COMPACTNESS PROPERTY OF THE LINEARIZED BOLTZMANN OPERATOR FOR A MIXTURE OF POLYATOMIC GASES

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Abstract. In this paper we concern ourselves with a kinetic description of gas mixtures for polyatomic molecules. In particular, we will consider the Boltzmann equation that models a mixture of polyatomic gases of n species $(\mathcal{A}_i)_{i=1,\cdots,n}$. At the microscopic level, one additional argument of the distribution function is introduced which is the parameter I denoting the continuous internal energy. Under some convenient assumptions on the collision cross-section $\mathcal{B}_{ij}$, we prove that the linearized Boltzmann operator $\mathcal{L}$ of this model is a Fredholm operator. For this, we write $\mathcal{L}$ as a perturbation of the collision frequency multiplication operator, and we prove that the perturbation operator $\mathcal{K}$ is compact. The result is established after inspecting the kernel form of $\mathcal{K}$ and proving it to be L^2 integrable over its domain using elementary arguments.

Keywords. multicomponent polyatomic gases, Boltzmann equation, linearized operator, Fredholm property

AMS subject classifications. 76P05; 82B40; 82C40; 45B05

1. Introduction

After an intensive kinetic study of the behaviour of monoatomic gases, the study of polyatomic gases is recently witnessing a remarkable progress as it is very important in applications. In fact, it is indispensable to take into consideration the presence of polyatomic particles in the gas flows upon the study of of non-equilibrium processes. In this paper, we implement a polyatomic model for a mixture of gases assuming the internal energy to be continuous. In such model, each polyatomic gas species i is represented by a distribution function f_i, where the phase space is generated by time t, space x, velocity v and a continuous variable I. The variable I describes the storage of vibrational and rotational energies inside each molecule. The idea of modeling the internal energy by a continuous variable was introduced in [12], which was dedicated for modeling polyatomic single gases based on the Borgnakke-Larsen parameterization process. This process is based on randomly proportioning the total energy in each interaction between the internal energy and kinetic energy.

In general, the Boltzmann equation for polyatomic gases with continuous internal energy can be modeled by hydrodynamic asymptotics [1,2,7,18], where for the collision operator transport coefficients are computed and proved to be consistent with experiments [19,20] for single-species collision kernel. In the spirit of hydrodynamics, a kinetic model describing a mixture of polyatomic gases was introduced in [18], for which the Chapman-Enskog method was developed in [3]. In the same perturbation framework, the authors of [22] established the global well-posedness for bounded mild solutions near global equilibria on torus. It is noteworthy to remark that many of the achieved results for mixtures of polyatomic gases were dedicated to the study of the BGK [6–8] and ES-BGK models [13].

The aim of our work is to prove the Fredholm property of the linearized Boltzmann operator for a mixture of polyatomic gases. In the monoatomic single gas setting,

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Grad [26] proved that the linearized Boltzmann operator is a Fredholm operator, by writing it as a compact perturbation of a multiplication operator (see also [16, 21]). For monoatomic mixtures, $\mathcal{L}$ was proved to be Fredholm in [4, 11, 27]. Coercivity estimates on the spectral gap of the linearized Boltzmann operator were also obtained [17]. In [15], the compactness property for $\mathcal{K}$ was proved in the case of diatomic molecules. For a single polyatomic gas in general, [14] and [5] proved $\mathcal{K}$ is compact by different approaches. For the case of a single resonant polyatomic gas, the compactness was proved in [10]. In this work, we aim to generalize the work [14] for a mixture of polyatomic gases using a suitable variables to write the perturbation operator under a kernel form. For this, we prove that $\mathcal{L}$ is written as a compact perturbation of the collision frequency ν multiplication operator, and we prove as well that ν is coercive. This implies that $\mathcal{L}$ is a Fredholm operator.

The plan of the paper is as follows: In Section 2, we give a brief recall on the collision model [12], which describes the microscopic state of a mixture of polyatomic gases, and give an equivalent formulation of the collision operator. In Section 3, we define the linearized operator $\mathcal{L}$, which is obtained by approximating the distribution function f around the Maxwellian M. The main aim of this paper is to prove that the linearized Boltzmann operator is a Fredholm operator, which is achieved in Section 4. In particular, we write $\mathcal{L}$ as $\mathcal{L}=\mathcal{K}-\nu$ Id and we prove that $\mathcal{K}$ is compact, and ν is coercive. As a result, $\mathcal{L}$ is viewed as a compact perturbation of the multiplication operator ν Id. To prove $\mathcal{K}$ is compact, we write $\mathcal{K}$ as $\mathcal{K}_3+\mathcal{K}_2-\mathcal{K}_1$, and we prove each $\mathcal{K}_l$, with $l=1,\cdots,3$, to be a Hilbert-Schmidt operator. In Section 5, we give the monotonicity property of the collision frequency, which helps to locate the essential spectrum of $\mathcal{L}$.

2. The Classical model

For modeling the Boltzmann equation for a mixture of polyatomic gases, we generalize in this paper the model in [12], which is dedicated for a single polyatomic gas. For this, we present first the conservation equations, which lead by parameterization to the pre-post collisional relations. Throughout the paper, c denotes a generic constant.

2.1. Boltzmann Equation We denote as usual by (v,v_*), (I,I_*) and (v',v'_*), (I',I'_*) the pre-collisional and post-collisional velocity and internal energy pairs respectively. In this model, the internal energies are assumed to be continuous. The following conservation of momentum and total energy equations hold:

$$\begin{aligned} m_i v + m_j v_* &= m_i v' + m_j v'_* \\ \frac{m_i}{2}v^2 + \frac{m_j}{2}v_*^2 + I + I_* &= \frac{m_i}{2}v'^2 + \frac{m_j}{2}v_*'^2 + I' + I'_*, \end{aligned} \tag{2.1}$$

where m_i and m_j are the respective particle mass of species $\mathcal{A}_i$ and $\mathcal{A}_j$. From the above equations, we can deduce the following equation representing the conservation of total energy in the center of mass reference frame:

$$\frac{\mu_{ij}}{2}|v-v_*|^2 + I + I_* = \frac{\mu_{ij}}{2}|v'-v'_*|^2 + I' + I'_* = E,$$

with E denoting the total energy, and $\mu_{ij} = \frac{m_i m_j}{m_i+m_j}$ is the reduced mass. The Borgnakke-Larsen parameterization procedure [9] used to express the post-collisional quantities in terms of the pre-collisional quantities is based on introducing s the parameter $R \in [0,1]$ which represents the portion allocated to the kinetic energy after collision out of the total energy, and the parameter $r\in[0,1]$ which represents the distribution of the post internal energy among the two colliding molecules. That is,

$$\frac{\mu_{ij}}{2}(v'-v'_*)^2 = RE$$
$$I'+I'_* = (1-R)E,$$

and

$$I' = r(1-R)E$$
$$I'_* = (1-r)(1-R)E.$$

In addition, we can express the post-collisional velocities in terms of the other quantities by the following relations

$$v' \equiv v'(v,v_*,I,I_*,\sigma,R) = \frac{m_i v + m_j v_*}{m_i+m_j} + \frac{m_j}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma$$
$$v'_* \equiv v'_*(v,v_*,I,I_*,\sigma,R) = \frac{m_i v + m_j v_*}{m_i+m_j} - \frac{m_i}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}}\,\sigma,$$

where $\sigma = \frac{v'-v'*}{|v'-v'*|}\in S^2$ is an additional parameter. In addition, we define the parameters $r'\in[0,1]$ and $R'\in[0,1]$ for the pre-collisional terms in the same manner as r and R. In particular

$$\frac{\mu_{ij}}{2}(v-v_*)^2 = R'E$$
$$I+I_* = (1-R')E,$$

and

$$I = r'(1-R')E$$
$$I_* = (1-r')(1-R')E.$$

The Boltzmann equation for a mixture of n species of polyatomic gases is

$$\partial_t f_i + v.\nabla_x f_i = \sum_{j=1}^{n} Q_{ij}(f_i,f_j), \quad 1\le i\le n, \tag{2.2}$$

where $f_i = f_i(t,x,v,I)\ge 0$ is the distribution function of the species $\mathcal{A}_i$, with $t\ge 0, x\in\mathbb{R}^3, v\in\mathbb{R}^3$, and $I\ge 0$. The operator Q_{ij} is the quadratic Boltzmann operator given as

$$Q_{ij}(f_i,f_j)(v,I) = \int_{(0,1)^2\times S^2\times\mathbb{R}_+\times\mathbb{R}^3}\left(\frac{f'_i f'_{j*}}{I'^{\alpha_i} I'^{\alpha_j}_*} - \frac{f_i f_{j*}}{I^{\alpha_i} I^{\alpha_j}_*}\right)\mathcal{B}_{ij}\times r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j}$$
$$I^{\alpha_i} I^{\alpha_j}_*(1-R)R^{1/2}\,\mathrm{d}R\mathrm{d}r\mathrm{d}\sigma\mathrm{d}I_*\,\mathrm{d}v_*, \tag{2.3}$$

