DISCRETE COAGULATION-FRAGMENTATION SYSTEM WITH TRANSPORT AND DIFFUSION.

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Résumé. On démontre l'existence de solutions pour deux systèmes infinis d'équations de coagulation-fragmentation. Dans un premier cas, on rajoute un terme de transport au système classique de coagulation-fragmentation et dans un second cas on rajoute un terme de transport et un terme de diffusion. Dans les deux cas les particules possèdent la même vitesse que le fluide et dans le second cas les coefficients de diffusion sont égaux. On résout dans un premier temps un problème tronqué en taille puis on passe à la limite en utilisant des lemmes de compacité.

Abstract. We prove the existence of solutions to two infinite systems of equations obtained by adding a transport term to the classical discrete coagulation-fragmentation system and in a second case by adding transport and spacial diffusion. In both case, the particles have the same velocity as the fluid and in the second case the diffusion coefficients are equal. First a truncated system in size is solved and after we pass to the limit by using compactness properties.

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1. Introduction.

Coagulation and fragmentation processes describe the mecanism by which clusters can coalesce with other clusters to form a larger cluster and can fragment to form two smaller pieces. The clusters are usually identified to their size which can be a positive number in the case of a continous model or an integer in the case of a discrete model. In this paper, only discrete models will be considered. $c_i(t,x)$ will denote the concentration of clusters containing i particles (i-mers, denoted by P_i in the sequel) at time t and position x. i will be called the size variable. More precisely, the coagulation corresponds to the chemical reaction

$$P_i + P_j \to P_{i+j}.$$

Two clusters of size i and j will give a bigger cluster of size $i+j$. The velocity of this reaction is equal to $a_{i,j}c_ic_j$, where $(a_{i,j})_{(i,j)\in\mathbb{N}^*\times\mathbb{N}^*}$ is called the coagulation kernel.

The fragmentation corresponds to the inverse reaction

$$P_{i+j} \to P_i + P_j.$$

One cluster containing $i+j$ particles gives two clusters containing respectively i and j particles. The velocity of this reaction is equal to $b_{i,j}c_{i+j}$, where $(b_{i,j})_{(i,j)\in\mathbb{N}^*\times\mathbb{N}^*}$ is called the fragmentation kernel.
When the two equations are in competition, we obtain the equilibrium

$$P_i + P_j \rightleftarrows P_{i+j}.$$

The velocity $v_{i,j}$ of this reaction is $v_{i,j} = a_{i,j}c_ic_j - b_{i,j}c_{i+j}$. As a mathematical point of view, this problem has been widely studied in ([1], [2], [3]).
In the same time, these particles are in a mouving fluid and the Fick law reads ([9], [1])

$$\partial_t c_i + div(j_i) = Q_i(c), \tag{1}$$

where j_i is the flux due to the motion of the clusters of size i. The flux is decomposed into two terms as $j_i = u_ic_i - d_i\nabla c_i$. u_i is the velocity of the clusters of size i and d_i is the velocity of diffusion of the particles of size i.
The term u_ic_i is due to the displacement of the fluid and the term $d_i\nabla c_i$ is due to the diffusion of the particles in the fluid. These two terms depend on the geometry and on the velocity of the flow.
If the fluid is viscous enough, the diffusion term of the flux is preponderant compared to the term due to the transport, (1) becomes

$$\partial_t c_i - d(i)\Delta c_i = Q_i(c).$$

This case has been investigated in ([6], [7], [10]). An existence theorem is proved in ([7]) when the diffusion coefficients $d(i)$ are equal and in ([6]), the asymptotic behaviour of the solutions in time is studied. In ([11]) an existence theorem is established when the size variable is continous.
When the transport term is preponderant compared to the diffusion term, one obtains

$$\partial_t c_i + div(uc_i) = Q_i(c). \tag{2}$$

This case has been studied in ([4], [8]) for continous models.
When both phenomena are comparable, the equation (1) reads

$$\partial_t c_i + div(uc_i) - d(i)\Delta c_i = Q_i(c). \tag{3}$$

$Q_i(c)$ is a source term coming from the velocity of the reaction defined by

$$Q_i(c) = \sum_{k=1}^{i-1} a_{k,i-k}c_kc_{i-k} + 2\sum_{k=1}^{\infty} b_{k,i}c_{i+k} - 2\sum_{k=1}^{\infty} a_{k,i}c_kc_i - \sum_{k=1}^{i-1} b_{k,i-k}c_i. \tag{4}$$

The term $\sum_{k=1}^{i-1} a_{k,i-k}c_kc_{i-k}$ accounts the formation of clusters P_i by coalescence of two smaller clusters. $2\sum_{k=1}^{\infty} b_{k,i}c_{i+k}$ represents the gain of clusters P_i by fragmentation of

larger clusters. $2\sum_{k=1}^{\infty} a_{k,i}c_k c_i$ accounts the depletion of clusters P_i by coagulation with another cluster, and $\sum_{k=1}^{i-1} b_{k,i-k}c_i$ represents the fragmentation of clusters P_i into two smaller clusters.
In this paper, we will assume that the coefficients $a_{i,k}$ and $b_{i,k}$ fulfill the conditions

$$a_{i,k} = a_{k,i} > 0,\ b_{i,k} = b_{k,i} > 0,\ a_{i,k} = o(k)\ b_{i,k} = o(k), \tag{5}$$
$$\sup_k \frac{a_{i,k}}{k} < \infty,\ \sup_k \frac{b_{i,k}}{k} < \infty.$$

$u(t,x)$ is the velocity of the fluid in which are the clusters. All the clusters are supposed to have the velocity of the fluid which is assumed to be incompressible.
This paper is organized as follows. The second section is devoted to the case 2. An existence theorem is proved when all the clusters have the velocity of the fluid. The third section deals with the case 3 where an existence theorem is established.

2. The case of pure transport.

2.1. Setting of the problem.

Consider the equation for any $i \in \mathbb{N}^*$,

$$\partial_t c_i(t,x) + div(uc_i)(t,x) = Q_i(c)(t,x),\ t > 0,\ x \in \mathbb{R}^D, \tag{6}$$
$$c_i(0,x) = c_i^0(x),\ x \in \mathbb{R}^D. \tag{7}$$

where Q_i has been defined in (4). We will consider weak solutions to the problem (6-7) in the following sense

Definition 1. $(c_i)_{i\in\mathbb{N}^*}$ *is a weak solution to the problem (6-7) if for any* $i \in \mathbb{N}^*$, $c_i \in \mathcal{C}^0(\mathbb{R}_+ \times \mathbb{R}^D)$ *and*

$$-\int_{\mathbb{R}^D} c_i^0(x)\varphi(0,x)dx - \int_{\mathbb{R}_+}\int_{\mathbb{R}^D} u\, c_i(t,x)\nabla_x\varphi dsdx$$
$$-\int_{\mathbb{R}_+}\int_{\mathbb{R}^D} c_i(t,x)\partial_t\varphi dsdx = \int_{\mathbb{R}_+}\int_{\mathbb{R}^D} Q_i(c)(t,x)\varphi(t,x)dxds,$$

for each $\varphi \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^D)$ *with compact support in* $\mathbb{R}_+ \times \mathbb{R}^D$.

