the "macroscopic" system (2.12), in the sense of [22], [16]. As already shown by these authors, the waves of the relaxation system (2.13) must be faster than the waves of system (2.12), that is the subcharacteristic condition. Here we need for the following condition (see [6]):

$$\forall U \in \Omega, \quad \sigma(F_d'(U)) \subset \left]\frac{\lambda_d^-}{2}, \frac{\lambda_d^+}{2}\right[, \qquad d = 1, 2 \tag{2.20}$$

which is equivalent to

$$\forall U \in \Omega, \quad \forall l \in \{1, \ldots, L\}, \quad \sigma(M_l'(U)) \subset]0, +\infty[. \tag{2.21}$$

It implies entropy properties that are detailed below.

2.3. BGK model for the bitemperature Euler system. In this section, we use the model above for the development of a numerical method for the bitemperature Euler system, generalizing the procedure in [8]. We restrict ourselves to the bidimensional case, but the procedure is avalaible in any space dimension.

2.3.1. Construction of the model. For $\alpha \in \{e, i\}$ we denote $F^\alpha(U^\alpha) = (\rho^\alpha u^\alpha, \rho^\alpha u^\alpha \otimes u^\alpha + p^\alpha \mathrm{I}, u^\alpha(\mathcal{E}^\alpha + p^\alpha))$ the flux of the conservative Euler system with the γ^α pressure law. The set of admissible states $\Omega^\alpha = \{U^\alpha \in \mathbb{R}^4,\ \rho^\alpha > 0,\ \varepsilon^\alpha > 0\}$ is convex. We consider the model (2.13) with (2.14), (2.18), (2.19) for each species: we have $K = 4$, $L = 4$ and we denote M^α the related maxwellian function defined by (2.19). The characteristic speeds $\lambda_d^\pm$ are the same for $\alpha = e$ and $\alpha = i$.

In order to approximate the nonconservative products, let us introduce a force term linked to the electric field $E(x,t) \in \mathbb{R}^2$:

$$\forall \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \quad N(E)\varphi = -(0, \varphi_1 E, \varphi_2 \cdot E).$$

For all $U^\alpha = (\rho^\alpha, \rho^\alpha u^\alpha, \mathcal{E}^\alpha) \in \mathbb{R}^4$ one has

$$\sum_{l=1}^{4} (N(E) M_l^\alpha(U^\alpha)) = N(E) U^\alpha = -(0, \rho^\alpha E, \rho^\alpha u^\alpha \cdot E). \tag{2.22}$$

Denoting $U^{\alpha,\varepsilon} = Pf^{\alpha,\varepsilon}$, the discrete BGK system for (2.5) is as follows ($1 \leq l \leq 4$):
(2.23)

$$\begin{cases} \partial_t f_l^{e,\varepsilon} + \displaystyle\sum_{d=1}^{2} v_{d,l}\partial_{x_d} f_l^{e,\varepsilon} + \frac{q^e}{m^e} N(E^\varepsilon) f_l^{e,\varepsilon} = \frac{1}{\varepsilon}\left(M_l^e(U^{e,\varepsilon}) - f_l^{e,\varepsilon}\right) + B_l^{ei}(f^{e,\varepsilon}, f^{i,\varepsilon}), \\ \partial_t f_l^{i,\varepsilon} + \displaystyle\sum_{d=1}^{2} v_{d,l}\partial_{x_d} f_l^{i,\varepsilon} + \frac{q^i}{m^i} N(E^\varepsilon) f_l^{i,\varepsilon} = \frac{1}{\varepsilon}\left(M_l^i(U^{i,\varepsilon}) - f_l^{i,\varepsilon}\right) + B_l^{ie}(f^{e,\varepsilon}, f^{i,\varepsilon}), \\ \partial_t E^\varepsilon = -\dfrac{1}{\varepsilon^2}\left(\dfrac{q^e}{m^e}\rho^{e,\varepsilon}u^{e,\varepsilon} + \dfrac{q^i}{m^i}\rho^{i,\varepsilon}u^{i,\varepsilon}\right), \\ \mathrm{div} E^\varepsilon = \dfrac{1}{\varepsilon^2}\left(\dfrac{q^e}{m^e}\rho^{e,\varepsilon} + \dfrac{q^i}{m^i}\rho^{i,\varepsilon}\right). \end{cases}$$

$q^e = -e$ and $q^i = Ze$ are respectively the electronic and ionic charges. The source terms $B^{\alpha\beta}$ model the interactions between ions and electrons, see [8]. They are such that if $\varepsilon \to 0$ then

$$PB^{\alpha\beta} \to (0, 0, 0, \nu^{\alpha\beta}(T^\beta - T^\alpha)). \tag{2.24}$$

When ε tends to 0, if a limit (f^e, f^i, E) exists, then, denoting $Pf^{\alpha,\varepsilon} = U^{\alpha,\varepsilon}$ and $Pf^\alpha = U^\alpha$, we have formally:

$$u^e = u^i = u, \qquad \frac{q^e}{m^e}\rho^e + \frac{q^i}{m^i}\rho^i = 0, \qquad f^\alpha = M^\alpha(U^\alpha), \qquad \alpha = e, i.$$

Consequently, quasineutrality is achieved: $\rho^e = \rho c^e$ and $\rho^i = \rho c^i$ and c^e, c^i are the constants defined in relations (2.2). Therefore $\mathcal{E}^e$ and $\mathcal{E}^i$ are given by (2.4) and if we set $\mathcal{U} = (\rho, \rho u, \mathcal{E}^e, \mathcal{E}^i)$, then $\mathcal{U}$, U^e and U^i are linked by (2.6). By applying the moment operator P to the two first set of equations of (2.23) and taking the limit $\varepsilon \to 0$, it comes, for $\alpha = e, i$:

$$\begin{cases} \partial_t \rho^\alpha + \mathrm{div}(\rho^\alpha u) = 0, & \text{(2.25a)} \\ \partial_t(\rho^\alpha u) + \mathrm{div}(\rho^\alpha u \otimes u) + \nabla p^\alpha - \dfrac{q^\alpha}{m^\alpha} E\rho^\alpha = 0, & \text{(2.25b)} \\ \partial_t \mathcal{E}^e + \mathrm{div}(u(\mathcal{E}^e + p^e)) - q^e m^e E\rho^e u = \nu^{ei}(T^i - T^e), & \text{(2.25c)} \\ \partial_t \mathcal{E}^i + \mathrm{div}(u(\mathcal{E}^i + p^i)) - q^i m^i E\rho^i u = -\nu^{ei}(T^i - T^e). & \text{(2.25d)} \end{cases}$$

By taking into account the fact that c_e and c_i are constant, the first equation is just the global mass conservation, that is the first equation in (2.5). By multiplying the moment equation (2.25b) for electrons by c_i and the same equation for ions by c_e, we obtain a generalized Ohm's law for E:

$$\frac{\rho^i q^i}{m^i} E = -\frac{\rho^e q^e}{m^e} E = -c^i \nabla p^e + c^e \nabla p^i.$$

Moreover, by adding equations (2.25b) for electrons and ions the force term vanishes and we obtain the second equation in (2.5). Hence $\mathcal{U} = (\rho, \rho u, \mathcal{E}^e, \mathcal{E}^i)$ is solution to the bitemperature Euler system (2.5).

Remark 2.1. The above considerations can be recast in a more general framework including continuous and discrete velocities, see [8], [4], [3] for one-dimensional cases. Here only the specific model that has been used numerically in the present article is developed.

2.3.2. Solutions admissibility. Let us now turn to the admissibility of solutions for the discrete velocity system (2.23). In that aim, we impose the subcharacteristic condition (2.21) for electrons and ions, namely, using notation (2.6):

$$\forall \mathcal{U} \in \Omega, \quad \frac{\lambda_d^-}{2} < u_d - a^\alpha < u_d + a^\alpha < \frac{\lambda_d^+}{2}, \quad \alpha = e, i, \quad d = 1, 2 \tag{2.26}$$

where $a^\alpha = \sqrt{\dfrac{\gamma^\alpha p^\alpha}{\rho^\alpha}}$ is the sound velocity of each species.

Remark 2.2. The condition (2.26) does not involve the global sound speed a defined in (2.7). Actually $a^e \leq a$ (*resp.* $a^i \leq a$) if and only if $\gamma^e(\gamma^e - 1)\varepsilon^e \leq \gamma^i(\gamma^i - 1)\varepsilon^i$ (*resp.* $\gamma^e(\gamma^e - 1)\varepsilon^e \geq \gamma^i(\gamma^i - 1)\varepsilon^i$). Hence if condition (2.26) is satisfied then one has also that

$$\forall \mathcal{U} \in \Omega \quad \frac{\lambda_d^-}{2} < u_d - a < u_d + a < \frac{\lambda_d^+}{2}, \quad \alpha = e, i, \quad d = 1, 2.$$

Note that the Maxwellian functions $M_l^\alpha(U)$ can be written as linear combinations of U^α and $F^\alpha(U^\alpha)$:

$$M_l^\alpha(U^\alpha) = \theta_l U^\alpha + \zeta_l F_1^\alpha(U^\alpha) + \chi_l F_2^\alpha(U^\alpha), \qquad 1 \leq l \leq 4, \qquad \alpha = e, i, \tag{2.27}$$

where θ_l, ζ_l and χ_l are real constants. Using the fact that $(Q_d^\alpha)'(U) = (\eta^\alpha)'(U) \circ (F_d^\alpha)'(U)$, it is easy to prove the following result:

LEMMA 2.3. *For $\alpha = e, i$ and $1 \leq l \leq L$ let G_l^α be the function defined by*

$$\forall U \in \Omega^\alpha, \quad G_l^\alpha(U) = \theta_l \eta^\alpha(U) + \zeta_l Q_1^\alpha(U) + \chi_l Q_2^\alpha(U). \tag{2.28}$$

Then one has

$$\forall U \in \Omega^\alpha, \quad (G_l^\alpha)'(U) = (\eta^\alpha)'(U) \circ (M_l^\alpha)'(U). \tag{2.29}$$

Our entropy result is based on the following proposition.