where we use the standard notations $f_{j*} = f_j(v_*,I_*), f'_i = f_i(v',I')$, and $f'_{j*} = f_j(v'_*,I'_*)$, and $\alpha_k > -1$ for $k=1,\cdots,n$ is the parameter related to the number of degrees of freedom D_k of the $k-th$ species as

$$\alpha_k = \frac{D_k-5}{2}, \quad 1\le i\le n.$$

The function $\mathcal{B}_{ij}=\mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma)$ is the collision cross-section. In the following, we give some assumptions on $\mathcal{B}_{ij}$. In general, $\mathcal{B}_{ij}$ is assumed to be an almost everywhere positive function satisfying the following microreversibility conditions:

$$\begin{aligned}\mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma)&=\mathcal{B}_{ji}(v_*,v,I_*,I,1-r,R,\sigma)\\ \mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma)&=\mathcal{B}_{ij}(v',v'_*,I',I'_*,r',R',\sigma'),\end{aligned}\tag{2.4}$$

where $\sigma'=\frac{v-v_*}{|v-v_*|}$. Assumption (2.4) clearly implies that

$$\tilde{\mathcal{B}}_{ij}=r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j}I^{\alpha_i}I_*^{\alpha_j}\mathcal{B}_{ij},$$

is also microreversible

$$\begin{aligned}\tilde{\mathcal{B}}_{ij}(v,v_*,I,I_*,r,R,\sigma)&=\tilde{\mathcal{B}}_{ji}(v_*,v,I_*,I,1-r,R,\sigma)\\ \tilde{\mathcal{B}}_{ij}(v,v_*,I,I_*,r,R,\sigma)&=\tilde{\mathcal{B}}_{ij}(v',v'_*,I',I'_*,r',R',\sigma').\end{aligned}$$

2.2. Main Assumptions on the Collision Cross-section $\mathcal{B}_{ij}$

Together with the above assumption (2.4), we assume the following boundedness assumptions on the collision cross section $\mathcal{B}_{ij}$. In fact, we assume two classes of assumptions for a given $\gamma_{ij}\geq 0$ and for $-1<\gamma_{ij}<0$. For $\gamma_{ij}\geq 0$, we assume

$$\gamma_{ij}\geq 0,\quad \Phi_{ij}(r,R)\left(\frac{\mu_{ij}}{2}|v-v_*|^{\gamma_{ij}}+I^{\gamma_{ij}/2}+I_*^{\gamma_{ij}/2}\right)\leq\mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma),\tag{2.5}$$

and

$$\gamma_{ij}\geq 0,\quad \mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma)\leq\Psi_{ij}(r,R)\left(\frac{\mu_{ij}}{2}|v-v_*|^{\gamma_{ij}}+I^{\gamma_{ij}/2}+I_*^{\gamma_{ij}/2}\right).\tag{2.6}$$

On the other hand, for $-1<\gamma_{ij}<0$, we prove that the $i-th$ component of $\mathcal{K}$ remains compact under the following upper bound assumption on $\mathcal{B}_{ij}$

$$-1<\gamma_{ij}<0,\quad \mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma)\leq\Psi_{\gamma_{ij}}(r,R)\,E^{\gamma_{ij}/2}\tag{2.7}$$

where Φ_{ij} and Ψ_{ij} are positive functions such that $\Phi_{ij}>0$ on a set of non-zero measure and

$$\Phi_{ij}\leq\Psi_{ij},$$

and

$$\Phi_{ij}(r,R)=\Phi_{ij}(1-r,R),\quad \Psi_{ij}(r,R)=\Psi_{ij}(1-r,R).\tag{2.8}$$

In addition, we assume the following conditions for Ψ_{ij}:

$$\Psi_{ij}^2(r,R)r^{\alpha_i+\alpha_j-1-\gamma_{ij}}(1-r)^{\alpha_j-1}R(1-R)^{\alpha_i+2\alpha_j-\gamma_{ij}}\in L^1((0,1)^2),\tag{2.9}$$

and

$$\Psi_{ij}^2(r,R)(1-r)^{2\alpha_j-1-\gamma_{ij}}r^{\alpha_i-1}R(1-R)^{\alpha_i+2\alpha_j-\gamma ij}\in L^1((0,1)^2).\tag{2.10}$$

In fact, (2.10) is needed for the compactness of $\mathcal{K}_3$ which we do not prove in this paper. In addition, though assumptions (2.9) and (2.10) seem to be strict, yet they cover several physical models presented below. Furthermore, one may notice that as the value of α_i or α_j increases, conditions (2.9) and (2.10) cover a wider class of functions Ψ_{ij}. We give now models for $\mathcal{B}_{ij}$ that satisfy condition (2.9).

Models of the Cross section $\mathcal{B}_{ij}$
Suppose that $\gamma_{ij} < \alpha_i + \alpha_j$, and $\alpha_j > 0$, then the following models for mixtures, extended from the single gas models suggested in [23]

$$\mathcal{B}_{ij} = \Big(\frac{\mu_{ij}}{2}|v-v_*|^{\gamma_{ij}} + I^{\gamma_{ij}/2} + I_*^{\gamma_{ij}/2}\Big), \tag{2.11}$$

$$\mathcal{B}_{ij} = \Big(R^{\gamma_{ij}/2}|v-v_*|^{\gamma_{ij}} + (1-R)^{\gamma_{ij}/2}(I+I_*)^{\gamma_{ij}/2}\Big), \tag{2.12}$$

and

$$\mathcal{B}_{ij} = \Big(\frac{\mu_{ij}}{2}R^{\gamma_{ij}/2}|v-v_*|^{\gamma_{ij}} + (r(1-R)I)^{\gamma_{ij}/2} + ((1-r)(1-R)I_*)^{\gamma_{ij}/2}\Big) \tag{2.13}$$

satisfy (2.6) by taking for model (2.11)

$$\Phi_{ij}(r,R) = \Psi_{ij}(r,R) = \Psi_{ij}(r,R) = 1,$$

for model (2.12)

$$\Phi_{ij}(r,R) = \min\{R,(1-R)\}^{\gamma_{ij}/2}, \text{ and } \Psi_{ij}(r,R) = \max\{R,(1-R)\}^{\gamma_{ij}/2},$$

and for model (2.13)

$$\Phi_{ij}(r,R) = \min\{R,(1-R)\}^{\gamma_{ij}/2}\min\{r,(1-r)\}^{\gamma_{ij}/2}, \text{and } \Psi_{ij}(r,R) = \max\{R,1-R\}^{\gamma_{ij}/2}.$$

3. The Linearized Boltzmann Operator

The Maxwellian function which represents the equilibrium state of the $i-th$ species of the gas and is denoted by $M_i(v,I)$, and given by

$$M_i(v,I) = \frac{n_i}{(2\pi)^{\frac{3}{2}}\Gamma(\alpha_i+1)}\frac{1}{(\kappa T)^{\alpha_i+\frac{5}{2}}}I^{\alpha_i}e^{-\frac{1}{\kappa T}(\frac{m_i}{2}(v-u)^2+I)},$$

where κ is the Boltzmann constant, and n_i, u, and T are the number of molecules per unit volume, the hydrodynamic velocity, and the temperature respectively. Without loss of generality, we will consider in the sequel a normalized version of M_i, by assuming $\kappa T = n_i = 1$ and $u = 0$. In particular, we will linearize the Boltzmann equation around the following global Maxwellian function

$$M_i(v,I) = \frac{1}{(2\pi)^{\frac{3}{2}}\Gamma(\alpha_i+1)}I^{\alpha_i}e^{-\frac{m_i}{2}v^2-I}. \tag{3.1}$$

We look for a solution f_i around M_i (3.1) having the form

$$f_i(t,x,v,I) = M_i(v,I) + M_i^{1/2}(v,I)g_i(t,x,v,I).$$

The linearization of the Boltzmann operator around M_i leads to introduce the linearized Boltzmann operator $\mathcal{L}$ applied on $g = (g_1,\cdots,g_n)$, with $\mathcal{L}g = ((\mathcal{L}g)_1,\cdots,(\mathcal{L}g)_n)$, where

$$(\mathcal{L}g)_i = \sum_{j=1}^{n} M_i^{-\frac{1}{2}}[Q_{ij}(M_i, M_j^{\frac{1}{2}}g_j) + Q_{ij}(M_i^{1/2}g_i, M_j)].$$

In particular, $(\mathcal{L}g)_i$ writes

$$([\mathcal{L}g)_i = \sum_{j=1}^{n} M_i^{-\frac{1}{2}} \int_\Delta \Big[\frac{M_i'(M_{j*}')^{\frac{1}{2}}}{I'^{\alpha_i}\sqrt{I_*'^{\alpha_j}}} \frac{g_{j*}'}{\sqrt{I_*'^{\alpha_j}}} - \frac{M_i M_{j*}^{\frac{1}{2}}}{I^{\alpha_i}\sqrt{I_*^{\alpha_j}}} \frac{g_{j*}}{\sqrt{I_*^{\alpha_j}}} + \frac{M_{j*}'(M_i')^{\frac{1}{2}}}{I_*'^{\alpha_j}\sqrt{I'^{\alpha_i}}} \frac{g_i'}{\sqrt{I'^{\alpha_i}}} - \frac{M_{j*}M_i^{1/2}}{I_*^{\alpha_j}\sqrt{I^{\alpha_i}}} \frac{g_i}{\sqrt{I^{\alpha_i}}} \Big] \times r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*.$$