In this section the main result is,

Theorem 1. *Assume that the coefficients* $a_{i,k}$ *and* $b_{i,k}$ *satisfy the assumptions (5), that the initial data satifies for any* $i \in \mathbb{N}^*$

$$c_i^0 \geq 0,\ c_i^0 \in C^1(\mathbb{R}^D),\ (i,x) \mapsto \partial_x c_i^0(x) \in L^\infty(\mathbb{R}^D \times \mathbb{N}),\ \rho^0 = \sum_{i=1}^{+\infty} i c_i^0 \in L^\infty(\mathbb{R}^D),$$

and that the velocity $u \in C^1(\mathbb{R}_+ \times \mathbb{R}^D)$ is bounded, fulfills the incompressibility condition $div(u) = 0$, and is such that $\partial_x u$ is bounded.
Then, the system (6-7) has a weak solution $(c_i)_{i\in\mathbb{N}}$ in the sense of Definition 1.

As in ([7]-[14]), we shall proceed into two steps. First, a truncated problem in size will be solved by a fix point argument and in a second step we will pass to the limit.

2.2. **Resolution of a truncated problem.**

Let $N \in \mathbb{N}$ be given. In this part, the sizes greater than N are neglected by considering the following problem for $i \in \{1 \cdots N\}$.

(8) $\partial_t c_i^N(t,x) + div(uc_i^N)(t,x) = (G_i^N - P_i^N)(c^N)(t,x), \;\; t>0, \;\; x \in \mathbb{R}^D,$

(9) $c_i^N(0;x) = c_i^{0,N}(x), \; x \in \mathbb{R}^D,$

where

$$c_i^{0,N} = c_i^0 \quad \text{if} \quad i \le N \quad c_i^{0,N} = 0 \quad \text{else}$$

$$G_i^N(c) = \sum_{k=1}^{i-1} a_{k,i-k} c_k c_{i-k} + 2\sum_{k=1}^{N-i} b_{k,i} c_{i+k} \quad \text{and} \quad P_i^N(c) = \nu_i^N(c) c_i \tag{10}$$

with

$$\nu_i^N = \sum_{k=1}^{i-1} b_{k,i-k} + 2\sum_{k=1}^{N-i} a_{k,i} c_{k,i}.$$

Proposition 1. *Under the hypotheses of Theorem 1, the system (8-9) possesses a unique solution on $\mathbb{R}_+ \times \mathbb{R}^D$.*

The solution of the truncated problem (8-9) will be the fix point of a mapping Γ. First, the following Lemma whose the proof is given in ([7]) will be used,

Lemma 1. *For all N-uple $\{c_1.....c_N\}$ such that, for all i, $c_i \geqslant 0$, it holds that*

$$\nu_i^N(c) \ge 0, \quad G_i^N(c) \ge 0, \quad |\nu_i^N| \le A_N \sup_{i=1..N} c_i + B_N,$$
$$|G_i^N(c)| \leqslant (\sup_{i=1..N} c_i^N)^2 A_N + B_N \sup_{i=1..N} c_i^N.$$

Consider Y_N the solution to the Cauchy problem

$$\frac{dY_N}{dt} = A_N Y_N^2 + B_N Y_N \tag{11}$$
$$Y_N(0) = R_0 \qquad t > 0 \tag{12}$$

Let T_N be the time of existence of the solution and $T \in]0, T_N[$. Consider the space

$$E = \{c \in [C^0([0;T] \times \mathbb{R}^D)]^N \cap (L^\infty([0,T] \times \mathbb{R}^d))^N; c \ge 0;$$
$$\|c(t,.)\|_{[L^\infty(\mathbb{R}^d)]^N} \le Y_N(t), t \in [0,T]\}.$$

Consider the following iteration

$$\partial_t d_i^{n+1}(t,x) + u(t,x).\nabla_x d_i^{n+1}(t,x) = [G_i^N(d_i^n) - \nu(d_i^n) d_i^{n+1}](t,x), \tag{13}$$

$$d_i^{n+1}(0,x) = c_i^0(x). \tag{14}$$

Let Γ be the map defining this iteration.

Lemma 2. *For the same assumptions as for Lemma 1, E is stable by Γ.*

Proof of Lemma 2.
The system (8-9) is solved on a caracteristic. Consider the vector field $X(s,t,x)$ satisfying

$$\partial_s X(s,t,x) = u(s, X(s,t,x)), \tag{15}$$

$$X(t,t,x) = x. \tag{16}$$

Then, the solution to the system (8-9) writes for any $i \in \{1 \cdots N\}$.

$$c_i^N(s, X(s,t,x)) = c_i^0(X(0,t,x)) + \int_0^t [G_i^N(c^N) - \nu_i^N c_i^N](s, X(s,t,x))ds. \tag{17}$$

So, by considering the quantity

$$d_i^{n+1}(\tau, X(\tau,t,x)) \exp\Big(\int_0^\tau \nu_i^N(d^n)(s, X(s,t,x))ds\Big),$$

the solution to the system (13-14) reads for any $i \in \{1 \cdots N\}$

$$\begin{aligned} d_i^{n+1}(t,x) &= c_i^0(X(0,t,x)) \exp(-\int_0^t \nu_i^N(d^n)(s, X(s,t,x))ds \\ &+ \int_0^t \exp(-\int_s^t \nu_i^N(d^n)(\sigma, X(\sigma,t,x))d\sigma)) G_i^N(d_i^n(s, X(s,t,x))ds. \end{aligned} \tag{18}$$

Then, by continuity of d_i^n, d_i^{n+1} is also continous. Moreover, the nonnegativity of c_i^0 and $G_i^N(d^n)$ implies according to (18) the nonnegativity of d_i^{n+1}. On the other hand, as $\nu_i^N \geq 0$, (18) leads to

$$d_i^{n+1}(t,x) \leq c_i^0(X(0,t,x)) + \int_0^t G_i^N(d^n)(s, X(s,t,x))ds.$$

But, as $d^n \in E$ and by using Lemma 1, it holds that

$$d_i^{n+1}(t,x) \leq c_i^0(X(0,t,x)) + \int_0^t \frac{d}{ds} Y^N(s)ds.$$

So, finally, we get that $d_i^{n+1}(t,x) \leqslant Y^N(t)$. $\square$

We are going to show that this map is a contraction for the norm

$$|||c||| = \sup_{s\in[0,T]} e^{-\omega_N s} |||c(s,)||_{[L^\infty(\mathbb{R}^d)]^N}$$

where ω_N is a constant which shall be chosen big enough so that Γ is a contraction and whose the choice shall be precized during the proof of the following Lemma.

