Fluid model for plasma physics under an imposed magnetic field: resonances

Olivier Lafitte

LAGA, Université de Paris 13, Sorbonne Paris Cité

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Partie de ce travail en collaboration avec

- B. Despres (LJLL) et L.M. Imbert-Gerard (CIMS)
- O. Maj (NMPP, IPP, Garching)
- Travail soutenu en partie par le Max Planck Institute

Contents

Model 1 used: the fluid model for electrons

- Imposed magnetic field $\vec{B}_0(x) = B_0(x)e_z$, electron density $n_0(x)$. Two frequencies in this study
  plasma frequency: $(\omega_p(x))^2 = \frac{e^2 n_0(x)}{m\varepsilon_0}$,
  cyclotron frequency: $\omega_c(x) = \frac{e|B_0(x)|}{m}$.
- Maxwell equations

$$\left\{ \begin{array}{l} \nabla \wedge E = -\partial_t B \\ c^2 \nabla \wedge B = \frac{1}{\varepsilon_0} J + \partial_t E \end{array} \right.$$

- Relation on the electric current (from $\vec{f} = q(\vec{E} + \vec{v} \wedge \vec{B})$):

$$\partial_t J = \varepsilon_0 (\omega_p(x))^2 E - \omega_c(x) J \wedge e_z - \nu J$$

Relation between the velocity of electrons and the current: $J = -n_0(x) e v$.

Model 2 used: the fluid model for electrons and ions

Multispecies: ions of charge Z_i and of masses m_i

$$J = -n_0(x)ev_e + \sum_i Z_i n_i(x) e v_i = -n_0(x)ev_e + \sum_i J_i$$

Electroneutrality $n_0(x) = \sum_i Z_i n_i(x)$.
Electrodynamics

$$\partial_t v_i = \frac{Z_i e}{m_i}(E + B_0(x) v_i \wedge b) - \nu v_i$$

$$\Leftrightarrow \partial_t J_i = \frac{Z_i^2 e^2 n_i(x)}{m_i} E + \frac{Z_i e B_0(x)}{m_i} J_i \wedge b - \nu J_i$$

Deduce, with $p_i = \frac{Z m_e}{m_i}$

$$\partial_t J + \nu J = \sum_i \frac{e^2 Z_i n_i(x)(1+p_i)}{m_e} E + \omega_c(x) \sum_i (1+p_i) J_i \wedge b - \omega_c(x) J \wedge b$$

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Effective dielectric tensor (cold plasma model, cf BD)

Recall that B_0 depends only on x.
Fourier transform in time $e^{-i\omega t}$. Express v_e, J_i in terms of E ($en_0(x)v_e = \Sigma(\omega, x)E$, $J_i = \Sigma_i(\omega, x)E$, $\Sigma(\omega, x), \Sigma_i(\omega, x)$ matrices).
Deduce $J = -en_0(x)v_e + \sum_i J_i = [\sum_i \Sigma_i(\omega, x) - \Sigma(\omega, x)]E$.
Obtention of Σ, Σ_i: each one is obtained independently using

$$-i\omega J_i + \nu J_i - p_i\omega_c J_i \wedge b = \varepsilon_0 \frac{Z_i^2 e^2 n_i(x)}{\varepsilon_0 m_i} E$$

Resonances, for $\nu = 0$, at x s. th. $\omega = p_i\omega_c(x)$.

$$-i\omega J + \nu J + \omega_c J \wedge b = (\Omega_p(x))^2 E + \omega_c(x) \sum_i (1 + p_i) J_i \wedge b$$

Effective dielectric tensor $\epsilon^\nu(\omega, x) \Rightarrow$ PDE on E:

$$\nabla \wedge \nabla \wedge E = \varepsilon_0\mu_0\omega^2\epsilon^\nu(\omega, x)E := \varepsilon_0\mu_0\omega^2(Id - \frac{1}{i\omega\varepsilon_0}\sigma(\omega, x))E.$$