Thanks to the conservation of total energy (2.1) we have $\frac{M_i}{I^{\alpha_i}} \frac{M_{j*}}{I_*^{\alpha_j}} = \frac{M_i'}{I'^{\alpha_i}} \frac{M_{j*}'}{I_*'^{\alpha_j}}$, and so $(\mathcal{L}g)_i$ has the following form

$$\begin{aligned}
(\mathcal{L}g)_i =& \\
&-\sum_{j=1}^{n} I^{-\frac{\alpha_i}{2}} \int_\Delta \frac{g_{j*}}{I_*^{\alpha_j/2}} \frac{M_i^{1/2}}{I^{\alpha_i/2}} \frac{M_{j*}^{1/2}}{I_*^{\alpha_j/2}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \\
&\qquad \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_* \\
&-\sum_{j=1}^{n} I^{-\alpha_i} \int_\Delta g_i \frac{M_{j*}}{I_*^{\alpha_j}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_* \\
&+\sum_{j=1}^{n} I^{-\frac{\alpha_i}{2}} \int_\Delta \frac{g_{j*}'}{(I_*')^{\alpha_j/2}} \frac{M_{j*}^{1/2}}{I_*^{\alpha_j/2}} \frac{(M_i')^{1/2}}{(I')^{\alpha_i/2}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \\
&\qquad \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_* \\
&+\sum_{j=1}^{n} I^{-\frac{\alpha_i}{2}} \int_\Delta \frac{g_i'}{I'^{\alpha_i/2}} \frac{M_{j*}^{1/2}}{I_*^{\alpha_j/2}} \frac{(M_{j*}')^{1/2}}{(I_*')^{\alpha_j/2}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \\
&\qquad \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*.
\end{aligned}$$

Here, Δ refers to the open set $(0,1)^2 \times S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$. In addition, $\mathcal{L}$ can be written in the form

$$\mathcal{L} = \mathcal{K} - \nu \, \mathrm{Id},$$

where the i-th component of ν is

$$\nu_i = \sum_{j=1}^{n} I^{-\alpha_i} \int_\Delta \frac{M_{j*}}{I_*^{\alpha_j}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*, \tag{3.2}$$

which represents the collision frequency of the i-th species. We write also the i-th component of $\mathcal{K}g$ as $[\mathcal{K}]_i$ as $(\mathcal{K})_i = (\mathcal{K}_3)_i + (\mathcal{K}_2)_i - (\mathcal{K}_1)_i$ with

$$(\mathcal{K}_1 g)_i = \sum_{j=1}^{n} I^{-\frac{\alpha_i}{2}} \int_\Delta \frac{g_{j*}}{I_*^{\alpha_j/2}} \frac{M_i^{1/2}}{I^{\alpha_i/2}} \frac{M_{j*}^{1/2}}{I_*^{\alpha_j/2}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*, \tag{3.3}$$

$$(\mathcal{K}_2 g)_i = \sum_{j=1}^{n} I^{-\frac{\alpha_i}{2}} \int_{\Delta} \frac{g'_{j*}}{(I'_*)^{\alpha_j/2}} \frac{M_{j*}^{1/2}}{I_*^{\alpha_j/2}} \frac{(M'_i)^{1/2}}{(I')^{\alpha_i/2}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*, \tag{3.4}$$

and

$$(\mathcal{K}_3 g)_i = \sum_{j=1}^{n} I^{-\frac{\alpha_i}{2}} \int_{\Delta} \frac{g'_{i}}{(I')^{\alpha_i/2}} \frac{M_{j*}^{1/2}}{I_*^{\alpha_j/2}} \frac{(M'_{j*})^{1/2}}{(I'_*)^{\alpha_j/2}} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} R^{1/2} I^{\alpha_i} I_*^{\alpha_j} \mathcal{B}_{ij} \, \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}v_*. \tag{3.5}$$

The i-th operator $(\mathcal{L})_i$ of the linearized operator $\mathcal{L}$ is a symmetric operator, with kernel

$$\ker(\mathcal{L}) = \{\mathbf{e}_k, \mathbf{m}v_1, \mathbf{m}v_2, \mathbf{m}v_3, \frac{\mathbf{m}}{2}v^2 + \mathbf{I}\}, \quad k = 1, \cdots, n,$$

where $\mathbf{e_k} = (\delta_{ik})_{i=1,\cdots,n}$, $\mathbf{m} = (m_1, \cdots, m_n)$, and $\mathbf{I} = (I, \cdots, I) \in \mathbb{R}_+^n$. Since $\mathcal{L}$ is symmetric and ν Id is self-adjoint on

$$Dom(\nu \text{ Id}) = \{g \in L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n : \nu g \in L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n\},$$

then $\mathcal{K}$ is symmetric. In the following section, we prove that $\mathcal{K}$ is a compact operator on $L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$. Hence, $\mathcal{L}$ is a self adjoint operator on Dom $(\mathcal{L}) = \text{Dom}(\nu \text{ Id})$. In section 5 we prove that ν is coercive. Therefore, $\mathcal{L}$ is a compact perturbation of the Fredholm operator νId, and thus $\mathcal{L}$ is a Fredholm operator on $L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$.

4. Main Result We state the following theorem, which is the main result of the paper.

Theorem 4.1.

1. *For $\gamma_{ij} \geq 0$, and under assumptions (2.6),(2.8), (2.9), and (2.10), the operator $\mathcal{K}$ whose i-th component is defined in (3.3)-(3.5) is a Hilbert-Schmidt (and thus compact) operator from $L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$, and the multiplication operator by ν is coercive. As a result, the linearized Boltzmann operator $\mathcal{L}$ is an unbounded self adjoint Fredholm operator from $Dom(\mathcal{L}) = Dom(\nu \ Id) \subset L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$.*
2. *For $-1 < \gamma_{ij} < 0$, under assumptions (2.7),(2.8), (2.9), and (2.10), $\mathcal{K}$ is a compact operator from $L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$ to $L^2(\mathbb{R}^3 \times \mathbb{R}_+)^n$.*

Proof. We give the proof of compactness of $\mathcal{K}$ for both cases of γ_{ij} ($\gamma_{ij} \geq 0$ and $-1 < \gamma_{ij} < 0$) right after the following corollary. In addition, we prove that ν is coercive for $\gamma_{ij} \geq 0$ in Section 5. As a result, by Theorem 4.3 in [24], $\mathcal{L}$ is a Fredholm operator for $\gamma_{ij} \geq 0$ and under assumptions (2.5),(2.6), (2.8), and (2.9). □

Corollary 4.1. *For $\gamma_{ij} \geq 0$, and for every $i \in \{1, \cdots, n\}$, there exists $C > 0$ such that, for each $\phi \in L^2(\nu_i dv dI)$, the following coercivity estimate holds*

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} \phi(\mathcal{L}(\phi))_i dv dI \geq C \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\phi - \mathbf{P}_i \phi)^2 \nu_i(v, I) dv dI,$$

where $\mathbf{P}$ is the orthogonal projection on $\ker(\mathcal{L})_i$. The proof of the corollary is similar to that in the monatomic case [25]. Therefore, we only give the proof of Theorem 1.

We carry out the proof of the coercivity of ν Id in Section 5, and we dedicate the rest of this section to the proof of the compactness of $\mathcal{K}$.

Proof of compactness of $\mathcal{K}$ Throughout the proof, we prove the compactness of each $\mathcal{K}_l$ with $l=1,\cdots,3$ separately.

Compactness of $\mathcal{K}_1$. The compactness of $\mathcal{K}_1$ is straightforward as $\mathcal{K}_1$ already possesses a kernel form. Thus, we can inspect the operator kernel of $(\mathcal{K}_1)_i$ (3.3) to be

$$k_1^{ij}(v,I,v_*,I_*)=\frac{(m_i m_j)^{\frac{3}{4}}}{\Gamma(\alpha_i+1)^{1/2}\Gamma(\alpha_j+1)^{1/2}(2\pi)^{\frac{3}{2}}} \int_{(0,1)^2\times S^2} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}I^{\alpha_i/2}I_*^{\alpha_j/2} \mathcal{B}_{ij}e^{-\frac{m_j}{4}v_*^2-\frac{m_i}{4}v^2-\frac{1}{2}I_*-\frac{1}{2}I}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma,$$

and therefore

$$(\mathcal{K}_1 g)_i(v,I)=\sum_{j=1}^n\int_{\mathbb{R}^3\times\mathbb{R}_+} g_j(v_*,I_*)k_1^{ij}(v,I,v_*,I_*)\,\mathrm{d}I_*\mathrm{d}v_* \quad \forall (v,I)\in\mathbb{R}^3\times\mathbb{R}_+.$$

We give the following lemma that yields to the compactness of $\mathcal{K}_1$.

LEMMA 4.1. *Using assumptions* (2.6), (2.7), (2.9), *and* (2.10) *on* $\mathcal{B}_{ij}$, *the function* $k_1^{ij}\in L^2(\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+)$.