Lemma 3. *There is a nonnegative constant ω_N depending only on N such that the mapping Γ is a contraction from E into itself for the norm $|||\quad|||$.*

Proof of lemma 3
By considering two consecutive terms of the iteration (13-14) and by substracting them, it holds that

$$\begin{aligned}\partial_t[d_i^{n+1}-d_i^n](t,x)+u(t,x).\nabla_X(d_i^{n+1}-d_i^n)(t,x)+\nu^N(d_i^n)(t,x)(d_i^{n+1}-d_i^n)(t,x)\\=[G_i^N(d^n)-G_i^N(d^{n-1})](t,x)+d_i^n(t,x)(\nu_i^N(d_i^{n-1})-\nu_i^N(d_i^n))(t,x).\end{aligned}$$

So, $(d_i^{n+1}-d_i^n)(t,x)$ writes

$$\begin{aligned}(d_i^{n+1}-d_i^n)(t,x)=\int_0^t[G_i^N(d^n)-G_i^N(d^{n-1})](\tau,X(\tau,t,x))d\tau\\+\int_0^t d_i^n(\tau,X(\tau,t,x))[\nu_i^N(d^{n-1})-\nu_i^N(d^n)](\tau,X(\tau,t,x))\\\exp(-\int_\tau^t\nu_i^N(d^n)(s,X(s,t,x))dsd\tau.\end{aligned}$$

According to the expression of G_i^N and P_i^N, there exists a nonnegative constant $C(N,T)$ such that

$$\begin{aligned}|G_i^N(c)-G_i^N(d)|\le C(N,T)\|c-d\|_{[L^\infty(\Omega)]^N},\\|P_i^N(c)-P_i^N(d)|\le C(N,T)\|c-d\|_{[L^\infty(\Omega)]^N}.\end{aligned}\tag{19}$$

So,

$$|||(d_i^{n+1}-d_i^n)|||\leqslant C(N,T)|||d_i^n-d_i^{n-1}|||[\frac{1-e^{-\omega_N t}}{\omega_N}].$$

Hence, by choosing ω_N big enough such that $k=\frac{2C(N,T)}{\omega_N}<1$, the result is proved. □

For the proof of Proposition 1, the following Lemma will be used

Lemma 4. *For G_i^N and P_i^N defined in (10), it holds that* $\sum_{i=1}^N[iG_i^N(c)-iP_i^N(c)]=0$.

For the proof, see ([7]). □

Proof of Proposition 1.
From previously, $(d_i^n)_{n\in\mathbb{N}}$ is a converging sequence in E and Γ is a contraction. Therefore the problem (8-9) has a unique solution on $[0,T]\times\mathbb{R}^d$ for $T\in]0;T_N[$.
In order to get a global solution on $\mathbb{R}_+\times\mathbb{R}^d$, consider

$$\rho^N=\sum_{i=1}^N ic_i^N,\quad \rho=\sum_{i=1}^{+\infty}ic_i\tag{20}$$

which represents total mass of particles which react.
Multiply in (8-9), the equation number i by i, sum on i until N and use Lemma 4, leads to

$$\partial_t \rho^N(t,x) + u(t,x).\nabla_x \rho^N(t,x) = 0, \tag{21}$$

$$\rho^N(0,x) = \rho_0^N(x) = \sum_{i=1}^{N} i c_i^0(x). \tag{22}$$

So, by considering the vector field $X(s,t,x)$ defined by the system (15-16), it holds that

$$0 \leq \rho^N(t,x) = \rho_0^N(X(0,t,x)) \leq \rho_0(X(0,t,x)) \leq \|\rho^0\|_{L^\infty(\mathbb{R}^D)}. \tag{23}$$

Hence, $(\forall i \in \{1;N\}), c_i^N \leq \|\rho^0\|_{L^\infty(\mathbb{R}^D)}$. Next, we choose $R_0 = \|\rho_0\|_{L^\infty(\mathbb{R}^D)}$ and we solve the equation (6) for any $i \in \{1....N\}$ on $[T,2T]$ with the Cauchy data equal to $c_i(T,\cdot)$. Then, a reiteration of this process gives global existence of the solution on $\mathbb{R}_+ \times \mathbb{R}^D$.$\square$

2.3. **Solution of the problem.**

The aim is now to pass to the limit when N tends to $+\infty$ in the system (8-9).

Proposition 2. *For any compact set $[0;T] \times K$ of $\mathbb{R}_+ \times \mathbb{R}^D$ and for any $i \in \mathbb{N}$, the sequence $(c_i^N)_{N\in\mathbb{N}}$ is strongly compact in $C^0([0;T] \times K)$.*

Proof of Proposition 2.
In order to apply the Ascoli theorem, we shall control $\partial_t c_i^N$ and $\partial_x c_i^N$. Consider $\overline{B}_R$ the closed ball of $\mathbb{R}^D$ with a radius equal to $R > 0$. Hence, by differentiating the relation (17) with respect to the space variable x_j, it holds that

$$\begin{aligned} \partial_{x_j}(c_i^N)(t,x) &= \partial_X[c_i^0(X(0,t,x))] \\ &+ \int_0^t \partial_{x_j}[Q_i^N(c)](s,X(s,t,x))\partial_{x_j}X(s,t,x)ds. \end{aligned} \tag{24}$$

Now, in order to estimate $\partial_t X(s,t,x)$ and $\partial_x X(s,t,x)$, let us show the following Lemma.

Lemma 5. *If $u \in C^1$ is bounded on $[0;T] \times \mathbb{R}^D$ and if $\partial_x u$ is bounded on $[0;T] \times \mathbb{R}^D$ then $\partial_x X$ and $\partial_t X$ are bounded on $[0;T]^2 \times \mathbb{R}^D$.*

Proof of Lemma 5.
By integrating (15), X writes

$$X(s,t,x) = x + \int_t^s u(\sigma, X(\sigma,t,x))d\sigma. \tag{25}$$

So by differentiating (25) with respect to the space variable x and by using that $\partial_x u$ is bounded on $[0,T] \times \mathbb{R}^D$, there exists $M > 0$ such that

$$|\partial_x X(s,t,x)| \leq 1 + M(\int_s^t |\partial_x X(\sigma,t,x)|d\sigma).$$

So, according to the Gronwall lemma, $\partial_x X(s,t,x)$ is bounded on $[0;T]^2 \times \mathbb{R}^D$. Analogously, the same result holds for $\partial_t X(s,t,x)$. $\square$

End of the proof Proposition 2.
In order to control the term $\partial_{x_j} Q_i$ consider the quantity

$$\partial_x [G_i^N(c^N)] = \sum_{k=1}^{i-1} a_{k,i-k} \Big(\partial_x c_k^N \, c_{i-k}^N + c_k^N \, \partial_x c_{i-k}^N \Big) + 2 \sum_{k=1}^{N} b_{k,i} \partial_x c_{i+k}^N. \tag{26}$$