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Hybrid resonance (case of electrons only)

One gets

$$\epsilon^\nu(\omega,x) = \begin{pmatrix} 1-\frac{(\omega+i\nu)\omega_p^2}{\omega((\omega+i\nu)^2-\omega_c^2)} & i\frac{\omega_c\omega_p^2}{\omega((\omega+i\nu)^2-\omega_c^2)} \\ -i\frac{\omega_c\omega_p^2}{\omega((\omega+i\nu)^2-\omega_c^2)} & 1-\frac{(\omega+i\nu)\omega_p^2}{\omega((\omega+i\nu)^2-\omega_c^2)} \end{pmatrix} \quad (1)$$

Definition of hybrid resonance: diagonal terms of ϵ^0 vanish. Why? System depending only on x: Fourier mode in y $e^{i\frac{\omega}{c}\sin\theta_0 y}$ (Normal incidence)

$$c\sin\theta_0 B_3^\nu(x) - \epsilon_{11}^\nu E_1^\nu(x) - \epsilon_{12}^\nu E_2^\nu(x) = 0.$$

$E_1^0(x)$ **is singular at** $x = x_h$ **such that** $\epsilon_{11}^0(x) = 0$**.** If x_h^ν solution of $(\omega+i\nu)\omega_p(x)^2 = \omega((\omega+i\nu)^2-\omega_c(x)^2) \Rightarrow x_h^\nu$ **regular singular point for the ODE**.

Oblique incidence: $c(ik_2B_3^\nu - ik_3B_2^\nu) - \epsilon_{11}^\nu E_1^\nu - \epsilon_{12}^\nu E_2^\nu = 0.$

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Maxwell equations on the electromagnetic field (electrons only)

Assumption: one seeks solutions of the form $f(x)e^{ik_0 \sin\theta_0 y - i\omega t}$, $k_0 = \frac{\omega}{c}$. System (normal incidence):

$$\begin{cases} i\omega B_1 = ik_0 \sin\theta_0 E_3 \\ i\omega B_2 = -E_3' \\ i\omega B_3 = E_2' - ik_0 \sin\theta_0 E_1 \\ c^2(ik_0 \sin\theta_0 B_3) = j_1 - i\omega E_1 \\ c^2(-B_3') = j_2 - i\omega E_2 \\ c^2(B_2' - ik_0 \sin\theta_0 B_1) = j_3 - i\omega E_3 \\ -i\omega j_1 = \omega_p^2(x)E_1 - \omega_c(x)j_2 - \nu j_1 \\ -i\omega j_2 = \omega_p^2(x)E_2 + \omega_c(x)j_1 - \nu j_2 \\ -i\omega j_3 = \omega_p^2(x)E_3 - \nu j_3 \end{cases}$$

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c^2(B_2' - ik_0 \sin\theta_0 B_1) = j_3 - i\omega E_3 \\
-i\omega j_1 = \omega_p^2(x) E_1 - \omega_c(x) j_2 - \nu j_1 \\
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-i\omega j_3 = \omega_p^2(x) E_3 - \nu j_3
\end{cases}
$$

System: oblique incidence, electrons: $(e^{ik_2y+ik_3z-i\omega t}f(x))$

$$\begin{cases} i\omega B_1 = ik_2E_3 - ik_3E_2 \\ i\omega B_2 = ik_3E_1 - E_3' \\ i\omega B_3 = E_2' - ik_2E_1 \\ c^2(ik_2B_3 - ik_3B_2) = j_1 - i\omega E_1 \\ c^2(ik_3E_1 - B_3') = j_2 - i\omega E_2 \\ c^2(B_2' - ik_2B_1) = j_3 - i\omega E_3 \\ -i\omega j_1 = \omega_p^2(x)E_1 - \omega_c(x)j_2 - \nu j_1 \\ -i\omega j_2 = \omega_p^2(x)E_2 + \omega_c(x)j_1 - \nu j_2 \\ -i\omega j_3 = \omega_p^2(x)E_3 - \nu j_3 \end{cases}$$

