

# GENERAL MOMENT SYSTEM FOR PLASMA PHYSICS BASED ON MINIMUM ENTROPY PRINCIPLE

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**ABSTRACT.** In plasma physics domain, the electrons transport can be described from kinetic and hydrodynamical models. Both methods present disadvantages and thus cannot be considered in practical computations for Inertial Confinement Fusion (ICF). That is why we propose in this paper a new model which is intermediate between these two descriptions. More precisely, the derivation of such models is based on an angular closure in the phase space and retains only the energy of particles as a kinetic variable. The closure of the moment system is obtained from a minimum entropy principle. The resulting continuous model is proved to satisfy fundamental properties. Moreover the model is discretized w.r.t the energy variable and the semi-discretized scheme is shown to satisfy conservation properties and entropy decay.

**1. Introduction.** A variety of classical problems in kinetic theory leads to use the Fokker-Planck-Landau equation to describe the evolution of different species of charged particles undergoing binary collisions ([9, 14]). For these collisions, the interaction potential is the long-range Coulomb interactions. More precisely the solutions of the kinetic equations are non-negative distribution functions  $f_\alpha(t, x, v)$  specifying the density of each species  $\alpha$  with velocity  $v$  at time  $t$  and position  $x$ . Here the plasma consists of electrons and ion species (see [35]). In this paper, the plasma is studied during the time scale corresponding to the electron frequency. This time scale being small compared to the characteristic time of ions motion, the

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ions can be considered as immobile. To approximate the solution of such problems, many computational methods have been developed up to now.

In [6, 7, 13], deterministic schemes have been developed to approximate the solution of the Fokker-Planck-Landau equation by using a phase space grid. Then in [19], the evolution of electromagnetic fields was inserted in the scheme to describe exactly the electrons transport. This scheme presents many fundamental properties such as conservation laws and positivity of the distribution function. However kinetic models for electrons transport are too much expensive to be used for Inertial Confinement Fusion (ICF). To reduce the computational time, plasmas can be described by fluid models. For example in [10, 11, 12], the authors consider a bi-fluid compressible Euler model coupled with the Poisson equation. Moreover for ITER application, the isothermal two-fluid Euler-Lorentz system coupled with a quasi-neutrality constraint has been studied in [4] by introducing an asymptotic preserving scheme. However, for the new high energy target drivers, the kinetic effects are too important to neglect them.

In the present paper, we propose an alternative approach by considering an intermediate description between fluid and kinetic level like in [31]. The velocity variable is written in spherical coordinates and the model is written by considering moment systems with respect to the angular variable. But the electron/electron collision operator being non linear, the moments extraction is complicated. That is why physicists approximate this operator by assuming that the main contribution of the distribution function comes from its isotropic part ([36]). But this approached model does not conserve the realizability domain defined as the set of vectors that are the moments of positive distribution functions.

There exists many moment models whose difference comes from the choice of the closure which approximates the full distribution function. This closure is essential to get a closed moment system and to assure some physics properties. Indeed, a well-known moment model called  $P_N$  (briefly reviewed in [8]) leads to solutions that are inconsistent with a positive concentration of particles. For example, the  $P_N$  model used in [29] does not satisfy positivity of the underlying distribution function of electrons and entropy dissipation. This comes from the definition of the closure that is based on a truncation of the spherical harmonic expansion w.r.t the angular variable. For the  $P_N$  model, modifications are proposed in ([22]) to obtain a non-negative distribution function. For example, in [26, 27], the authors propose an asymptotic limit approximation to prescribe boundary conditions for the neutronic transport equation.

In this context, the motivation of our current work is to use a moments closure based on an entropy minimization principle which inherits many of the fundamental properties such as the conservation laws and entropy dissipation. This closure leads to a new moment model called  $M_N$  model. This model has been firstly introduced in [18] in a reduced form to find an approximation of radiative transfer and energy evolution equation. The subsequent system of conservation laws is closed by minimizing the radiative entropy (see also [20] for extension). Moreover, a moment model  $M_1$  has been proposed in [31] by considering the three first moments. The benefit of the  $M_1$  model lies in that it satisfies fundamental properties. Indeed mass and energy are conserved, the distribution function stays positive and the entropy decreases. Nevertheless proofs presented in [31] are very specific to the  $M_1$  model. Indeed theorems and proofs are based on the knowledge of known coefficients

to define the distribution function  $f$ . Here these coefficients are unknown which implies to use a different approach.