A bound on c_i^N independent of N is first researched in $L^\infty((0,T);\mathbb{R}^D)$.
As $\rho_0^N = \displaystyle\sum_{i=1}^{N} i c_i^0$ and $\rho_0 = \displaystyle\sum_{i=1}^{\infty} i c_i^0$, with $\rho_0 \in L^\infty(\mathbb{R}^D)$, it holds that

$$c_i^N \leqslant \|\rho_0^N\|_{L^\infty(\mathbb{R}^D)} \leqslant \|\rho_0\|_{L^\infty(\mathbb{R}^D)} \leqslant M_0. \tag{27}$$

Now, let us control the term $\displaystyle\sum_{k=1}^{i-1} a_{k,i-k} \partial_x c_k^N(t,x) c_{i-k}^N(t,x)$ of the right-hand side of the equation (26).

$$\Big| \sum_{k=1}^{i-1} a_{k,i-k} \partial_x c_k^N(t,x) c_{i-k}^N(t,x) \Big| \le \sum_{k=1}^{i-1} \frac{a_{k,i-k}}{i-k} |\partial_x c_k^N(t,x)| (i-k) c_{i-k}^N(t,x)$$

From assumption (5), there is $A > 0$ such that

$$\sup_{k \in \{1, i-1\}} \Big| \frac{a_{k,i-k}}{i-k} \Big| \le A.$$

Hence,

$$\Big| \sum_{k=1}^{i-1} a_{k,i-k} \partial_x c_k^N(t,x) c_{i-k}^N(t,x) \Big| \le A \sup_{i \in \mathbb{N}} \sup_{x \in \mathbb{R}^D} |\partial_x c_k^N(t,x)| \sum_{k=1}^{i-1} (i-k) c_{i-k}^N.$$

But, by definition of ρ^N, it comes that

$$\Big| \sum_{k=1}^{i-1} (i-k) c_{i-k}^N \Big| \le \rho^N.$$

Then (23) leads to

$$\Big| \sum_{k=1}^{i-1} (i-k) c_{i-k}^N \Big| \le \|\rho\|_{L^\infty(\mathbb{R}^D)}.$$

Therefore, by using (27), we get the estimate

$$\Big| \sum_{k=1}^{i-1} a_{k,i-k} \partial_x c_k^N(t,x) c_{i-k}^N(t,x) \Big| \le A M_0 \big(\sup_{i \in \mathbb{N}} \sup_{x \in \mathbb{R}^D} |\partial_x c_i^N(t,x)| \big).$$

where M_0 is a nonnegative constant. In the same way, we obtain

$$|\sum_{k=1}^{i-1} a_{k,i-k}\partial_x c_{i-k}^N(t,x)c_k^N(t,x)| \leq AM_0(\sup_{i\in\mathbb{N}}\sup_{x\in\mathbb{R}^D}|\partial_x c_i^N(t,x)|),$$

$$|\sum_{k=1}^{N-1} b_{k,i}\partial_x(c_{i+k}^N(t,x))| \leq B(\sup_{i\in\mathbb{N}}\sup_{x\in\mathbb{R}^D}|\partial_x c_i^N(t,x)|),$$

where B is a constant independent of the variables k,i,t,x. So finally, there exists a constant C independent of the variables t,x,i,N such that

$$|\partial_x[G_i^N(c^N)](t,x)| \leqslant C(\sup_{i\in\mathbb{N}}\sup_{x\in\mathbb{R}^D}|\partial_x c_i^N(t,x)|).$$

The same result holds for $P_i^N(c^N)$. So,

$$\sup_{x\in\mathbb{R}^D}\|\partial_x[c_i^N(t,x)]\|_{L^\infty(\mathbb{R}^D)} \leq C+\alpha\int_0^t \sup_{x\in\mathbb{R}^D}\|\partial_x[c_i^N(s,x)]\|_{L^\infty(\mathbb{R}^D)}ds,$$

where α and C are nonnegative constants independent of the variables t,x,N. From the Gronwall lemma, it holds that

$$(\forall i\in\mathbb{N}^*),(\forall t\in[0;T]),\|\partial_x c_i^N(t,x)\|_{L^\infty(\mathbb{R}^D)} \leq Ce^{\alpha t}.$$

By reasonning in the same way, we can prove that $\partial_t c_i^N$ is also controled. Then, by using the Ascoli Theorem, the sequence $(c_i^N)_{N\in\mathbb{N}}$ is strongly compact in $C^0([0;T]\times\overline{B}_R)$ for any i. □

Proof of Theorem 1.
By using a diagonal process, there is a subsequence $c^{\phi(N)}$ of c^N such that $(\forall i\in\mathbb{N})$, $c_i^{\phi(N)}$ is converging to a continous function c_i uniformly on all compact set of the form $[0;T]\times\overline{B}_R$. By arguing as in ([7]), it holds that $G_i^N(c^N)$ (resp. $P_i^N(c^N)$) converges to $G_i(c)$ (resp. $P_i(c)$) uniformly on $[0;T]\times\overline{B}_R$. So, we can pass to the limit in the weak form of (6-7). □

3. The case of transport and diffusion.

3.1. **Setting of the problem.**

In this section, consider the problem

$$\partial_t c_i(t,x)+div(uc_i)(t,x)-\Delta c_i(t,x)=Q_i(c)(t,x) \tag{28}$$

$$c_i(0;x)=c_i^0(x),\ x\in\Omega, \tag{29}$$

$$\frac{\partial c_i}{\partial\eta}(t,\sigma)=0\quad t>0,\ \sigma\in\partial\Omega, \tag{30}$$

where Ω is an open bounded set of class C^1 with $\partial\Omega$ as boundary and where $Q(c)_i$ has been defined in (4). It corresponds to the case (3) when the diffusion coefficients $d(i)$ are taken equal to one. The solutions of the problem (28-29-30) will be considered in the following sense.

Definition 2. *$(c_i)_{i\in\mathbb{N}}$ is a weak solution to the problem (28-29-30) if for any $i \in \mathbb{N}^*$ and any $T > 0$, $c_i \in L^2([0,T]\times\Omega)$ and*

$$-\int_{\mathbb{R}^D} c_i^0(x)\varphi(0,x)dx - \int_{\mathbb{R}_+}\int_{\mathbb{R}^D} c_i(t,x)\partial_t\varphi(t,x)dx$$
$$-\int_{\mathbb{R}_+}\int_{\mathbb{R}^D} c_i(s,x)\,u\cdot\nabla_x\varphi(s,x)dxds - \int_{\mathbb{R}_+}\int_{\mathbb{R}^D}\nabla_x c_i(s,x)\cdot\nabla_x\varphi(s,x)dxds$$
$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}^D} Q_i(c)\varphi(s,x)dxds$$

for all $\varphi \in \mathcal{C}^1(\mathbb{R}_+\times\mathbb{R}^D)$ with compact support in $\mathbb{R}_+\times\mathbb{R}^D$.

