STATIONARY MIXTURE BGK MODELS WITH THE CORRECT FICK COEFFICIENTS

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ABSTRACT. Unlike the single species gases, the transport coefficients such as Fick, Soret, Dufour coefficients arise in the hydrodynamic limit of multi-species gas mixtures. To the best of the authors' knowledge, no multi-component relaxational models is reported that produces all these values correctly. In this paper, we establish the existence of unique stationary mild solutions to the BGK models for gas mixtures which produces the correct Fick coefficients in the Navier-Stokes limit for inert gases [14], and for reactive gases [15] in a unified manner.

1. INTRODUCTION

The BGK model [7, 48] is a relaxation model of the original Boltzmann equation. It reproduces many crucial properties of the Boltzmann equation, such as conservation laws and H-theorem, at a much lower computational cost. But, the original BGK model has a well-known drawback. Since the viscosity and heat conductivity derived from the BGK model do not match with the Boltzmann data in the Navier-Stokes limit, it produces an incorrect Prandtl number, which is defined as the ratio between the viscosity and the thermal conductivity. In view of ensuring the reliability of theoretical analyzes and numerical simulations based on model equations, it is important to reproduce the correct transport coefficients. ES-BGK model [32, 2, 12, 13] and Shakov model [44] are the extended models of the original BGK model developed to overcome this drawback.

In the context of gas mixtures, extending the original BGK model is not straightforward. In the early stage of constructing the mixture BGK models, various models have been proposed by physicists [20, 21, 22, 23, 30, 39, 45]. All these models have the relaxation operators that consist of a sum of BGK-type operators, as in the case of the Boltzmann equation. None of these models, however, satisfy all the fundamental properties of the Boltzmann equation, such as conservation laws, H-theorem, state of equilibrium, indifferentiability, positivity of densities and temperatures. This was overcome by Andries, Aoki, and Perthame [1] by modeling the mixture BGK model using only a single relaxation operator per each species. It was the first mixture BGK model that was shown to satisfy all the above fundamental properties rigorously. Due to its consistency and simplicity, this model was widely employed by many researchers to derive mixture BGK models (1) that account for diverse physical scenarios [8, 9, 10, 15, 25, 26, 27, 28, 33, 40], or (2) that reproduce the right transport coefficients in the Navier-Stokes limit [14, 15, 16, 24, 27, 28, 46]. We also refer to a recent series of consistent BGK-type models [11, 29, 36], which readopted the initial approach, namely, the approach

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that use both the inter-species and intra-species relaxation operators. Unlike the earlier works, all the three models were proved to satisfy the fundamental properties, including the conservation laws, H-theorem, state of equilibrium, and positivity of densities and temperatures.

Unlike the single species gases, the Navier-Stokes limits give rise to various transport coefficients such as the viscosity, the thermal conductivity, as well as the Fick, Soret, and Dufour coefficients. These coefficients are fundamentally related to how the mass, momentum, and energy fluxes of the fluid system depend on the gradients in density, momentum, or temperature. Specifically, the Fick coefficient is the coefficient associated with the gradient of the density in the mass flux, while the Soret coefficient corresponds to the gradient of the temperature in the mass flux, and the Dufour coefficient is related to the gradient of the density in the energy flux. As in the case of single species gases, reproducing the right transport coefficients is one of the main issues in modeling of kinetic equations for gas mixtures. However, to the best of the authors' knowledge, there is no BGK-type relaxation model for inert and reactive gas mixtures that reproduce all the transport coefficients completely. For inert gas mixtures, this particular issue was partially addressed in [14, 16, 17, 24, 46]. The model [14] gives the exact Fick and Newton coefficients. The model [16] recovers the exact Fourier and Newton coefficients. The model in [24] gives the correct Fick and Newton coefficients. In [46], the authors derived BGK-type models that reproduce the correct Fick, Newton, and Fourier coefficients, but only two species gases are covered.

For the reactive BGK models, the issue of the transport coefficients for gas mixtures was first treated in [15] by considering the mechanical collisions and reactive collisions separately. In [15], the authors suggested the BGK-type model where the total relaxation operators are given by the sum of the reactive operator of [25] and the inert BGK model presented in [14]. Since all transport coefficients depend only on the mechanical operator in the slow reaction regime, the model [15] reproduces the right Fick coefficient, similar to the inert BGK model [14].

In this paper, we consider the following two stationary problems of BGK models for gas mixtures designed to yield the correct Fick coefficients in the Navier-Stokes limit:

• BGK model for inert gas mixture [14]:

$$v_1 \frac{\partial f_i}{\partial x} = \frac{\nu^M}{\tau}(\mathcal{M}_i - f_i) \qquad (i = 1, ..., N) \tag{1.1}$$

subject to the boundary data:

$$f_i(0, v) = f_{i,L}(v) \text{ on } v_1 > 0,\ f_i(1, v) = f_{i,R}(v) \text{ on } v_1 < 0,$$

• BGK model for reactive gas mixture [15]:

$$v_1 \frac{\partial f_i}{\partial x} = \frac{\nu^M}{\tau}(\mathcal{M}_i - f_i) + \frac{\nu_i^C}{\tau}(\mathcal{C}_i - f_i) \text{ on } [0, 1] \times \mathbb{R}^3, \qquad (i = 1, ..., 4) \tag{1.2}$$

subject to the boundary data:

$$f_i(0, v) = f_{i,L}(v) \text{ on } v_1 > 0,\ f_i(1, v) = f_{i,R}(v) \text{ on } v_1 < 0.$$

Note that, in (1.2), the number of species in the gases is restricted to 4. The velocity distribution function $f_i(x, v)$ represents the number density of ith molecule at the position $x \in [0, 1]$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and τ is the Knudsen number defined by the ratio of the mean free path and the characteristic length. The mechanical Maxwellian $\mathcal{M}_i$ is defined by

$$\mathcal{M}_i = \frac{n^{(i)}}{(2\pi kT^*/m_i)^{3/2}} \exp\left(-\frac{m_i|v-U_i|^2}{2kT^*}\right),$$

and the chemical Maxwellian $\mathcal{C}_i$ reads as

$$\mathcal{C}_i = \frac{\tilde{n}_i}{(2\pi k\tilde{T}/m_i)^{3/2}} \exp\left(-\frac{m_i|v-\tilde{U}|^2}{2k\tilde{T}}\right),$$

where k is the Boltzmann constant.
Here, m_i represents the mass for each species i and the parameters in the two Maxwellians $\mathcal{M}_i$ and $\mathcal{C}_i$ are defined in the following manner. (In the following, N is an arbitrary positive integer for (1.1) , but restricted to $N=4$ for (1.2))

First, we define the traditional macroscopic quantities:
(a) Single component macroscopic fields:

$$\begin{aligned}
\rho^{(i)} &:= m_i n^{(i)} := m_i \int_{\mathbb{R}^3} f_i dv,\\
\rho^{(i)}U^{(i)} &:= m_i \int_{\mathbb{R}^3} v f_i dv,\\
3k\rho^{(i)}T^{(i)} &:= m_i^2 \int_{\mathbb{R}^3} |v-U^{(i)}|^2 f_i dv.
\end{aligned}$$

(b) Global macroscopic fields:

$$n = \sum_{i=1}^N n^{(i)}, \qquad \rho = \sum_{i=1}^N \rho^{(i)}, \qquad U = \frac{1}{\rho}\sum_{i=1}^N \rho^{(i)}U^{(i)},$$
$$nkT = \sum_{i=1}^N n^{(i)}kT^{(i)} + \frac{1}{3}\sum_{i=1}^N \rho^{(i)}(|U^{(i)}|^2 - |U|^2).$$

Then, in order to define the parameters U_i and T^* in the mechanical Maxwellian $\mathcal{M}_i$, we consider the mass flux for ith species $\mathbf{J}_i$ which can be obtained from the Boltzmann equation using the Chapman-Enskog approach:

$$\mathbf{J}_i = \sum_{j=1}^N L_{ij}\nabla\left(\frac{-\mu_j}{T}\right) + L_{i,N+1}\nabla\left(\frac{1}{T}\right)$$

where μ_i is the chemical potential of the species i in the mixture:

$$\frac{\mu_i}{T} = -k\left(\ln(n^{(i)}) - \frac{3}{2}\ln\left(\frac{2\pi kT}{m_i}\right)\right).$$

Or equivalently, the mass flux can be written by using the phenomenological point of view as

$$\mathbf{J}_i = \sum_{j=1}^N D_{ij}\nabla n^{(j)} + D_{i,N+1}\nabla T$$

where D_{ij} and $D_{i,N+1}$ are called the Fick and Soret coefficients, respectively, which can be measured from experiments. We mention that the matrix $L=(L_{ij})$ is symmetric and non-positive (so-called Onsager relation ([14])) and there is the trivial one-to-one correspondence between L_{ij} and D_{ij}. The link between these two formulations has been shown by Kurochkin, Makarenko, and Tirskii ([37]). The Fick coefficients can be written as

$$D_{ij} = -\frac{nk_B L_{ij}}{n_i n_j}.$$

With these right transport coefficient L_{ij} (or the Fick coefficient D_{ij}), we define the non-positive symmetric matrix L^* whose elements are given by

$$L^*_{ij} = \frac{L_{ij}}{\sqrt{\rho_i \rho_j} T}. \tag{1.3}$$

The matrix L^* is always diagonalizable with an orthogonal matrix W (see Lemma 5 in [14]):

$$L^* = W^T K^* W.$$

Let k^*_r be the diagonal components of K^*. Then, up to some permutation, we have k^*_r being nonzero for $r = 1, \dots, N-1$ but zero for $r = N$. We set

$$\lambda_r = -k_r^{*-1} \text{ for } r = 1, \dots, N \quad \text{and } \lambda_N = 0.$$

Using this, we define U_i and T^* as below:

$$\underline{\mathrm{U}} := (U_1, \dots, U_N)^T := \mathbf{U} + \mathbf{N}^{-1} W^T \left(I - \frac{1}{\nu^M} \Delta \right) W \mathbf{N} (\overline{\mathbf{U}} - \mathbf{U}),$$

$$T^* := T - \frac{1}{3nk} \left\| W^T \left(I - \frac{1}{\nu^M} \Delta \right) W \mathbf{N} (\overline{\mathbf{U}} - \mathbf{U}) \right\|_2^2,$$

where $\mathbf{U} = (U, \dots, U)^T$, $\overline{\mathbf{U}} = (U^{(1)}, \dots, U^{(N)})^T$, $\mathbf{N} = \mathrm{diag}(\sqrt{\rho_1}, \dots, \sqrt{\rho_N})$, and $\Delta = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. This definition of $\mathcal{M}_i$ gives that the model reproduces the right Fick coefficients through the Chapman–Enskog expansion [14].