Proof. Applying Cauchy-Schwarz inequality we get

$$||k_1^{ij}||_{L^2}^2\le c\int_{\mathbb{R}^3\times\mathbb{R}_+}\int_{\Delta} I^{\alpha_i}I_*^{\alpha_j}\mathcal{B}_{ij}^2 e^{-\frac{m_j}{2}v_*^2-\frac{m_i}{2}v^2-I_*-I}\mathrm{d}I\mathrm{d}v dr dR d\sigma \mathrm{d}I_*\mathrm{d}v_*$$

For $\gamma_{ij}\ge 0$ we use assumptions (2.6) and either (2.9) or (2.10) to get

$$\begin{aligned}
||k_1^{ij}||_{L^2}^2 &\le c_0 + c\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} I^{\alpha_i}I_*^{\alpha_j}\left(|v-v_*|^{2\gamma_{ij}}+I^{\gamma_{ij}}+I_*^{\gamma_{ij}}\right)e^{-\frac{m_j}{2}v_*^2-\frac{m_i}{2}v^2-I_*-I} \\
&\qquad \mathrm{d}I\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_* \\
&\le c_0 + c\int_{\mathbb{R}^3} e^{-\frac{m_j}{2}v_*^2}\left[\int_{|v-v_*|\le 1} e^{-\frac{m_i}{2}v^2}\mathrm{d}v + \int_{|v-v_*|\ge 1}|v-v_*|^{\lceil 2\gamma_{ij}\rceil}e^{-\frac{m_i}{2}v^2}\mathrm{d}v\right]\mathrm{d}v_* \\
&\le c_0 + c\int_{\mathbb{R}^3} e^{-\frac{m_j}{2}v_*^2}\left[\int_{|v-v_*|\ge 1}\sum_{k=0}^{\lceil 2\gamma_{ij}\rceil}|v|^k|v_*|^{\lceil 2\gamma_{ij}\rceil-k}e^{-\frac{m_i}{2}v^2}\mathrm{d}v\right]\mathrm{d}v_* \\
&\le c_0 + c\sum_{k=0}^{\lceil 2\gamma_{ij}\rceil}\int_{\mathbb{R}^3}|v_*|^{\lceil 2\gamma_{ij}\rceil-k}e^{-\frac{m_j}{2}v_*^2}\left[\int_{\mathbb{R}^3}|v|^k e^{-\frac{m_i}{2}v^2}\mathrm{d}v\right]\mathrm{d}v_* < \infty,
\end{aligned}$$

where $\lceil 2\gamma_{ij}\rceil$ is the ceiling of $2\gamma_{ij}$, and c_0 is such that

$$c\int_{(\mathbb{R}_+\times\mathbb{R}^3)^2} I^{\alpha_i}I_*^{\alpha_j}(I^{\gamma_{ij}}+I_*^{\gamma_{ij}})e^{-\frac{m_j}{2}|v_*|^2-\frac{m_i}{2}|v|^2-I_*-I}\mathrm{d}I\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_*\le c_0.$$

On the other hand, for $-1<\gamma_{ij}<0$, we use assumption (2.7) to obtain

$$||k_1||_{L^2}^2 \leq c\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} I^{\alpha_i} I_*^{\alpha_j}(I+I_*+|v-v_*|)^{\gamma_{ij}} e^{-\frac{m_j}{2}|v_*|^2-\frac{m_i}{2}|v|^2-I_*-I}\mathrm{d}I\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_*,$$

and using the inequality

$$(I+I_*+|v-v_*|)^{\gamma_{ij}} \leq I^{\gamma_{ij}/2} I_*^{\gamma_{ij}/2}$$

we get,

$$||k_1||_{L^2}^2 \leq c\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} I^{\alpha_i+\gamma_{ij}/2} I_*^{\alpha_j+\gamma_{ij}/2} e^{-\frac{m_j}{2}|v_*|^2-\frac{m_i}{2}|v|^2-I_*-I}\mathrm{d}I\mathrm{d}I_* < \infty,$$

□

This implies that $(\mathcal{K}_1)_i$ is a Hilbert-Schmidt operator, and thus compact. We prove now the compactness of $(\mathcal{K}_2)_i$, by proving it to be a Hilbert-Schmidt operator as well.

Compactness of $(\mathcal{K}_2)_i$. Additional work is required to inspect the kernel form of $\mathcal{K}_2$, since the kernel is not obvious. $(\mathcal{K}_2)_i$ is written explicitely as

$$\begin{aligned}(\mathcal{K}_2 g)_i(v,I) = &\sum_{j=1}^n \frac{(m_i m_j)^{\frac{3}{4}}}{\Gamma(\alpha_i+1)^{1/2}\Gamma(\alpha_j+1)^{1/2}(2\pi)^{\frac{3}{2}}} \\ &\int_\Delta e^{-\frac{I_*}{2}-\frac{1}{2}r(1-R)\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)-\frac{m_j}{4}v_*^2} \\ &g_j\left(\frac{m_i v+m_j v_*}{m_i+m_j}-\frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma,\right. \\ &\left.(1-R)(1-r)\left[\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right]\right) \\ &e^{-\frac{m_i}{4}\left(\frac{m_i v+m_j v_*}{m_i+m_j}+\frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma\right)^2} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1} \\ &R^{1/2}(I'_*)^{-\alpha_j/2} I^{\alpha_i/2} I_*^{\alpha_j}\mathcal{B}_{ij}\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*.\end{aligned} \tag{4.1}$$

We seek first to write $[\mathcal{K}_2 g]_i$ in its kernel form. For this, the necessary change of variable is $(v_*,I_*)\mapsto(v'_*,I'_*)$ since g depends on (v'_*,I'_*) instead of a direct dependence on (v_*,I_*). We denote by the new variables (x,y) the variables (v'_*,I'_*) to avoid confusions. In particular, the new coordinates (x,y) are

$$\begin{aligned} x &= \frac{m_i v+m_j v_*}{m_i+m_j}-\frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma, \\ y &= (1-R)(1-r)\left[\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right]. \end{aligned} \tag{4.2}$$

We define now $h^{ij}_{v,I,r,R,\sigma}$; where for simplicity the index will be omitted; as

$$\begin{aligned} h:\mathbb{R}^3\times\mathbb{R}_+ &\longmapsto h(\mathbb{R}^3\times\mathbb{R}_+)\subset\mathbb{R}^3\times\mathbb{R}_+ \\ (v_*,I_*) &\longmapsto (x,y)=\Bigg(\frac{m_iv+m_jv_*}{m_i+m_j}-\frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma, \\ &\qquad (1-R)(1-r)\left[\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right]\Bigg), \end{aligned} \tag{4.3}$$

for fixed v,I,r,R, and σ. The function h is invertible, and (v_*,I_*,v',I') can be expressed in terms of (x,y) as

$$v_*=\frac{m_i+m_j}{m_j}x+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma-\frac{m_i}{m_j}v, \quad v'=x+\sqrt{\frac{2}{\mu_{ij}}}\sqrt{Ray}\sigma,$$

and

$$I_*=ay-I-\frac{\mu_{ij}}{2}\left(\frac{m_i+m_j}{m_j}x-\left(\frac{m_i}{m_j}+1\right)v+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma\right)^2, \quad I'=\frac{r}{1-r}y,$$

where $a=\frac{1}{(1-r)(1-R)}$.

LEMMA 4.2. *The Jacobian of h (4.3) is*

$$J=\left|\frac{\partial v_*\partial I_*}{\partial x\partial y}\right|=\left(\frac{m_j}{m_i+m_j}\right)^3(1-r)(1-R).$$

Proof. We pass by the intermediate change of variable

$$(v_*,I_*)\mapsto(v_*,E)\mapsto(x,y).$$

The Jacobian of the first map is unity as we recall that E is given in terms of I_* as $E=\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*$. For the second map, we shall write (x,y) in terms of (v_*,E) as

$$\begin{aligned} x&=\frac{m_iv+m_jv_*}{m_i+m_j}-\frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}E}\sigma, \\ y&=(1-R)(1-r)E. \end{aligned}$$