The main result of this part is

Theorem 2. *Let Ω be an open and bounded set of class C^1. Assume that the kinetic coefficients $(a)_{(i,j)\in\mathbb{N}^*\times\mathbb{N}^*}$ and $(b)_{(i,j)\in\mathbb{N}^*\times\mathbb{N}^*}$ satisfy the assumptions (5), that the initial data satisfies*

$$c_i^0 \geq 0,\ i\in\mathbb{N}^*,\ c_i^0 \in L^2(\mathbb{R}_+\times\Omega),\ \rho^0 = \sum_{i=1}^{+\infty} ic_i^0 \in L^\infty(\Omega).$$

and that the velocity of the fluid $u \in H^1(\mathbb{R}_+; H_0^1(\Omega)\cap L^\infty(\Omega))$ fulfills the incompressibility condition $div(u) = 0$.
Therefore the system (28-29-30) has a weak solution on $\mathbb{R}_+\times\mathbb{R}^D$ in the sense of Definition 2.

As previously, we shall proceed into two parts. First, we shall solve a truncated problem and after we will pass to the limit.

3.2. **Resolution of a truncated problem.**

Let $N \in \mathbb{N}$ be given. The sizes greater than N are removed, by considering the following problem for any $i \in \{1\cdots N\}$,

$$\partial_t c_i^N + u\cdot\nabla_x c_i^N - \Delta c_i^N = Q_i(c^N), \tag{31}$$
$$c_i^N(0;x) = c_i^{0,N}(x) \quad i\in\mathbb{N},\quad t>0,\quad x\in\Omega, \tag{32}$$
$$\frac{\partial}{\partial\eta}c_i^N(t,\sigma) = 0,\quad t>0,\quad \sigma\in\partial\Omega. \tag{33}$$

Proposition 3. *Under the assumptions of Theorem 2, the problem (31-32-33) has a unique solution defined on $\mathbb{R}_+\times\mathbb{R}^D$.*

Consider the following iteration for any $i \in \{1\cdots N\}$,

$$\partial_t d_i^{n+1} - \Delta d_i^{n+1} + u(t,x)\cdot\nabla_x d_i^{n+1} + \nu_i^N(d_i^n)d_i^{n+1} = G_i^N(d^n), \tag{34}$$
$$d_i^{n+1}(0;x) = c_i^0(x), \tag{35}$$
$$\frac{\partial}{\partial\eta}d_i^{n+1}(t,\sigma) = 0,\ t>0,\ \sigma\in\partial\Omega. \tag{36}$$

Let S be the mapping defining this iteration.

Lemma 6. *If for any $i \in \{1...N\}$, $c_i \geq 0$, then $S(c)_i \geq 0$.*

Proof of Lemma 6.
Consider $c_i \geq 0$ and put $d_i = S(c)_i$. Consider $f \in C^1(\mathbb{R})$ such that f is nondecreasing on $]0;+\infty[$ and $f(t) = 0$ for $t \in \mathbb{R}_-$. Multiply the equation (34) satisfied by d_i by $f(-d_i)$ and integrate on $[0;t] \times \Omega$ leads to $(\forall t \in [0;T])$,

$$\int_0^t \int_\Omega (\partial_t d_i)\, f(-d_i)\, dxdt - \int_0^t \int_\Omega \Delta d_i\, f(-d_i) dxds + \int_0^t \int_\Omega u \cdot \nabla_x d_i\, f(-d_i) dxdt$$

$$+ \int_0^t \int_\Omega \nu_i^N(c)\, d_i\, f(-d_i) = \int_0^t \int_\Omega G_i^N(c) f(-d_i) dxds. \tag{37}$$

But, from the Green formula and by using (36), it holds that

$$- \int_0^t \int_\Omega \Delta d_i\, f(-d_i) dxdt = \int_0^t \int_\Omega \nabla_x d_i \cdot \nabla_x f(-d_i) dxdt.$$

Let F be the primitive function of f such that $F(0) = 0$. So, F is nondecreasing on $]0;+\infty[$ and is identically 0 on $]-\infty;0]$. Hence, we get

$$\int_0^t \int_\Omega u_j(s,x) \partial_{x_j} d_i(s,x) f(-d_i)(s,x) dxds = - \int_0^t \int_\Omega u_j(s,x) \partial_{x_j} F(-d_i)(s,x) dsdx.$$

So, by using the Green formula, it comes that

$$\begin{aligned} \int_\Omega u_j(s,x) \partial_{x_j} d_i(s,x) f(-d_i)(s,x) dxds &= - \int_\Gamma u_j(s,\sigma) F(-d_i)(s,\sigma) d\sigma \\ &+ \int_\Omega \frac{\partial u_j}{\partial x_j} F(-d_i)(s,x) dx. \end{aligned}$$

But, as $u \in H_0^1(\Omega)$ and $div(u) = 0$, we get

$$\int_0^t \int_\Omega u \cdot \nabla_x d_i\, f(-d_i) dxdt = 0.$$

So, (37) gives the inequality,

$$\int_\Omega F(-d_i)(x,v) dx - \int_\Omega F(-d_i)(0,v) dx + \int_0^t \int_\Omega f'(-d_i) |\nabla_x d_i|^2 dxdt$$

$$+ \int_0^t \int_\Omega \nu_i^N(-d_i)(s,x)\, (-d) f(-d_i)(s,x) dxds \leq 0. \tag{38}$$

But, as $d_i(0;x) = c_i^0(x)$, F is identically 0 on $]-\infty;0]$ and $F(-d)(0;x) = 0$. Moreover, $\nu_i^N(c)$ and $K(x) = xf(x)$ are nonnegative quantities. So,

$$\int_0^t \int_\Omega \nu^N(-d)(s,x)(-d) f(-d)(s,x) dxds \geq 0.$$

f being nondecreasing $f'(-d_i) \geq 0$, we get from (38),

$$\int_\Omega F(-d_i)(t,x) dx \leq 0.$$

$F(-d_i)$ being nonnegative, we obtain that $F(-d_i)$ is zero *a.e.* $F(-d_i)$ being continous, $F(-d_i)$ is zero on $[0;T] \times \mathbb{R}^D$. So, by definition of F, $d \geq 0$ on $[0;T] \times \Omega$. $\square$

A bound in L^∞ on the sequence d_i^n is now researched. It is given by the following lemma.