Then, the parameters $\tilde{n}_i, \tilde{U}$, and $\tilde{T}$ of the reactive Maxwellian $\mathcal{C}_i$ are defined through the following procedure. First, we define $\tilde{n}_1$ as the unique root of the equation:

$$\frac{\nu_3^C \nu_4^C}{\nu_1^C \nu_2^C} \frac{\nu_1^C \tilde{n}_1 [\nu_2^C n^{(2)} + \nu_1^C (\tilde{n}_1 - n^{(1)})]}{[\nu_3^C n^{(3)} - \nu_1^C (\tilde{n}_1 - n^{(1)})][\nu_4^C n^4 - \nu_1^C (\tilde{n}_1 - n^{(1)})]} \exp\left(-\frac{\Delta E}{kF(\tilde{n}_1)} \right) = \left(\frac{\mu^{12}}{\mu^{34}} \right)^{\frac{3}{2}}, \tag{1.4}$$

where F is given by

$$F(y) := \frac{\sum_{i=1}^4 \nu_i^C n^{(i)} \left[\frac{1}{2} m_i (|U^{(i)}|^2 - |\tilde{U}|^2) + \frac{3}{2} kT^{(i)} \right] + \Delta E \nu_1^C (y - n^{(1)})}{\frac{3}{2} k \sum_{i=1}^4 \nu_i^C n^{(i)}},$$

and ΔE represents the energy threshold for chemical reactions. We mention that the solution $\tilde{n}_1$ was sought under the following constraints [25]:

$$\tilde{n}_1 > 0, \quad \tilde{n}_1 > n^{(1)} - \frac{\nu_2^C}{\nu_1^C} n^{(2)}, \quad \tilde{n}_1 < n^{(1)} + \frac{\nu_3^C}{\nu_1^C} n^{(3)}, \quad \tilde{n}_1 < n^{(1)} + \frac{\nu_4^C}{\nu_1^C} n^{(4)},$$

$$\tilde{n}_1 > n^{(1)} - \frac{1}{\nu_1^C} \frac{1}{\Delta E} \sum_{i=1}^4 \nu_i^C n^{(i)} \Big[\frac{1}{2} m_i (|U^{(i)} - \tilde{U}|^2) + \frac{3}{2} kT^{(i)} \Big].$$

Since the left-hand-side of (1.4) is a strictly increasing function of $\tilde{n}^1$ with its range $(0,\ \infty)$, the root of (1.4) always exists uniquely (see [25]). With such $\tilde{n}_1$, we define $\tilde{U}, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4$ and $\tilde{T}$ through the following relations:

$$\begin{aligned}
&\tilde{n}_i := n^{(i)} + \lambda_i \frac{\nu_1^C}{\nu_i^C}(\tilde{n}_1 - n^{(1)}), \quad i = 2,3,4,\\
&\tilde{U} := \sum_{i=1}^4 \nu_i^C m_i n^{(i)} U^{(i)} \Big/ \sum_{i=1}^4 \nu_i^C m_i n^{(i)},\\
&\tilde{T} := F(\tilde{n}_1).
\end{aligned}$$

Finally, the mechanical collision frequency ν^M and the chemical collision frequency ν_i^C are defined by

$$\begin{aligned}
\nu^M &= \max\left(\frac{nkT}{\eta}, \max \lambda_r\right),\\
\nu_1^C &= \nu_{12}^{34} \frac{2}{\sqrt{2\pi}} \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT}\right) n^{(2)}, \quad \nu_2^C = \nu_{12}^{34} \frac{2}{\sqrt{2\pi}} \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT}\right) n^{(1)},\\
\nu_3^C &= \nu_{12}^{34} \frac{2}{\sqrt{2\pi}} \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT}\right)\left(\frac{\mu_{12}}{\mu_{34}}\right) \exp\left(\frac{\Delta E}{kT}\right) n^{(4)},\\
\nu_4^C &= \nu_{12}^{34} \frac{2}{\sqrt{2\pi}} \Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT}\right)\left(\frac{\mu_{12}}{\mu_{34}}\right) \exp\left(\frac{\Delta E}{kT}\right) n^{(3)},
\end{aligned}$$

where η is the shear viscosity and ν_{12}^{34} is the microscopic chemical collision frequency. In this work, we assume that η is a positive continuous function of macroscopic densities and temperatures and ν_{12}^{34} is a fixed constant.

We note that although they may seem excessively simplified, boundary value problems of gas in a slab facilitate the introduction of diverse flow regimes encountered in rarefied gas dynamics [19], and are crucial in the analytical study of the Knudsen layer [43].

We check the literature on the existence theory of BGK models. We start with a review of BGK models for single-species gases. For the original BGK model, the existence of weak solutions was obtained in [41]. The result [42] showed the uniqueness and existence of mild solutions in a weighted L^∞ space. The author of [53] proved the existence of L^p solutions. The solutions near equilibrium was studied in [49]. Recently, large amplitude problem of the BGK models was considered in [5]. For results on the ES-BGK model, we refer to [50, 51, 52]. The results on the Shakhov model can be found in [3, 38]. The stationary problem of BGK models in a slab can be found in [6, 18, 47]. Then, we also review a few existence results of the BGK models for gas mixtures. The existence of a unique mild solutions in a weighted L^∞ space to the models [1, 36] were established in [35]. The classical solution of the model [36] near global equilibrium was studied in [4]. Recently, in [34], the existence of stationary BGK models for chemical reacting gases [26, 25] was proved. The result of [34] also covers the inert mixture BGK model [1].

To the best of our knowledge, there is no existence theory of BGK models for inert or reactive gas mixtures that reproduce the correct Fick coefficients, which is the main motivation of the current paper. The main line of the proof of the current paper is based on the argument developed in [34]. However, the auxiliary parameters have the complicated structures involving $W^T\Delta W$. The matrix $W^T\Delta W$ arises to reproduce the correct

Fick coefficients. Due to such complicated structure, several novel difficulties arise. For example, it is quite tricky to get Lipschitz continuity of the auxiliary parameters. For this, we rewrite $W^T \Delta W$ as a rational function of each components of L^*. Then, we show that the denominator of the rational function is a multiplication of non-zero eigenvalues of L^*. This reformulation, together with restricting their domain to a compact set, yields the desired Lipschitz continuity of the auxiliary parameters.

We remark that the two models treated in this paper give incorrect Soret and Dufour coefficients. In [14], the authors mention that the model (1.1) gives zero cross kinetic coefficients. And so does the model (1.2), because the two models share the same mechanical operator, as pointed out in [15].

The paper is organized as follows: In section 2, we define some notations used throughout this paper and present our main result. In Section 3, we define our solution space and estimate the macroscopic parameters on the space. Section 4 is devoted to proving that our solution operator maps the solution space into itself. Finally, in Section 5, we show that our solution map is contractive on the solution map, which completes the our main result.

2. Main result

In this section, we present our main result. We first need to introduce notations and norms and set conventions:

- Every constant denoted by C will be generically defined. The values of C may differ line by line but are computable in principle.

- We use $C_{a,b,\cdots}$ to denote a positive constant depending on $a, b, \cdots$. However, we fix $C_{\ell,u}$ to denote constants depending only on quanitities in (2.1),(2.2), and (2.3).

- We define the norm $\|\cdot\|_{L^1_2}$ by
$$\|f\|_{L^1_2} = \int_{\mathbb{R}^3} |f(x,v)|(1+|v|^2)dv.$$

- For an arbitrary $m \times n$ matrix A, we define
$$\|A\|_F := \left(\sum_{i=1}^m \sum_{j=1}^n (a_{ij})^2 \right)^{1/2}, \quad \|A\|_2 := \sup \left\{ \|Ax\|_2 \middle| \|x\|_2 = 1, \ x \in \mathbb{R}^n \right\}.$$

- Throughout this paper, we use the abbreviate notation:
$$f_{i,LR} = f_{i,L} 1_{v_1>0} + f_{i,R} 1_{v_1<0}.$$

- We define the following quantities for brevity $(i = 1, ..., N)$
$$\begin{aligned} a_{i,u} &= 2\int_{\mathbb{R}^3} f_{i,LR} dv, \quad c_{i,u} = 2\int_{\mathbb{R}^3} f_{i,LR}|v|^2 dv, \\ a_{i,s} &= \int_{\mathbb{R}^3} \frac{1}{|v_1|} f_{i,LR} dv, \quad c_{i,s} = \int_{\mathbb{R}^3} \frac{1}{|v_1|} f_{i,LR}|v|^2 dv. \end{aligned} \tag{2.1}$$

- We define

$$a_{i,l} = \frac{1}{8}a_{i,u}, \quad c_{i,l} = \frac{1}{8}c_{i,u},$$
$$a_u = \max_i\{a_{i,u}\}, \ a_l = \min_i\{a_{i,l}\}, \ c_u = \max_i\{c_{i,u}\}, \ c_l = \min_i\{c_{i,l}\}. \tag{2.2}$$

- We also define the following quantity which will serve as a lower bound for the temperature:

$$\gamma_{i,l} = \frac{1}{16}\left(\int_{v_1>0} f_{i,L}|v_1|dv\right)\left(\int_{v_1<0} f_{i,R}|v_1|dv\right), \ \gamma_l = \min_i\{\gamma_{i,l}\}. \tag{2.3}$$

We now define our mild solutions to (1.2):

Definition 2.1. A pair of functions $F = (f_1, f_2, f_3, f_4) \in (L^\infty([0,1]_x; L^1_2(\mathbb{R}^3_v)))^4$ is said to be a mild solution for (1.2) if f_i satisfies the following equations: for each $i = 1,2,3,4$

$$\begin{aligned} f_i&(x,v) \\ &= \left(e^{-\frac{1}{\tau|v_i|}\int_0^x \overline{\nu}_i(y)dy} f_{i,L}(v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \overline{\nu}_i(z)dz}(\nu^M\mathcal{M}_i + \nu_i^C\mathcal{C}_i)dy\right)1_{v_1>0} \\ &\quad + \left(e^{-\frac{1}{\tau|v_1|}\int_x^1 \overline{\nu}_i(y)dy} f_{i,R}(v) + \frac{1}{\tau|v_1|}\int_x^1 e^{-\frac{1}{\tau|v_1|}\int_x^y \overline{\nu}_i(z)dz}(\nu^M\mathcal{M}_i + \nu_i^C\mathcal{C}_i)dy\right)1_{v_1<0}, \end{aligned}$$

where we used the notation $\overline{\nu}_i = \nu^M + \nu_i^C$.

Now we are ready to state our main result.

Theorem 2.2. *Assume the inflow data $f_{i,LR}$ satisfies*

$$f_{i,LR}, \ \frac{1}{|v_1|}f_{i,LR} \in L^1_2(\mathbb{R}^3)$$

and

$$\int_{\mathbb{R}^2} f_{i,L}v_j dv_2dv_3 = \int_{\mathbb{R}^2} f_{i,R}v_j dv_2dv_3 = 0,$$

for $j = 2,3$. Then there exists a constant $L > 0$, depending only on the constants defined in (2.1), (2.2), and (2.3), such that if $\tau > L$, then there exists unique mild solution $F = (f_1, f_2, f_3, f_4)$ for (1.2) satisfying

$$a_{i,l} \le \int_{\mathbb{R}^3} f_i(x,v)dv \le a_{i,u}, \qquad c_{i,l} \le \int_{\mathbb{R}^3} |v|^2 f_i(x,v)dv \le c_{i,u}$$

and

$$\left(\int_{\mathbb{R}^3} f_i dv\right)\left(\int_{\mathbb{R}^3} |v|^2 f_i dv\right) - \left(\int_{\mathbb{R}^3} v_1 f_i dv\right)^2 \ge \gamma_l.$$

Remark 2.3. In this result, there are no restrictions on the microscopic chemical collision frequencies ν_{12}^{34}. The model (1.2) can be reduced to the one presented in [14] by setting $\nu_{12}^{34} = 0$. Thus, our result implies the existence of the solution to the boundary value problem for the stationary BGK model of an inert gas mixture [14] with four gas species, and the analysis in the setting $\nu_{12}^{34} = 0$ can be trivially extended to the case for an arbitrary number of gas species.

In the proof, the assumption that the Knudsen number τ is sufficiently large is used to show the contractivity of the solution operator in the Banach fixed point framework. Developing an analytical argument valid for small τ is left for future work, which is physically relevant since in the BGK model, the distribution function does not deviate too far from a Maxwellian. Another interesting future direction is to set the matrix L^* to be τ-dependent and consistent with the fact that diffusion and velocity disappear in the Euler limit.