Multispecies, oblique incidence (restructuring)

$$
\begin{cases}
i\omega B_2 = ik_3 E_2 - E_3' \\
c^2(B_2' - ik_2 B_1) = j_3 - i\omega E_3 \\
\\
(-i\omega + \nu) j_3 = \omega_p^2(x) E_3 \\
i\omega B_1 = (ik_2 E_3 - ik_3 E_2) \\
\\
i\omega B_3 = E_2' - ik_2 E_1 \\
c^2(ik_3 B_1 - B_3') = j_2 - i\omega E_2 \\
\\
j_1 - i\omega E_1 = c^2(ik_2 B_3 - ik_3 B_2) \\
(-i\omega + \nu) J_{1,i} - p_i \omega_c J_{2,j} = \varepsilon_0 (\omega_p^i)^2 E_1 \\
p_i \omega_c J_{1,i} + (-i\omega + \nu) J_{2,j} = \varepsilon_0 (\omega_p^i)^2 E_2 \\
(-i\omega + \nu) j_1 + \omega_c(x) j_2 = \varepsilon_0 \Omega_p^2(x) E_1 + \omega_c \sum_i (1 + p_i) J_{i,2} \\
(-i\omega + \nu) j_2 - \omega_c(x) j_1 = \varepsilon_0 \Omega_p^2(x) E_2 - \omega_c \sum_i (1 + p_i) J_{i,1}
\end{cases}
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Hybrid singularities

$2n-1$ last equations $\Rightarrow \varepsilon_0^{-1}(j_1, j_2, J_{1,i}, J_{2,i}), E_1$ in terms of $(E_2, ik_2B_3 - ik_3B_2)$.

Assembling the system:

$$\begin{pmatrix} M_i & 0 & \begin{pmatrix} (\omega_p^i)^2 \\ 0 \end{pmatrix} \\ \omega_c \begin{pmatrix} 0 & -1-p_i \\ 1+p_i & 0 \end{pmatrix} & M & \begin{pmatrix} (\Omega_p)^2 \\ 0 \end{pmatrix} \\ (1,0) & 0 & -i\omega \end{pmatrix}$$

'Physical' remarks:

- $p_i << 1 \to \Omega_p^2 \simeq \omega_p^2$
- $M = \begin{pmatrix} -i\omega+\nu & \omega_c \\ -\omega_c & -i\omega+\nu \end{pmatrix}$,
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Replace E_1 by its value from the last equation:

$$\begin{pmatrix} M_i & \frac{(\omega_p^i)^2}{i\omega}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \omega_c \begin{pmatrix} 0 & -1-p_i \\ 1+p_i & 0 \end{pmatrix} & M + \frac{(\Omega_p)^2}{i\omega}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

Electrons only: determinant:
$d_\nu(x) = \omega_c(x)^2 + (i\omega - \nu)^2 + \omega_p^2 \frac{i\omega-\nu}{i\omega}$.
$d_0(x)$ vanishes at x_h such that $\omega^2 = (\omega_p^2 + \omega_c^2)(x_h)$.
Does not vanish at x_c.
Definition: a hybrid singularity is a x such that the determinant of the system vanish for $\nu = 0$
Assumption: simple hybrid singularity: $D_0(x_{hi}) = 0, D_0'(x_{hi}) \neq 0$.
For simplicity: concentrate on electrons only, oblique incidence.