Lemma 7. *Let $d = S(c)$. Then for any $i \in \{1, N\}$, d_i satisfies*

$$\begin{aligned} \|d_i(t,\cdot)\|_{L^\infty(\Omega)} &\leq \|c_i^0\|_{L^\infty(\Omega)} \\ &+ \int_0^t [A_N(\|c(s,\cdot)\|_{L^\infty(\Omega)^N})^2 + B_N\|c(s,\cdot)\|_{L^\infty(\Omega)^N}]ds. \end{aligned} \tag{39}$$

Proof of lemma 7.
Let us put

$$f(t) = \int_0^t [A_N(\|c(s,\cdot)\|_{L^\infty(\Omega)^N})^2 + B_N\|c(s,\cdot)\|_{L^\infty(\Omega)^N}]ds.$$

Consider now a function G such that $G \in C^1(\mathbb{R})$, G is nondecreasing on $]0;+\infty[$, $(\forall s \leq 0)$, $G(s) = 0$. Let us put $K_i = \|c_i^0\|_{L^\infty(\Omega)}$, $H(s) = \int_0^s G(\sigma)d\sigma$. For any $i \in \{1, N\}$, introduce the function φ_i defined by

$$\varphi_i(t) = \int_\Omega H(d_i(t,x) - K_i - \int_0^t f(s)ds)dx.$$

For any $i \in \{1, N\}$ φ_i satisfies $\varphi_i(0) = 0$, $\varphi_i \in C^1(]0;+\infty[;\mathbb{R})$ $\varphi_i \geq 0$. Derive φ_i with respect to the time variable t leads to

$$\varphi_i'(t) = \int_\Omega G[d_i(t,x) - K_i - \int_0^t f(s)ds](\partial_t d_i(t,x) - f(t))dx.$$

As $\nu_i^N(d_i^n)d_i^{n+1} \geq 0$, it holds that

$$\partial_t d_i(t,x) - f(t) \leq \Delta d_i(t,x) - \sum_{j=1}^D u_j(t,x)\partial_{x_j} d_i(t,x).$$

So, we get

$$\begin{aligned} \varphi_i'(t) &\leq \int_\Omega G(d_i(t,x) - K_i - \int_0^t f(s)ds)\Delta d_i(t,x)dx \\ &- \sum_{j=1}^D \int_\Omega u_j(t,x)\partial_{x_j} d_i(t,x) G(d_i(t,x) - K_i - \int_0^t f(s)ds)dx. \end{aligned} \tag{40}$$

The definition of H implies that

$$\partial_{x_j} H\Big(d_i(t,x) - K - \int_0^t f(s)ds\Big) = \partial_{x_j} d_i(t,x) G\big(d_i(t,x) - K - \int_0^t f(s)ds\big).$$

By using the Green formula, $u \in H_0^1(\Omega)$ and $div(u) = 0$, it comes that

$$\sum_{j=1}^{D} \int_\Omega u_j(t,x)\partial_{x_j} H[d_i(t,x) - K - \int_0^t f(s)ds]dx = 0.$$

Moreover from the Green formula, it holds that

$$\int_\Omega G(d_i(t,x) - K_i - \int_0^t f(s)ds)\Delta d_i(t,x)dx$$
$$= \int_\Omega |\nabla_x d_i(t,x)|^2 G'\big(d_i(t,x) - K - \int_0^t f(s)ds\big)dx.$$

So, G' being nondecreasing, (40) leads to

$$\varphi_i'(t) \leq -\int_\Omega |\nabla_x d_i(t,x)|^2 G'((d_i(t,x) - K - \int_0^t f(s)ds)dx < 0.$$

Hence, φ_i is nonincreasing on $\mathbb{R}_+$. As, φ_i is nonnegative and satisfies $\varphi_i(0) = 0$, φ_i yields 0 everywhere. So, by definition of H, (39) holds. $\square$

We shall use an analogous method as in the previous part. Consider the Cauchy problem

$$\frac{dY_N}{dt} = A_N Y_N^2 + B_N Y_N, \tag{41}$$

$$Y_N(0) = R_0, \quad t > 0. \tag{42}$$

Let T_N be the time of existence of Y_N and consider $T \in [0; T_N]$. Define the space

$$E = \{c \in (L^\infty([0,T] \times \mathbb{R}^d))^N; c \geqslant 0; \|c(t,.)\|_{[L^\infty(\mathbb{R}^d)]^N} \leqslant Y_N(t), t \in [0,T]\}.$$

Lemma 8. *E is stable by S.*

<u>Proof of Lemma 8.</u>
Let $c \in E$ and $d = S(c)$. From Lemma 6, for any $i \in \{1...N\}$, $d_i \geq 0$. As $c \in E$, for any $i \in \{1, N\}$, $c_i(t,x) \leq Y_N(t)$. Then, from the maximum principle applied to the equation (34), it holds that

$$\|d_i(t,\cdot)\|_{L^\infty(\Omega)} \leq \|c_i^0\|_{L^\infty(\Omega)} + \int_0^t [A_N(\|c(s,\cdot)\|_{L^\infty(\Omega)^N})^2 + B_N\|c(s,\cdot)\|_{L^\infty(\Omega)^N}]ds.$$

As $c \in E$, the equation (41) leads to

$$\|d_i(t,\cdot)\|_{L^\infty(\Omega)} \leq \|c_i^0\|_{L^\infty(\Omega)} + \int_0^t \frac{dY_N(t)}{dt}(s)ds.$$

By choosing $R_0 > \|c_i^0\|_{L^\infty(\Omega)}$, the result follows. $\square$

E is equipped with the norm

$$|||c||| = \sup_{s\in[0;T]} |e^{-\omega_N s}\|c(s,\cdot)\|_{[L^2(\Omega)]^N}$$

and the constant ω_N will be chosen so that S is a contraction for this norm.

Lemma 9. *There is a constant ω_N depending only on N such that S is a contraction from E into itself for the norm $|||\quad|||$.*

<u>Proof Lemma 9</u>
Substract two consecutive terms of the iteration, multiply the last equation by $(d_i^{n+1}-d_i^n)(t,x)$ and integrate on $[0,T]\times\Omega$ leads to

$$\begin{aligned}
&\int_0^t\int_\Omega(d_i^{n+1}-d_i^n)\partial_t(d_i^{n+1}-d_i^n)dxds-\int_0^t\int_\Omega(d_i^{n+1}-d_i^n)\Delta(d_i^{n+1}-d_i^n)dxds\\
&+\int_0^t\int_\Omega(d_i^{n+1}-d_i^n)\,u\cdot\nabla_x(d_i^{n+1}-d_i^n)dxds+\int_0^t\int_\Omega\nu_i^N(d_i^n)(d_i^{n+1}-d_i^n)dxds\\
&\qquad=\int_0^t\int_\Omega(d_i^{n+1}-d_i^n)[G_i^N(d_i^n)-G_i^N(d_i^{n-1})]dxds\\
&\qquad+\int_0^t\int_\Omega d_i^n\,(d_i^{n+1}-d_i^n)[\nu^N(d^{n-1})-\nu^N(d^n))]dxds.
\end{aligned}\tag{43}$$

But, as $d_i^{n+1}(0,x)=d_i^n(0,x)=c_i^0(x)$, it holds that

$$\int_0^t\int_\Omega(d_i^{n+1}-d_i^n)\partial_t(d_i^{n+1}-d_i^n)(s,x)dxds=\frac{1}{2}\int_\Omega(d_i^{n+1}-d_i^n)^2(t,x)dx.$$