3. Fixed point Set-up

In this section, we define our solution space by

$$\Omega = \left\{ F = (f_1, \dots, f_4) \in (L^\infty([0,1]_x; L^1_2(\mathbb{R}^3_v)))^4 \mid F \text{ satisfies } (\mathcal{A}), (\mathcal{B}), (\mathcal{C}) \right\}$$

with the metric $d(F,G) = \sum_{i=1}^4 \sup\limits_{x\in[0,1]} ||f_i - g_i||_{L^1_2}$, where $(\mathcal{A}), (\mathcal{B})$, and $(\mathcal{C})$ denote

- $(\mathcal{A})$ f_i are non-negative,
- $(\mathcal{B})$ The macroscopic quantities satisfy the followings:

$$a_{i,l} \le \int_{\mathbb{R}^3} f_i(x,v)dv \le a_{i,u}, \quad c_{i,l} \le \int_{\mathbb{R}^3} |v|^2 f_i(x,v)dv \le c_{i,u},$$

- $(\mathcal{C})$ The following lower bounds hold:

$$\left(\int_{\mathbb{R}^3} f_i dv\right)\left(\int_{\mathbb{R}^3} |v|^2 f_i dv\right) - \left(\int_{\mathbb{R}^3} v f_i dv\right)^2 \ge \gamma_l.$$

Consequently, we define our solution map $\Phi : \Omega \to \Phi(\Omega)$ by $\Phi(F) = (\phi_1, \dots, \phi_4)$. Here, ϕ_i is defined as follows :

$$\begin{aligned}\phi_i(x,v) = \quad & \left(e^{-\frac{1}{\tau|v_i|}\int_0^x \overline{\nu}_i(y)dy} f_{i,L}(v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \overline{\nu}_i(z)dz}(\nu^M \mathcal{M}_i + \nu_i^C \mathcal{C}_i)dy\right)1_{v_1>0} \\ + \quad & \left(e^{-\frac{1}{\tau|v_1|}\int_x^1 \overline{\nu}_i(y)dy} f_{i,R}(v) + \frac{1}{\tau|v_1|}\int_x^1 e^{-\frac{1}{\tau|v_1|}\int_x^y \overline{\nu}_i(z)dz}(\nu^M \mathcal{M}_i + \nu_i^C \mathcal{C}_i)dy\right)1_{v_1<0}.\end{aligned}$$

For simplicity, we denote $\phi_i = \phi_i^+ + \phi_i^-$ by

$$\begin{aligned}\phi_i^+(x,v) =& \left(e^{-\frac{1}{\tau|v_i|}\int_0^x \overline{\nu}_i(y)dy} f_{i,L}(v) + \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \overline{\nu}_i(z)dz}(\nu^M \mathcal{M}_i + \nu_i^C \mathcal{C}_i)dy\right)1_{v_1>0}, \\ \phi_i^-(x,v) =& \left(e^{-\frac{1}{\tau|v_1|}\int_x^1 \overline{\nu}_i(y)dy} f_{i,R}(v) + \frac{1}{\tau|v_1|}\int_x^1 e^{-\frac{1}{\tau|v_1|}\int_x^y \overline{\nu}_i(z)dz}(\nu^M \mathcal{M}_i + \nu_i^C \mathcal{C}_i)dy\right)1_{v_1<0}.\end{aligned}$$

We note that a fixed point of the solution map is the mild solution to (1.2). In the following sections, we will show that the solution operator Φ has a unique fixed point in Ω by using the Banach fixed point argument. For this, we show (1) $\Phi(\Omega) = \Omega$ and (2) Φ is contractive on Ω. Before proving it, we first estimate the macroscopic parameters and the auxiliary parameters of the velocity distributions $F = (f_1, f_2, f_3, f_4) \in \Omega$.

3.1. **Actual single component parameters and global parameters.** We recall some elementary estimates of the actual macroscopic parameters.

Lemma 3.1 ([6, 34])**.** *For $F=(f_1,f_2,f_3,f_4)\in\Omega$, the single component parameters satisfy the following inequalities:*

$$|U^{(i)}|\le\frac{a_{i,u}+c_{i,u}}{2a_{i,l}}$$

and

$$\frac{m_i\gamma_l}{3ka_{i,u}^2}\le T^{(i)}\le\frac{m_ic_{i,u}}{3ka_{i,l}}.$$

Proof. For $|U^{(i)}|$, we have

$$|U^{(i)}|=\frac{|\rho^{(i)}U^{(i)}|}{\rho^{(i)}}=\frac{|\int_{\mathbb{R}^3}vf_1dv|}{\int_{\mathbb{R}^3}f_idv}.$$

Here, the Young's inequality gives

$$\begin{aligned}\left|\int_{\mathbb{R}^3}vf_idv\right|&\le\frac{\int_{\mathbb{R}^3}f_idv+\int_{\mathbb{R}^3}|v|^2f_idv}{2}\\&\le\frac{a_{i,u}+c_{i,u}}{2},\end{aligned}$$

which implies

$$|U^{(i)}|\le\frac{a_{i,u}+c_{i,u}}{2a_{i,l}}.$$

For $T^{(i)}$, we obtain

$$\begin{aligned}T^{(i)}&=\frac{(3kn^{(i)}T^{(i)}+\rho^{(i)}|U^{(i)}|^2)-|\rho^{(i)}U^{(i)}|^2(\rho^{(i)})^{-1}}{3kn^{(i)}}\\&=\frac{m_i\int_{\mathbb{R}^3}|v|^2f_idv-m_i|\int_{\mathbb{R}^3}vf_idv|^2|(\int_{\mathbb{R}^3}f_idv)^{-1}}{3k\int_{\mathbb{R}^3}f_idv}.\end{aligned}$$

For the lower bound, we have

$$T^{(i)}=\frac{m_i(\int_{\mathbb{R}^3}f_idv)(\int_{\mathbb{R}^3}|v|^2f_idv)-m_i|\int_{\mathbb{R}^3}vf_idv|^2}{3k(\int_{\mathbb{R}^3}f_idv)^2}\ge\frac{m_i\gamma_l}{3ka_{i,u}^2}.$$

Finally, the upper bound is obtained as follows:

$$T^{(i)}=\frac{m_i\int_{\mathbb{R}^3}|v|^2f_idv-m_i|\int_{\mathbb{R}^3}vf_idv|^2|(\int_{\mathbb{R}^3}f_idv)^{-1}}{3k\int_{\mathbb{R}^3}f_idv}\le\frac{m_i\int_{\mathbb{R}^3}|v|^2f_idv}{3k\int_{\mathbb{R}^3}f_idv}\le\frac{m_ic_{i,u}}{3ka_{i,l}}.$$

□

Then, we also obtain some estimates of the global macroscopic parameters.

Lemma 3.2 ([6, 34])**.** *For $F=(f_1,f_2,f_3,f_4)\in\Omega$, the global macroscopic parameters U and T satisfy*

$$|U|\le\max_{1\le i\le 4}\left\{\frac{a_{i,u}+c_{i,u}}{2a_{i,l}}\right\}$$

and

$$\min_{1\leq i\leq 4}\left\{\frac{m_i\gamma_l}{3ka_{i,u}^2}\right\}\leq T\leq \frac{c_u}{12ka_l}\sum_{i=1}^4 m_i.$$

We will denote this lower bound as T_l *and this upper bound as* T_u*.*

Proof. For the bound of U, we have

$$|U|\leq \frac{1}{\rho}\sum_{i=1}^4\rho^{(i)}|U^{(i)}|\leq \max_{1\leq i\leq 4}|U^{(i)}|\leq \max_{1\leq i\leq 4}\left\{\frac{a_{i,u}+c_{i,u}}{2a_{i,l}}\right\}.$$

For the lower bound of T, we observe

$$\sum_{i=1}^4\rho^{(i)}(|U^{(i)}|^2-|U|^2)=\sum_{i=1}^4\rho^{(i)}(|U^{(i)}-U|^2)\geq 0,$$

which implies

$$T\geq \sum_{i=1}^4\frac{n^{(i)}}{n}T^{(i)}\geq \min_{1\leq i\leq 4}T^{(i)}\geq \min_{1\leq i\leq 4}\left\{\frac{m_i\gamma_l}{3ka_{i,u}^2}\right\}.$$

The upper bound of T is obtained by

$$T\leq \sum_{i=1}^4\frac{n^{(i)}}{n}T^{(i)}+\frac{1}{3nk}\sum_{i=1}^4\rho^{(i)}|U^{(i)}|^2=\frac{1}{3nk}\sum_{i=1}^4 m_i\int_{\mathbb{R}^3}|v|^2f_i dv\leq \frac{c_u}{12ka_l}\sum_{i=1}^4 m_i.$$

□

3.2. **Auxiliary parameters for mechanical and reactive Maxwellians.** Using the previous lemmas for the actual parameters, we estimate the auxiliary parameters for mechanical and reactive Maxwellians.

Lemma 3.3. *For* $F=(f_1,f_2,f_3,f_4)\in\Omega$*, the auxiliary velocity* $\underline{U}$ *of the Mechanical Maxwellian* $\mathcal{M}_i$ *satisfies the following inequality:*

$$\|\underline{U}\|_2\leq C\max_i\left\{\frac{a_{i,u}+c_{i,u}}{2a_{i,l}}\right\}.$$

Proof. By the definition of $\underline{\mathrm{U}}$, we easily have

$$\begin{aligned}\|\underline{\mathrm{U}}\|_2&\leq \|\mathbf{U}\|_2+\left\|\mathbf{N}^{-1}W^T\left(I-\frac{1}{\nu^M}\Delta\right)W\mathbf{N}(\overline{\mathbf{U}}-\mathbf{U})\right\|_2\\&\leq \|\mathbf{U}\|_2+\max\left|1-\frac{\lambda_r}{\nu^M}\right|\left\|\overline{\mathbf{U}}-\mathbf{U}\right\|_2\\&\leq C\max_i\left\{\frac{a_{i,u}+c_{i,u}}{2a_{i,l}}\right\}.\end{aligned}$$

□

Lemma 3.4. *For* $F=(f_1,\ f_2,\ f_3,\ f_4)\in\Omega$*, the auxiliary temperature* T^* *of the Mechanical Maxwellian* $\mathcal{M}_i$ *satisfies the following inequality:*

$$T_l\leq T^*\leq T_u,$$

where T_l and T_u are given in Lemma 3.2.

Proof. The upper bound is trivial by the definition of T^*:

$$T^* \le T \le T_u.$$

Using the orthogonality of W, we get

$$\begin{aligned}
T^* &= T - \frac{1}{3nk}\left\| W^T\left(I - \frac{1}{\nu^M}\Delta\right)W\mathbf{N}(\overline{\mathbf{U}} - \mathbf{U})\right\|_2^2 \\
&\ge T - \frac{1}{3nk}\max\left(1 - \frac{\lambda_r}{\nu^M}\right)^2 \left\|\mathbf{N}(\overline{\mathbf{U}} - \mathbf{U})\right\|_2^2 \\
&\ge T - \frac{1}{3nk}\left(1 - \frac{\min\lambda_r}{\nu^M}\right)^2 \left\|\mathbf{N}(\overline{\mathbf{U}} - \mathbf{U})\right\|_2^2.
\end{aligned}$$

By the definition of ν^M, we know $\nu^M > \max\lambda_r$ so that

$$T^* \ge T - \frac{1}{3nk}\left\|\mathbf{N}(\overline{\mathbf{U}} - \mathbf{U})\right\|_2^2 \ge T - \frac{1}{3nk}\left\|\mathbf{N}(\overline{\mathbf{U}} - \mathbf{U})\right\|_F^2,$$

where we used the fact $\|\cdot\|_2 \le \|\cdot\|_F$. We recall $\mathbf{N} = \mathrm{diag}(\sqrt{\rho_1}, \dots, \sqrt{\rho_4})$ to obtain

$$\begin{aligned}
T^* &\ge T - \frac{1}{3nk}\sum_{i=1}^4 \rho^{(i)}|U^{(i)} - U|^2 \\
&= \sum_{i=1}^4 n^{(i)}T^{(i)}/n \\
&\ge T_l.
\end{aligned}$$

□

Corresponding estimations for the auxiliary parameters $\tilde{n}_i$, $\tilde{U}$, and $\tilde{U}$ in the reactive Maxwellian $\mathcal{C}_i$ can be found in [34]. We recall the following lemma.