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System of ODEs

Elimination of (j_1, j_2):

$$\begin{cases} (-i\omega + \nu - \frac{\omega_p^2}{i\omega})\frac{j_1}{\varepsilon_0} - \omega_c \frac{j_2}{\varepsilon_0} = -c^2 \frac{\omega_p^2}{i\omega}(ik_2 B_3 - ik_3 B_2) \\ \omega_c \frac{j_1}{\varepsilon_0} + (-i\omega + \nu)\frac{j_2}{\varepsilon_0} = \omega_p^2 E_2 \end{cases}$$

Obtain:

$$\begin{cases} \varepsilon_0^{-1} j_1 = \frac{\omega_p^2}{d_\nu}[\frac{i\omega-\nu}{i\omega} c^2 (ik_2 B_3 - ik_3 B_2) + \omega_c E_2] \\ \varepsilon_0^{-1} j_2 = \frac{\omega_p^2}{d_\nu}[\frac{\omega_c}{i\omega} c^2 (ik_2 B_3 - ik_3 B_2) + (-i\omega + \nu - \frac{\omega_p^2}{i\omega}) E_2] \end{cases}$$

Recall $E_1 = \frac{1}{i\omega}\frac{j_1}{\varepsilon_0} - \frac{c^2 V_3}{i\omega}$.

Unknowns:

$(E_2, V_3 = ik_2B_3 - ik_3B_2, E_3, V_2 = ik_2B_2 + ik_3B_3)$

$$\begin{cases} \partial_x E_2 = i\omega B_3 + \mathbf{ik_2E_1} \\ \partial_x c^2 V_3 = i\omega(ik_2E_2 + ik_3E_3) + ik_3\varepsilon_0^{-1} j_3 + \varepsilon_0^{-1}\mathbf{ik_2j_2} \\ \partial_x E_3 = -i\omega B_2 + \mathbf{ik_3E_1} \\ \partial_x(c^2V_2) = -i\omega(ik_2E_3 - ik_3E_2) - |k|^2c^2B_1 + ik_2\epsilon_0^{-1}j_3 - \epsilon_0^{-1}\mathbf{ik_3j_2} \end{cases}$$

with

$$\begin{cases} \varepsilon_0^{-1} j_2 = \frac{\omega_p^2}{d_\nu}[\alpha_{11}c^2V_3 + \alpha_{12}E_2] \\ E_1 = \frac{\omega_p^2}{d_\nu}[\alpha_{21}c^2V_3 + \alpha_{22}E_2] - (i\omega)^{-1}c^2V_3 = \tilde{E}_1 - (i\omega)^{-1}c^2V_3 \end{cases}$$

$$\alpha_{11} = \alpha_{22} = \frac{\omega_c}{i\omega}, \alpha_{12} = -i\omega + \nu - \frac{\omega_p^2}{i\omega}, \alpha_{21} = \frac{1}{i\omega}\frac{i\omega - \nu}{i\omega}.$$

Note that $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = -\omega^{-2}d_\nu$.

System after transformations

$$\begin{cases} \partial_x E_2 = -\frac{i\omega}{|k|^2}(ik_2V_3 + ik_3V_2) - \frac{ik_2}{i\omega}c^2V_3 + \mathbf{ik_2\tilde{E}_1} \\ \partial_x c^2V_3 = i\omega(ik_2E_2 + ik_3E_3) + ik_3\frac{\omega_p^2}{-i\omega+\nu}E_3 + \varepsilon_0^{-1}\mathbf{ik_2j_2} \\ \partial_x E_3 = \frac{i\omega}{|k|^2}(ik_2V_2 - ik_3V_3) - \frac{ik_3}{i\omega}c^2V_3 + \mathbf{ik_3\tilde{E}_1} \\ \partial_x(c^2V_2) = (-i\omega - \frac{|k|^2c^2}{i\omega})(ik_2E_3 - ik_3E_2) + ik_2\frac{\omega_p^2}{-i\omega+\nu}E_3 - \epsilon_0^{-1}\mathbf{ik_3j_2} \end{cases}$$