The Green formula implies

$$-\int_\Omega(d_i^{n+1}-d_i^n)(s,x)\Delta(d_i^{n+1}-d_i^n)(s,x)dx=\int_\Omega|\nabla_x[d_i^{n+1}-d_i^n](s,x)|^2dx.$$

On the other hand, by using again the Green formula, $u(t,\cdot)\in H_0^1(\Omega)$ and $div(u)=0$, it comes that

$$\int_0^t\int_\Omega u\cdot\nabla_x(d_i^{n+1}-d_i^n)(d_i^{n+1}-d_i^n)dxdt=0.$$

From (43), we get that

$$\begin{aligned}
&\int_0^t\int_\Omega[\nabla_x(d_i^{n+1}-d_i^n)(s,x)]^2dxdt+\frac{1}{2}\int_\Omega(d_i^{n+1}-d_i^n)^2(t,x)dx\\
&\quad=\int_0^t\int_\Omega(d_i^{n+1}-d_i^n)(s,x)[G_i^N(d_i^n)-G_i^N(d_i^{n-1})](s,x)dxds\\
&+\int_0^t\int_\Omega d_i^n\,(d_i^{n+1}-d_i^n)(s,x)[\nu^N(d^{n-1})-\nu^N(d^n))](s,x)dxds.
\end{aligned}$$

Which leads to the inequality

$$\begin{aligned}
\frac{1}{2}\int_\Omega(d_i^{n+1}-d_i^n)^2(t,x)dx\quad&\le\quad\int_0^t\int_\Omega(d_i^{n+1}-d_i^n)[G_i^N(d_i^n)-G_i^N(d_i^{n-1})](s,x)dxds\\
&+\quad\int_0^t\int_\Omega d_i^n(d_i^{n+1}-d_i^n)(s,x)[\nu^N(d^{n-1})-\nu^N(d^n))](s,x)dxds.
\end{aligned}$$

According to (19) and as $d_i^n(t,x) \leq Y_N(t)$, we get

$$\int_\Omega (d_i^{n+1} - d_i^n)^2(t,x)dx \leq C(N,T)\int_0^t \int_\Omega \sup_{j\in\{1..N\}} [|d_j^n - d_j^{n-1}|](s,x)[d_i^n - d_i^{n-1}](s,x)dxds.$$

The Young inequality gives that

$$\begin{aligned}\int_\Omega (d_i^{n+1} - d_i^n)^2(t,x)dx &\leq \frac{C(N,T)}{2}\int_0^t \int_\Omega (\sup_{j\in\{1..N\}} [|d_j^n - d_j^{n-1}|](s,x))^2 dxds \\ &+ \frac{C(N,T)}{2}\int_0^t \int_\Omega ([d_i^n - d_i^{n-1}](s,x))^2 dxds.\end{aligned}$$

From the Gronwall Lemma, it holds that

$$\int_\Omega (d_i^{n+1} - d_i^n)^2(t,x)dx \leq \tilde{C}(N,T)(\int_0^t \int_\Omega \sup_{j\in\{1..N\}} [|d_j^n - d_j^{n-1}|]^2(s,x))dsdx.$$

Multiply the last inequality by $e^{-\omega_N t}$ leads to

$$\begin{aligned}e^{-\omega_N t}\int_\Omega (d_i^{n+1} - d_i^n)^2(t,x)dx &\leq e^{-\omega_N t}\tilde{C}(N,T)\frac{1-e^{\omega_N t}}{\omega_N} \\ &\sup_{s\in[0;T]} e^{-\omega_N s}\int_\Omega \sup_{i\in 1..N}(d_i^{n+1} - d_i^n)^2(s,x)dx.\end{aligned}$$

So, by putting $k = \dfrac{2\tilde{C}(N,T)}{\omega_N}$ and by choosing ω_N big enough so that $k < 1$, the result holds. $\square$

Proof of Proposition 3.
From lemma 9, for any $i \in \{1 \cdots N\}$, $(d_i^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E and so converges in E to the fixed point of S. Moreover, S being a contraction, this solution is unique on $[0;T]\times\Omega$. In order to get a global solution in time, we shall proceed as in section 2 by considering ρ and ρ^N defined in (20). ρ^N is solution to

$$\partial_t \rho^N(t,x) - \Delta\rho^N(t,x) + u(t,x)\cdot\nabla_x\rho^N(t,x) = 0, \tag{44}$$

$$\rho^N(0;x) = \rho_0^N(x) = \sum_{i=1}^N ic_i^0(x) \quad x\in\mathbb{R}^D, \tag{45}$$

$$\frac{\partial\rho^N}{\partial\eta}(t,\sigma) = 0, \quad t\in[0;T], \quad \sigma\in\partial\Omega. \tag{46}$$

The maximum principle proved in Lemma 7 yields

$$\rho^N(t,x) \leq \|\rho_0^N\|_\infty \leq \|\rho_0\|_\infty \leq M_0. \tag{47}$$

We obtain by definition of c_i^N, $c_i^N(t,x) \leq \|\rho_0\|_\infty$. Next, we choose $R_0 = \|\rho_0\|_\infty$ and we solve (31) on $[T,2T]$, with the Cauchy data at $t = T$ equal to $c_i(T,.)$. Then, a reiteration of this process gives global existence and uniqueness of the solution on $\mathbb{R}_+\times\mathbb{R}^D$.$\square$

3.3. **Solution of the problem.**

In order to pass to the limit in the truncated problem, we need to obtain compactness on the sequence of approximations. This is given by the following proposition.

Proposition 4. *For any $i \in \mathbb{N}$ and for any $T > 0$, the sequence $(c_i^N)_{N\in\mathbb{N}}$ is strongly compact in $L^2([0;T];L^2(\Omega))$.*

First, let us show that

Lemma 10. *For $i \in \mathbb{N}^*$ and for any $T > 0$, c_i^N is bounded in $L^2([0;T];H^1(\Omega))$.*

Proof of Lemma 10

Multiply equation (31) by $c_i^N(t,x)$ and integrate on $[0;T]\times\Omega$ leads to

$$\int_\Omega\int_0^T \frac{d}{dt}c_i^N(t,x)dtdx + \int_\Omega\int_0^T u\cdot\nabla_x c_i^N(t,x)c_i^N(t,x)dtdx$$

$$-\int_\Omega\int_0^T \Delta c_i^N(t,x)c_i^N(t,x)dtdx = \int_\Omega\int_0^T Q^N(c_i^N)(t,x)c_i^N(t,x)dtdx. \tag{48}$$

The boundary condition (33) gives

$$-\int_\Omega \Delta c_i^N(t,x)c_i^N(t,x)dtdx = \int_\Omega |\nabla_x c_i^N|^2(t,x)dx,$$