Lemma 3.5 ([34])**.** *There exist positive lower and upper bounds for $\tilde{n}_i, \tilde{T}$ and upper bound for $\tilde{U}$ depending only on the quantities given in* (2.1), (2.2)*, and* (2.3)*. Additionally, the value of ν^M is bounded below and above by some positive values depending only on the quantities given in* (2.1), (2.2)*, and* (2.3)*.*

To simplify the presentation, we denote the lower and upper bounds for $\tilde{n}_i, \tilde{T}$ as $\tilde{n}_{i\ell}$, $\tilde{n}_{iu}$, $\tilde{T}_\ell$, and $\tilde{T}_u$, respectively, and the upper bound for $\tilde{U}$ as $\tilde{U}_u$. Similarly, we represent the lower and upper bounds for ν^M as ν_ℓ^M and ν_u^M, respectively. These notations will be used consistently throughout the rest of the paper.

3.3. **Collision frequencies.** We note that the collision frequency ν^M consists of λ_r and ν. Here, the values λ_r are the inverses of the eigenvalues of L^*. To estimate them, we introduce the following lemma on eigenvalues of symmetric matrices.

Lemma 3.6. *(Hoffman and Wielandt* [31]*) For every symmetric matrix A, let $\eta_r(A)$ be r-th largest eigenvalue of A. Then the following Lipschitz continuity holds:*

$$|\eta_r(A) - \eta_r(B)| \le \|A - B\|_F.$$

From Lemma 3.6, we have the following estimates for λ_r.

Lemma 3.7. *There exist positive lower and upper bounds for λ_r depending only on the quantities given in* (2.1), (2.2), *and* (2.3) .

Proof. Since L^* is continuous with respect to the macroscopic parameters, Lemma 3.6 implies that λ_r is also continuous with respect to the macroscopic parameters with range $(0,\infty)$. By the assumption, $a_l \le n^{(i)} \le a_u$, and Lemma 3.1, we have the desired result. $\square$

We also denote such lower bound and upper bound as λ_m and λ_M, respectively.

Lemma 3.8. *The shear viscosity η is bounded below and above by positive constants depending only on the quantities given in* (2.1), (2.2) *and* (2.3).

Proof. We recall that in this paper η is assumed to be a positive continuous function of macroscopic densities and temperature. Thus, the same argument used in Lemma 3.7 gives the desired result. $\square$

We denote such lower bound and upper bound as η_m and η_M, respectively. We then have a positve lower bound and a positve upper bound of collision frequency ν^M.

Lemma 3.9. *The relaxation coefficient ν^M satisfies the following inequality:*

$$\nu^M_m \le \nu^M \le \nu^M_M,$$

where we used

$$\nu^M_m = \frac{4a_l kT_l}{\eta_M}, \qquad \text{and} \qquad \nu^M_M = \max\left(\frac{4a_u kT_u}{\eta_m}, \lambda_M\right).$$

For the chemical collision frequency ν^C_i, it also holds that

$$\nu^C_m \le \nu^C_i \le \nu^C_M,$$

where we used

$$\begin{aligned}
\nu^C_m =& \nu^{34}_{12}\frac{2}{\sqrt{2\pi}}\Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_u}\right)\min\left\{1, \left(\frac{\mu_{12}}{\mu_{34}}\right)\exp\left(\frac{\Delta E}{kT_u}\right)a_l\right\},\\
\nu^C_M =& \nu^{34}_{12}\frac{2}{\sqrt{2\pi}}\Gamma\left(\frac{3}{2}, \frac{\Delta E}{kT_l}\right)\max\left\{1, \left(\frac{\mu_{12}}{\mu_{34}}\right)\exp\left(\frac{\Delta E}{kT_l}\right)a_u\right\}.
\end{aligned}$$

Proof. By Lemma 3.2, 3.7, and 3.8, we have for

$$\frac{4a_l kT_l}{\eta_M} \le \nu^M := \max\left(\frac{nkT}{\eta}, \max \lambda_r\right) \le \max\left(\frac{4a_u kT_u}{\eta_m}, \lambda_M\right).$$

We can also obtain the desired inequality for ν^C_i directly from Lemma 3.2. $\square$

4. Φ maps Ω into itself

In this section, we show that for any $F \in \Omega$, $\Phi(F)$ satisfies the conditions $(\mathcal{A}), (\mathcal{B})$ and $(\mathcal{C})$.

Lemma 4.1. *Let $F \in \Omega$. it holds that*

$$\phi_i \ge 0.$$

Thus, $\Phi(F)$ satisfies $(\mathcal{A})$.

Proof. By Lemma 3.3 and 3.4, we have

$$\begin{aligned}\mathcal{M}_i &= n^{(i)}\left(\frac{m_i}{2\pi kT^*}\right)^{3/2}\exp\left(-\frac{m_i|v-U_i|^2}{2kT^*}\right)\\ &\geq a_{i,l}\left(\frac{m_i}{2\pi kT_u}\right)^{3/2}\exp\left(-\frac{m_i|v-U_i|^2}{2kT^*}\right)\\ &> 0.\end{aligned}$$

Similarly, by Lemma 3.5, we also have

$$\begin{aligned}\mathcal{C}_i &= n^{(i)}\left(\frac{m_i}{2\pi k\tilde{T}}\right)^{3/2}\exp\left(-\frac{m_i|v-\tilde{U}|^2}{2k\tilde{T}}\right)\\ &\geq a_{i,l}\left(\frac{m_i}{2\pi k\tilde{T}_u}\right)^{3/2}\exp\left(-\frac{m_i|v-\tilde{U}|^2}{2k\tilde{T}}\right)\\ &> 0.\end{aligned}$$

Hence, we have

$$\phi_i \geq e^{-\frac{1}{\tau|v_1|}\int_0^x \overline{\nu}_i(y)dy}f_{i,L}(v)1_{v_1>0}+e^{-\frac{1}{\tau|v_1|}\int_x^1 \overline{\nu}_i(y)dy}f_{i,R}(v)1_{v_1<0}\geq 0.$$

□

Lemma 4.2. *Let $F\in\Omega$. We have*

$$\mathcal{M}_i(1+|v|^2),\ \mathcal{C}_i(1+|v|^2)\leq C_{\ell,u}\exp\left(-C_{\ell,u}|v|^2\right).$$

Proof. By Lemma 3.3 and 3.4, we have

$$\begin{aligned}\mathcal{M}_i &= n^{(i)}\left(\frac{m_i}{2k\pi T^*}\right)^{3/2}\exp\left(-\frac{m_i|v-U_i|^2}{2kT^*}\right)\\ &\leq C_{\ell,u}\exp\left(\frac{-m_i|v-U_i|^2}{2kT^*}\right)\\ &\leq C_{\ell,u}\exp\left(\frac{m_i|U_i|^2}{2kT^*}\right)\exp\left(\frac{-m_i|v|^2}{4kT^*}\right)\\ &\leq C_{\ell,u}\exp\left(-C_{\ell,u}|v|^2\right).\end{aligned}$$

For $|v|^2\mathcal{M}_i$, we know

$$\begin{aligned}|v|^2\mathcal{M}_i &\leq C_{\ell,u}\exp\left(-C_{\ell,u}|v|^2\right)|v|^2.\\ &\leq C_{\ell,u}\exp\left(-C_{\ell,u}|v|^2\right)\end{aligned}$$

where we use $x^2e^{-x^2}<C$ for some $C>0$. The proof for $\mathcal{C}_i$ is similar. We omit it. □

Lemma 4.3. *For $f_{i,L},f_{i,R}\in L^1_2(\mathbb{R}^3_v)$, if τ is sufficiently large, then we have*

$$\int_{v_1>0}e^{-\frac{1}{\tau|v_1|}\int_0^x\overline{\nu}_i(y)dy}f_{i,L}(v)\begin{pmatrix}1\\|v_1|\\|v|^2\end{pmatrix}dv\geq\frac{1}{4}\int_{v_1>0}f_{i,L}(v)\begin{pmatrix}1\\|v_1|\\|v|^2\end{pmatrix}dv$$

and

$$\int_{v_1<0} e^{-\frac{1}{\tau|v_1|}\int_x^1 \bar{\nu}_i(y)dy} f_{i,R}(v)\begin{pmatrix}1\\|v_1|\\|v|^2\end{pmatrix}dv \geq \frac{1}{4}\int_{v_1<0} f_{i,R}(v)\begin{pmatrix}1\\|v_1|\\|v|^2\end{pmatrix}dv.$$

Proof. We choose sufficiently small $r>0$ such that

$$\int_{v_1\geq r} f_{i,L}(v)dv \geq \frac{1}{2}\int_{v_1>0} f_{i,L}(v)dv.$$

By Lemma 3.9, we obtain that

$$\begin{aligned}\int_{v_1>0} e^{-\frac{1}{\tau|v_1|}\int_0^x \bar{\nu}_i(y)dy} f_{i,L}(v)dv &\geq e^{-\frac{C_{\ell,u}}{\tau r}}\int_{v_1>r} f_{i,L}dv\\ &\geq \frac{1}{4}\int_{v_1>0} f_{i,L}dv,\end{aligned}$$

for sufficiently large τ. The second inequality can be proved by the same argument. We omit it. □

Lemma 4.4. *Assume $F\in\Omega$ and $f_{i,L}, f_{i,R}\in L^1_2(\mathbb{R}^3_v)$. Then we have*

$$a_{i,l} \leq \int_{\mathbb{R}^3}\phi_i\ dv, \quad c_{i,l}\leq\int_{\mathbb{R}^3}|v|^2\phi_i\ dv. \tag{4.1}$$

Further, if $\tau>0$ is sufficiently large, then we have

$$\int_{\mathbb{R}^3}\phi_i dv \leq a_{i,u}, \quad \int_{\mathbb{R}^3}|v|^2\phi_i dv\leq c_{i,u}. \tag{4.2}$$

Therefore, $\Phi(f)$ satisfies $(\mathcal{B})$.

Proof. We know

$$\phi_i \geq e^{-\frac{1}{\tau|v_1|}\int_0^x \bar{\nu}_i(y)dy} f_{i,L}(v)1_{v_1>0} + e^{-\frac{1}{\tau|v_1|}\int_x^1 \bar{\nu}_i(y)dy} f_{i,R}(v)1_{v_1<0}.$$

This, together with Lemma 4.3, directly gives (4.1). Then, in order to obtain (4.2), we consider

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_i^+ dv =& \int_{v_1>0} e^{-\frac{1}{\tau|v_1|}\int_0^x \bar{\nu}_i(y)dy} f_{i,L}(v)dv\\ &+\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{1}{\tau|v_1|}\int_y^x \bar{\nu}_i(z)dz}(\nu^M\mathcal{M}_i+\nu_i^C\mathcal{C}_i)dydv\\ =&I+II.\end{aligned}$$

For I, we easily know

$$\int_{v_1>0} e^{-\frac{1}{\tau|v_1|}\int_0^x \bar{\nu}_i(y)dy} f_{i,L}(v)dv \leq \int_{v_1>0} f_{i,L}(v)dv \leq \frac{a_{i,u}}{2}. \tag{4.3}$$

We compute the upper bound of II as like:

$$\begin{aligned}
&\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{1}{\tau|v_1|}\int_y^x \bar{\nu}_i}(z)dz(\nu^M\mathcal{M}_i+\nu_i^C\mathcal{C}_i)dv \\
&\quad\le C_{\ell,u}\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}e^{-\bar{\nu}_m(x-y)/\tau|v_1|}e^{-C_{\ell,u}|v|^2}dydv \\
&\quad\le C_{\ell,u}\bigg(\int_0^x\int_{v_1>0}\frac{1}{\tau|v_1|}e^{-\bar{\nu}_m(x-y)/\tau|v_1|}e^{-C_{\ell,u}|v_1|^2}dv_1dy\bigg) \\
&\qquad\times\bigg(\int_{\mathbb{R}^2}e^{-C_{\ell,u}(|v_2|^2+|v_3|^2)}dv_2dv_3\bigg) \\
&\quad\le C_{\ell,u}\int_0^x\int_{v_1>0}\frac{1}{\tau|v_1|}e^{-\bar{\nu}_m(x-y)/\tau|v_1|}e^{-C_{\ell,u}|v_1|^2}dv_1dy \\
&\quad\le C_{\ell,u}\overline{II}.
\end{aligned}\tag{4.4}$$

where $\bar{\nu}_m=\nu_m^M+\nu_m^C$ and we used Lemma 3.9. To estimate $\overline{II}$, we divide the domain of integration into there subsets:

$$\begin{aligned}
\overline{II}&=\bigg\{\int_0^x\int_{0<v_1<\frac{1}{\tau}}+\int_0^x\int_{\frac{1}{\tau}<v_1<\tau}+\int_0^x\int_{\tau<v_1}\bigg\}\frac{1}{\tau|v_1|}e^{-\bar{\nu}_m(x-y)/\tau|v_1|}e^{-C_{\ell,u}|v_1|^2}dv_1dy \\
&=A+B+C.
\end{aligned}$$

