Cercignani conjecture for the Landau equation

Laurent Desvillettes, IMJ-PRG, Université Paris Cité

November 7, 2023

Landau, 36: For $f := f(v) \geq 0$ number density of charged particles of velocity $v \in \mathbb{R}^3$,

$$Q(f)(v) = \nabla \cdot \int_{\mathbb{R}^3} f(v) f(w) \, |v-w|^{-1} \, \Pi(v-w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right) dw,$$

where

$$\Pi_{ij}(z) := \delta_{ij} - \frac{z_i z_j}{|z|^2}$$

is the i, j-component of the orthogonal projection Π onto

$$z^{\perp} := \{ y \,/\, y \cdot z = 0 \}$$

.

Landau operator, other potentials

$$Q(f)(v) = \nabla \cdot \int_{\mathbb{R}^3} f(v)f(w)\,|v-w|^{\gamma+2}\,\Pi(v-w)\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right) dw$$

Hard potentials: $\gamma \in]0,1[$

Maxwell molecules: $\gamma = 0$

Moderately soft potentials: $\gamma \in [-2,0[$

Very soft potentials: $\gamma \in]-4,-2[$ (includes the Coulomb case $\gamma = -3$)

Spatially homogeneous Landau equation

Unknown:

$$f := f(t, v) \geq 0,$$

Equation:

$$\frac{\partial f}{\partial t}(t, v) = Q(f(t, \cdot))(v).$$

Existence, moments and smoothness

Hard potentials : LD, Villani 00, Chen, Li, Xu 08,10, Morimoto, Pravda-Starov, Xu 13; Moments and smoothness are created

Maxwell molecules : Villani 98; Moments are propagated, smoothness is created, many explicit computations are possible

Moderately soft potentials : Fournier, Guérin 09, Wu 14, LD 15, Alonso, Bagland, Lods 23 Moments are propagated, smoothness is created

Spatially homogeneous Landau equation

Unknown:

$$f := f(t, v) \geq 0,$$

Equation:

$$\frac{\partial f}{\partial t}(t, v) = Q(f(t, \cdot))(v).$$

Existence, moments and smoothness

Very soft potentials (including Coulomb): LD 15, Carrapatoso, LD, He 17, Silvestre 17, Gualdani, Golse, Imbert, Vasseur 20, Ben Porath 22, Golse, Imbert, Vasseur 22, LD, He, Jiang 23, Alonso, Bagland, LD, Lods 23, Golding, Gualdani, Loher 23 Moments are propagated, some amount of smoothness is created, no theory of strong global solutions for general initial data yet;

Spatially inhomogeneous: Guo 02, Alexandre, Villani 04

Landau operator, weak formulation

$$\int_{\mathbb{R}^3} Q(f)(v)\,\varphi(v)\,dv$$

$$= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(w)\,|v-w|^{\gamma+2}$$

$$\Big(\nabla\varphi(v)-\nabla\varphi(w)\Big)^T\,\Pi(v-w)\left(\frac{\nabla f(v)}{f(v)}-\frac{\nabla f(w)}{f(w)}\right)dvdw.$$

Conservation of mass, momentum and energy:

$$\int_{\mathbb{R}^3} Q(f)(v)\,\begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix}\,dv = 0.$$

Entropy inequality (first part of H-theorem) for the operator

Entropy production: $f := f(v)$

$$D(f) := -\int_{\mathbb{R}^3} Q(f)(v) \ln f(v)\, dv$$

$$= \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\, f(w)\, |v-w|^{\gamma+2}$$

$$\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right)^T \Pi(v-w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right) dvdw$$

$$= \frac{1}{2} \sum_{i<j} \iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\, f(w)\, |v-w|^{\gamma} \left|(v_i - w_i)\left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w)\right)\right.$$

$$\left. -(v_j - w_j)\left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w)\right)\right|^2 dvdw \geq 0.$$

Case of equality in the entropy inequality (second part of H-theorem) for the operator

For reasonable $f \geq 0$,

$$D(f) = 0 \qquad \Rightarrow \qquad f(v) = f_{eq}(v) := \frac{\rho}{(2\pi T)^{3/2}} \exp\left(- \frac{|v-u|^2}{2T} \right),$$

for some $\rho \geq 0, u \in \mathbb{R}^3, T > 0$

Entropy inequality (first part of H-theorem) for the (spatially homogeneous) equation