PROPOSITION 2.4. *([28], [10]) Let η^α, Q^α be the entropy pair defined in (2.9). Suppose that the subcharacteristic condition (2.26) is satisfied. Then M_l^α is bijective and one can define the kinetic entropies, for $1 \leq l \leq 4$ and $\alpha = e, i$,:*

$$H_l^\alpha(f_l^\alpha) = G_l^\alpha((M_l^\alpha)^{-1}(f_l^\alpha)). \tag{2.30}$$

The kinetic entropies enjoy the following properties:

- *for $l = 1, ..., 4$, the function H_l^α is convex.* *(E0)*
- $\displaystyle\sum_{l=1}^{4} H_l^\alpha(M_l^\alpha(U^\alpha)) = \eta^\alpha(U^\alpha).$ *(E1)*
- $\displaystyle\sum_{l=1}^{4} V_l H_l^\alpha(M_l^\alpha(U^\alpha)) = Q^\alpha(U^\alpha).$ *(E2)*
- *for all f, by denoting $U_f = P(f)$, one has* $\displaystyle\sum_{l=1}^{4} H_l^\alpha(M_l^\alpha(U_f)) \leq \sum_{l=1}^{4} H_l^\alpha(f_l).$ *(E3)*

Such kinetic entropies are said to be entropies compatible with the macroscopic entropy η^α.

Then $\mathcal{U}$ is an admissible solution of the bitemperature Euler system, that is the following theorem can be stated:

THEOREM 2.5. *Suppose that the subcharacteristic condition* (2.26) *is satisfied and that* $U^{\alpha,\varepsilon}, U^\alpha \in \Omega_\alpha$ *for all* $\varepsilon > 0$, $\alpha \in \{e,i\}$. *Let* $\mathcal{U}$ *be a solution of bitemperature Euler system* (2.5) *obtained by passing to the limit in* (2.23)*. Then,* $\mathcal{U}$ *satisfies the following entropy inequality:*

$$\partial_t \eta(\mathcal{U}) + \mathrm{div} Q(\mathcal{U}) \leq -\frac{\nu^{ei}}{k_B T^i T^e}(T^i - T^e)^2. \tag{2.31}$$

Proof. First, in (2.23), take the scalar product of the equation over f_l^α by the gradient $(H_l^\alpha)'(f_l^\alpha)$, and sum over l. The following equation is obtained, where $\alpha, \beta \in \{e,i\}$ and $\alpha \neq \beta$:

$$\begin{aligned}\partial_t\left(\sum_{l=1}^4 H_l^\alpha(f_l^{\alpha,\varepsilon})\right) &+ \sum_{l=1}^4 V_l\cdot\nabla_x\left(H_l^\alpha(f_l^{\alpha,\varepsilon})\right) + \frac{q^\alpha}{m^\alpha}\sum_{l=1}^4 (H_l^\alpha)'(f_l^{\alpha,\varepsilon})N(E)f_l^{\alpha,\varepsilon}\\ &= \frac{1}{\varepsilon}\sum_{l=1}^4 (H_l^\alpha)'(f_l^{\alpha,\varepsilon})(M_l^\alpha(U^{\alpha,\varepsilon}) - f_l^{\alpha,\varepsilon}) + \sum_{l=1}^4 (H_l^\alpha)'(f_l^{\alpha,\varepsilon})B_l^{\alpha\beta}(f_l^{\alpha,\varepsilon}, f_l^{\beta,\varepsilon}).\end{aligned}$$

By convexity of H_l^α (property (E0)) and property (E3), the first term of the right-hand-side satisfies the following inequality:

$$\sum_{l=1}^4 (H_l^\alpha)'(f_l^{\alpha,\varepsilon})(M_l^\alpha(U^{\alpha,\varepsilon}) - f_l^{\alpha,\varepsilon}) \leq \sum_{l=1}^4 (H_l^\alpha(M_l^\alpha(U^{\alpha,\varepsilon})) - H_l^\alpha(f_l^{\alpha,\varepsilon})) \leq 0.$$

Hence, one gets:

$$\begin{aligned}\partial_t\left(\sum_{l=1}^4 H_l^\alpha(f_l^{\alpha,\varepsilon})\right) + V_l\cdot\nabla_x\left(H_l^\alpha(f_l^{\alpha,\varepsilon})\right) &+ \frac{q^\alpha}{m^\alpha}\sum_{l=1}^4 (H_l^\alpha)'(f_l^{\alpha,\varepsilon})N(E)f_l^{\alpha,\varepsilon}\\ &\leq \sum_{l=1}^4 (H_l^\alpha)'(f_l^{\alpha,\varepsilon})B_l^{\alpha\beta}(f_l^{\alpha,\varepsilon}, f_l^{\beta,\varepsilon}).\end{aligned} \tag{2.32}$$

By passing formally to the limit $\varepsilon \to 0$, one has $f_l^\alpha = M_l^\alpha(U^\alpha)$ and thanks to properties (E1) and (E2), the inequality (2.32) becomes:

$$\begin{aligned}\partial_t \eta^\alpha(U^\alpha) + \mathrm{div} Q^\alpha(U^\alpha) &+ \frac{q^\alpha}{m^\alpha}\sum_{l=1}^4 (H_l^\alpha)'(M_l^\alpha(U^\alpha))N(E)M_l^\alpha(U^\alpha)\\ &\leq \sum_{l=1}^4 (H_l^\alpha)'(M_l^\alpha(U^\alpha))B_l^{\alpha\beta}(M_l^\alpha(U^\alpha), M_l^\beta(U^\beta)).\end{aligned} \tag{2.33}$$

Note that applying lemma 2.3 gives

$$\forall l \in \{1,2,3,4\}, \quad (H_l^\alpha)'(M_l^\alpha(U^\alpha)) = (\eta^\alpha)'(U^\alpha) \tag{2.34}$$

and by a straightforward computation :

$$(\eta^\alpha)'(U^\alpha)N(E)U^\alpha = 0. \tag{2.35}$$

Hence, it comes that the third term of the left-hand-side of equation (2.33) is equal to zero. Moreover, we have

$$\frac{\partial \eta^\alpha}{\partial \mathcal{E}^\alpha}(U^\alpha) = -\frac{1}{k_B T^\alpha} \tag{2.36}$$

so by using again equations (2.34) and (2.24), one finds:

$$\sum_{l=1}^{4} (H_l^\alpha)'(M_l^\alpha(U^\alpha)) B_l^{\alpha\beta}(M_l^\alpha(U^\alpha), M_l^\beta(U^\beta)) = -\frac{\nu^{ei}}{k_B T^\alpha}(T^\beta - T^\alpha).$$

By summing over α, we obtain estimate (2.31). ∎

3. A first order numerical scheme for the bitemperature Euler system.

In this section, we use the discrete BGK model presented in the previous section to design a finite volume scheme for system (2.5), following the ideas in [8]. We restrict ourselves to a cartesian grid. Denote Δx_1 and Δx_2 the space steps, Δt the time step, and $j = (j_1, j_2) \in \mathbb{Z}^2$. Denoting $e_1 = (1, 0)$, $e_2 = (0, 1)$, and for any unknown $v(x_1, x_2, t)$, v_j^n denotes its approximate value at time t^n in cell $C_j =]x_{1,j_1-\frac{1}{2}}, x_{1,j_1+\frac{1}{2}}[\times]x_{2,j_2-\frac{1}{2}}, x_{2,j_2+\frac{1}{2}}[$.

An approximate solution $(\mathcal{U}_j^n)_{j\in\mathbb{Z}^2}$ of (2.5) at time t_n being known we set

$$U_j^{\alpha,n} = (c^\alpha \rho_j^n, c^\alpha \rho_j^n u_j^n, \mathcal{E}_j^{\alpha,n}), \quad j \in \mathbb{Z}^2 \quad \alpha = e, i. \tag{3.1}$$

We then approximate the discrete kinetic system (2.23).