The Jacobian is therefore given as

$$\begin{aligned} J=\left|\frac{\partial x\partial y}{\partial v_*\partial E}\right| &= \begin{vmatrix} \frac{\partial x_1}{\partial v_{*1}} & \frac{\partial x_1}{\partial v_{*2}} & \frac{\partial x_1}{\partial v_{*3}} & \frac{\partial x_1}{\partial E} \\ \frac{\partial x_2}{\partial v_{*1}} & \frac{\partial x_2}{\partial v_{*2}} & \frac{\partial x_2}{\partial v_{*3}} & \frac{\partial x_2}{\partial E} \\ \frac{\partial x_3}{\partial v_{*1}} & \frac{\partial x_3}{\partial v_{*2}} & \frac{\partial x_3}{\partial v_{*3}} & \frac{\partial x_3}{\partial E} \\ \frac{\partial y}{\partial v_{*1}} & \frac{\partial y}{\partial v_{*2}} & \frac{\partial y}{\partial v_{*3}} & \frac{\partial y}{\partial E} \end{vmatrix} = \begin{vmatrix} \frac{m_j}{m_i+m_j} & 0 & 0 & -\frac{m_i}{(m_i+m_j)}\sqrt{\frac{R}{2\mu_{ij}E}}\sigma_1 \\ 0 & \frac{m_j}{m_i+m_j} & 0 & -\frac{m_i}{(m_i+m_j)}\sqrt{\frac{R}{2\mu_{ij}E}}\sigma_2 \\ 0 & 0 & \frac{m_j}{m_i+m_j} & -\frac{m_i}{(m_i+m_j)}\sqrt{\frac{R}{2\mu_{ij}E}}\sigma_3 \\ 0 & 0 & 0 & (1-r)(1-R) \end{vmatrix} \\ &= \left(\frac{m_j}{m_i+m_j}\right)^3(1-r)(1-R), \end{aligned}$$

since the latter matrix is upper triangular. □

Noticing that the Jacobian J depends on r and R only, instead of an additional dependence on v, I and σ yields to less complications in the proof of the L^2 integrability of the kernel of $\mathcal{K}_2$. The positivity of I_* restricts the variation of the variables (x,y) in integral (4.1) to the space

$$H^{v,I}_{R,r,\sigma} = h(\mathbb{R}^3 \times \mathbb{R}_+) = \\ \left\{ (x,y) \in \mathbb{R}^3 \times \mathbb{R}_+ : ay - I - \frac{\mu_{ij}}{2} \left(\frac{m_i + m_j}{m_j} x - \left(\frac{m_i}{m_j} + 1 \right) v + \frac{m_i}{m_j \sqrt{\mu_{ij}}} \sqrt{2Ray}\sigma \right)^2 > 0 \right\}.$$

In fact, $H^{v,I}_{R,r,\sigma}$ can be explicitly expressed as

$$H^{v,I}_{R,r,\sigma} = \Bigg\{ (x,y) \in \mathbb{R}^3 \times \mathbb{R}_+ : \\ x \in B_{\left[v - \frac{m_i}{(m_i+m_j)\sqrt{\mu_{ij}}} \sqrt{2Ray}\sigma \right]} \left(\frac{m_j}{(m_i+m_j)} \sqrt{\frac{2(ay-I)}{\mu_{ij}}} \right) \text{ and } y \in ((1-r)(1-R)I, +\infty) \Bigg\}.$$

Therefore, equation (4.1) becomes

$$\begin{aligned}(\mathcal{K}_2 g)_i = \sum_{j=1}^n & \frac{(m_i m_j)^{\frac{3}{4}}}{(2\pi)^{\frac{3}{2}} \Gamma(\alpha_i+1)^{1/2} \Gamma(\alpha_j+1)^{1/2}} \int_{(0,1)^2 \times S^2} \int_{H^{v,I}_{R,r,\sigma}} J_{ij} \mathcal{B}_{ij} \\ & e^{-\frac{ay-I}{2} + \frac{\mu_{ij}}{4} \left(\frac{m_i+m_j}{m_j} x - \left(\frac{m_i}{m_j} + 1 \right) v + \frac{m_i}{m_j \sqrt{\mu_{ij}}} \sqrt{2Ray}\sigma \right)^2} \\ & e^{-\frac{r}{2(1-r)} y - \frac{m_j}{4} \left(\frac{m_i+m_j}{m_j} x + \frac{m_i}{m_j \sqrt{\mu_{ij}}} \sqrt{2Ray}\sigma - \frac{m_i}{m_j} v \right)^2} e^{-\frac{1}{4} \left(x + \sqrt{\frac{2Ray}{\mu_{ij}}} \sigma \right)^2} \\ & r^{\alpha_i} (1-r)^{\alpha_j} (1-R)^{\alpha_i+\alpha_j+1} R^{1/2} y^{-\alpha_j/2} I^{\alpha_i/2} I_*^{\alpha_j} g_j(x,y) \mathrm{d}x \mathrm{d}y \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma.\end{aligned} \tag{4.4}$$

We now point out the kernel form of $[\mathcal{K}_2]_i$ and prove after by the help of assumption (2.6) that it belongs to $L^2(\mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+)$. Indeed, we recall the definition of Δ, with $\Delta := (0,1)^2 \times S^2 \times \mathbb{R}_+ \times \mathbb{R}^3$, and we define $H^{v,I}$ to be

$$H^{v,I} := \Bigg\{ (R,r,\sigma,x,y) \in \Delta : R \in (0,1),\, r \in (0,1),\, \sigma \in S^2, \\ x \in B_{\left[v - \frac{m_i}{(m_i+m_j)\sqrt{\mu_{ij}}} \sqrt{2Ray}\sigma \right]} \left(\frac{m_j}{(m_i+m_j)} \sqrt{\frac{2(ay-I)}{\mu_{ij}}} \right), y \in ((1-r)(1-R)I, +\infty) \Bigg\}.$$

We remark that $H^{v,I}_{R,r,\sigma}$ is a slice of $H^{v,I}$, and we define the slice $H^{v,I}_{x,y} \subset (0,1) \times (0,1) \times S^2$ such that

$$H^{v,I} = H^{v,I}_{x,y} \times \mathbb{R}^3 \times \mathbb{R}_+ \text{ which is equivalent to } H^{v,I} = (0,1) \times (0,1) \times S^2 \times H^{v,I}_{R,r,\sigma}.$$

In other words,

$$H^{v,I}_{x,y} = \{ (r,R,\sigma) \in (0,1) \times (0,1) \times S^2 : (y,x,\sigma,r,R) \in H^{v,I} \}.$$

Then by Fubini theorem, it holds that

$$
\begin{aligned}
(\mathcal{K}_2 g)_i(v,I) = & \sum_{j=1}^{n} \frac{(m_i m_j)^{\frac{3}{4}}}{(2\pi)^{\frac{3}{2}}\Gamma(\alpha_i+1)^{1/2}\Gamma(\alpha_j+1)^{1/2}} \\
& \int_{H^{v,I}} J_{ij}\mathcal{B}_{ij} e^{-\frac{ay-I}{2}+\frac{\mu_{ij}}{4}\left(\frac{m_i+m_j}{m_j}x-\left(\frac{m_i}{m_j}+1\right)v+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma\right)^2} \\
& \quad e^{-\frac{r}{2(1-r)}y-\frac{m_j}{4}\left(\frac{m_i+m_j}{m_j}x+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma-\frac{m_i}{m_j}v\right)^2} e^{-\frac{m_i}{4}\left(x+\sqrt{\frac{2Ray}{\mu_{ij}}}\sigma\right)^2} \\
& r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}y^{-\alpha_j/2}I^{\alpha_i/2}I_*^{\alpha_j}g_j(x,y)\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}x\mathrm{d}y \\
= & \sum_{j=1}^{n} \frac{(m_i m_j)^{\frac{3}{4}}}{(2\pi)^{\frac{3}{2}}\Gamma(\alpha_i+1)^{1/2}\Gamma(\alpha_j+1)^{1/2}} \int_{\mathbb{R}^3\times\mathbb{R}_+}\int_{H^{v,I}_{x,y}} J_{ij}\mathcal{B}_{ij} \\
& \quad e^{-\frac{ay-I}{2}+\frac{\mu_{ij}}{4}\left(\frac{m_i+m_j}{m_j}x-\left(\frac{m_i}{m_j}+1\right)v+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma\right)^2} \\
& \quad e^{-\frac{r}{2(1-r)}y-\frac{m_j}{4}\left(\frac{m_i+m_j}{m_j}x+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma-\frac{m_i}{m_j}v\right)^2} e^{-\frac{m_i}{4}\left(x+\sqrt{\frac{2Ray}{\mu_{ij}}}\sigma\right)^2} \\
& \quad r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}y^{-\alpha_j/2}I^{\alpha_i/2}I_*^{\alpha_j}g_j(x,y)\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

The kernel of $(\mathcal{K}_2)_i$ is thus inspected and written explicitly in the following lemma.

LEMMA 4.3. *Using assumptions* (2.6), (2.7), *and* (2.9) *on* $\mathcal{B}_{ij}$, *the kernel of* $[\mathcal{K}_2]_i$ *given by*

$$
\begin{aligned}
k_2^{ij}(v,I,x,y) = & \frac{(m_i m_j)^{\frac{3}{4}}}{(2\pi)^{\frac{3}{2}}\Gamma(\alpha_i+1)^{1/2}\Gamma(\alpha_j+1)^{1/2}} \\
& \int_{H^{v,I}_{x,y}} e^{-\frac{m_i}{4}\left(x+\sqrt{\frac{2Ray}{\mu_{ij}}}\sigma\right)^2} e^{-\frac{ay-I}{2}+\frac{\mu_{ij}}{4}\left(\frac{m_i+m_j}{m_j}x-\left(\frac{m_i}{m_j}+1\right)v+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma\right)^2} \\
& \quad e^{-\frac{r}{2(1-r)}y-\frac{m_j}{4}\left(\frac{m_i+m_j}{m_j}x+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma-\frac{m_i}{m_j}v\right)^2} \\
& \quad J_{ij}\mathcal{B}_{ij}r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}y^{-\alpha_j/2}I^{\alpha_i/2}I_*^{\alpha_j}\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma
\end{aligned}
$$

is in $L^2(\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+)$.