Introduce $A = \begin{pmatrix} \alpha_{22} & \alpha_{21} \\ \alpha_{12} & \alpha_{11} \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} \alpha_{22} & \alpha_{21} \\ -\alpha_{12} & -\alpha_{11} \end{pmatrix}$. System

$$\partial_x \begin{pmatrix} E_2 \\ c^2V_3 \end{pmatrix} = ik_2B_{11}\begin{pmatrix} E_2 \\ c^2V_3 \end{pmatrix} + ik_3B_{12}\begin{pmatrix} E_3 \\ c^2V_2 \end{pmatrix} + ik_2\frac{\omega_p^2}{d_\nu}A\begin{pmatrix} E_2 \\ c^2V_3 \end{pmatrix}$$

$$\partial_x \begin{pmatrix} E_3 \\ c^2V_2 \end{pmatrix} = ik_3B_{21}\begin{pmatrix} E_2 \\ c^2V_3 \end{pmatrix} + ik_2B_{22}\begin{pmatrix} E_3 \\ c^2V_2 \end{pmatrix} + ik_3\frac{\omega_p^2}{d_\nu}\tilde{A}\begin{pmatrix} E_2 \\ c^2V_3 \end{pmatrix}$$

Contents

System for $(E_2, W := cB_3)$

$$\frac{d}{dx}\begin{pmatrix} E_2 \\ W \end{pmatrix} = \begin{pmatrix} -ik_2\frac{\epsilon^\nu_{12}}{\epsilon^\nu_{11}} & \left(\frac{\omega}{c}\right)^2 - \frac{k_2^2}{\epsilon^\nu_{11}} \\ -\frac{\epsilon^\nu_{11}\epsilon^\nu_{22}-\epsilon^\nu_{12}\epsilon^\nu_{21}}{\epsilon^\nu_{11}} & -ik_2\frac{\epsilon^\nu_{21}}{\epsilon^\nu_{11}} \end{pmatrix}\begin{pmatrix} E_2 \\ W \end{pmatrix}$$

For $\nu = 0$, singularity at $x = x_h$.

For $\nu = 0$, matrix bounded at $x = x_c$, with a turning point.

Main result:

i) one can construct solutions in the neighborhood of both points.

ii) heating of the plasma only occurs at $x = x_h$

Main hypothesis: ω_c and ω_p are locally analytic at $x = x_h$.

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Approximate solution in the neighborhood of x_h

$$\frac{d}{dx}\begin{pmatrix} E_2 \\ W \end{pmatrix} = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & -a_\nu \end{pmatrix}\begin{pmatrix} E_2 \\ W \end{pmatrix}.$$

ODE of order 2: use $c_\nu \neq 0$ in a neighborhood of x_h
$\Rightarrow E_2 = (c_\nu)^{-1}(W' + a_\nu W)$.

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{c_\nu}\frac{d}{dx}W\right) = \left(\frac{a_\nu^2}{c_\nu} + b_\nu - \left(\frac{a_\nu}{c_\nu}\right)'\right)W, \qquad x \in \mathbb{R}.$$

New unknown $h(x) := (-c_\nu)^{-\frac{1}{2}}W(x)$. $\Rightarrow$

$$\frac{d^2 h_\nu}{dx^2} = \left(a_\nu^2 + b_\nu c_\nu - c_\nu\left(\frac{a_\nu}{c_\nu}\right)' + \sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)''\right)h_\nu, \qquad x \in \mathbb{R}.$$

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Note $a_\nu \epsilon_{11}^\nu(x)$ bounded and not zero in a neighborhood of x_h, $i\partial_\nu \epsilon_{11}^\nu(x_h) > 0, \partial_x \epsilon_{11}^0(x_h) > 0$.
$\Rightarrow \epsilon_{11}^\nu(x) = 0 \Leftrightarrow x = x_h^\nu$ (local), $\frac{x_h^\nu - x_h}{\nu} = ia + O(\nu)$.
Estimates

$$0 < c_1 \le |(x - x_h^\nu)(a_\nu^2 + b_\nu c_\nu)(x)| \le c_2$$
$$0 < c_1 \le |(\frac{a_\nu}{c_\nu})'| \le c_2.$$

and the most singular term of the ODE is $(x - X_\nu)^{-\frac{1}{2}}((x - X_\nu)^{\frac{1}{2}})'' = -\frac{1}{4(x - X_\nu)^2}$.