In kinetic theory, the idea of using minimization problem under moments constraints has been analyzed by Levermore in the framework of gas dynamics ([28]). The main motivation was to construct BGK models ([2]) leading to the correct Prantl number at the Navier-Stokes level. Moreover the well-posedness of such problems has been clarified later by Junk and Schneider ([25, 33]). In the present case, the aim is different. Here the energy of particles is a free parameter. Then we integrate only the kinetic equation expressed in spherical coordinate with respect to the angle variable and we return only the energy of particles as a kinetic variable in the frame of ions considered completely frozen.

At the numerical point of view, the entropic average used to define the distribution function on interfaces allows to show that the semi-discretized scheme is entropic.

The remainder of the paper is organized as follows. In section 2, we briefly review some backgrounds about the collision operator properties and present the kinetic equation considered. In section 4, we introduce a new electron/electron collision operator that is devoted to approach the full Landau collision operator. This new model is based on a linearization of the Landau operator around the equilibrium state of the electron/ion collision operator. Next, this new operator is proved to preserve the realizability domain. Section 5 deals with the derivation of a semi-discrete scheme, where the time is kept as a continuous variable whereas the energy variable is discrete. The definition of the approximated solution on the dual mesh through an entropic average allows to get the entropic dissipation property of collision operator. Next, the N-moment model is derived from the continuous kinetic equation in section 6 and is shown to preserve previous property. In section 7, the semi-discrete scheme for the moment system is proved to be entropic. Moreover we present at the end of this part by a test case, showing the interest of the  $M_2$  and  $M_3$  models compared to the  $M_1$  model. Comparisons with  $P_N$  models are also given. Finally we finish in the last section by conclusions to this work.

**2. Presentation of the kinetic model.** In this section we recall some backgrounds about classical kinetic models for plasma physics where the ions are considered as frozen. Firstly, the general kinetic model is presented. Then some fundamental properties of collision operators are explained.

**2.1. Classical kinetic models.** In this paper, the time evolution of the electrons is described by a distribution function  $f(t, x, v)$ ,  $x \in \mathbb{R}^3$ ,  $v \in \mathbb{R}^3$  solution of a kinetic equation.

**2.1.1. hydrodynamic quantities.** From the distribution function, we define the density, the macroscopic velocity and the temperature of the electrons as follows

$$n = \int_{\mathbb{R}^3} f(t, x, v) dv, \quad nu = \int_{\mathbb{R}^3} v f(t, x, v) dv, \quad T = \frac{2}{3} \int_{\mathbb{R}^3} (v - u)^2 f(t, x, v) dv.$$

**2.1.2. Classical kinetic equations.** In the present paper, the distribution function  $f$  satisfies the kinetic equation

$$\partial_t f + v \partial_x f = C(f, f), \tag{2.1}$$

where the collision operator  $C(f, f)$  introduced in [9] is defined by

$$C(f, f) = C_{ee}(f, f) + C_{ei}(f).$$

The operator  $C_{ee}$  stands for the electron-electron collision operator

$$C_{ee}(f, f) = \nabla_v \cdot \left( \int_{\mathbb{R}^3} \Phi(V) [f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv' \right), \quad (2.2)$$

where  $V = v - v'$  is the relative velocity of electrons and  $\Phi$  is an operator acting on the relative velocity  $V$

$$\Phi(V) = \frac{1}{|V|^3} (|V|^2 Id - V \otimes V), \quad (2.3)$$

where  $Id$  is the unit tensor.