Proof. Applying Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
\|k_2^{ij}\|_{L^2}^2 \le & c\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{H^{v,I}_{x,y}} \\
& e^{-\left[ay-I-\frac{\mu_{ij}}{2}\left(\frac{m_i+m_j}{m_j}x-\left(\frac{m_i}{m_j}+1\right)v+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma\right)^2\right]} e^{-\frac{r}{(1-r)}y} \\
& e^{-\frac{m_j}{2}\left(\frac{m_i+m_j}{m_j}x+\frac{m_i}{m_j\sqrt{\mu_{ij}}}\sqrt{2Ray}\sigma-\frac{m_i}{m_j}v\right)^2} e^{-\frac{m_i}{2}\left(x+\sqrt{\frac{2Ray}{\mu_{ij}}}\sigma\right)^2} \\
& r^{2\alpha_i}(1-r)^{2\alpha_j}(1-R)^{2\alpha_i+2\alpha_j+2}Ry^{-\alpha_j}I^{\alpha_i}I_*^{2\alpha_j}J_{ij}^2\mathcal{B}_{ij}^2\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R\mathrm{d}y\mathrm{d}x\mathrm{d}I\mathrm{d}v.
\end{aligned}
$$

By means of h^{-1}, we get

$$
\begin{aligned}
\|k_2^{ij}\|_{L^2}^2 \le & c\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{(0,1)^2\times S^2} E^{-\alpha_j} e^{-I_*-r(1-R)\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)-\frac{m_j}{2}v_*^2} \\
& e^{-\frac{1}{m_i}\left(\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma\right)^2} r^{2\alpha_i}(1-r)^{\alpha_j} \\
& (1-R)^{2\alpha_i+\alpha_j+2}RI^{\alpha_i}I_*^{2\alpha_j}J_{ij}\mathcal{B}_{ij}^2(v,v_*,I,I_*,r,R,\sigma)\mathrm{d}\sigma\mathrm{d}r\mathrm{d}R\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}I\mathrm{d}v.
\end{aligned}
$$

Furthermore, if $\gamma_{ij}\geq 0$, we use assumption (2.6) on $\mathcal{B}_{ij}$ together with the inequality

$$(v-v_*)^{2\gamma_{ij}}+I^{\gamma_{ij}}+I_*^{\gamma_{ij}}\leq cE^{\gamma_{ij}},$$

and if $-1<\gamma_{ij}<0$, we use assumption (2.7). In both cases, using the inequality

$$I^{\alpha_i}\leq\Big(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\Big)^{\alpha_i}=E^{\alpha_i},$$

we get

$$\begin{aligned}||k_2^{ij}||^2_{L^2}\leq & c\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{(0,1)^2\times S^2}\Psi_{ij}^2(r,R)E^{\alpha_i-\alpha_j+\gamma_{ij}}\\ & e^{-I_*-r(1-R)\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)-\frac{m_j}{2}v_*^2}\\ & e^{-\frac{m_i}{2}\left(\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma\right)^2}\\ & r^{2\alpha_i}(1-r)^{\alpha_j}(1-R)^{2\alpha_i+\alpha_j+2}RJ_{ij}I_*^{2\alpha_j}\,d\sigma\mathrm{d}r\mathrm{d}R\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}I\mathrm{d}v.\end{aligned}\tag{4.5}$$

Perform the change of variable $I\longmapsto E=I+I_*+\frac{\mu_{ij}}{2}|v-v_*|^2$, then as $dI=dE$, (4.5) becomes

$$\begin{aligned}||k_2^{ij}||^2_{L^2}\leq & c\int_{(0,1)^2\times S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\Psi_{ij}^2(r,R)E^{\alpha_i-\alpha_j+\gamma_{ij}}\\ & e^{-I_*-\frac{m_j}{2}v_*^2-r(1-R)E-\frac{m_i}{2}\left(\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}E}\sigma\right)^2}\\ & r^{2\alpha_i}(1-r)^{\alpha_j}(1-R)^{2\alpha_i+\alpha_j+2}RJ_{ij}I_*^{2\alpha_j}\,dI\mathrm{d}v\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\\ = & c\int_{(0,1)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Psi_{ij}^2(r,R)E^{\alpha_i-\alpha_j+\gamma_{ij}}e^{-I_*-\frac{m_j}{2}v_*^2-r(1-R)E}\times\\ & \left[\int_{S^2}\int_{\mathbb{R}^3}e^{-\frac{m_i}{2}\left(\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}E}\sigma\right)^2}\mathrm{d}v\mathrm{d}\sigma\right]\times\\ & r^{2\alpha_i}(1-r)^{\alpha_j-1}(1-R)^{2\alpha_i+\alpha_j+1}RI_*^{2\alpha_j}\,\mathrm{d}E\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R.\end{aligned}$$

Let $\tilde{V}=\frac{m_iv+m_jv_*}{m_i+m_j}+\frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}E}\sigma$, then

$$\begin{aligned}||k_2^{ij}||^2_{L^2}\leq & c\int_{(0,1)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Psi_{ij}^2(r,R)E^{\alpha_i-\alpha_j+\gamma_{ij}}e^{-I_*-\frac{m_j}{2}v_*^2-r(1-R)E}\\ & \left[\int_{S^2}\int_{\mathbb{R}^3}e^{-\frac{m_i}{2}\tilde{V}^2}\mathrm{d}\tilde{V}\mathrm{d}\sigma\right]\\ & r^{2\alpha_i}(1-r)^{\alpha_j-1}(1-R)^{2\alpha_i+\alpha_j+1}RI_*^{2\alpha_j}\,\mathrm{d}E\mathrm{d}I_*\mathrm{d}v_*\mathrm{d}r\mathrm{d}R.\end{aligned}$$

Therefore,

$$\begin{aligned}||k_2^{ij}||^2_{L^2}\leq & c\int_{(0,1)^2}\Psi_{ij}^2(r,R)\left[\int_{\mathbb{R}_+}E^{\alpha_i-\alpha_j+\gamma_{ij}}e^{-r(1-R)E}\mathrm{d}E\right]\\ & r^{2\alpha_i}(1-r)^{\alpha_j-1}(1-R)^{2\alpha_i+\alpha_j+1}Rdr\mathrm{d}R\\ \leq & c\int_{(0,1)^2}\Psi_{ij}^2(r,R)r^{\alpha_i+\alpha_j-1-\gamma_{ij}}(1-r)^{\alpha_j-1}(1-R)^{\alpha_i+2\alpha_j-\gamma_{ij}}Rdr\mathrm{d}R,\end{aligned}$$

with $c>0$. The lemma is thus proved. □ This implies that $\mathcal{K}_2$ is a Hilbert-Schmidt operator.

Compactness of $(\mathcal{K}_3)_i$. The proof of the compactness of $(\mathcal{K}_3)_i$ (3.5) is very similar to that of $(\mathcal{K}_2)_i$. The operator $(\mathcal{K}_3)_i$ which has the explicit form

$$
\begin{aligned}
(\mathcal{K}_3 g)_i(v,I) = \sum_{j=1}^{n} & \frac{(m_i m_j)^{\frac{3}{4}}}{\Gamma(\alpha_i+1)^{1/2}\Gamma(\alpha_j+1)^{1/2}(2\pi)^{\frac{3}{2}}} \\
& \int_{\Delta} g_j\Bigg(\frac{m_i v + m_j v_*}{m_i+m_j} + \frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma, \\
& \qquad r(1-R)\left[\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right]\Bigg)\times \\
& e^{-\frac{I_*}{2}-\frac{1}{2}(1-r)(1-R)\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)-\frac{m_j}{4}v_*^2-\frac{m_j}{4}\left(\frac{m_i v+m_j v_*}{m_i+m_j}\right.} \\
& e^{\left.-\frac{m_i}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma\right)^2} \\
& r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}I^{\alpha_i/2}I_*^{\alpha_j}(I')^{-\alpha_i/2}\mathcal{B}_{ij}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*.
\end{aligned}
$$

inherits the same form as $(\mathcal{K}_2)_i$, with a remark that the new coordinates $(x,y)\in\mathbb{R}^3\times\mathbb{R}_+$ the we use to obtain the kernel form of $(\mathcal{K}_3)_i$ are

$$
\begin{aligned}
x &= \frac{m_i v + m_j v_*}{m_i+m_j} + \frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma, \\
y &= r(1-R)\left[\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right].
\end{aligned}
$$

The Jacobian of the transformation

$$
\begin{aligned}
h: \mathbb{R}^3\times\mathbb{R}_+ &\longmapsto h(\mathbb{R}^3\times\mathbb{R}_+)\subset\mathbb{R}^3\times\mathbb{R}_+ \\
(v_*,I_*) &\longmapsto (x,y) = \Bigg(\frac{m_i v + m_j v_*}{m_i+m_j} + \frac{m_j}{(m_i+m_j)}\sqrt{\frac{2R}{\mu_{ij}}\left(\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right)}\sigma, \\
& \qquad\qquad r(1-R)\left[\frac{\mu_{ij}}{2}(v-v_*)^2+I+I_*\right]\Bigg),
\end{aligned}
$$

is calculated to be

$$
\tilde{J} = \frac{1}{r(1-R)}\left(\frac{m_i+m_j}{m_j}\right)^3.
$$

For the kernel of $\mathcal{K}_3$ to be L^2 integrable, the final computations require

$$
\int_{(0,1)^2} \Psi_{ij}^2(r,R)r^{2\alpha_j-1-\gamma_{ij}}(1-r)^{\alpha_i-1}R(1-R)^{\alpha_j+2\alpha_i+1} < \infty. \tag{4.6}
$$

Applying the change of variable $r\mapsto 1-r$, and using the symmetry assumption (2.8) of Ψ_{ij}, (4.6) is satisfied by (2.10).