Entropy inequality: $f := f(t,v) \geq 0$

$$\frac{d}{dt} H(f(t,\cdot)) = -D(f(t,\cdot)) \leq 0,$$

with the (negative of) the entropy

$$H(f) := \int_{\mathbb{R}^3} f(v) \ln f(v)\, dv.$$

Traditional formulation:

$$D(f) \geq C\,(H(f) - H(f_{eq}))$$

for some $C \geq 0$ depending only on $\int_{\mathbb{R}^3} f(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv$ and an upper bound of $H(f)$.

Expected to be false for soft potentials (because of weights) ;

Known to be true in the case of (over)Maxwell molecules LD, Villani 00 ;

Belongs to the folklore of specialists in the case of hard potentials, but only when C is allowed to depend on extra quantities (L^p_q norms for p, q well chosen).

Exponential convergence towards equilibrium with explicit rate

When the conjecture holds, using the conserved quantities and the entropy estimate

$$\frac{d}{dt} H(f(t,\cdot)) = -D(f(t,\cdot)),$$

the inequality

$$D(f) \geq C\,(H(f) - H(f_{eq}))$$

implies that

$$\frac{d}{dt}\Big(H(f(t,\cdot)) - H(f_{eq})\Big) \leq -C\,\Big(H(f(t,\cdot)) - H(f_{eq})\Big),$$

and

$$H(f(t,\cdot)) - H(f_{eq}) \leq \Big[H(f(0,\cdot)) - H(f_{eq})\Big]\, e^{-C\,t}.$$

Exponential convergence to equlibrium in L^1 norm follows thanks to Cziszar-Kullback-Pinsker inequality.

Some references on Cercignani's conjecture

Conjecture Cercignani 82,

Negative results Wennberg 97, Bobylev, Cercignani 99

Positive results for the Boltzmann equation Carlen, Carvalho 92, 94, Toscani, Villani 99, 00, Villani 03

Positive results for the Landau equation LD, Villani 00

Positive and negative results for LFD/Nordheim equation Alonso, Bagland, LD, Lods 21, 22, Borsoni 23

Use of Gross logarithmic Sobolev inequality

For normalized $f := f(v) \geq 0$,

$$\int_{\mathbb{R}^3} \left| \frac{\nabla f}{f}(v) + v \right|^2 f(v)\, dv \geq C\,(H(f) - H(f_{eq})).$$

As a consequence, Cercignani's conjecture holds as soon as

$$D(f) := \frac{1}{2} \sum_{i<j} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v-w|^\gamma \, \Bigg| (v_i - w_i) \left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w) \right)$$

$$- (v_j - w_j) \left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w) \right) \Bigg|^2 f(v)\, f(w)\, dvdw$$

$$\geq C \int_{\mathbb{R}^3} \left| \frac{\nabla f}{f}(v) + v \right|^2 f(v)\, dv$$

Typical estimate for Landau operator with hard potentials

Taken from LD14: For all $f := f(v) > 0$ (sufficiently smooth for the terms in the estimate to make sense) satisfying

$$\int_{\mathbb{R}^3} f(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix},$$

one has

$$\int_{\mathbb{R}^3} \frac{|\nabla f(v)|^2}{f(v)}\, dv \le 3072\, \Delta(f)^{-2} \left\{ 8448 + 48\, \sqrt{1+\pi}\, D(f)\, ||f||_{L^2(\mathbb{R}^3)} \right\},$$

where

$$\Delta(f) := \text{ Det} \left(\int_{\mathbb{R}^3} f(w)\, (1+|w|^2)^{-1/2} \begin{bmatrix} 1 & w_i & w_j \\ w_i & w_i^2 & w_i\, w_j \\ w_j & w_i\, w_j & w_j^2 \end{bmatrix} dw \right).$$

Problems: Huge numerical constants ; dependence on (an upper bound of) $H(f)$ (or $||f||_{L^2(\mathbb{R}^3)}$) to control concentration of f on planes.