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Lemma

There exists $R_\nu(x)$, analytic in a ball of center x_h and of radius δ,sur that the equation on h_ν writes

$$\frac{d^2 h_\nu}{dx^2} = \left(-\frac{1}{4(x-X_\nu)^2} + \frac{R_\nu(x)}{x-X_\nu}\right) h_\nu, \qquad x \in \mathbb{R} \tag{2}$$

Lemma

For $\lambda^2 = -4R$, $\sqrt{x-X_\nu} J_0(\lambda\sqrt{x-X_\nu})$ is a solution of $\frac{d^2 h_\nu}{dx^2} = \left(-\frac{1}{4(x-X_\nu)^2} + \frac{R}{x-X_\nu}\right) h_\nu, \qquad x \in \mathbb{R}.$

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Eikonal equation

• **Model solutions**: Let J_0 and Y_0 be the Bessel functions of the first kind and of the second kind. The functions $\sqrt{x}J_0(\sqrt{x})$ (and $\sqrt{x}Y_0(x)$) are solutions of

$$U'' = (-\frac{1}{4x^2} - \frac{1}{4x})U.$$

• **Stretching**: If u is solution of a Sturm equation $u''(x) = p(x)u$, and if ρ is a given analytic (in $V(x_h)$) function, such that $\rho(x_h) = 0$, $\rho'(x_h) \neq 0$, then $v(x) = (\rho')^{-\frac{1}{2}}u(\rho(x))$ solves

$$v'' = [(\rho')^2 p(\rho) + (\rho')^{\frac{1}{2}}((\rho')^{-\frac{1}{2}})'']v.$$

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$$\frac{(\rho_\nu'(x))^2}{4\rho_\nu^2(x)}(1 + \rho_\nu(x)) = \frac{1}{4(x - X_\nu)^2} - \frac{R_\nu(x)}{x - X_\nu}.$$

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The solutions of the ODE

Introduce the special function $T_0(z) = Y_0(z) - \frac{2}{\pi}\ln z J_0(z)$, which admits an analytic expansion in z^2. Define

$$\begin{aligned} U_\nu(x) &= \sqrt{\tfrac{\rho_\nu}{\rho'_\nu}} J_0(\lambda_\nu \sqrt{\rho_\nu(x)}), \\ V_\nu(x) &= \sqrt{\tfrac{\rho_\nu}{\rho'_\nu}} [Y_0(\lambda_\nu \sqrt{\rho_\nu(x)}) - \tfrac{2}{\pi}(\ln \lambda_\nu) J_0(\lambda_\nu \sqrt{\rho_\nu(x)})] \\ &= \sqrt{\tfrac{\rho_\nu}{\rho'_\nu}} [T_0(\lambda_\nu \sqrt{\rho_\nu(x)}) + \tfrac{2}{\pi}(\ln \sqrt{\rho_\nu(x)}) J_0(\lambda_\nu \sqrt{\rho_\nu(x)})]. \end{aligned}$$

They solve $U'' = (-\frac{1}{4(x-X_\nu)^2} + \frac{R_\nu(x)}{x-X_\nu})U + s_\nu U$. where s_ν smooth.

Proposition

There exists two 'smooth' functions A_ν et B_ν, such that a solution of (2) writes

$$h_\nu(x) = A_\nu(x) U_\nu(x) + B_\nu(x) V_\nu(x).$$

Solution of the Eikonal equation and of the ODE

Lemma

One has $\rho_\nu(x) = (x - X_\nu)\sigma_\nu(x)$, *where* $\sigma_\nu(z)$ *is the unique solution of* $\sigma_\nu'(x) = \sigma_\nu(x)F_\nu(x, \sigma_\nu(x))$, *in the neighborhood of* x_h^ν *such that* $\sigma_\nu(x_h^\nu) = 1$

In this lemma

$$F_\nu(x,a) = \frac{4\left[R_\nu(X_\nu)a - R_\nu(x)\right]}{\sqrt{(1-4R_\nu(x)(x-X_\nu))} + \sqrt{(1-4R_\nu(X_\nu)(x-X_\nu)a)}}.$$

Use of Duhamel principle yields
$(A_\nu', B_\nu')^T(x) = \pi\mathcal{M}_\nu(x)s_\nu(A_\nu, B_\nu)^T(x)$,

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integral equation with initial datum. Quod erat demonstrandum.