The electron-ion collision operator  $C_{ei}(f)$  is defined by

$$C_{ei}(f) = \nabla_v \cdot [\Phi(V) \nabla_v f(v)]. \quad (2.4)$$

**2.2. Properties of the collision operator.** Next we present the conservation laws and entropy dissipation property of the collision operators  $C_{ee}$  and  $C_{ei}$ . The electron-electron collision operator satisfies mass, momentum and energy conservation properties

$$\int_{\mathbb{R}^3} C_{ee}(f, f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0, \quad t \geq 0,$$

while the electron-ion collision operator satisfies only mass and energy conservation

$$\int_{\mathbb{R}^3} C_{ei}(f) \begin{pmatrix} 1 \\ |v|^2 \end{pmatrix} dv = 0, \quad t \geq 0.$$

They both dissipate the entropy i.e.

$$\int_{\mathbb{R}^3} C_{ei}(f) \log f \, dv \leq 0 \quad \text{and} \quad \int_{\mathbb{R}^3} C_{ee}(f, f) \log f \, dv \leq 0, \quad (2.5)$$

which implies that the Boltzmann entropy

$$\mathcal{H}(f) = \int_{\mathbb{R}^3} (f \log f - f) \, dv$$

is a Lyapounov function for (2.1).

**Property 1.** [(i)]

1. *The equilibrium state of the electron-ion collision operator  $C_{ei}$  (i.e.  $C_{ei}(f) = 0$ ) is given by the set of isotropic functions  $f = f(|v|)$ .*
2. *The equilibrium state of the electron-electron collision operator  $C_{ee}$  (i.e.  $C_{ee}(f, f) = 0$ ) is given by the Maxwellian distribution function such that*

$$\exists(n, m_e, T, u) \in \mathbb{R}_+^3 \times \mathbb{R}^3 / f(v) = n \left( \frac{m_e}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left( -\frac{m_e(v-u)^2}{2k_B T} \right),$$

where  $k_B$  is the Boltzmann constant,  $n$  is the density,  $T$  is the temperature and  $u_e$  represents the mean velocity.

3. *The equilibrium state considered by both collision operators is given by the isotropic Maxwellian distribution function such that*

$$\exists(m_e, n, T) \in \mathbb{R}_+^3 / f(v) = n \left( \frac{m_e}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left( -\frac{m_e v^2}{2k_B T} \right).$$

**3. Setting of the problem.** In this section, we explain the construction of the  $M_N$  model and the definition of the realizability domain. Next in subsection 3.3 we show that the classical approximation of electron/electron collision operator does not conserve the realizability domain. This is one of the main motivation of this paper.

**3.1. Notations.** If  $\mathbb{S}^2$  is the unit sphere,  $\Omega = v/|v|$  represents the direction of propagation of the particles and  $\mu = \Omega_x = \cos \theta$ ,  $\theta \in [0, \pi]$ . In this paper, we chose a one dimensional direction of propagation, i.e. we take  $\mu \in [-1, 1]$  as the direction of propagation instead of  $\Omega$ . By setting  $\zeta = |v|$ , the distribution function  $f$  writes in spherical coordinates:  $f(x, \zeta, \mu)$ . Hence the  $N$  first moments with respect to  $\mu$  are defined by

$$f^i = 2\pi\zeta^2 \int_{-1}^1 f(\zeta, \mu)\mu^i d\mu = \zeta^2 \langle f\mu^i \rangle \quad \forall i \in \{0, N\}, \quad (3.1)$$

where  $\langle \cdot \rangle$  is defined for any function  $\Psi$  by

$$\langle \Psi \rangle = 2\pi \int_{-1}^1 \Psi(\mu) d\mu.$$

We use also the following notation

$$F^i(\zeta) = \frac{f^i(\zeta)}{\zeta^2}. \quad (3.2)$$

This means that  $F^0$  represents the isotropic part of the distribution function  $f$  ( $F^0 = \langle f \rangle$ ).

In this paper, moment systems will be constructed from kinetic equations. One fundamental property for these systems is the preservation of the realizability domain

$$\mathcal{A} = \left\{ \bar{g} = \begin{pmatrix} g^0 \\ \vdots \\ g^N \end{pmatrix} \in \mathbb{R}^N / \exists g \geq 0 \in L^1([-1, 1]) \text{ and } g^i = \zeta^2 \langle \mu^i g(\mu) \rangle \forall i \in \{0, N\} \right\}. \quad (3.3)$$

**3.3** guarantees that the moments can be recovered from a nonnegative distribution function.

**3.2. The  $M_N$  closure.** One important step of the present work, is the construction of a moment system from a kinetic equation. But the extraction of these moments leads to system of equations having more unknowns than equations because a supplementary moment appears. That is why, a closure must be defined such that the highest moment writes in function of the previous ones. The closure chosen is the  $M_N$  closure that is based on an entropy minimization principle. The advantage of such a closure compared to the classical  $P_N$  closure is the nonnegativity of the underlying distribution function.