First step: we set the $f_j^{\alpha,n}$ as

$$f_j^{\alpha,n} = M^\alpha(U_j^{\alpha,n}), \qquad j \in \mathbb{Z}^2, \qquad \alpha = e, i. \tag{3.2}$$

Second step: we solve the linear set of transport equations $\partial_t f^\alpha + \sum_{d=1}^{2} \Lambda_d \partial_{x_d} f^\alpha = 0$ by the upwind scheme and apply the moment operator P. With the usual notation

$$\forall \lambda \in \mathbb{R}, \quad \lambda^+ = \max(\lambda, 0), \quad \lambda^- = \max(-\lambda, 0), \quad \Lambda_d^\pm = \mathrm{diag}(v_{d,l}^\pm \mathrm{I})_{1\le l\le L},$$

we define $\forall j \in \mathbb{Z}^2$,

$$f_j^{\alpha,n+\frac{1}{2}} = f_j^{\alpha,n} - \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\left(h_{j+\frac{e_d}{2}}^{\alpha,n} - h_{j-\frac{e_d}{2}}^{\alpha,n}\right), \quad h_{j+\frac{e_d}{2}}^{\alpha,n} = \Lambda_d^+ f_j^{\alpha,n} - \Lambda_d^- f_{j+e_d}^{\alpha,n}. \tag{3.3}$$

Then we define $U_j^{\alpha,n+\frac{1}{2}}$ as $U_j^{\alpha,n+\frac{1}{2}} = P(f_j^{\alpha,n+\frac{1}{2}})$. Therefore

$$\begin{aligned} U_j^{\alpha,n+\frac{1}{2}} &= U_j^{\alpha,n} - \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\left(F_{j+\frac{e_d}{2}}^{\alpha,n} - F_{j-\frac{e_d}{2}}^{\alpha,n}\right), \\ F_{j+\frac{e_d}{2}}^{\alpha,n} &= \mathcal{F}_d^\alpha(U_j^{\alpha,n}, U_{j+e_d}^{\alpha,n}) \\ \mathcal{F}_d^\alpha(U, V) &= P\Lambda_d^+ M^\alpha(U) - P\Lambda_d^- M^\alpha(V) \end{aligned}$$

which, by the compatibility conditions (2.16), is consistent with F^α.

In the case of the model (2.17), (2.18), (2.19) we find

$$\begin{cases} \text{If} \quad 0 \le \lambda_d^- < \lambda_d^+, & \mathcal{F}_d^\alpha(U,V) = F_d^\alpha(U), \\ \text{If} \quad \lambda_d^- < \lambda_d^+ \le 0, & \mathcal{F}_d^\alpha(U,V) = F_d^\alpha(V), \\ \text{If} \quad \lambda_d^- < 0 < \lambda_d^+, & \mathcal{F}_d^\alpha(U,V) = \dfrac{\lambda_d^+ F_d^\alpha(U) - \lambda_d^- F_d^\alpha(V)}{\lambda_d^+ - \lambda_d^-} + \dfrac{\lambda_d^+\lambda_d^-(V-U)}{2(\lambda_d^+ - \lambda_d^-)}, \end{cases} \tag{3.4}$$

which corresponds to the classical HLL scheme for conservation laws [29]. We recall that this scheme preserves the positivity of density and temperature under appropriate CFL conditions, see [19].

Remark 3.1. It is easy to see that $F^{\alpha,n}_{j+\frac{e_d}{2},1} = c^\alpha F^n_{j+\frac{e_d}{2},1}$ where $F^n_{j+\frac{e_d}{2},1}$ is as follows.

$$\begin{cases} \text{If} \quad 0 \le \lambda_d^- < \lambda_d^+, & F^{\alpha,n}_{j+\frac{e_d}{2},1} = \rho_j^n u_{d,j}^n \\ \text{If} \quad \lambda_d^- < \lambda_d^+ \le 0, & F^{\alpha,n}_{j+\frac{e_d}{2},1} = \rho_{j+e_d}^n u_{d,j+e_d}^n \\ \text{If} \quad \lambda_d^- < 0 < \lambda_d^+, & F^{\alpha,n}_{j+\frac{e_d}{2},1} = \dfrac{\lambda_d^+ \rho_j^n u_{d,j}^n - \lambda_d^- \rho_{j+e_d}^n u_{d,j+e_d}^n}{\lambda_d^+ - \lambda_d^-} + \dfrac{\lambda_d^+\lambda_d^-(\rho_{j+e_d}^n - \rho_j^n)}{2(\lambda_d^+ - \lambda_d^-)}. \end{cases}$$

Hence $\rho_j^{\alpha,n+\frac{1}{2}} = c^\alpha \rho_j^{n+\frac{1}{2}}$, with

$$\rho_j^{n+\frac{1}{2}} = \rho_j^n - \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\left(F^n_{j+\frac{e_d}{2},1} - F^n_{j-\frac{e_d}{2},1}\right). \tag{3.5}$$

Our formalism allows us to prove a discrete entropy inequality. Still for model (2.17), (2.18), (2.19), the upwind scheme (3.3) is monotone if and only if

$$\forall d \in \{1,2\}, \quad \lambda_d \frac{\Delta t}{\Delta x_d} \le 1 \quad \text{with} \quad \lambda_d = \max(|\lambda_d^-|, |\lambda_d^+|). \tag{3.6}$$

If conditions (2.26) and (3.6) are satisfied then there exist discrete entropy fluxes $\mathcal{G}^{\alpha,n}_{j+\frac{e_d}{2},l} = \overline{\mathcal{G}}^\alpha_{d,l}(f^{\alpha,n}_{j,l}, f^{\alpha,n}_{j+e_d,l})$ for $d = 1,2$ such that

$$\frac{H_l^\alpha(f_{j,l}^{\alpha,n+\frac{1}{2}}) - H_l^\alpha(f_{j,l}^{\alpha,n})}{\Delta t} + \sum_{d=1}^{2} \frac{\mathcal{G}^{\alpha,n}_{j+\frac{e_d}{2},l} - \mathcal{G}^{\alpha,n}_{j-\frac{e_d}{2},l}}{\Delta x_d} \le 0. \tag{3.7}$$

Namely, consistently with the exact entropy flux $V_l H_l^\alpha$:

$$\overline{\mathcal{G}}^\alpha_{d,l}(f^{\alpha,n}_{j,l}, f^{\alpha,n}_{j+e_d,l}) = v^+_{d,l} H_l^\alpha(f^{\alpha,n}_{j,l}) - v^-_{d,l} H_l^\alpha(f^{\alpha,n}_{j+e_d,l}), \quad d = 1,2.$$

We have then

LEMMA 3.2. *We consider the model* (2.17), (2.18), (2.19) *and we suppose that conditions* (2.26) *and* (3.6) *are satisfied. Then the following discrete entropy inequality holds:*

$$\sum_\alpha \frac{\eta^\alpha\left(U_j^{\alpha,n+\frac{1}{2}}\right) - \eta^\alpha(U_j^{\alpha,n})}{\Delta t} + \sum_{d=1}^{2} \frac{\mathcal{Q}^{\alpha,n}_{j+\frac{e_d}{2}} - \mathcal{Q}^{\alpha,n}_{j-\frac{e_d}{2}}}{\Delta x_d} \le 0 \tag{3.8}$$

where

$$\mathcal{Q}^{\alpha,n}_{j+\frac{e_d}{2}} = \sum_{\alpha=e,i} \sum_{l=1}^{4} \overline{\mathcal{G}}^\alpha_{d,l}(M_l^\alpha(U_j^{\alpha,n}), M_l^\alpha(U^{n,\alpha}_{j+e_d})) = \mathcal{Q}_d(\mathcal{U}_j^n, \mathcal{U}_{j+1}^n). \tag{3.9}$$

Proof. Sum equation (3.7) over l and over α. Thanks to properties (E3) and (E1), it comes:

$$\sum_{l=1}^{4} H_l^\alpha\left(f_{j,l}^{\alpha,n+\frac{1}{2}}\right) \geq \sum_{l=1}^{4} H_l^\alpha\left(M_l^\alpha\left(\sum_{l=1}^{4} f_{j,l}^{\alpha,n+\frac{1}{2}}\right)\right) = \eta^\alpha\left(\sum_{l=1}^{4} f_{j,l}^{\alpha,n+\frac{1}{2}}\right),$$

which gives the conclusion. ∎

Third step: we take into account the force terms and the source terms. For all $j \in \mathbb{Z}^2$, $\alpha, \beta \in \{e, i\}$ and $\beta \neq \alpha$, we define

$$f_{j,l}^{\alpha,n+\frac{3}{4}} = f_{j,l}^{\alpha,n+\frac{1}{2}} - \Delta t \frac{q^\alpha}{m^\alpha} N(E_j^{n+1}) f_{j,l}^{\alpha,n+1} + \Delta t B_l^{\alpha\beta}(f_j^{\alpha,n+1}, f_j^{\beta,n+1}), \quad 1 \leq l \leq 4 \tag{3.10}$$

and

$$U_j^{\alpha,n+1} = P(f_j^{\alpha,n+\frac{3}{4}}). \tag{3.11}$$

One obtains the following equations for $\alpha, \beta \in \{e, i\}$ and $\alpha \neq \beta$, $\rho_j^{n+\frac{1}{2}}$ being defined in (3.5):

$$\rho_j^{\alpha,n+1} = c^\alpha \rho_j^{n+\frac{1}{2}} \tag{3.12}$$

$$\rho_j^{\alpha,n+1} u_j^{\alpha,n+1} = \rho_j^{\alpha,n} u_j^{\alpha,n} - \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\left(F_{j+\frac{e_d}{2},2}^{\alpha,n} - F_{j-\frac{e_d}{2},2}^{\alpha,n}\right) + \frac{\Delta t\, q^\alpha}{m^\alpha} E_j^{n+1} \rho_j^{\alpha,n+1}$$

$$\begin{aligned}\mathcal{E}_j^{\alpha,n+1} = \mathcal{E}_j^{\alpha,n} &- \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\left(F_{j+\frac{e_d}{2},3}^{\alpha,n} - F_{j-\frac{e_d}{2},3}^{\alpha,n}\right) \\ &+ E_j^{n+1} \cdot u_j^{n+1} \frac{\Delta t\, q^\alpha}{m^\alpha} \rho_j^{\alpha,n+1} + \Delta t \nu^{ei}(T_j^{\beta,n+1} - T_j^{\alpha,n+1}).\end{aligned}$$