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Explicit solution of the global system

$$\partial_x F - ik_2(B_{11} + \frac{\omega_p^2}{d_\nu} A)F = ik_3 B_{12} G$$

$$\partial_x G - ik_2 B_{22} G = ik_3(B_{21} + \frac{\omega_p^2}{d_\nu} \tilde{A})F.$$

Sketch:

• From the equation on F, obtain an equation on $c^2 V_3$ (plays the role of W) with source term depending on G.
System $(A'_\nu, B'_\nu)^T(x) = \pi \mathcal{M}_\nu(x) s_\nu (A_\nu, B_\nu)^T(x) + ik_3 \mathcal{B}_{22}(G)(x)$

• One obtained $F = F^{free} + ik_3 \mathcal{T}(G)(x)$.

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Another way of studying the problem:

$$\partial_x G = ik_2 B_{22} G + ik_3 B_{21} F + \frac{k_3}{k_2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [\partial_x F - ik_2 B_{11} F - ik_3 B_{12} G].$$

Equation on $H = G - \frac{k_3}{k_2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F$:

$$\begin{aligned} \partial_x H = \; & ik_2 B_{22} H + ik_3 \left(B_{22} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B_{21} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B_{11}\right) F \\ & -i \frac{k_3^2}{k_2} B_{12} \left(H + \frac{k_3}{k_2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F\right). \end{aligned}$$

The singularity generated by F is only in the source term for H.

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Resonant heating

Relevant quantity

$$Q^\nu(a,b) = \Im \int_a^b (\underline{\underline{\varepsilon}}^\nu(x)\vec{E}^\nu(x), (\vec{E}^\nu)^*(x))dx$$
$$= -\Im(W^\nu(b)(E_2^\nu)^*(b)) + \Im(W^\nu(b)(E_2^\nu)^*(a))$$

where a and b are fixed.

In the case of the cyclotron frequency, no singularity and limit zero when (a,b) is a small neighborhood of x_c.

For the hybrid frequency, note that

$$\lim_{\nu\to 0_\pm} \ln(\rho_\nu(x)) := (\ln(\rho_0(x)))^\pm = \begin{cases} \ln(\rho_0(x)), x > x_h \\ \ln(-\rho_0(x)) \mp i\pi, x < x_h \end{cases}.$$

Hence $\mathcal{M}_\nu(x) \to \mathcal{M}_\pm(x)$, is L^1_{loc} and one notes the following approximations:

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Singular limit

$$E_1^\nu - \frac{1}{\pi}\frac{\epsilon_{12}^\nu}{(\epsilon_{11}^\nu)^2-|\epsilon_{12}^\nu|^2} W_1^\nu \frac{\rho_\nu'(x)}{\rho_\nu(x)} - \frac{i\theta}{\epsilon_{11}^\nu}[A_\nu W_1^\nu + B_\nu(W_3^\nu + \frac{1}{\pi}W_1^\nu \ln\frac{\rho_\nu(x)}{4})] \rightarrow A_0^\pm K_1(x) + B_0^\pm K_2(x) + B_0^\pm S(x)(\ln\rho(x))^\pm .$$

Limit of $Q^\nu(a,b)$ for $a < x_h - \delta_0 < x_h + \delta_0 + b$ and $\nu \to 0_+$:

$$\frac{1}{\pi}|B_0(x_h)|^2|\epsilon_{12}^0(x_h)|^2(sign(\partial_\nu \epsilon_{11}^\nu)(x_h,0)).$$

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