For A, we have

$$\begin{aligned}
A&=\int_{0<v_1<\frac{1}{\tau}}\int_0^x\frac{1}{\tau|v_1|}e^{\bar{\nu}_m(x-y)/\tau|v_1|}e^{-C_{\ell,u}|v_1|^2}dydv_1 \\
&=\frac{1}{\bar{\nu}_m}\int_{0<v_1<\frac{1}{\tau}}\Big(1-e^{-\bar{\nu}_m x/\tau|v_1|}\Big)e^{-C_{\ell,u}|v_1|^2}dv_1 \\
&\le\frac{1}{\bar{\nu}_m}\int_{0<v_1<\frac{1}{\tau}}1\ dv_1 \\
&\le\frac{1}{\bar{\nu}_m}
\end{aligned}$$

where we used $1-e^{-\frac{\nu_1}{|v_1|}}\le 1$ and $e^{-C_{\ell,u}|v_1|^2}\le 1$. For B, it holds that

$$\begin{aligned}
B&\le\frac{1}{\bar{\nu}_m}\int_{\frac{1}{\tau}<v_1<\tau}\Big(1-e^{-\bar{\nu}_m x/\tau|v_1|}\Big)e^{-C_{\ell,u}|v_1|^2}dv_1 \\
&\le\int_{\frac{1}{\tau}<v_1<\tau}\frac{1}{\tau|v_1|}dv_1 \\
&=\frac{2}{\tau}\ln\tau.
\end{aligned}$$

where we used $1-e^{-x}\le x$. For C, we obtain

$$\begin{aligned}
C &\le \int_{\tau<v_1}\int_0^x \frac{1}{\tau|v_1|}e^{-\bar{\nu}_m(x-y)/\tau|v_1|}e^{-C_{\ell,u}|v_1|^2}dydv_1\\
&\le \frac{1}{\tau^2}\int_{\mathbb{R}} e^{-C_{\ell,u}|v_1|^2}dv_1\\
&\le C_{\ell,u}\frac{1}{\tau^2}.
\end{aligned}$$

We sum all the estimates for A,B and C to obtain

$$II\le C_{\ell,u}\left\{\frac{1}{\tau}+\frac{\ln\tau}{\tau}+\frac{1}{\tau^2}\right\}\le C_{\ell,u}\left\{\frac{\ln\tau+1}{\tau}\right\}, \tag{4.5}$$

which, together with (4.3), gives

$$\int_{\mathbb{R}^3}\phi_I^+dv\le \frac{a_{i,l}}{2}+C_{\ell,u}\left\{\frac{\ln\tau+1}{\tau}\right\}.$$

Similarly, we have

$$\int_{\mathbb{R}^3}\phi_I^-dv\le \frac{a_{i,l}}{2}+C_{\ell,u}\left\{\frac{\ln\tau+1}{\tau}\right\}.$$

By choosing sufficiently large $\tau>0$, we obtain the desired result (4.2). □

Lemma 4.5. *Let $F\in\Omega$ and $f_{i,L},f_{i,R}\in L^1_2(\mathbb{R}^3_v)$. Assume that*

$$\int_{\mathbb{R}^2}f_{i,L}v_jdv_2dv_3=\int_{\mathbb{R}^2}f_{i,R}v_jdv_2dv_3=0.$$

Then, for $j=2,3$, we have

$$\left|\int_{\mathbb{R}^3}\phi_iv_jdv\right|\le C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right).$$

Proof. We integrate ϕ_i^+ with respect to $v_2dv_2dv_3$:

$$\begin{aligned}
\int_{\mathbb{R}^2}\phi_i^+v_2dv_2dv_3 &= e^{-\frac{1}{\tau|v_1|}\int_0^x\bar{\nu}_i(y)dy}\int_{\mathbb{R}^2}f_{i,L}(v)v_2dv_2dv_3\\
&\quad+\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x\bar{\nu}_i(z)dz}\left(\int_{\mathbb{R}^2}(\nu^M\mathcal{M}_i+\nu_i^C\mathcal{C}_i)v_2dv_2dv_3\right)dy\\
&=\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x\bar{\nu}_i(z)dz}\left(\int_{\mathbb{R}^2}(\nu^M\mathcal{M}_i+\nu_i^C\mathcal{C}_i)v_2dv_2dv_3\right)dy.
\end{aligned} \tag{4.6}$$

where we used our assumption on $f_{i,L}$. By the similar way in (4.4), we have

$$\begin{aligned}
\int_{\mathbb{R}^2}(\nu^M\mathcal{M}_i+\nu_i^C\mathcal{C}_i)v_2dv_2dv_3 &\le C_{\ell,u}e^{-C_{\ell,u}|v_1|^2}\int_{\mathbb{R}^2}e^{-C_{\ell,u}(|v_2|^2+|v_3|^2)}|v_2|dv_2dv_3\\
&\le C_{\ell,u}e^{-C_{\ell,u}|v_1|^2}.
\end{aligned}$$

Substituting this in (4.6) and then integrating on $v_1>0$, we get

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_i^+v_2dv&\le C_{\ell,u}\int_0^x\int_{v_1>0}\frac{1}{\tau|v_1|}e^{-\frac{1}{\tau|v_1|}\int_y^x\bar{\nu}_i(z)dz}e^{-C_{\ell,u}|v_1|^2}dv_1dy\\&\le C_{\ell,u}\bigg\{\frac{\ln\tau+1}{\tau}\bigg\},\end{aligned}$$

where we obtain the last inequality from (4.4) and (4.5). Likewise, we have

$$\int_{\mathbb{R}^3}\phi_i^-v_2dv\le C_{\ell,u}\bigg\{\frac{\ln\tau+1}{\tau}\bigg\}.$$

□

Lemma 4.6. *Let $F\in\Omega$ and $f_{i,L},f_{i,R}\in L^1_2(\mathbb{R}^3_v)$. Assume that*

$$\int_{\mathbb{R}^2}f_{i,L}v_jdv_2dv_3=\int_{\mathbb{R}^2}f_{i,R}v_jdv_2dv_3=0.$$

Then, for sufficiently large $\tau>0$, we have

$$\bigg(\int_{\mathbb{R}^3}\phi_idv\bigg)\bigg(\int_{\mathbb{R}^3}\phi_i|v|^2dv\bigg)-\bigg|\int_{\mathbb{R}^3}\phi_ivdv\bigg|^2\ge\gamma_l,$$

which means $\Phi(f)$ satisfies $(\mathcal{C})$.

Proof. The Cauchy-Schwarz inequality gives

$$\begin{aligned}&\bigg(\int_{\mathbb{R}^3}\phi_idv\bigg)\bigg(\int_{\mathbb{R}^3}\phi_i|v|^2dv\bigg)-\bigg|\int_{\mathbb{R}^3}\phi_ivdv\bigg|^2\\&\ge\bigg(\int_{\mathbb{R}^3}\phi_i|v|dv\bigg)^2-\bigg|\int_{\mathbb{R}^3}\phi_ivdv\bigg|^2\\&\ge\bigg(\int_{\mathbb{R}^3}\phi_i|v_1|dv\bigg)^2-\bigg|\int_{\mathbb{R}^3}\phi_ivdv\bigg|^2.\end{aligned}$$

Decomposing the last term in the last line, we have

$$\begin{aligned}&\bigg(\int_{\mathbb{R}^3}\phi_i|v_1|dv\bigg)^2-\bigg|\int_{\mathbb{R}^3}\phi_ivdv\bigg|^2\\&=\bigg(\int_{\mathbb{R}^3}\phi_i|v_1|dv\bigg)^2-\bigg(\int_{\mathbb{R}^3}\phi_iv_1dv\bigg)^2-R,\end{aligned}$$

where $R=|\int\phi_iv_2dv|^2+|\int\phi_iv_3dv|^2$. Here, Lemma 4.5 gives

$$R\le C_{\ell,u}\bigg(\frac{\ln\tau+1}{\tau}\bigg).$$

Also, we know

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3}\phi_i|v_1|dv\right)^2-\left(\int_{\mathbb{R}^3}\phi_i v_1 dv\right)^2\\
&\geq\left(\int_{\mathbb{R}^3}\phi_i(|v_1|+v_1)dv\right)\left(\int_{\mathbb{R}^3}\phi_i(|v_1|-v_1)dv\right)\\
&=4\left(\int_{v_1>0}\phi_i|v_1|dv\right)\left(\int_{v_1<0}\phi_i|v_1|dv\right),
\end{aligned}$$

where the last term can be estimated by Lemma 4.3 as follows:

$$\begin{aligned}
&4\left(\int_{v_1>0}\phi_i|v_1|dv\right)\left(\int_{v_1<0}\phi_i|v_1|dv\right)\\
&\geq 4\left(\int_{v_1>0}e^{-\frac{1}{\tau|v_1|}\int_0^x\bar{\nu}_i(y)dy}f_{i,L}(v)|v_1|dv\right)\left(\int_{v_1<0}e^{-\frac{1}{\tau|v_1|}\int_x^1\bar{\nu}_i(y)dy}f_{i,R}(v)|v_1|dv\right)\\
&\geq\frac{1}{4}\left(\int_{v_1>0}f_{i,L}(v)|v_1|dv\right)\left(\int_{v_1<0}f_{i,R}(v)|v_1|dv\right)\\
&=4\gamma_l.
\end{aligned}$$

Summarizing all the above estimates, we obtain that for sufficiently large $\tau>0$

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3}\phi_i dv\right)\left(\int_{\mathbb{R}^3}\phi_i|v|^2dv\right)-\left|\int_{\mathbb{R}^3}\phi_i v dv\right|^2\\
&\geq 4\gamma_l-C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right)\\
&\geq\gamma_l.
\end{aligned}$$

□

Combining Lemmas 4.1, 4.4, and 4.6, we have the following proposition.

Proposition 4.1. *Let $F\in\Omega$ and $f_{i,L},f_{i,R}\in L^1_2(\mathbb{R}^3_v)$. Assume that*

$$\int_{\mathbb{R}^2}f_{i,L}v_j dv_2dv_3=\int_{\mathbb{R}^2}f_{i,R}v_j dv_2dv_3=0,$$

and $\tau>0$ is sufficiently large. Then, under the assumption of Theorem 2.2, we have

$$\Phi(\Omega)\subset\Omega.$$

5. Φ is contractive in Ω

In this section, we show the map $\Phi:\Omega\to\Omega$ is a contraction map. Throughout this section, we assume that the inflow boundary data satisfies

$$f_{i,L},f_{i,R}\in L^1_2(\mathbb{R}^3_v),$$

and

$$\int_{\mathbb{R}^2}f_{i,L}v_j dv_2dv_3=\int_{\mathbb{R}^2}f_{i,R}v_j dv_2dv_3=0.$$

We start with proving that the actual macroscopic parameters have the Lipschitz continuity.