More precisely, the  $M_N$  closure is obtained by solving the following entropy minimization problem:  $f^i(\zeta)$ ,  $i \in \{0, N\}$  being given

$$\min_{g \geq 0} \{ \mathcal{H}(g), \zeta \in \mathbb{R}_+, / \int_{-1}^1 \mu^i g(\mu, \zeta) d\mu = f^i(\zeta), i \in \{0; N\} \}. \quad (3.4)$$

Therefore according to [28, 32], if  $(f^i(\zeta))_{i \in \{0;N\}} \in \mathcal{A}$ , then  $f$  solution to (3.4) reads

$$f(t, x, \zeta, \mu) = \exp(\bar{\alpha}(t, x, \zeta) \bar{\mu}) , \quad (3.5)$$

$$\text{with } \bar{\alpha} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_N \end{pmatrix} \text{ and } \bar{\mu} = \begin{pmatrix} \mu^0 \\ \vdots \\ \mu^N \end{pmatrix} .$$

**3.3. Conservation of the realizability domain.** Consider here the case of a space homogeneous situation with one species of particle, i.e. the plasma consists only of electrons. In this case, equation (2.1) is reduced to

$$\partial_t f = C_{ee}(f, f) . \quad (3.6)$$

The moments extraction for the electron-electron collision operator  $C_{ee}(f, f)$  is very complicated because of its nonlinearity. That is why in Plasma physics, classical approximations for the operator lead to consider that the main contribution for the electron-electron collision operator comes from the isotropic part of the distribution function. This means that the collision operator  $C_{ee}(f, f)$  is approached by

$$Q_{ee}^0 = \frac{1}{\zeta^2} \partial_\zeta \left( \zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta (F^0(\zeta)) - F^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right) , \quad (3.7)$$

with

$$\tilde{J}(\zeta, \zeta') = \frac{2}{3} \inf \left( \frac{1}{\zeta^3}, \frac{1}{\zeta'^3} \right) \zeta'^2 \zeta^2 . \quad (3.8)$$

This approximation (3.7) denoted by  $Q_{ee}^0$  has been presented in [36] and used in [5, 15, 16] to study the homogeneous Fokker-Planck-Landau equation for isotropic distribution functions. However, the following example 1 shows that this model does not preserve the realizability domain  $\mathcal{A}$ .

Indeed, consider the distribution function as in (3.5) and extract the two first moments of (3.6) with respect to  $\mu$ , where  $C_{ee}(f, f)$  has been replaced by  $Q_{ee}^0$ . Hence, we get

$$\begin{cases} \partial_t f^0 = Q_{ee}^0 , \\ \partial_t f^1 = 0 . \end{cases} \quad (3.9)$$

For more details, the expression (3.7) and the derivation of the system (3.9) are explained in Appendix A.

In the particular case of the  $N = 1$ , the realizability domain  $\mathcal{A}$  is shown to be equal to

$$\mathcal{B} = \left\{ \bar{g} = \begin{pmatrix} g^0 \\ g^1 \end{pmatrix} \in \mathbb{R}^2, g^0 > 0 \text{ and } |g^1| < g^0 \right\} \cup \{(0, 0)\} .$$

**Example 1.** Let us chose the following initial data for  $f^0$  and  $f^1$

$$f^0(t=0) = \frac{1}{3} \chi_{[0,3]}(\zeta) \text{ and } f^1(t=0) = \frac{1}{4} \chi_{[0,3]}(\zeta) .$$

Because of the electron-electron collision operator effect,  $f^0$  converges to a centered Maxwellian. Moreover, the  $M_1$  model preserves mass and energy. So the obtained Maxwellian presents the same mass ( $\int_0^\infty f^0 d\zeta = 1$ ) and the same energy

( $\int_0^\infty f^0 \zeta^2 d\zeta = 3$ ) as the initial distribution function. So the asymptotic steady state for  $f^0$  reads

$$f^0 = \sqrt{\frac{2}{\pi}} \exp\left(\frac{-\zeta^2}{2}\right) \zeta^2 .$$

Therefore figure 1 shows that the realizability domain is not preserved anymore.