Subsequently, it is necessary to ensure that the quasineutrality constraints are satisfied, which correspond to Maxwell-Gauss and Maxwell-Ampère equations in the limit $\varepsilon \to 0$:

$$\frac{q^e}{m^e} \rho_j^{e,n+1} + \frac{q^i}{m^i} \rho_j^{i,n+1} = 0, \quad \frac{q^e}{m^e} \rho_j^{e,n+1} u_j^{e,n+1} + \frac{q^i}{m^i} \rho_j^{i,n+1} u_j^{i,n+1} = 0.$$

By remark 3.1 the first condition is satisfied and $\rho_j^{n+1} = \rho^{e,n+1,j} + \rho_j^{i,n+1} = \rho_j^{n+\frac{1}{2}}$. The second condition is equivalent to $u_j^{i,n+1} = u_j^{e,n+1} = u_j^{n+1}$. As a consequence if $\mathcal{U}_j^{n+1} = (\rho_j^{n+1}, \rho_j^{n+1} u_j^{n+1}, \mathcal{E}_j^{e,n+1}, \mathcal{E}_j^{i,n+1})$ then $U_j^{e,n+1}$ and $U_j^{i,n+1}$ satisfy (3.1), so our notation is consistent. By applying these properties to equation (3.12) for $\alpha = e, i$, one gets:

$$\begin{aligned}c^e \rho_j^{n+1} u_j^{n+1} &= c^e \rho_j^n u_j^n - \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\left(F_{j+\frac{e_d}{2},2}^{e,n} - F_{j-\frac{e_d}{2},2}^{e,n}\right) + \frac{\Delta t\, q^e}{m^e} E_j^{n+1} \rho_j^{e,n+1}, \\ c^i \rho_j^{n+1} u_j^{n+1} &= c^i \rho_j^n u_j^n - \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\left(F_{j+\frac{e_d}{2},2}^{i,n} - F_{j-\frac{e_d}{2},2}^{i,n}\right) + \frac{\Delta t\, q^i}{m^i} E_j^{n+1} \rho_j^{i,n+1}.\end{aligned}$$

Hence, by multiplying the first equation by c^i and the second equation by c^e, and then by substracting one to the other, one obtains, analoguously to the continuous case, the discrete generalized Ohm law:

$$E_j^{n+1}\frac{q^i}{m^i}\rho_j^{i,n+1} = -E_j^{n+1}\frac{q^e}{m^e}\rho_j^{e,n+1} = \sum_{d=1}^{2}\frac{1}{\Delta x_d}(\delta^n_{j+\frac{e_d}{2}} - \delta^n_{j-\frac{e_d}{2}}),$$

where nonconservative products $\delta^n_{j+\frac{e_d}{2}}$ are defined by:

$$\delta^n_{j+\frac{e_d}{2}} = -c^i F^{e,n}_{j+\frac{e_d}{2},2} + c^e F^{i,n}_{j+\frac{e_d}{2},2} \in \mathbb{R}^2.$$

Remark that this approximation of nonconservative products is consistent:

$$\delta^n_{j+\frac{e_d}{2}} = \delta_d(\mathcal{U}_j^n, \mathcal{U}_{j+e_d}^n), \qquad \delta(\mathcal{U},\mathcal{U}) = (-c^i p^e + c^e p^i)\mathrm{I}.$$

Finally, the numerical scheme for the total bitemperature Euler system writes:

$$\left\{\begin{aligned}
&\rho_j^{n+1} = \rho_j^n - \sum_{d=1}^{2}\frac{\Delta t}{\Delta x_d}\left(F^n_{j+\frac{e_d}{2},1} - F^n_{j-\frac{e_d}{2},1}\right),\\
&\rho_j^{n+1}u_j^{n+1} = \rho_j^n u_j^n - \sum_{d=1}^{2}\frac{\Delta t}{\Delta x_d}\left(F^n_{j+\frac{e_d}{2},2} - F^n_{j-\frac{e_d}{2},2}\right),\\
&\mathcal{E}_j^{e,n+1} = \mathcal{E}_j^{e,n} - \sum_{d=1}^{2}\frac{\Delta t}{\Delta x_d}\left(F^n_{j+\frac{e_d}{2},3} - F^n_{j-\frac{e_d}{2},3}\right)\\
&\qquad\qquad - u_j^{n+1}\cdot\sum_{d=1}^{2}\frac{\Delta t}{\Delta x_d}\left(\delta^n_{j+\frac{e_d}{2}} - \delta^n_{j-\frac{e_d}{2}}\right) + \Delta t\nu^{ei}(T_j^{i,n+1} - T_j^{e,n+1}),\\
&\mathcal{E}_j^{i,n+1} = \mathcal{E}_j^{i,n} - \sum_{d=1}^{2}\frac{\Delta t}{\Delta x_d}\left(F^n_{j+\frac{e_d}{2},4} - F^n_{j-\frac{e_d}{2},4}\right)\\
&\qquad\qquad + u_j^{n+1}\cdot\sum_{d=1}^{2}\frac{\Delta t}{\Delta x_d}\left(\delta^n_{j+\frac{e_d}{2}} - \delta^n_{j-\frac{e_d}{2}}\right) - \Delta t\nu^{ei}(T_j^{i,n+1} - T_j^{e,n+1}),
\end{aligned}\right. \tag{3.13}$$

with $F^n_{j+\frac{e_d}{2},1}$ defined in Remark 3.1 and

$$F^n_{j+\frac{e_d}{2},2} = \sum_{\alpha=e,i} F^{\alpha,n}_{j+\frac{e_d}{2},2}, \quad F^n_{j+\frac{e_d}{2},3} = F^{e,n}_{j+\frac{e_d}{2},3}, \quad F^n_{j+\frac{e_d}{2},4} = F^{i,n}_{j+\frac{e_d}{2},3}.$$

More precisely:

$$\delta^n_{j+\frac{e_d}{2}} = \left|\begin{array}{l}
\left(-c_i p^{e,n}_{j+e_d} + c_e p^{i,n}_{j+e_d}\right) e_d \text{if} \quad \lambda_d^- < \lambda_d^+ \le 0,\\
\left(-c_i p^{e,n}_{j} + c_e p^{i,n}_{j}\right) e_d \text{if} \quad 0 \le \lambda_d^- < \lambda_d^+,\\
\left(\frac{\lambda_d^+}{\lambda_d^+ - \lambda_d^-}(-c^i p_j^{e,n} + c^e p_j^{i,n}) - \frac{\lambda_d^-}{\lambda_d^+ - \lambda_d^-}(-c^i p^{e,n}_{j+e_d} + c^e p^{i,n}_{j+e_d})\right) e_d\\
\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad \lambda_d^- < 0 < \lambda_d^+ .
\end{array}\right.$$

Consequently, equations over partial energies can be rewritten:

$$\begin{cases} \mathcal{E}_j^{e,n+1} = \mathcal{E}_j^{e,n} - \displaystyle\sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d} \left(F^n_{j+\frac{e_d}{2},3} - F^n_{j-\frac{e_d}{2},3} \right) \\ \qquad\qquad - \displaystyle\sum_{d=1}^{2} u_{d,j}^{n+1} \frac{\Delta t}{\Delta x_d} \left(\delta^n_{j+\frac{e_d}{2},d} - \delta^n_{j-\frac{e_d}{2},d} \right) + \Delta t \nu^{ei} (T_j^{i,n+1} - T_j^{e,n+1}), \\ \mathcal{E}_j^{i,n+1} = \mathcal{E}_j^{i,n} - \displaystyle\sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d} \left(F^n_{j+\frac{e_d}{2},4} - F^{i,n}_{j-\frac{e_d}{2},4} \right) \\ \qquad\qquad + \displaystyle\sum_{d=1}^{2} u_{d,j}^{n+1} \frac{\Delta t}{\Delta x_d} \left(\delta^n_{j+\frac{e_d}{2},d} - \delta^n_{j-\frac{e_d}{2},d} \right) - \Delta t \nu^{ei} (T_j^{i,n+1} - T_j^{e,n+1}). \end{cases}$$

By using the following expression for temperature,

$$T^\alpha = \frac{1}{C_v^\alpha} \left(-\frac{1}{2}|u|^2 + \frac{\mathcal{E}^\alpha}{\rho^\alpha} \right), \qquad C_v^\alpha = \frac{k_B}{m^\alpha(\gamma^\alpha - 1)}, \qquad \alpha \in \{e, i\},$$

one obtains an explicit expression of electronic and ionic energies $\mathcal{E}_j^{e,n+1}$, $\mathcal{E}_j^{i,n+1}$ as the solution of a linear 2×2 system which determinant is:

$$1 + \Delta t \, \nu^{ei} \left(\frac{1}{\rho_j^{e,n+1} C_v^e} + \frac{1}{\rho_j^{i,n+1} C_v^i} \right) \neq 0.$$

Remark 3.3. By summing the expressions for $\mathcal{E}_j^{e,n+1}$ and $\mathcal{E}_j^{i,n+1}$ we observe that the approximation of $(\rho, \rho u, \mathcal{E} = \mathcal{E}^e + \mathcal{E}^i)$ is conservative, and in the case $\gamma^e = \gamma^i$ it coincides with the HLL scheme. As a consequence the positivity of ρ and of the total temperature $T = \dfrac{ZT^e + T^i}{Z+1}$ are preserved ([19]).