Lemma 5.1. *For any* $F=(f_1,f_2,f_3,f_4),\ G=(g_1,g_2,g_3,g_4)\in\Omega$, *the single component macroscopic parameters satisfy:*

$$|n_F^{(i)}-n_G^{(i)}|,|U_F^{(i)}-U_G^{(i)}|,|T_F^{(i)}-T_G^{(i)}|\leq C_{\ell,u}\sup_{x\in[0,1]}||f_i-g_i||_{L^1_2},$$

Furthermore, we have for global macroscopic parameters,

$$|n_F-n_G|,|U_F-U_G|,|T_F-T_G|\leq C_{\ell,u}d(F,G)$$

Proof. We can easily get

$$|n_F^{(i)}-n_G^{(i)}|=\int_{\mathbb{R}^3}|f_i-g_i|dv\leq\sup_{x\in[0,1]}||f_i-g_i||_{L^1_2}.$$

For the bulk velocity $U^{(i)}$, we see

$$\begin{aligned}|U_F^{(i)}-U_G^{(i)}|&\leq\frac{1}{\rho_F^{(i)}}|\rho_F^{(i)}U_F^{(i)}-\rho_G^{(i)}U_G^{(i)}|+\frac{1}{\rho_F^{(i)}}|\rho_F^{(i)}-\rho_G^{(i)}|\ |U_G^{(i)}|\\&\leq\frac{m_i}{\rho_F^{(i)}}\int_{\mathbb{R}^3}|f_i-g_i|\ |v|dv+\frac{m_i|U_G^{(i)}|}{\rho_F^{(i)}}\int_{\mathbb{R}^3}|f_i-g_i|dv\\&\leq C_{\ell,u}\sup_{x\in[0,1]}||f_i-g_i||_{L^1_2},\end{aligned}$$

where we used $\rho_F^{(i)}\geq m_ia_{i,l}$ in the second line. For $T^{(i)}$, we split it into two parts:

$$\begin{aligned}|T_F^{(i)}-T_G^{(i)}|&\leq\frac{1}{n_F^{(i)}}|n_F^{(i)}T_F^{(i)}-n_G^{(i)}T_G^{(i)}|+\frac{1}{n_F^{(i)}}|n_F^{(i)}-n_G^{(i)}|\ |T_G^{(i)}|\\&\leq\frac{m_i}{3kn_F^{(i)}}\int_{\mathbb{R}^3}\Big|f_i|v-U_F^{(i)}|^2-g_i|v-U_G^{(i)}|^2\Big|dv+\frac{m_i|T_G^{(i)}|}{n_F^{(i)}}\int_{\mathbb{R}^3}|f_i-g_i|dv\\&=I+II.\end{aligned}$$

We compute I as

$$\begin{aligned}&\int_{\mathbb{R}^3}\Big|f_i|v-U_F^{(i)}|^2-g_i|v-U_G^{(i)}|^2\Big|dv\\&\quad\leq\int_{\mathbb{R}^3}\Big|(f_i-g_i)|v-U_F^{(i)}|^2+g_i(|v-U_F^{(i)}|^2-|v-U_G^{(i)}|^2)\Big|dv\\&\quad=\int_{\mathbb{R}^3}\Big|(f_i-g_i)|v-U_F^{(i)}|^2+g_i(2v-U_F^{(i)}-U_G^{(i)})(U_F^{(i)}-U_G^{(i)})\Big|dv\\&\quad\leq C_{\ell,u}\int_{\mathbb{R}^3}|f_i-g_i|(1+|v|^2)+|g_i|(1+|v|)|U_F^{(i)}-U_G^{(i)}|dv\\&\quad\leq C_{\ell,u}\sup_{x\in[0,1]}||f_i-g_i||_{L^1_2},\end{aligned}$$

which, together with Lemma 5.1 and the fact $n_F^{(i)}\geq a_{i,l}$, implies that

$$II\leq C_{\ell,u}\sup_{x\in[0,1]}||f_i-g_i||_{L^1_2}.$$

The estimates for the global macroscopic parameters can be obtained directly from above estimates. We omit it. $\square$

Then, we also need the Lipschitz continuity for the auxiliary parameters and the collision frequencies. As we mentioned in Section 1, the auxiliary parameters for mechanical Maxwellians include $W^T\Delta W$ that arises to fit the correct Fick coefficients. By the definition, $W^T\Delta W$ can be seen as the pseudo-inverse of the matrix $-L^*$. But, it is known that the operation of pseudo-inverse is not continuous generally. To overcome this problem, we show that $W^T\Delta W$ can be rewritten as a rational function of each components of L^* with a non-zero denominator.

Lemma 5.2. *Each component of $W^T\Delta W$ can be written as a rational function of the components of L^*. Also , all their denominators are equal to $(\lambda_1\lambda_2\lambda_3)^2$.*

Proof. By the definitions of W and Δ, we know $W^T\Delta W$ is the pseudo-inverse of $-L^*$. Using the well-known property of the pseudo-inverse, we have

$$\begin{aligned} W^T\Delta W &= (-L^*)^+ \\ &= -\lim_{\delta\downarrow 0}((L^*)^TL^*+\delta I)^{-1}(L^*)^T \\ &= -\lim_{\delta\downarrow 0}\frac{C_\delta(L^*)^T}{\det((L^*)^TL^*+\delta I)} \end{aligned}$$

where C_δ is a adjoint matrix of $(L^*)^TL^*+\delta I$. Since $\lambda_4=0$ and hence $\det((L^*)^TL^*)=0$, we obtain for the denominator that

$$\det((L^*)^TL^*+\delta I)=\prod_{i=1}^{4}(\lambda_i^2+\delta)=p(L^*)\delta+O(\delta^2)$$

where $p(L^*)$ is a polynomial on the each component of L^* and is equal to $\lambda_1^2\lambda_2^2\lambda_3^2$. Denote each $ij-$ component of $C_\delta L^T$ by

$$q_0^{ij}(L^*)+q_1^{ij}(L^*)\delta+\cdots+q_{n-1}^{ij}(L^*)\delta^{n-1}$$

where $q_k^{ij}(L^*)$ are polynomials on the each component of L^*. Then, we get

$$(W^T\Delta W)_{ij}=-\lim_{\delta\downarrow 0}\frac{q_0^{ij}(L^*)+q_1^{ij}(L^*)\delta+\cdots+q_{n-1}^{ij}(L^*)\delta^{n-1}}{p(L^*)\delta+O(\delta^2)}$$

Since the limit exists, $q_0^{ij}(L^*)=0$ and we have the desire result. ☐

Using this, we obtain the Lipschitz continuity of $W^T\Delta W$. Consequently, we also obtain the Lipschitz continuity of U_i, T^* and ν^M.

Lemma 5.3. *For $F,G\in\Omega$, if $\tau>0$ is sufficiently large, then the following inequality holds:*

$$\left\|W_F^\top\Delta_FW_F-W_G^\top\Delta_GW_G\right\|_F\le C_{\ell,u}d(F,G).$$

Hence, we have

$$|U_{F,i}-U_{G,i}|,|T_f^*-T_g^*|\le C_{\ell,u}d(F,G),$$

and

$$|\nu_F^M-\nu_G^M|\le C_{\ell,u}d(F,G).$$

Proof. Consider the matrix $W_F^\top \Delta_F W_F - W_G^\top \Delta_G W_G$ componentwisely. We have from Lemma 3.7 and 5.2 that

$$\begin{aligned}
\left|(W_F^\top \Delta_F W_F - W_G^\top \Delta_G W_G)_{ij}\right| &= \left|\frac{q_1^{ij}(L_F^*)}{p(L_F^*)} - \frac{q_1^{ij}(L_G^*)}{p(L_G^*)}\right| \\
&= \left|\frac{q_1^{ij}(L_F^*)p(L_G^*) - q_1^{ij}(L_G^*)p(L_F^*)}{p(L_F^*)p(L_G^*)}\right| \\
&\leq \frac{\left|q_1^{ij}(L_F^*)p(L_G^*) - q_1^{ij}(L_G^*)p(L_F^*)\right|}{\lambda_m^{12}}.
\end{aligned} \tag{5.1}$$

By the definition (1.3) of L^* and Lemma 3.1, we obtain that every component of L_F^* and L_G^* is bounded below and above by constants depending only on the quantities defined in (2.1), (2.2) and (2.3). Thus, we have

$$\begin{aligned}
\left|q_1^{ij}(L_F^*)p(L_G^*) - q_1^{ij}(L_G^*)p(L_F^*)\right| \leq& \left|q_1^{ij}(L_F^*)p(L_G^*) - q_1^{ij}(L_F^*)p(L_F^*)\right| \\
&+ \left|q_1^{ij}(L_F^*)p(L_F^*) - q_1^{ij}(L_G^*)p(L_F^*)\right| \\
=& \left|q_1^{ij}(L_F^*)\right| \left|p(L_G^*) - p(L_F^*)\right| + \left|p(L_F^*)\right| \left|q_1^{ij}(L_F^*) - q_1^{ij}(L_G^*)\right| \\
\leq& C_{l,u} \left\|L_F^* - L_G^*\right\|_F .
\end{aligned}$$

This, together with (5.1), gives

$$\begin{aligned}
\left|(W_F^\top \Delta_F W_F - W_G^\top \Delta_G W_G)_{ij}\right| &\leq \frac{C_{\ell,u}}{\lambda_m^{12}} \left\|L_F^* - L_G^*\right\|_F \\
&\leq C_{\ell,u} \sum_{i=1}^4 \left(|n_F^{(i)} - n_G^{(i)}| + |T_F - T_G|\right) \\
&\leq C_{\ell,u} d(F,G),
\end{aligned}$$

where we used the definition of L^*, (1.3), and Lemma 5.1. Consequently, we obtain

$$\begin{aligned}
\|\underline{\mathrm{U}}_F - \underline{\mathrm{U}}_G\|_F \leq& \|\mathbf{U}_F - \mathbf{U_G}\|_F + \bigg\|\mathbf{N}_F^{-1} W_F^T \left(I - \frac{1}{\nu_F^M}\Delta_F\right) W_F \mathbf{N}_F (\overline{\mathbf{U}}_\mathbf{F} \\
&- \mathbf{U_F}) - \mathbf{N}_G^{-1} W_G^T \left(I - \frac{1}{\nu_G^M}\Delta_G\right) W_G \mathbf{N}_G (\overline{\mathbf{U}}_\mathbf{G} - \mathbf{U_G})\bigg\|_F \\
\leq& 2\|\mathbf{U_F} - \mathbf{U_G}\|_F + \left\|\overline{\mathbf{U}}_\mathbf{F} - \overline{\mathbf{U}}_\mathbf{G}\right\|_F \\
&+ \left\|\frac{1}{\nu_F^M}\mathbf{N}_F^{-1} W_F^T \Delta_F W_F \mathbf{N}_F (\overline{\mathbf{U}}_\mathbf{F} - \mathbf{U_F}) - \frac{1}{\nu_G^M}\mathbf{N}_G^{-1} W_G^T \Delta_G W_G \mathbf{N}_G (\overline{\mathbf{U}}_\mathbf{G} - \mathbf{U_G})\right\|_F ,
\end{aligned}$$

where we recall that

$$\begin{aligned}
\mathbf{U} &= (U, ..., U)^T, \\
\overline{\mathbf{U}} &= (U^{(1)}, ..., U^{(N)})^T, \\
\mathbf{N} &= \mathrm{diag}(\sqrt{\rho_1}, ..., \sqrt{\rho_N}).
\end{aligned}$$

From Lemma 5.1, we know that all $\mathbf{U}$, $\bar{\mathbf{U}}$, and $\mathbf{N}$ have the Lipschitz continuity. Also, it follows from Lemma 3.1 and 3.2 that

$$\left\|\frac{1}{\nu^M}\right\|, \ \left\|\mathbf{N}^{-1}\right\|, \ \left\|W^T\Delta W\right\|, \ \left\|\bar{\mathbf{U}}\right\|, \ \|\mathbf{U}\| \leq C_{l,u}.$$

With the above facts, it follows from the elementary and tedious computations that

$$\|\underline{\mathrm{U}}_F - \underline{\mathrm{U}}_G\|_F \leq C_{\ell,u} d(F,G).$$

The proof for T^* is almost same. We omit it. Then, for ν^M we have from Lemma 5.1

$$\begin{aligned}
|\nu_F^M - \nu_G^M| &\leq \left|\frac{n_F kT_F}{\eta_F} - \frac{n_G kT_G}{\eta_G}\right| + |\max\lambda_{F,r} - \max\lambda_{G,r}| \\
&\leq C_{\ell,u} \sup_{x\in[0,1]} \left\|f_j - g_j\right\|_{L^1_2} + |\max\lambda_{F,r} - \max\lambda_{G,r}|.
\end{aligned}$$

Here, Lemma 3.6 gives

$$|\max\lambda_{F,r} - \max\lambda_{G,r}|^2 \leq \left\|W_F^T\Delta_F W_F - W_G^T\Delta_G W_G\right\|_F^2 \leq C_{\ell,u} \sup_{x\in[0,1]} \left\|f_j - g_j\right\|_{L^1_2},$$

which completes the proof. □

For the auxiliary parameters for reactive Maxwellian, we recall the following lemma in [34].