New type of estimate for Landau operator with hard potentials

For all $f := f(v) > 0$ (sufficiently smooth for the terms in the estimate to make sense) satisfying

$$\int_{\mathbb{R}^3} f(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix},$$

the following holds:

$$\int_{\mathbb{R}^3} \left| \frac{\nabla f}{f}(v) + v \right|^2 f(v)\, dv \leq 200\, ||f||^2_{L^2_6(\mathbb{R}^3)}\, D(f),$$

as soon as

$$||f||^2_{L^2_6(\mathbb{R}^3)}\, D(f) \leq 0.062,$$

Consequence for exponential convergence

Lemma: Suppose that $H \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ and $D \in C(\mathbb{R}_+;\mathbb{R}_+)$ are such that

$$-H' = D.$$

We also assume that for some real numbers $q, c_0 > 0$

$$D \le q \qquad \Rightarrow \qquad D \ge c_0\, H.$$

Then, for all $t \ge H(0)/q$, the following estimate holds:

$$H(t) \le H(0)\, \exp(c_0\, H(0)/q)\, e^{-c_0\, t}.$$

Corollary: The solutions of the (spatially homogeneous) Landau equation with hard potentials (with reasonable initial data) converge exponentially fast, with explicit rate, towards equilibrium.

Simplified assumptions

For all $f := f(v) > 0$ (sufficiently smooth for the terms in the estimate to make sense) satisfying

$$\int_{\mathbb{R}^3} f(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix},$$

and such that (for $i = 1, 2, 3$)

$$I_i(f) := \int_{\mathbb{R}^3} f(v)\, v_i^2\, dv = 1,$$

the following simplified estimate holds:

$$\int_{\mathbb{R}^3} \left| \frac{\nabla f}{f}(v) + v \right|^2 f(v)\, dv \leq 15\, ||f||_{L^2(\mathbb{R}^3)}^{1/2}\, D(f).$$

Idea of the proof

$$D(f) = \frac{1}{2} \sum_{i<j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |q^f_{i,j}(v,w)|^2 \, |v-w|^\gamma \, f(v)\, f(w)\, dvdw$$

where

$$q^f_{i,j}(v,w) := (v_i - w_i) \left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w) \right) - (v_j - w_j) \left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w) \right).$$

Multiplying this definition by $w_i\, f(w)$ and integrating on $\mathbb{R}^3$ in the w-variable, one gets (for $i \neq j$)

$$I_i(f)\, \frac{\partial_j f}{f}(v) + v_j = - \int_{\mathbb{R}^3} w_i\, f(w)\, q^f_{i,j}(v,w)\, dw.$$

Multiplying then by $f(w)$ and integrating on $\mathbb{R}^3$ in the w-variable, one gets

$$v_i\, \frac{\partial_j f}{f}(v) - v_j\, \frac{\partial_i f}{f}(v) = \int_{\mathbb{R}^3} f(w)\, q^f_{i,j}(v,w)\, dw.$$

Idea of the proof, simplified case

If we assume that (for $i = 1, 2, 3$) $I_i(f) := \int_{\mathbb{R}^3} f(v)\, v_i^2\, dv = 1$, then

$$\frac{\partial_j f}{f}(v) + v_j = - \int_{\mathbb{R}^3} w_i\, f(w)\, q^f_{i,j}(v,w)\, dw,$$

so that

$$\int_{\mathbb{R}^3} \left| \frac{\nabla f}{f}(v) + v \right|^2 f(v)\, dv \le \sum_{i=1}^3 \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} w_i\, f(w)\, q^f_{i,i+1}(v,w) \right|^2 f(v)\, dv$$

$$\le \sum_{i=1}^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |q^f_{i,i+1}(v,w)|^2\, |v-w|^{\gamma}\, f(v)\, f(w)\, dvdw$$

$$\times \sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} |v-w|^{-\gamma} w_i^2\, f(w) dw$$

$$\le 2\,(1+\gamma) \left(\frac{4\pi}{\gamma\,(3-2\gamma)}\, ||f||_{L^2(\mathbb{R}^3)} \right)^{\frac{\gamma}{3-\gamma}} D(f).$$

Conclusion and Perspectives

- Reasonable (not huge) constants
- Short proof
- Treats all hard potentials
- Transfers somewhat to Landau Fermi-Dirac equation

- Maybe possible transfer to Landau equation with soft potential
- Not much hope to be usable for Boltzmann-like equations