THEOREM 3.4. *We suppose that conditions* (2.26) *and* (3.6) *are satisfied. The numerical scheme* (3.13) *is entropy dissipative: with the notation* (3.9)

$$\frac{\eta(\mathcal{U}_j^{n+1}) - \eta(\mathcal{U}_j^n)}{\Delta t} + \sum_{d=1}^{2} \frac{\mathcal{Q}^n_{j+\frac{e_d}{2}} - \mathcal{Q}^n_{j-\frac{e_d}{2}}}{\Delta x_d} \leq - \frac{\nu^{ei}}{k_B T_j^{i,n+1} T_j^{e,n+1}} (T_j^{i,n+1} - T_j^{e,n+1})^2. \tag{3.14}$$

Proof. We have

$$U_j^{\alpha,n+1} = U_j^{\alpha,n+\frac{1}{2}} - \Delta t \frac{q^\alpha}{m^\alpha} N(E_j^{n+1}) U_j^{\alpha,n+1} + \Delta t \nu^{\alpha\beta} (T_j^{\beta,n+1} - T_j^{\alpha,n+1}) e_4,$$

with $e_4 = (0, 0, 0, 1)$. Multiply this equation by $(\eta^\alpha)'(U_j^{\alpha,n+1})$. $(\eta^\alpha)'$ being a convex function, one gets:

$$\eta^\alpha(U_j^{\alpha,n+1}) - \eta^\alpha(U_j^{\alpha,n+\frac{1}{2}}) \leq (\eta^\alpha)'(U_j^{\alpha,n+1})(U_j^{\alpha,n+1} - U_j^{\alpha,n+\frac{1}{2}}). \tag{3.15}$$

Using properties (2.35) and (2.36) and summing equation (3.15) over α, it comes:

$$\sum_\alpha \frac{\eta^\alpha(U_j^{\alpha,n+1}) - \eta^\alpha(U_j^{\alpha,n+\frac{1}{2}})}{\Delta t} \leq - \frac{\nu^{ei}}{k_B T_j^{i,n+1} T_j^{e,n+1}} (T_j^{i,n+1} - T_j^{e,n+1})^2. \tag{3.16}$$

Finally, by combining (3.8) and (3.16), and using the fact that $U_j^{e,n+1}$ and $U_j^{i,n+1}$ satisfy (3.1), discrete entropy inequality (3.14) is obtained. ∎

4. Second-order extension. In this section, we extend our scheme to the second order. The second order in time is reached by Heun's method. We focus our attention to second order in space. Like in [27], a piecewise affine reconstruction is used to determine intermediate values in subcells, but here this viewpoint leads to practical computations that are not required in the conservative case. Let us first recall the viewpoint for a one-dimensional system of conservation laws

$$\partial_t U + \partial_x F(U) = 0.$$

Assume that a first-order conservative scheme has been chosen:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\right)$$

with $F^n_{j+\frac{1}{2}} = \mathcal{F}(U_j^n, U_{j+1}^n)$ and $\mathcal{F}(U,U) = F(U)$. Define a piecewise affine reconstruction:

$$\forall x \in C_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[, \qquad U^n(x) = U_j^n + \sigma_j^n(x - x_j), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}). \tag{4.1}$$

Once the reconstruction has been chosen, the values at the interfaces are

$$U^+_{j+\frac{1}{2}} = (U^n(x_{j+\frac{1}{2}}))^+ = U^n_{j+1} - \sigma^n_{j+1}\frac{\Delta x}{2}, \qquad U^-_{j+\frac{1}{2}} = (U^n(x_{j+\frac{1}{2}}))^- = U^n_j + \sigma^n_j \frac{\Delta x}{2}. \tag{4.2}$$

Then modify the first-order scheme in the following manner:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U^-_{j+\frac{1}{2}}, U^+_{j+\frac{1}{2}}) - \mathcal{F}(U^-_{j-\frac{1}{2}}, U^+_{j-\frac{1}{2}})\right). \tag{4.3}$$

The stability properties of the first order scheme, such as positivity preservation, are satisfied by (4.3) under a half-CFL condition. This is due to the fact that this scheme can be interpreted as a first-order scheme defined on half-cells $C_j^- =]x_{j-\frac{1}{2}}, x_j[$ and $C_j^+ =]x_j, x_{j+\frac{1}{2}}[$, see [27] and also [11]: taking $U^+_{j-\frac{1}{2}}$ in C_j^- and $U^-_{j+\frac{1}{2}}$ in C_j^+ as initial values at time t_n, one gets:

$$U_j^{n+1,-} = U^+_{j-\frac{1}{2}} - \frac{2\Delta t}{\Delta x}\left(\mathcal{F}(U^+_{j-\frac{1}{2}}, U^-_{j+\frac{1}{2}}) - \mathcal{F}(U^-_{j-\frac{1}{2}}, U^+_{j-\frac{1}{2}})\right)$$

$$U_j^{n+1,+} = U^-_{j+\frac{1}{2}} - \frac{2\Delta t}{\Delta x}\left(\mathcal{F}(U^-_{j+\frac{1}{2}}, U^+_{j+\frac{1}{2}}) - \mathcal{F}(U^+_{j-\frac{1}{2}}, U^-_{j+\frac{1}{2}})\right).$$

Then, the scheme (4.3) is obtained by

$$U_j^{n+1} = \frac{1}{2}\left(U_j^{n+1,-} + U_j^{n+1,+}\right).$$

This procedure is extended in the case of a two-dimensional triangular mesh in [27]. More developments, particularly on the limitation procedure can be found in [25], [9], [14]. It is important to note that the effective computation of the numerical fluxes at the interface of two subcells is not needed in the conservative case. It is just useful to interpretate the scheme as a combination of first order schemes. One can also add others subcells in order to realize positivity requirements, but without additional computational cost, see [9].

To treat the nonconservative case, we want to use the same ideas. We treat directly the case of the two-dimensional cartesian grid. Contrarily to the conservative case, this algorithm necessitates the computation of the numerical fluxes at the interface of two subcells. This is a key point that leads us to detail our procedure.

Each cell C_j is divided into four subcells, according to figure 1.

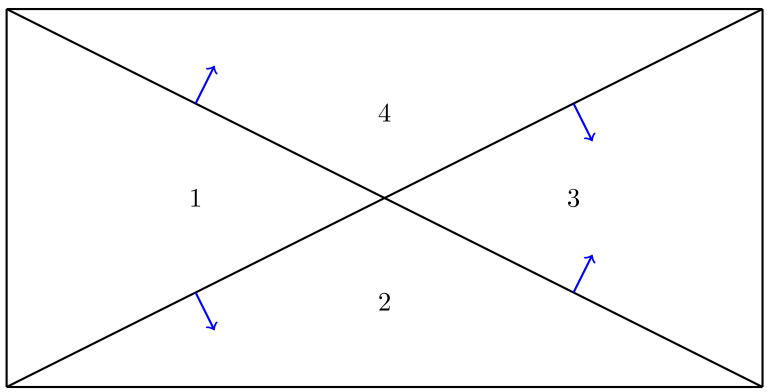

Fig. 1. *For each cell C_j: subdivision into 4 triangles $T_j^{(i)}$ ($i \in \{1,2,3,4\}$), and corresponding unit normal vectors.*

Let $(\mathcal{U}_j^n)_j$ be the approximate solution at time t^n. $\mathcal{U}^n$ is reconstructed to second-order by using slopes $\sigma_j^n = (\sigma_{1,j}^n, \sigma_{2,j}^n)$, $j \in \mathbb{Z}^2$:

$$\forall x \in C_j, \qquad \mathcal{U}(x) = \mathcal{U}_j^n + (x - x_j) \cdot \sigma_j^n.$$

Then, we define four constant states:

$$\begin{aligned}
\mathcal{U}_j^{(1)} &= \mathcal{U}_j^n - \frac{\Delta x_1}{2}\sigma_{1,j}^n, \qquad & \mathcal{U}_j^{(2)} &= \mathcal{U}_j^n - \frac{\Delta x_2}{2}\sigma_{2,j}^n,\\
\mathcal{U}_j^{(3)} &= \mathcal{U}_j^n + \frac{\Delta x_1}{2}\sigma_{1,j}^n, \qquad & \mathcal{U}_j^{(4)} &= \mathcal{U}_j^n + \frac{\Delta x_2}{2}\sigma_{2,j}^n .
\end{aligned}$$

The state $\mathcal{U}_j^{(i)}$ is the initial value at time t_n in the subcell $T_j^{(i)}$ of C_j. We apply a first-order scheme to this new triangular mesh. We follow the same lines as in section 3 except that we need to use the upwind scheme on triangles instead of rectangles. The positivity and entropy properties of this first order approximation are the same as in the rectangular case.