Use the Green formula together with $u \in H_0^1(\Omega)$ and $div(u) = 0$ leads to,

$$\sum_{j=1}^N \int_\Omega (u_j(t,x)\partial_{x_j}[(c_i^N)^2](t,x)dx = 0. \tag{49}$$

From (48), we get that

$$\int_\Omega [c_i^N(T,x)]^2dx + \int_0^T\int_\Omega |\nabla_x c_i^N(t,x)|^2dxdt \leq \int_\Omega (c_i^0(x))^2dx + \int_\Omega\int_0^T c_i^NQ_i^N(t,x)dtdx. \tag{50}$$

On the other hand, as $c_i^N(t,x) \leq \|\rho_0\|_\infty \leq M_0$, it comes that

$$\int_\Omega\int_0^T |c_i^N(t,x)Q_i^N(c^N)(t,x)|dtdx \leq M_0\int_\Omega\int_0^T |Q_i^N(c^N)(t,x)|dtdx.$$

By using the assumptions 5 of Theorem 2, we get a L^∞ bound on $Q_i^N(c^N)$. So,

$$\int_0^T\int_\Omega |\nabla_x c_i^N(t,x)|^2dxdt \leq \int_\Omega (c_i^0(x))^2dx + M_0M_1T\,mes(\Omega)$$

and $\int_\Omega (c_i^0(x))^2dx$ being also bounded, the result holds. □

Proof of Proposition 4.

We shall apply the Aubin-Simon Lemma ([13]). From Lemma 10, c_i^N is bounded in

$L^2([0;T];H^1(\Omega))$. It remains to show that $\partial_t c_i^N$ is bounded in $L^1([0;T];H^{-1}(\Omega))$. Let $\varphi \in C_c^\infty(\Omega)$. Multiply (31) by φ and integrate on Ω leads to

$$\begin{aligned} \int_\Omega \partial_t c_i^N(t,x)\varphi(x)dx &= \int_\Omega([G_i^N - P_i^N](c^N)(t,x)\varphi(x)dx + \int_\Omega \Delta c_i^N(t,x)\varphi(x)dx \\ &- \int_\Omega u(t,x)\cdot\nabla_x c_i^N(t,x)\varphi(x)dx. \end{aligned} \tag{51}$$

φ being compactly supported, the Green formula gives that

$$\int_\Omega \Delta c_i^N(t,x)\varphi(x)dx = -\int_\Omega \nabla_x c_i^N(t,x)\cdot\nabla_x\varphi(x)dx.$$

From the Cauchy-Schwartz inequality, it comes that

$$\int_\Omega \Delta c_i^N(t,x)\varphi(x)dx \le (\int_\Omega |\nabla_x c_i^N(t,x)|^2dx)^{\frac{1}{2}}\|\varphi\|_{H^1(\Omega)}.$$

On the other hand, from the Green formula,

$$\sum_{j=1}^D\int_\Omega u_j(t,x)\partial_{x_j}c_i^N(t,x)\varphi(x)dx = -\sum_{j=1}^D\int_\Omega c_i^N(t,x)\partial_{x_j}(u_j(t,x)\varphi(x))dx. \tag{52}$$

But, as $\partial_{x_j}(u_j(t,x)\varphi(x)) = \partial_{x_j}(u_j(t,x))\varphi(x) + \partial_{x_j}(\varphi(x))(u_j)(t,x)$ and as $div(u)=0$, the equation (52) reads

$$\sum_{j=1}^D\int_\Omega u_j(t,x)\partial_{x_j}c_i^N(t,x)\varphi(x)dx = -\sum_{j=1}^D\int_\Omega c_i^N(t,x)\partial_{x_j}(\varphi(x))(u_j)(t,x)dx.$$

From (47), we get $(\forall(t,x)\in[0;T]\times\mathbb{R}^D, c_i^N(t,x)\le M_0$. So,

$$|\sum_{j=1}^D\int_\Omega u_j(t,x)\partial_{x_j}c_i^N(t,x)\varphi(x)dx| \le M_0[\sum_{j=1}^D\int_\Omega(u_j(t,x))^2]dx]^{\frac{1}{2}}\|\varphi\|_{H^1(\Omega)}.$$

$[G_i^N - P_i^N](c^N)$ being bounded in $L^\infty([0;T]\times\Omega)$, there is a nonnegative constant M_1 independant with respect to the quantities t,x,i and N such that

$$|\int_\Omega[G_i^N - P_i^N](c^N)(t,x)\varphi(x)dx| \le M_1\sqrt{mes(\Omega)}\|\varphi\|_{H^1(\Omega)}.$$

Hence, from (51), it holds that

$$\begin{aligned} |\int_\Omega \partial_t c_i^N(t,x)\varphi(x)dx| &\le (\int_\Omega|\nabla_x c_i^N t,x)|^2dx)^{\frac{1}{2}} + M_0[\sum_{j=1}^D\int_\Omega(u_j(t,x))^2dx)\|\varphi\|_{H^1(\Omega)} \\ &+ M_1\sqrt{mes(\Omega)}\|\varphi\|_{H^1(\Omega)}. \end{aligned}$$

By using the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, the previous inequality holds for any $\varphi \in H_0^1(\Omega)$. So,

$$\begin{aligned}\int_0^T \|\partial_t c_i^N(t,x)\|_{H^{-1}(\Omega)} dt &\leq M_0 \int_0^T \|u\|_{[L^2(\Omega)]^D} dt \\ &+ \int_0^T (\int_\Omega |\nabla_x c_i^N|^2 dx)^{\frac{1}{2}} dt + TM_1\sqrt{mes(\Omega)}.\end{aligned}$$

From the Cauchy-Schwartz inequality, it holds that

$$\int_0^T (\int_\Omega |\nabla_x c_i^N|^2 dx)^{\frac{1}{2}} dt \leq \sqrt{T}\|c_i^N\|_{L^2(0,T;H^1(\Omega))},$$

$$\int_0^T \|u\|_{[L^2(\Omega)]^D} dt \leq \|u\|_{L^2(0,T;L^2(\Omega))}\sqrt{T}.$$

But, c_i^N being bounded in $L^2(0,T;H^1(\Omega))$, $\partial_t c_i^N$ is then bounded in $L^1(0,T;H^{-1}(\Omega))$. So, from Aubin-Simon Lemma ([13]), c_i^N is strongly compact in $L^2(0,T;L^2(\Omega))$. $\square$

Proof of Theorem 2.
From a diagonal process there is a subsequence of $(c_i^N)_{N\in\mathbb{N}}$ (still denoted $(c_i^N)_{N\in\mathbb{N}}$) such that $(\forall T>0), (\forall i \in \mathbb{N}^*)$ $c_i^N \to c_i$ in $L^2([0;T]\times\mathbb{R}^D)$ strongly. By arguing as in ([7]), we can prove that $G_i^N(c^N)$ (resp. $P_i(c^N)$) converges to $G_i(c)$ (resp. $P_i(c)$) in L^2. So we can pass to the limit in the weak form of (31, 32, 33).

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