Lemma 5.4. *Let $F, G \in \Omega$. Assume that $\tau > 0$ is sufficiently large. Then we have*

$$|\tilde{n}_{F,i} - \tilde{n}_{G,i}|, |\tilde{U}_F - \tilde{U}_G|, |\tilde{T}_F - \tilde{T}_G| \leq C_{\ell,u} d(F,G),$$

and

$$|\nu_{F,i}^C - \nu_{G,i}^C| \leq C_{\ell,u} d(F,G).$$

Proof. See the reference [34]. □

We obtain the Lipschitz continuity of both the Maxwellians.

Lemma 5.5. *Let $F, G \in \Omega$. Assume $\tau > 0$ is sufficiently large. Then the following inequalities hold:*

$$|\mathcal{M}_{F,i} - \mathcal{M}_{G,i}| \leq C_{\ell,u} d(F,G),$$

and

$$|\mathcal{C}_{F,i} - \mathcal{C}_{G,i}|, \ \leq C_{\ell,u} d(F,G).$$

Proof. For simplicity, we only consider the first inequality. By Taylor expansion, we can write $\mathcal{M}_{F,i} - \mathcal{M}_{G,i}$ as

$$\begin{aligned}
\mathcal{M}_{F,i} - \mathcal{M}_{G,i} =& (n_F^{(i)} - n_G^{(i)}) \int_0^1 \frac{\partial \mathcal{M}(\theta)}{\partial n} d\theta \\
&+ (U_{F,i} - U_{G,i}) \int_0^1 \frac{\partial \mathcal{M}(\theta)}{\partial U} d\theta \\
&+ (T_F^* - T_G^*) \int_0^1 \frac{\partial \mathcal{M}(\theta)}{\partial T} d\theta \\
&= A + B + C,
\end{aligned}$$

where we used the notation

$$\mathcal{M}(\theta) = \frac{n_\theta}{2\pi kT_\theta/m_i}^{3/2} \exp\left(-\frac{m_1|v-U_{i,\theta}|^2}{2kT_\theta}\right)$$

and

$$\frac{\partial \mathcal{M}(\theta)}{\partial X} = \frac{\partial \mathcal{M}(\theta)}{\partial X}(m_i, n_\theta, U_{i,\theta}, T_\theta)$$

with $(n_\theta, U_\theta, T_\theta) = (1-\theta)(n_F^{(i)}, U_{F,i}, T_F^*) + \theta(n_G^{(i)}, U_{G,i}, T_G^*)$. For A, we see

$$\frac{\partial \mathcal{M}(\theta)}{\partial n} = \frac{1}{n_\theta}\mathcal{M}(\theta),$$

to obtain

$$\left|\frac{\partial \mathcal{M}(\theta)}{\partial \rho}\right| \leq C_{\ell,u} e^{-C_{\ell,u}|v|^2}.$$

For B, we have

$$\frac{\partial \mathcal{M}(\theta)}{\partial U} = \frac{m_i(v-U_\theta)}{kT_\theta}\mathcal{M}(\theta),$$

and hence,

$$\begin{aligned}\left|\frac{\partial \mathcal{M}(\theta)}{\partial U}\right| &\leq C_{\ell,u}(1+|v|)\mathcal{M}(\theta)\\ &\leq C_{\ell,u} e^{-C_{\ell,u}|v|^2}.\end{aligned}$$

Finally, for C, we know

$$\frac{\partial \mathcal{M}(\theta)}{\partial T} = \left\{-\frac{3}{2T_\theta} + \frac{m_i|v-U_\theta|^2}{2kT_\theta^2}\right\}\mathcal{M}(\theta),$$

which gives

$$\left|\frac{\partial \mathcal{M}(\theta)}{\partial T}\right| \leq C_{\ell,u}(1+|v|^2)e^{-C_{\ell,u}|v|^2} \leq C_{\ell,u} e^{-C_{\ell,u}|v|^2}.$$

Thus, we obtain

$$|\mathcal{M}_{F,i} - \mathcal{M}_{G,i}| \leq C_{\ell,u}\left\{|n_F^{(i)} - n_G^{(i)}| + |U_{F,i} - U_{G,i}| + |T_F^* - T_G^*|\right\}e^{-C_{\ell,u}|v|^2}.$$

which, together with Lemma 5.1 and Lemma 5.3, gives $|\mathcal{M}_{F,i} - \mathcal{M}_{G,i}| \leq C_{\ell,u} d(F,G)$. □

Proposition 5.1. *Let $F, G \in \Omega$. If τ is sufficiently large, then there exist a $\alpha \in (0,1)$ such that*

$$d(\Phi(F), \Phi(G)) \leq \alpha d(F,G).$$

Proof. We only compute $|\phi^+_{F,i} - \phi^+_{G,i}|$ because $|\phi^-_{F,i} - \phi^-_{G,i}|$ can be considered by the same argument. First, we split $\phi^+_{F,i} - \phi^+_{G,i}$ into three terms as

$$\begin{aligned}
\phi^+_{F,i} - \phi^+_{G,i} &= \left\{ e^{-\frac{1}{\tau|v_1|}\int_0^x \bar\nu_{F,i}(y)dy} - e^{-\frac{1}{\tau|v_1|}\int_0^x \bar\nu_{G,i}(y)dy} \right\} f_{i,L}(v) \\
&\quad + \frac{1}{\tau|v_1|}\bigg(\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{F,i}(z)dz} \nu_F^M(y)\mathcal{M}_{F,i}dy \\
&\quad - \int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{G,i}(z)dz} \nu_G^M(y)\mathcal{M}_{G,i}dy \bigg) \\
&\quad + \frac{1}{\tau|v_1|}\bigg(\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{F,i}(z)dz} \nu_{F,i}^C(y)\mathcal{C}_{F,i}dy \\
&\quad - \int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{G,i}(z)dz} \nu_{G,i}^C(y)\mathcal{C}_{G,i}dy \bigg) \\
&= I + II + III.
\end{aligned}$$

By the mean value theorem, there exists $0 < \theta < 1$ such that

$$\begin{aligned}
I &= \left\{ e^{-\frac{1}{\tau|v_1|}\int_0^x \bar\nu_{F,i}(y)dy} - e^{-\frac{1}{\tau|v_1|}\int_0^x \bar\nu_{F,i}(y)dy} \right\} f_{i,L}(v) \\
&= -\frac{1}{\tau|v_1|} e^{-\frac{1}{\tau|v_1|}\int_0^x (1-\theta)\bar\nu_{F,i}(y)+\theta\bar\nu_{G,i}(y)dy} \int_0^x \big(\bar\nu_{F,i}(y) - \bar\nu_{G,i}(y)\big)\,dy\ f_{i,L}(v).
\end{aligned}$$

Lemma 5.3 and 5.4 give that

$$|\bar\nu_{F,i} - \bar\nu_{G,i}| \le d(F,G),$$

and hence, we have

$$\begin{aligned}
|I| &\le \frac{1}{\tau|v_1|}\left(e^{-\frac{1}{\tau|v_1|}\int_0^x \bar\nu_m dy} \int_0^x |\bar\nu_{F,i} - \bar\nu_{G,i}|dy \right) f_{i,L}(v) \\
&\le \frac{C_{\ell,u}}{\tau|v_1|} e^{-\frac{\bar\nu_m}{\tau|v_1|}} f_{i,L} d(F,G) \qquad (5.2)\\
&\le \frac{C_{\ell,u}}{\tau|v_1|} f_{i,L} d(F,G).
\end{aligned}$$

We rewrite II as the following three terms:

$$\begin{aligned}
&\frac{1}{\tau|v_1|}\left| \int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{F,i}(z)dz} \nu_F^M(y)\mathcal{M}_{F,i}dy - \int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{G,i}(z)dz} \nu_F^M(y)\mathcal{M}_{F,i}dy \right| \\
&+ \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{G,i}(z)dz} |\nu_F^M(y) - \nu_G^M(y)|\mathcal{M}_{F,i}dy \qquad (5.3)\\
&+ \frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar\nu_{G,i}(z)dz} \nu_G^M(z)(\mathcal{M}_{F,i} - \mathcal{M}_{G,i})dy.
\end{aligned}$$

For the first term of (5.3), we use the same argument for I to obtain

(5.4)
$$\begin{aligned}
&\frac{1}{\tau|v_1|}\left|\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar{\nu}_{F,i}(z)dz}\nu_F^M(y)\mathcal{M}_{F,i}dy-\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar{\nu}_{G,i}(z)dz}\nu_F^M(y)\mathcal{M}_{F,i}dy\right|\\
&\leq \frac{1}{\tau|v_1|}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{1}{\tau|v_1|}\int_y^x(1-\theta)\bar{\nu}_{F,i}(z)+\theta\bar{\nu}_{G,i}(z)dz}\int_y^x|\bar{\nu}_{F,i}(z)-\bar{\nu}_{G,i}(z)|dz\ \bar{\nu}_{F,i}(y)\mathcal{M}_{F,i}dy\\
&\leq \frac{C_{\ell,u}}{\tau|v_1|}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{\bar{\nu}_m}{\tau|v_1|}}\mathcal{M}_{F,i}dy\cdot d(F,G)\\
&\leq \frac{C_{\ell,u}}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{2\tau|v_1|}}\mathcal{M}_{F,i}dy\cdot d(F,G)
\end{aligned}$$

where we used that $xe^{-x}<C$ for some $C>0$. The second term of (5.3) is estimated as below:

$$\begin{aligned}
&\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar{\nu}_{G,i}(z)dz}|\nu_F^M(y)-\nu_G^M(y)|\mathcal{M}_{F,i}dy\\
&\leq \frac{C_{\ell,u}}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}\mathcal{M}_{F,i}dy\cdot d(F,G).
\end{aligned}\tag{5.5}$$

For the third term of (5.3), Lemma 5.5 implies that

$$\begin{aligned}
&\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{1}{\tau|v_1|}\int_y^x \bar{\nu}_{G,i}(z)dz}\nu_G^M(z)(\mathcal{M}_{F,i}-\mathcal{M}_{G,i})dy\\
&\leq \frac{C_{\ell,u}}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}e^{-C_{\ell,u}|v|^2}dy\cdot d(F,G).
\end{aligned}\tag{5.6}$$

Combining (5.4), (5.5), and (5.6), we obtain

$$\begin{aligned}
II\leq C_{\ell,u}\cdot d(F,G)\cdot\bigg(&\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}\mathcal{M}_{F,i}dy\\
&+\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}e^{-C_{\ell,u}|v|^2}dy\bigg).
\end{aligned}\tag{5.7}$$

As like II, a similar argument gives

$$\begin{aligned}
III\leq C_{\ell,u}\cdot d(F,G)\cdot\bigg(&\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}\mathcal{C}_{F,i}dy\\
&+\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}e^{-C_{\ell,u}|v|^2}dy\bigg).
\end{aligned}\tag{5.8}$$

Summing up (5.2),(5.7), and (5.8), we have

$$\begin{aligned}
|\phi^+_{F,i}-\phi^+_{G,i}|\leq C_{\ell,u}\cdot d(F,G)\cdot\bigg(&\frac{1}{\tau|v_1|}f_{i,L}+\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}(\mathcal{M}_{F,i}+\mathcal{C}_{F,i})dy\\
&+\frac{1}{\tau|v_1|}\int_0^x e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}e^{-C_{\ell,u}|v|^2}dy\bigg).
\end{aligned}$$