We denote $T_\mu = T_j^{(i)}$, $\mathcal{U}_\mu = \mathcal{U}_j^{(i)}$. We set

$$U_\mu^{\alpha,n} = (c^\alpha \rho_\mu^n, c^\alpha \rho_\mu^n u_\mu^n, \mathcal{E}_\mu^{\alpha,n}), \quad f_\mu^{\alpha,n} = M^\alpha(U_\mu^{\alpha,n}), \quad \alpha \in \{e,i\}.$$

Then we solve the linear transport set of transport equations $\partial_t f^\alpha + \sum_{d=1}^{2} \Lambda_d \partial_{x_d} f^\alpha = 0$ by the upwind scheme. For a triangle T_μ, the adjacent triangles are denoted T_{μ_1}, T_{μ_2}, T_{μ_3}, the outward unit normal vector from T_μ to T_{μ_k} is denoted n_k, the edge between T_μ and T_{μ_k} is denoted Γ_k. The upwind scheme then writes as

(4.4)
$$f_{\mu,l}^{\alpha,n+\frac{1}{2}} = f_{\mu,l}^{\alpha,n} - \frac{\Delta t}{|T_\mu|} \sum_{k=1}^{3} \Big((V_l \cdot n_k)^+ f_{\mu,l}^n - (V_l \cdot n_k)^- f_{\mu_k,l}^n \Big) |\Gamma_k|, \quad l \in \{1,2,3,4\}$$

which can be rewritten

$$f_{\mu,l}^{\alpha,n+\frac{1}{2}} = f_{\mu,l}^{\alpha,n} - \Delta t \sum_{k=1}^{3} \Phi_{k,l}(f_{\mu,l}^{\alpha,n}, f_{\mu_k,l}^{\alpha,n}, n_k),$$

where for $f, g \in \mathbb{R}^4$ and $n \in \mathbb{R}^2$,

$$\Phi_{k,l,\mu}(f,g,n) = \big((V_l \cdot n)^+ f - (V_l \cdot n)^- g\big) \frac{|\Gamma_k|}{|T_\mu|}.$$

LEMMA 4.1. *Let λ_1 and λ_2 be defined in* (3.6). *The upwind scheme* (4.4) *is monotone if and only if the following CFL condition holds:*

$$\Delta t \max_{1 \le d \le 2} \frac{\lambda_d}{\Delta x_d} \le \frac{1}{4}\,. \tag{4.5}$$

Proof. For a given triangle T_μ with edges Γ_k and outward unit normal vectors n_k we have to satisfy the condition

$$\forall l \in \{1,2,3,4\}, \qquad \frac{\Delta t}{|T_\mu|} \sum_{k=1}^{3} (V_l . n_k)^+ \, |\Gamma_k| \le 1.$$

It is necessary to compute the quantities $G = \frac{|\Gamma_k|}{|T_\mu|} V_l \cdot n_k$, for each type of interface. In the setting chosen here, there exist four types of edges:

- Vertical edges ($n = e_1$): $G = 4\dfrac{v_{1,l}}{\Delta x_1}$.
- Horizontal edges ($n = e_2$): $G = 4\dfrac{v_{2,l}}{\Delta x_2}$.
- Diagonal edges similar to the ones between subcells 1 and 2 on figure 1: $G = 2\left(\dfrac{v_{1,l}}{\Delta x_1} - \dfrac{v_{2,l}}{\Delta x_2}\right)$.
- Diagonal edges similar to the ones between subcells 1 and 4 on figure 1: $G = 2\left(\dfrac{v_{1,l}}{\Delta x_1} + \dfrac{v_{2,l}}{\Delta x_2}\right)$.

The result is then achieved straightforwardly. ∎

The remaining steps for the subcell T_μ are the same as in the cartesian case, in particular the homogeneity property of remark 3.1 is still available. Macroscopic fluxes for species α can be defined as

$$\forall (U,V) \in \mathbb{R}^4, \qquad \mathcal{F}_{k,\mu}^{\alpha}(U,V,n_k) = \sum_{l=1}^{4} \Phi_{k,l,\mu}(M_l^\alpha(U), M_l^\alpha(V), n_k)$$

and we obtain

$$\begin{cases} \rho_\mu^{n+1} = \rho_\mu^n - \Delta t \sum_{k=1}^{3} \mathcal{F}_{k,\mu,1}^n, \\ \rho_\mu^{n+1} u_\mu^{n+1} = \rho_\mu^n u_\mu^n - \Delta t \sum_{k=1}^{3} \mathcal{F}_{k,\mu,2}^n, \\ \mathcal{E}_\mu^{e,n+1} = \mathcal{E}_\mu^{e,n} - \Delta t \sum_{k=1}^{3} \mathcal{F}_{k,\mu,3}^n + \Delta t u_\mu^{n+1} \cdot \sum_{k=1}^{3} \delta_{k,\mu}^n + \Delta t \nu^{ei} (T_\mu^{i,n+1} - T_\mu^{e,n+1}), \\ \mathcal{E}_\mu^{i,n+1} = \mathcal{E}_\mu^{i,n} - \Delta t \sum_{k=1}^{3} \mathcal{F}_{k,\mu,4}^n - \Delta t u_\mu^{n+1} \cdot \sum_{k=1}^{3} \delta_{k,\mu}^n - \Delta t \nu^{ei} (T_\mu^{i,n+1} - T_\mu^{e,n+1}), \end{cases}$$

where

$$\mathcal{F}_{k,\mu,1}^n = \sum_\alpha \mathcal{F}_{k,\mu,1}^\alpha (U_\mu^{\alpha,n}, U_{\mu_k}^{\alpha,n}, n_k), \qquad \mathcal{F}_{k,\mu,2}^n = \sum_\alpha \mathcal{F}_{k,\mu,2}^\alpha (U_\mu^{\alpha,n}, U_{\mu_k}^{\alpha,n}, n_k),$$
$$\mathcal{F}_{k,\mu,3}^n = \mathcal{F}_{k,\mu,3}^e (U_\mu^{e,n}, U_{\mu_k}^{e,n}, n_k), \qquad \mathcal{F}_{k,\mu,4}^n = \mathcal{F}_{k,\mu,3}^i (U_\mu^{i,n}, U_{\mu_k}^{i,n}, n_k),$$

and

$$\delta_{k,\mu}^n = -c^i \mathcal{F}_{k,\mu,2}^e (U_\mu^{e,n}, U_{\mu_k}^{e,n}, n_k) + c^e \mathcal{F}_{k,\mu,2}^i (U_\mu^{i,n}, U_{\mu_k}^{i,n}, n_k) \in \mathbb{R}^2.$$

Computation of partial energies is similar to the first-order scheme, by the resolution of 2×2 system.

Finally, denoting $\mathcal{U}_j^{(i),n+1}$ the value obtained in subcell number $T_j^{(i)}$, solution at time t^{n+1} is defined by:

$$\mathcal{U}_j^{n+1} = \frac{1}{4} \sum_{i=1}^{4} \mathcal{U}_j^{(i),n+1}.$$

Again if $\gamma^e = \gamma^i$, the positivity of ρ and of the total temperature are preserved under appropriate reconstruction and CFL condition.

5. Numerical results. In this section, the second-order method developed previously is validated by a series of test cases: 1D Riemann problem extended to 2D, 2D Riemann problem with four states and an implosion test case.

For all test cases, the following physical parameters are fixed: Boltzmann constant $k_B = 1.3807 \times 10^{-23}$ J.K^{-1}, electronic particular mass $m^e = 9,1094 \times 10^{-31}$ kg, ionic particular mass $m^i = 1.6726 \times 10^{-27}$ kg and elementary electric charge $e = -q^e = q^i = 1.6022 \times 10^{-19}$ C. Ionization rate Z is fixed at 1.

The first problem we have to deal with is the choice of the velocities $\lambda_d^\pm$. As a matter of fact, due to the physical values involved: high temperatures, strong differences between electronic and ionic masses, the theoretical condition (2.26) largely overestimates the needed values. Hence the computed solutions are highly diffusive, even for refined grids. This is due to the fact that there is a high difference between the electronic and ionic sound velocities. Consequently we choose to use the global sound velocity:

$$(5.1) \qquad \forall \mathcal{U} \in \Omega, \quad \frac{\lambda_d^-}{2} < u_d - a < u_d + a < \frac{\lambda_d^+}{2}, \quad d = 1, 2$$

where a is defined in (2.7).

5.1. 1D to 2D. The goal of our first test case is to establish the consistency of the 2D code with already obtained 1D results. In [8] and [13] the one-dimensional first order version of the scheme presented here is compared to other first order schemes. It is noticed that in the presence of shocks, that is when the nonconservative products $u \cdot \nabla(c_i p_e - c_e p_i)$ are not well defined, the values of ionic and electronic temperatures are sensitive to the choice of the discretisation method. In particular, the 1D first order version of the scheme presented here is in good agreement with the DVM and the kinetic relaxed method, with physically meaningful results. In the present work, we want to verify that the values of discontinuous temperatures remain the same when 1D and 2D versions of the the scheme are applied, and also when we move from first to second order. The second order 1D scheme is constructed with the same ideas as the 2D one.

Let $(\overline{\rho}, \overline{\rho}\,\overline{u}, \overline{\mathcal{E}_e}, \overline{\mathcal{E}_i}) \in \mathbb{R}^4$ a solution of the 1D bitemperature Euler system. For $\omega = (\cos\theta, \sin\theta)$ fixed, we define for $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$:

$$\rho(x,t) = \overline{\rho}(x\cdot\omega, t), \quad u(x,t) = \overline{u}(x\cdot\omega,t)\,\omega, \quad \mathcal{E}_\alpha(x,t) = \overline{\mathcal{E}_\alpha}(x\cdot\omega,t), \quad \alpha = e,i.$$

This defines a solution of the 2D system (2.5).

All quantities are in SI units. In order to prove that γ_i and γ_e are allowed to be distinct we choose $\gamma_e = 5/3$, $\gamma_i = 7/5$. We set $\overline{\rho}(x,0) = 1$, $\overline{u}(x,0) = 0$, and electronic and ionic initial temperatures are:

$$\begin{aligned}
&\overline{T^e}(x,0) = 2.3\times 10^6 \quad \text{if} \quad x < \frac{1}{2}, \quad \overline{T^e}(x,0) = 2.3\times 10^7 \quad \text{else},\\
&\overline{T^i}(x,0) = 1.7406\times 10^6 \quad \text{if} \quad x < \frac{1}{2}, \quad \overline{T^i}(x,0) = 1.7406\times 10^7 \quad \text{else}.
\end{aligned}$$

The rotation angle is $\theta = -\pi/12$. Final simulation time is set equal to $t = 4.0901 \times 10^{-7}$. In this test case, we set $\nu^{ei} = 4\times 10^9$, so that the ionic and electronic temperatures remain distinct. The 1D test is performed on a 800 points uniform mesh of [0,1], while the 2D test is performed on a 800×800 uniform mesh of $[0,1]\times[0,1]$.