5. Properties of the Collision Frequency We give in this section some properties of ν. The first is the coercivity property, which implies that $\mathcal{L}$ is a Fredholm operator, and we prove the monotonicity of ν which depends on the choice of the collision cross section $\mathcal{B}$. The latter property is used for locating the essential spectrum of $\mathcal{L}$.

PROPOSITION 5.1 (Coercivity of ν Id). *With the assumption (2.5), there exists $c(\alpha_i,\alpha_j)>0$ such that*

$$\nu_i(v,I)\geq \sum_{j=1}^{n} c(\alpha_i,\alpha_j)(|v|^{\gamma_{ij}}+I^{\gamma_{ij}/2}+1),\quad i=1,\dots,n,$$

for any $\gamma_{ij}\geq 0$. As a result, the multiplication operator ν_i Id is coercive.

Proof. The collision frequency (3.2) is

$$\nu_i(v,I)=\sum_{j=1}^{n}\frac{m_j^{\frac{3}{2}}}{\Gamma(\alpha_j+1)(2\pi)^{\frac{3}{2}}}\int_{\Delta}\mathcal{B}_{ij}I_*^{\alpha_j}r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}e^{-I_*-\frac{m_j}{2}v_*^2}\,\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*,$$

where by (2.5) we get

$$\begin{aligned}\nu_i(v,I)&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)\int_{S^2}\int_{\mathbb{R}^3}\left(|v-v_*|^{\gamma_{ij}}+I^{\gamma_{ij}/2}\right)e^{-\frac{m_j}{2}v_*^2}\,\mathrm{d}v_*\\&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)\left(I^{\gamma_{ij}/2}+\int_{\mathbb{R}^3}||v|-|v_*||^{\gamma_{ij}}e^{-\frac{m_j}{2}v_*^2}\,\mathrm{d}v_*\right),\end{aligned}$$

where c is a generic constant. We consider the two cases, $|v|\geq 1$ and $|v|\leq 1$. If $|v|\geq 1$ we have

$$\begin{aligned}\nu_i(v,I)&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)\left(I^{\gamma_{ij}/2}+\int_{|v_*|\leq\frac{1}{2}|v|}(|v|-|v_*|)^{\gamma_{ij}}e^{-\frac{m_j}{2}v_*^2}\,\mathrm{d}v_*\right)\\&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)\left(I^{\gamma_{ij}/2}+|v|^{\gamma_{ij}}\int_{|v_*|\leq\frac{1}{2}}e^{-\frac{m_j}{2}v_*^2}\mathrm{d}v_*\right)\\&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)(|v|^{\gamma_{ij}}+I^{\gamma_{ij}/2}+1).\end{aligned}$$

For $|v|\leq 1$,

$$\begin{aligned}\nu_i(v,I)&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)\left(I^{\gamma_{ij}/2}+\int_{|v_*|\geq 2}(|v_*|-|v|)^{\gamma_{ij}}e^{-\frac{m_j}{2}v_*^2}\,\mathrm{d}v_*\right)\\&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)\left(I^{\gamma_{ij}/2}+\int_{|v_*|\geq 2}e^{-\frac{m_j}{2}v_*^2}\,\mathrm{d}v_*\right)\\&\geq\sum_{j=1}^{n}c(\alpha_i,\alpha_j)(1+I^{\gamma_{ij}/2}+|v|^{\gamma_{ij}}).\end{aligned}$$

□

The result is thus proved. We give now the following proposition, which is a generalization of the work of Grad [26], where he proved that the collision frequency of monatomic single gases is monotonic based on the choice of the collision cross section $\mathcal{B}_{ij}$.

PROPOSITION 5.2 (monotonicity of ν_i). *Under the assumption that*

$$\sum_{j=1}^{n}\int_{(0,1)^2\times S^2} r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}\mathcal{B}_{ij}(|V|,I,I_*,r,R,\sigma)\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma$$

is increasing (respectively decreasing) in $|V|$ and I for every I_*, the collision frequency of the i-th species ν_i is increasing (respectively decreasing), where $|V|=|v-v_*|$.

In particular, for Maxwell molecules, where $\sum_{j=1}^{n}\mathcal{B}_{ij}$ is constant in $|V|$ and I, ν_i is constant. On the other hand, for collision cross-sections of the form

$$\mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma)=\Phi_{ij}(r,R)\Big(|v-v_*|^{\gamma_{ij}}+I^{\gamma_{ij}/2}+I_*^{\gamma_{ij}/2}\Big),$$

the integral (5.2) is increasing, and thus ν_i is increasing, where $\gamma_{ij}\geq 0$, and Φ_{ij} is a positive function such that

$$\Phi_{ij}(r,R)=\Phi_{ji}(1-r,R),$$

and

$$\Phi_{ij}(r,R)r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}\in L^1((0,1)^2).$$

In fact, if Φ_{ij} for instance satisfies

$$\Phi_{ij}^2(r,R)r^{\alpha_i+\alpha_j-1-\gamma_{ij}}(1-r)^{\alpha_j-1}(1-R)^{\alpha_i+2\alpha_j+1}R\in L^1((0,1)^2)$$

then this collision cross section satisfies our main assumptions (2.5) and (2.6).

Proof. We remark first that ν_i is a radial function in $|v|$ and I. In fact, we perform the change of variable $V=v-v_*$ in the integral (3.2), where the expression of ν_i becomes

$$\nu_i(|v|,I)=\sum_{j=1}^{n}\frac{m_j^{\frac{3}{2}}}{\Gamma(\alpha_j+1)^{1/2}(2\pi)^{\frac{3}{2}}}\int_{\Delta}\mathcal{B}_{ij}(|V|,I,I_*,r,R,\sigma)e^{-\frac{m_j}{2}(v-V)^2-I_*}\\ I_*^{\alpha_j}r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}V, \tag{5.1}$$

where $\Delta=(0,1)^2\times S^2\times\mathbb{R}_+\times\mathbb{R}^3$. The integration in V in the above integral (5.1) is carried out in the spherical coordinates of V, with fixing one of the axes of the reference frame along v, and therefore, the above integral will be a function of $|v|$ and I.

The partial derivative of ν_i in the v_k direction, where $k=1,2,3$, is

$$\frac{\partial\nu_i}{\partial v_k}=\sum_{j=1}^{n}\int_{\Delta}\frac{v_k-v_{*k}}{|v-v_*|}\frac{\partial\mathcal{B}_{ij}}{\partial|v-v_*|}(|v-v_*|,I,I_*,r,R,\sigma)e^{-\frac{m_j}{2}v_*^2-I_*}\\ \frac{m_j^{\frac{3}{2}}I_*^{\alpha_j}r^{\alpha_i}(1-r)^{\alpha_j}(1-R)^{\alpha_i+\alpha_j+1}R^{1/2}}{\Gamma(\alpha_j+1)^{1/2}(2\pi)^{\frac{3}{2}}}\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma\mathrm{d}I_*\mathrm{d}v_*. \tag{5.2}$$

Perform the change of variable $V = v - v_*$ in (5.2), then

$$\frac{\partial \nu_i}{\partial v_k} = \sum_{j=1}^{n} \int_{\Delta} \frac{V}{|V|} \frac{\partial \mathcal{B}_{ij}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) e^{-\frac{1}{2}(v-V)_*^2 - I_*} \frac{m_j^{\frac{3}{2}} I_*^{\alpha_j} r^{\alpha_i} (1-r)^{\alpha_j} (1-R)^{\alpha_i+\alpha_j+1} R^{1/2}}{\Gamma(\alpha_j+1)^{1/2} (2\pi)^{\frac{3}{2}}} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}V$$

and thus,

$$\sum_{k=1}^{3} v_k \frac{\partial \nu_i}{\partial v_k} = \sum_{j=1}^{n} \int_{\Delta} \frac{v \cdot V}{|V|} \frac{\partial \mathcal{B}_{ij}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) e^{-\frac{1}{2}(v-V)_*^2 - I_*} \frac{m_j^{\frac{3}{2}} I_*^{\alpha_j} r^{\alpha_i} (1-r)^{\alpha_j} (1-R)^{\alpha_i+\alpha_j+1} R^{1/2}}{\Gamma(\alpha_j+1)^{1/2} (2\pi)^{\frac{3}{2}}} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}V \tag{5.3}$$