Multiplying it by $(1+|v|^2)$ and integrating over $v_1>0$, we have

$$
\begin{aligned}
||\phi^+_{F,i}-\phi^+_{G,i}||_{L^1_2} \le C_{\ell,u}\cdot d(F,G)\cdot\bigg(&\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}f_{i,L}(1+|v|^2)dydv\\
&+\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}(\mathcal{M}_{F,i}+\mathcal{C}_{F,i})(1+|v|^2)dydv\\
&+\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}e^{-C_{\ell,u}|v|^2}(1+|v|^2)dydv\bigg).
\end{aligned}
$$

We apply Lemma 4.2 to obtain

$$
\begin{aligned}
||\phi^+_{F,i}-\phi^+_{G,i}||_{L^1_2} \le C_{\ell,u}\cdot d(F,G)\cdot\bigg(&\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}f_{i,L}(1+|v|^2)dydv\\
&+\int_{v_1>0}\int_0^x \frac{1}{\tau|v_1|}e^{-\frac{\bar{\nu}_m}{\tau|v_1|}(x-y)}e^{-C_{\ell,u}|v|^2}dydv\bigg).
\end{aligned}
$$

where we used the fact $x^2e^{-x^2}<Ce^{-x^2/2}$. A similar computation on $\bar{I}I$ in Lemma 4.4, together with the assumption that $f_{i,L}/|v|\in L^1_2(\mathbb{R}^3_v)$, gives that

$$
\begin{aligned}
||\phi^+_{F,i}-\phi^+_{G,i}||_{L^1_2} &\le C_{\ell,u}\left[\frac{a_{i,s}+c_{i,s}}{\tau}+\left(\frac{\ln\tau+1}{\tau}\right)\right]d(F,G)\\
&\le C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right)d(F,G).
\end{aligned}
$$

By the same way, it can be obtained that

$$
||\phi^-_{F,i}-\phi^-_{G,i}||_{L^1_2} \le C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right)d(F,G).
$$

Summarizing the above estimates and taking supremum over $x\in[0,1]$ on the both sides, we have

$$
d(\Phi(F),\Phi(G)) \le C_{\ell,u}\left(\frac{\ln\tau+1}{\tau}\right)d(F,G).
$$

Finally, if we choose τ large enough so that $C_{\ell,u}(\ln\tau+1)/\tau<1$, then we get the desired result. □

Proposition 4.1 and 5.1 imply that we can apply the Banach fixed point theorem on our solution map, which completes the proof of Theorem 2.2.

Data Availability No data was used for the research described in the article.

Conflicts of Interest The authors declare that they have no conflicts of interest.

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References

[1] Andries, P., Aoki, K., and Perthame, B.: A consistent BGK-type model for gas mixtures, J. Stat. Phys. 106, 993 (2002)

[2] Andries, P., Le Tallec, P., Perlat, J. P., and Perthame, B.: The Gaussian-BGK model of Boltzmann equation with small Prandtl number, European Journal of Mechanics-B/Fluids, 19(6), (2000), 813-830.

[3] Bae, G. C., and Yun, S. B.: The Shakhov model near a global Maxwellian. Nonlinear Analysis: Real World Applications, 70, (2023), 103742.

[4] Bae, G.-C., Klingenberg, C., Pirner, M., and Yun, S.-B.: BGK model of the multi-species uehling uhlenbeck equation, Kinetic and Related Models, 14 (1) : 25-44, (2021).

[5] Bae, G.-C., Ko, G., Lee, D., and Yun, S.-B.: Large amplitude problem of BGK model: Relaxation to quadratic nonlinearity. arXiv preprint arXiv:2301.09857. (2023).

[6] Bang, J., and Yun, S. B.: Stationary solutions for the ellipsoidal BGK model in a slab. Journal of Differential Equations, 261(10), (2016), 5803-5828.

[7] Bhatnagar, P. L., Gross, E. P and Krook, M.: A model for collision processes in gases, Physical Revis, **94** (1954), 511-525

[8] Bisi, M., Cáceres, M. J.: A BGK relaxation model for polyatomic gas mixtures. Communications in Mathematical Sciences, 14(2):297–325, 2016.

[9] Bisi, M., Monaco, R., and Soares, A. J.: A BGK model for reactive mixtures of polyatomic gases with continuous internal energy. J. Phys. A 51(12) 125501, 1–29, (2018)

[10] Bisi, M., Spiga, G.: On a kinetic BGK model for slow chemical reactions, Kinet. Relat. Models **4** (2011), no. 1, 153–167.

[11] Bobylev, A. V., Bisi, M., Groppi, M., Spiga, G., Potapenko, I. F.: A general consistent BGK model for gas mixtures. Kinet. Relat. Models 11(6):1377–1393, 2018.

[12] Brull, S., and Schneider, J.: A new approach of the Ellipsoidal statistical model, Cont. Mech. Thermodyn., 20(2), 63–74, 2008.

[13] Brull, S., and Schneider, J.: On the ellipsoidal statistical model for polyatomic gases, Cont. Mech. Thermodyn., 20(8), 489–508, 2009.

[14] Brull, S., Pavan, V., and Schneider. J.: Derivation of a BGK model for mixtures. Eur. J. Mech. B-Fluid, 33:74–86, 2012.

[15] Brull, S., and Schneider, J.: Derivation of a BGK model for reacting gas mixtures. Commun. Math. Sci 12 (2014), no.7, 1199-1223.

[16] Brull, S.: An ellipsoidal statistical model for gas mixtures. Communications in Mathematical Sciences, 2015, 13.1: 1-13.

[17] Brull, S: An Ellipsoidal Statistical Model for a monoatomic and polyatomic gas mixture Communication in Math. Science, 19, 8, 2177-2194, (2021)

[18] Brull, S., and Yun, S.-B. Stationary flows of the es-bgk model with the correct prandtl number. arXiv preprint arXiv:2012.08490 (2020).

[19] Cercignani, C.: Rarefied gas dynamics: from basic concepts to actual calculations (Vol. 21). Cambridge university press (2000).

[20] Garzó, V., Santos, A., and Brey, J. J. A kinetic model for a multicomponent gas. Physics of Fluids A: Fluid Dynamics, 1(2) (1989), 380-383.

[21] Goldman E., and Sirovich, L.: Equations for gas mixtures, Phys. Fluids, 10 (1967), 1928–1940.

[22] Greene, J. M.: Improved Bhatnagar–Gross–Krook model for electron–ion collisions, Phys. Fluids, 16 (1973), 2022–2023.

[23] Gross E. P., and Krook, M.: Model for collision processes in gases: Small–amplitude oscillations of charged two–component systems, Phys. Rev., 102 (1956), 593–604.

[24] Groppi, M., Monica, S., and Spiga, G.: A kinetic ellipsoidal BGK model for a binary gas mixture. Europhysics Letters, 96(6) (2011), 64002.

[25] Groppi, M., Rjasanow, S., and Spiga, G.: A kinetic relaxation approach to fast reactive mixtures: shock wave structure, J. Stat. Mech.-Theory Exp. (2009), P10010

[26] Groppi, M., and Spiga, G.: A BGK-type approach for chemically reacting gas mixtures, Phys. Fluids, 16(12):4273-4284, (2004).
[27] Groppi, M., and Spiga, G.: An ES-BGK model for the kinetic analysis of a chemically reacting gas mixture. MATCH Commun. Math. Comput. Chem, 69(2), (2013) 197-214.
[28] Groppi, M., and Spiga, G.: A Note on a Kinetic Relaxation Model for a Reactive Gas Mixture. MATCH-COMMUNICATIONS IN MATHEMATICAL AND IN COMPUTER CHEMISTRY, 77(2) (2017), 493-506.
[29] Haack, J. R., Hauck, C. D., Murillo, M. S. (2017). A conservative, entropic multispecies BGK model. Journal of Statistical Physics, 168(4), 826-856.
[30] Hamel, B. B.: Kinetic model for binary gas mixtures, Phys. Fluids, 8 (1965), 418–425.
[31] Hoffman, A., and Wielandt, H.:The variation of the spectrum of a normal matrix, Duke Math. J. 20 , no. 1, 37–39, (1953).
[32] Holway Jr, L. H. New statistical models for kinetic theory: methods of construction. The physics of fluids, 9(9), (1966), 1658-1673.
[33] Hwang, B. H., Lee, M. S., and Yun, S. B.: Relativistic BGK model for gas mixtures. arXiv preprint arXiv:2306.05349. (2023)
[34] Kim, D., Lee, M.-S., and Yun, S.-B: Stationary BGK models for chemically reacting gas in a slab. Journal of Statistical Physics 184.2 (2021): 1-33.
[35] Klingenberg, C., and Pirner, M.: Existence, uniqueness and positivity of solutions for BGK models for mixtures J. Differ. Equ. 264 (2018), 702–27
[36] Klingenberg, C., Pirner, M., and Puppo, G., A consistent kinetic model for a two-component mixture with an application to plasma. arXiv preprint arXiv:1806.09462. (2018)
[37] V.I.Kurochkin, S.F.Makarenko, G.A.Tirskii, Transport coefficients and the Onsager relations in the kinetic theory of dense gas mix- tures, J. Appl. Mech. Tech. Phys., 25, 218-225, 1984.
[38] Latyshev, A. V., and Yushkanov, A. A.: Analytic solution of boundary value problems for the Shakhov equation with the collision frequency proportional to the molecule velocity. Fluid dynamics, 38(4), (2003), 632-645.
[39] Morse, T. F.: Kinetic model equations for a gas mixture, Phys. Fluids, 7 (1964), 2012–2013.
[40] Monaco, R., and M. Pandolfi B.: A BGK-type model for a gas mixture with reversible reactions. New Trends in Mathematical Physics (World Scientific, Singapore, 2004).
[41] Perthame, B. Global existence to the BGK model of Boltzmann equation. J. Differential Equations 82, 1 (1989), 191–205.
[42] Perthame, B., and Pulvirenti, M. Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model. Arch. Rational Mech. Anal. 125, 3 (1993), 289–295.
[43] Y. Sone, K. Aoki and I. Yamashita, A study of unsteady strong condensation on a plane condensed phase with special interest in formation of steady profile, Rarefied Gas Dynamics (eds. V. Boffi and C. Cercignani) vol. 2, Teuber, 1986, pp.363-383.
[44] Shakhov E.M.: Generalization of the krook kinetic relaxation equation, Fluid Dyn. 3 (5) (1968) 95–96.
[45] Sirovich, L.: Kinetic modeling of gas mixtures, Phys. Fluids, 5 (1962), 908–918.
[46] Todorova, B. N., and Steijl, R, Derivation and numerical comparison of Shakhov and Ellipsoidal Statistical kinetic models for a monoatomic gas mixture. European Journal of Mechanics-B/Fluids, 76 (2019), 390-402.
[47] Ukai, S. Stationary solutions of the BGK model equation on a finite interval with large boundary data. In Proceedings of the Fourth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics (Kyoto, 1991) (1992), vol. 21, pp. 487–500.
[48] Welander, P.: On the temperature jump in a rarefied gas, Ark. Fys. 7, 507-553 (1954).
[49] Yun, S.-B. Cauchy problem for the Boltzmann-BGK model near a global Maxwellian. J. Math. Phys. 51, 12 (2010), 123514, 24.
[50] Yun, S.-B. Classical solutions for the ellipsoidal BGK model with fixed collision frequency. J. Differential Equations 259, 11 (2015), 6009–6037.
[51] Yun, S.-B. Ellipsoidal BGK model near a global Maxwellian. SIAM J. Math. Anal. 47, 3 (2015), 2324–2354
[52] Yun, S.-B. Ellipsoidal BGK model for polyatomic molecules near Maxwellians: a dichotomy in the dissipation estimate. J. Differential Equations 266, 9 (2019), 5566–5614.

[53] Zhang, X., and Hu, S. L_p solutions to the Cauchy problem of the BGK equation. J. Math. Phys. 48, 11 (2007), 113304, 17