In figure 2 we present the total 2D density ρ (left) and electronic temperature (right) for the second order scheme. Then we compare 1D results with 2D values on a segment along the propagation direction $\omega = (\cos\theta, \sin\theta)$ passing by the center of the unit square. We focus on the electronic and ionic temperatures. The first order and second order 1D plateaux are identical, see figures 3 left (electronic) and right (ionic). The 1D and 2D results also coincide, see figures 4 left (electronic) and right (ionic).

5.2. Four interfaces Riemann problem. For this second test case, consider, on domain $[0,1]\times[0,1]$, a partition in four quadrants of identical size. A constant state is chosen as initial data on each quadrant. Initial velocity is equal to zero over the whole domain and initial densities are as follows:

$$\begin{cases}
\rho(x_1,x_2,0) = 1 \text{ kg.m}^{-3}, \text{ if } x_1 < 0.5 \text{ and } x_2 < 0.5,\\
\rho(x_1,x_2,0) = 0.125 \text{ kg.m}^{-3}, \text{ if } x_1 < 0.5 \text{ and } x_2 > 0.5,\\
\rho(x_1,x_2,0) = 0.125 \text{ kg.m}^{-3}, \text{ if } x_1 > 0.5 \text{ and } x_2 < 0.5,\\
\rho(x_1,x_2,0) = 1 \text{ kg.m}^{-3}, \text{ if } x_1 > 0.5 \text{ and } x_2 > 0.5,
\end{cases}$$

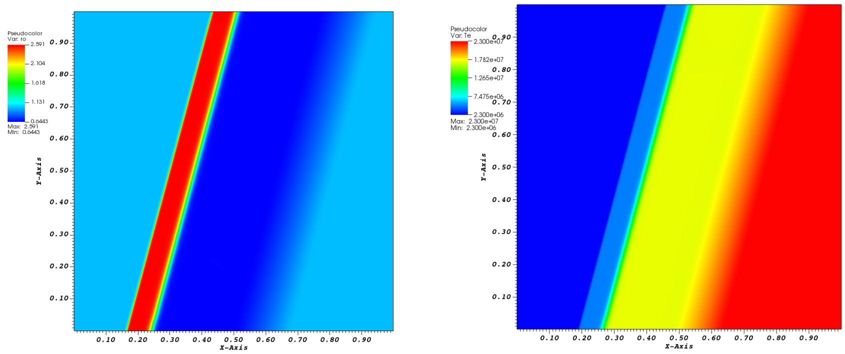

FIG. 2. *Shock tube test case with* $\nu^{ei} = 4 \times 10^9$*, 800 by 800 points. Left: total density, right : electronic temperature.*

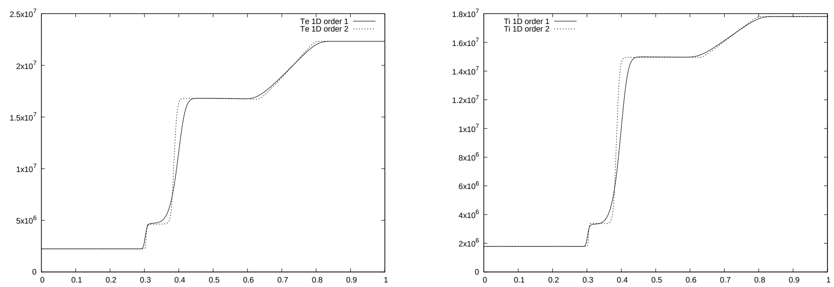

FIG. 3. *Shock tube test case with* $\nu^{ei} = 4 \times 10^9$*, 800 by 800 points. 1D results. Left: electronic temperature, right: ionic temperature.*

and initial electronic and ionic temperatures are defined by:

$$\begin{cases} T^e(x_1,x_2,0) = 293 \text{ K} \ , T^i(x_1,x_2,0) = 273 \text{ K} \ , \ \text{if } x_1 < 0.5 \text{ and } x_2 < 0.5, \\ T^e(x_1,x_2,0) = 220 \text{ K} \ , T^i(x_1,x_2,0) = 200 \text{ K} \ , \ \text{if } x_1 < 0.5 \text{ and } x_2 > 0.5, \\ T^e(x_1,x_2,0) = 220 \text{ K} \ , T^i(x_1,x_2,0) = 200 \text{ K} \ , \ \text{if } x_1 > 0.5 \text{ and } x_2 < 0.5, \\ T^e(x_1,x_2,0) = 293 \text{ K} \ , T^i(x_1,x_2,0) = 273 \text{ K} \ , \ \text{if } x_1 > 0.5 \text{ and } x_2 > 0.5. \end{cases}$$

Here $\gamma^e = \gamma^i = 5/3$.

We compute the solution on a 2000×2000 grid. Final time is $t = 0.0001$. Moreover, we set $\nu^{ei} = 100 \text{ s}^{-1}$. Electronic temperature is presented in figure 5.

We proceed to cut the solution displayed on figure 5 along two different axis. The first one is along axis $x_1 = 0.05$ and is displayed on figure 6 (left). The second one is made along the axis $x_2 = 0.95$ and is visible on figure 6 (right). We retrieve the solutions of the associate one-dimensional Riemann problems.

5.3. Implosion test case. In this test case, consider an implosion-type problem, introduced in [20]. The physical domain is the square $[-1,1] \times [-1,1]$. We set $\gamma^e = \gamma^i = 5/3$. Initial data for this Riemann problem is as follows: $\rho = 1 \text{ kg.m}^{-3}$, $u = 0$

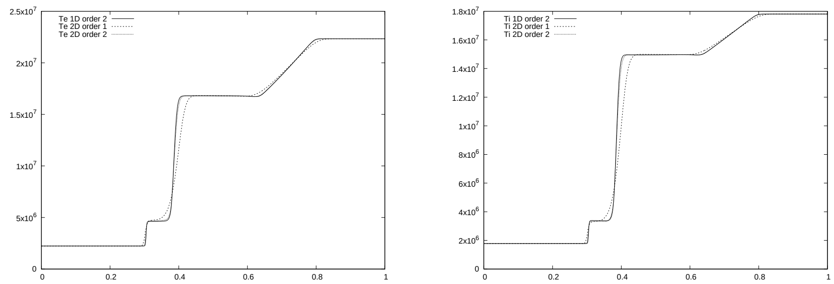

FIG. 4. *Shock tube test case with* $\nu^{ei} = 4 \times 10^9$*, 800 by 800 points. 1D Vs 2D results along the propagation direction. Left: electronic temperature, right: ionic temperature.*

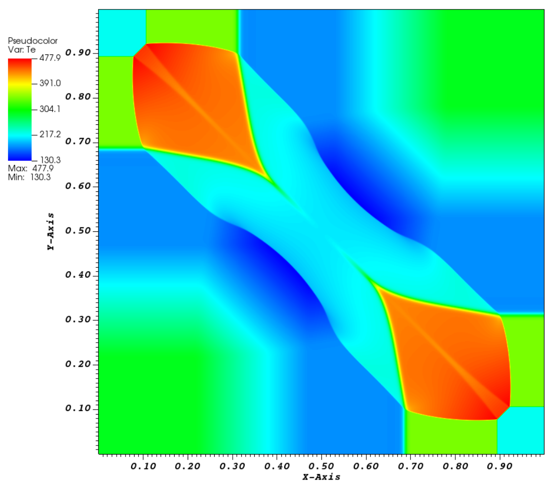

FIG. 5. *Electronic temperature at time* $t = 0.0001s$ *for a four interfaces Riemann problem with* $\nu^{ei} = 100\ s^{-1}$*, with a grid of 2000 by 2000 points.*

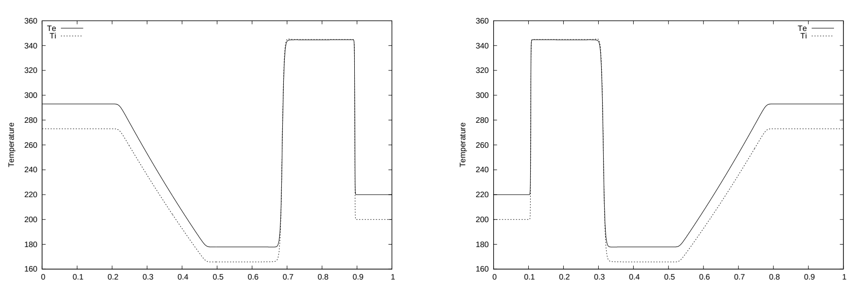

FIG. 6. *Electronic and ionic temperatures at time* $t = 0.0001s$ *for a four interfaces Riemann problem with* $\nu^{ei} = 100\ s^{-1}$*, with a grid of 2000 by 2000 points along axis* $x_1 = 0.05$ *(left) and along axis* $x_2 = 0.95$ *(right).*