Applying Fubini's theorem, we write (5.3) as

$$\sum_{k=1}^{3} v_k \frac{\partial \nu_i}{\partial v_i} = \sum_{j=1}^{n} \int_{\mathbb{R}_+ \times \mathbb{R}^3} \left[\int_{(0,1)^2 \times S^2} \frac{\partial \mathcal{B}_{ij}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \right] \times \frac{m_j^{\frac{3}{2}} I_*^{\alpha_j} r^{\alpha_i} (1-r)^{\alpha_j} (1-R)^{\alpha_i+\alpha_j+1} R^{1/2}}{\Gamma(\alpha_j+1)^{1/2} (2\pi)^{\frac{3}{2}}} \frac{v \cdot V}{|V|} e^{-\frac{1}{2}(v-V)^2 - I_*} \mathrm{d}I_* \mathrm{d}V.$$

The partial derivative of ν_i along I is

$$\begin{aligned} \frac{\partial \nu_i}{\partial I} &= \sum_{j=1}^{n} \int_{\Delta} \frac{m_j^{\frac{3}{2}} I_*^{\alpha_j} r^{\alpha_i} (1-r)^{\alpha_j} (1-R)^{\alpha_i+\alpha_j+1} R^{1/2}}{\Gamma(\alpha_j+1)^{1/2} (2\pi)^{\frac{3}{2}}} \frac{\partial \mathcal{B}_{ij}}{\partial I}(|V|, I, I_*, r, R, \sigma) e^{-\frac{m_j}{2}(v-V)^2 - I_*} \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \mathrm{d}I_* \mathrm{d}V \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}^3 \times \mathbb{R}_+} \frac{m_j^{\frac{3}{2}} I_*^{\alpha_j} r^{\alpha_i} (1-r)^{\alpha_j} (1-R)^{\alpha_i+\alpha_j+1} R^{1/2}}{\Gamma(\alpha_j+1)^{1/2} (2\pi)^{\frac{3}{2}}} \left[\int_{(0,1)^2 \times S^2} \frac{\partial \mathcal{B}_{ij}}{\partial I}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma \right] e^{-\frac{m_j}{2}(v-V)^2 - I_*} \mathrm{d}I_* \mathrm{d}V. \end{aligned} \tag{5.4}$$

When $v \cdot V > 0$, the exponential in the integral (5.3) is greater than when $v \cdot V < 0$, and so the term $v \cdot V$ doesn't affect the sign of the partial derivatives of ν_i. Therefore, the sign of the partial derivative of ν_i along $|v|$ has the same sign as

$$\sum_{j=1}^{n} \int_{(0,1)^2 \times S^2} \frac{m_j^{\frac{3}{2}} (1-r)^{\alpha_j} (1-R)^{\alpha_j}}{\Gamma(\alpha_j+1)^{1/2}} \frac{\partial \mathcal{B}_{ij}}{\partial |V|}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma.$$

It's clear as well that the partial derivative of ν_i with respect to I (5.4) has the same sign as

$$\sum_{j=1}^{n} \int_{(0,1)^2 \times S^2} \frac{m_j^{\frac{3}{2}} (1-r)^{\alpha_j} (1-R)^{\alpha_j}}{\Gamma(\alpha_j+1)^{1/2}} \frac{\partial \mathcal{B}_{ij}}{\partial I}(|V|, I, I_*, r, R, \sigma) \mathrm{d}r \mathrm{d}R \mathrm{d}\sigma.$$

As a result, for a collision cross-section $\mathcal{B}_{ij}$ satisfying the condition that the integral

$$\int_{(0,1)^2\times S^2} \frac{m_j^{\frac{3}{2}}(1-r)^{\alpha_j}(1-R)^{\alpha_j}}{\Gamma(\alpha_j+1)^{1/2}}\mathcal{B}_{ij}(|V|,I,I_*,r,R,\sigma)\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma$$

is increasing (respectively decreasing) in $|V|$ and I, the collision frequency is increasing (respectively decreasing). □

In this appendix, we aim to write the collision operator (2.3) in an equivalent form. The derivation of the latter is a result of subsequent changes of variables, see (.5). The final result sought is the Jacobian of the following map:

$$\begin{aligned} T:\mathbb{R}^6\times\mathbb{R}_+^2\times(0,1)^2\times S^2 &\to \mathbb{R}^6\times\mathbb{R}_+^2\times\mathbb{R}^3\times\mathbb{R}_+ \\ (v,v_*,I,I_*,r,R,\sigma) &\mapsto (v,G,E,I,v',I'), \end{aligned} \tag{.5}$$

where $g=v-v_*$ and $G=\frac{m_i v+m_j v_*}{m_i+m_j}$. For this transformation, the following Jacobians are elementary:

$$J_{(v,v_* I,I_*,r,R,\sigma)\mapsto(g,G,I,I_*,r,R,\sigma)}=1,$$

and

$$J_{(g,G,I,I_*,r,R,\sigma)\mapsto(g,G,I,E,r,R,\sigma)}=1. \tag{.6}$$

Equation (.6) is due to the fact that only E is a function of I_*. What remains in deducing the Jacobian of T is calculating the Jacobian of the transformation $(g,G,I,E,r,R,\sigma)\mapsto(v,G,I,E,v',I')$. As an intermediate step we define

$$\lambda=\frac{m_j}{m_i+m_j}\sqrt{\frac{2RE}{\mu_{ij}}},\quad \mu=r(1-R),$$

which induces the Jacobian

$$J_{(g,G,I,E,r,R,\sigma)\mapsto(g,G,I,E,\lambda,\mu,\sigma)}=\frac{1}{2}\frac{m_j}{m_i+m_j}\frac{(1-R)}{\sqrt{R}}\sqrt{\frac{2E}{\mu_{ij}}}.$$

Thus the final sub-transformation is $(g,G,I,E,\lambda,\mu,\sigma)\mapsto(v,G,I,E,v',I')$, where specifically,

$$v'=G+\lambda\sigma,\quad \text{and}\quad I'=\mu E.$$

It's clear that

$$J_{(g,G,I,E,\lambda,\mu,\sigma)\mapsto(g,G,I,E,\lambda,I',\sigma)}=E$$

and for v' we have

$$J_{(g,G,I,E,\lambda,I',\sigma)\mapsto(g,G,I,E,v',I')}=\lambda^2=\left(\frac{m_j}{m_i+m_j}\right)^2\frac{2RE}{\mu_{ij}},$$

since (λ,σ) is the spherical representation of $v'-G$. As $v=\frac{m_j}{m_i+m_j}g+G$, then the Jacobian

$$J_{(g,G,I,E,v',I')\mapsto(v,G,I,E,v',I')}=\left(\frac{m_j}{m_i+m_j}\right)^3.$$

Finally, combining the preceding transformations, the Jacobian of T is

$$J_T = \frac{\sqrt{2}}{\mu_{ij}^{\frac{3}{2}}}\left(\frac{m_j}{m_i+m_j}\right)^6 R^{\frac{1}{2}}(1-R)E^{\frac{5}{2}}.$$

In other words,

$$\mathrm{d}v\mathrm{d}v_*\mathrm{d}I\mathrm{d}I_*\mathrm{d}r\mathrm{d}R\mathrm{d}\sigma = \frac{\sqrt{2}}{\mu_{ij}^{\frac{3}{2}}}\left(\frac{m_j}{m_i+m_j}\right)^6 R^{\frac{1}{2}}(1-R)E^{\frac{5}{2}}\mathrm{d}v\mathrm{d}G\mathrm{d}I\mathrm{d}E\mathrm{d}v'\mathrm{d}I'.$$

The equivalent model of (2.3), based on the above computations is therefore

$$Q_{ij}(f,f)(v,I) = \int_{(\mathbb{R}^3\times\mathbb{R}_+)^2}\left(\frac{f_i' f_{j*}'}{I'^{\alpha_i} I_*'^{\alpha_j}} - \frac{f_i f_{j*}}{I^{\alpha_i} I_*^{\alpha_j}}\right) W_{ij}(v,I,v',I',G,E)\,\mathrm{d}G\mathrm{d}E\ \mathrm{d}v'\mathrm{d}I',$$

where

$$W_{ij}(v,I,v',I',G,E) = \frac{\mu_{ij}^{\frac{3}{2}}}{\sqrt{2}}\left(\frac{m_i+m_j}{m_j}\right)^6 (I'I)^{\alpha_i}(I_*I_*')^{\alpha_j}E^{-\frac{5}{2}-\alpha_i-\alpha_j}\mathcal{B}_{ij}(v,v_*,I,I_*,r,R,\sigma), \tag{.7}$$

where $I_* = I_*(v,I,G,E)$, $I_*' = I_*'(v',I',G,E)$, $v_*' = v_*'(G,v')$, $v_* = v_*(G,v)$, $\sigma = \sigma(v',G)$, $R = R(v',E,G)$, and $r = r(I',v',E,G)$. Moreover, W_{ij} in (.7) is clearly microreversible, and the measure $\mathrm{d}E\mathrm{d}G\mathrm{d}v\mathrm{d}I\mathrm{d}v'\mathrm{d}I'$ is obviously invariant if time is reversed.

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