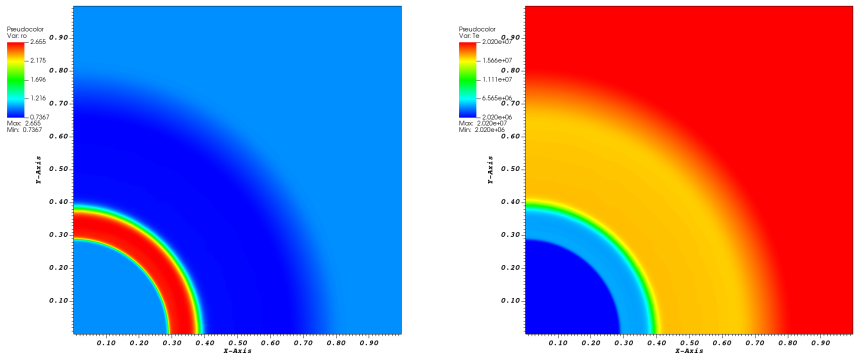

FIG. 7. *Total density (left) and electronic temperature (right) at time $t = 4.0901 \times 10^{-7}s$ for a implosion test case with ν^{ei} given by the NRL formula with a grid of 500 by 500 points.*

m.s^{-1} and temperatures are given by:

$$\begin{aligned} &T^e(x_1,x_2,0) = 2,3\times 10^6 K, \quad T^i(x_1,x_2,0) = 1.7406\times 10^6 K \quad \text{ if } (x_1)^2+(x_2)^2 < \frac{1}{4},\\ &T^e(x_1,x_2,0) = 2,3\times 10^7 K, \quad T^i(x_1,x_2,0) = 1.7406\times 10^7 K \quad \text{ otherwise.}\end{aligned}$$

The relaxation frequency ν^{ei} is chosen realistically, according to the formulae given by the NRL formulary [21].

Thanks to symmetry properties of the problem, it is only necessary to solve it on the quarter domain $[0,1]\times[0,1]$, equipped with suitable boundary conditions. On figure 7, are given the isovalues of the total density and of the electronic temperature at time $t = 4.0901\times 10^{-7}$ s.

We compare our results to the ones in [20], pages 48-52, which have been obtained by replacing the nonconservative bitemperature Euler system by a conservative one with the hypothesis that the electrons have an isentropic behaviour. Qualitatively, the results are similar, including the numerical values taken by densities, velocities and temperatures. The difference lies only on the velocity of propagation of the waves. In order to clarify this point we write the system in polar coordinates for such a solution: the velocity is a scalar function $v(r)$ multiplied by the radial vector $(\cos\theta, \sin\theta)$ so that $|u| = |v|$. One has

$$\rho(x,t) = \overline{\rho}(r,t), \quad u(x,t) = v(r,t)(\cos\theta,\sin\theta), \quad \mathcal{E}^\alpha(x,t) = \overline{\mathcal{E}^\alpha}(r,t)$$

satisfying the following system:

$$\begin{cases} \partial_t\overline{\rho} + \partial_r\left(\overline{\rho}v\right) = -\dfrac{1}{r}\overline{\rho}v \\ \partial_t(\overline{\rho}v) + \partial_r\left(\overline{\rho}v^2 + \overline{p^e} + \overline{p^i}\right) = -\dfrac{1}{r}\overline{\rho}v^2 \\ \partial_t\overline{\mathcal{E}^e} + \partial_r\left(v(\overline{\mathcal{E}^e} + \overline{p^e})\right) + v\partial_r\left(c^e\overline{p^i} - c^i\overline{p^e}\right) = -\dfrac{1}{r}v(\overline{\mathcal{E}^e} + \overline{p^e}) + \nu_{ei}(\overline{T^i} - \overline{Te}) \\ \partial_t\overline{\mathcal{E}^i} + \partial_r\left(v(\overline{\mathcal{E}^i} + \overline{p^i})\right) - v\partial_r\left(c^e\overline{p^i} - c^i\overline{p^e}\right) = -\dfrac{1}{r}v(\overline{\mathcal{E}^i} + \overline{p^i}) + \nu_{ei}(\overline{T^e} - \overline{T^i}). \end{cases}$$

This one-dimensional system can be viewed as the 1D cartesian system with a source term, so we compute the solution by using a slight modification of the 1D cartesian

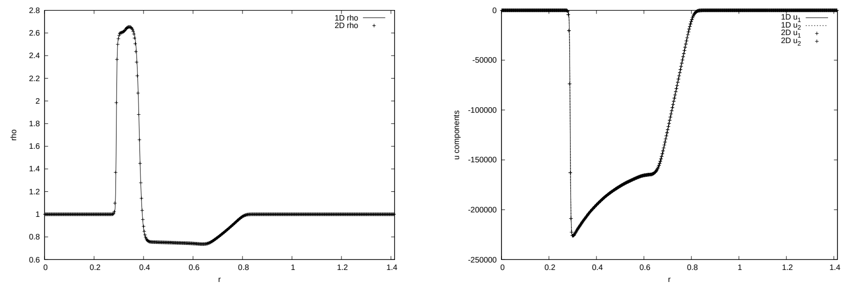

FIG. 8. *Total density (left) and velocity (right) along the first bisector at time* $t = 4.0901 \times 10^{-7}s$ *for an implosion test case with* ν^{ei} *given by the NRL formula with a grid of 500 by 500 points. Comparison with a 1D computation in polar coordinates.*

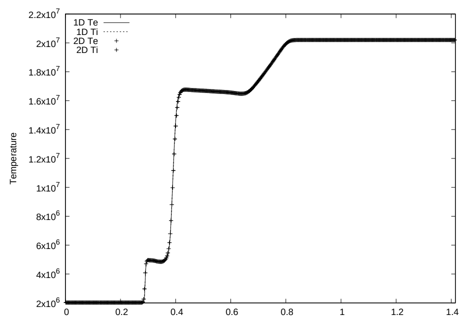

FIG. 9. *Electronic and ionic temperatures along the first bisector at time* $t = 4.0901 \times 10^{-7}s$ *for an implosion test case with* ν^{ei} *given by the NRL formula with a grid of 500 by 500 points. Comparison with a 1D computation in polar coordinates.*

scheme. We find the same results as the 2D computation, as shown on figures 8, 9 where a cut along the first bisector is provided: the total density and the components of the velocity are displayed on figure 8. On figure 9 on can observe that at final time, electronic and ionic temperatures have completely relaxed towards equilibrium : $T^i = T^e$. The discrepancy with the results of [20] can be due, either to the change of model, or, more probably to an error on the value of the final time of computation by those authors.

Finally we observe the peak of density at time $t = 8.798 \times 10^{-7}$sec, see figure 10.

6. Conclusion. In this article, a BGK-type discrete velocity underlying kinetic system for the 2D bitemperature Euler system has been constructed in order to approximate the bitemperature Euler system. It takes into account the force term induced by the electric field and it owns entropy dissipation properties that allow to prove that the numerical scheme is also entropy dissipative and therefore admissible in the sense defined in [8].

At first order and if $\gamma^e = \gamma^i$, we have shown that the total density, the velocity and the total energy provided by our scheme coincide with those provided by the HLL

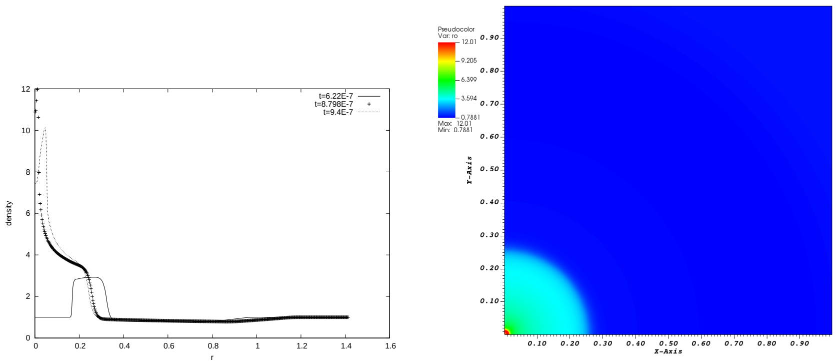

FIG. 10. *Implosion test case with ν^{ei} given by the NRL formula with a grid of 500 by 500 points. Left: density along the first bisector at 3 different times: the peak occurs for $t = 8.798 \times 10^{-7}$sec. Right: isovalues of the density when the peak occurs.*

scheme. Consequently positivity of density and internal total energy are preserved under suitable conditions. The novelty lies in the approximation of the nonconservative terms *via* a discrete Ohm's law for the ionic and electronic energies.

Due to the special structure of the system, we had to develop a new procedure to obtain a second order extension of this scheme able to preserve the positivity properties, along with the conservation of the density, momentum and total energy. The Euler bitemperature system was introduced in the context of Inertial Confinment Fusion, where high densities and temperatures are involved. During this work we did not have problems of non positivity, so we did not investigate the effective way to preserve these properties. This will be done in a forthcoming work.

Several test cases have been performed in order to show the good behaviour of the method in different situations. We proved that the 2D results are in perfect agreement with the one-dimensional known ones, validated in [8]. Moreover, for the implosion test, we compared our results with the ones obtained in [20] with a simplified conservative model. A discrepancy appeared, which led us to perform 1D computations in polar coordinates which seem to confirm our results.

In order to go towards more realistic applications, we aim to integrate magnetic fields in the bitemperature Euler model. In [12], starting from a kinetic system coupled with the Maxwell system in the transverse magnetic configuration, we have derived a bitemperature system and developed a Suliciu relaxation scheme. Hence we shall address the discrete BGK model including magnetic fields in a forthcoming paper.

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