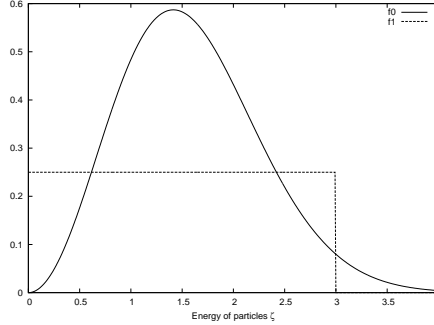


FIGURE 1. Representation of the moments  $f^0$  and  $f^1$  with respect to the energy  $\zeta$  at the steady state.

#### 4. New Kinetic continuous model.

**4.1. Approximation of the collision operator  $C_{ee}$ .** In order to preserve the realizability domain (proof in section 6.2), we consider a new collision operator based on a linearisation of  $C_{ee}$  around the equilibrium state of  $C_{ei}$ . The expression of this new collision operator is given by equation (4.1).

**Approximation of  $C_{ee}$**  The electron-electron collision operator  $Q_{ee}(f)$  and the electron-ion collision operator  $Q_{ei}(f)$  are given by

$$Q_{ee}(f) = \frac{1}{\zeta^2} \partial_\zeta \left( \zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right), \quad (4.1)$$

where  $\tilde{J}(\zeta, \zeta')$  is given in (3.8).

For  $C_{ei}$ , we do not use any approximation, since this operator is already linear. To be consistent with the notation  $Q_{ee}(f)$ , we use the notation  $Q_{ei}$  instead of  $C_{ei}$ . Its expression can be rewritten

$$Q_{ei}(f) = \frac{1}{\zeta^3} \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial f}{\partial \mu} \right). \quad (4.2)$$

The computation of (4.2) in spherical coordinate is established in Appendix B. So equation (2.1) can be approached in spherical coordinates by

$$\partial_t f + \zeta \mu \partial_x f = Q(f), \quad (4.3)$$

where  $Q(f) = Q_{ee}(f) + Q_{ei}(f)$  and  $Q_{ee}$ ,  $Q_{ei}$  are defined by (4.1, 4.2).

**Remark 1.** If  $f$  is isotropic, i.e.  $f = F^0$ , we obtain

$$Q(f) = Q(F^0) = \int_{-1}^1 C_{ee}(F^0, F^0) d\mu = \int_{-1}^1 C_{ee}(f, f) d\mu .$$

Now, let us check that the new model conserves fundamental properties.

**4.2. Properties of the model.** In this subsection, we present the fundamental properties satisfied by the new model, that is the conservation laws and a H-theorem.

**Proposition 1.** *The operator  $Q(f)$  satisfies mass and energy conservation properties and the entropy dissipation property, i.e.*

$$\left\langle \int_0^\infty \zeta^2 Q(f) \left( \frac{1}{\zeta^2} \right) d\zeta \right\rangle = 0 \quad \text{and} \quad \left\langle \int_0^\infty \zeta^2 Q(f) \log f d\zeta \right\rangle \leq 0.$$

*Proof.* Firstly, in order to recover the conservations property, consider the quantity

$$\left\langle \int_0^\infty \zeta^2 Q(f) \left( \frac{1}{\zeta^2} \right) d\zeta \right\rangle = \int_0^\infty \zeta^2 \langle Q_{ee}(f) \rangle \left( \frac{1}{\zeta^2} \right) d\zeta + \int_0^\infty \zeta^2 \langle Q_{ei}(f) \rangle \left( \frac{1}{\zeta^2} \right) d\zeta. \quad (4.4)$$

For the sake of clarity, we introduce the notation

$$Q_{ee}^0 = \zeta^2 \langle Q_{ee}(f) \rangle.$$

The first term of the right hand side of equation (4.4) can be rewritten as

$$\int_0^\infty Q_{ee}^0 \left( \frac{1}{\zeta^2} \right) d\zeta = \int_0^\infty \partial_\zeta \left( \zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta F^0(\zeta) - F^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right) \left( \frac{1}{\zeta^2} \right) d\zeta.$$

So,  $\int_0^\infty Q_{ee}^0 d\zeta = 0$ . Moreover, since

$$\begin{aligned} \int_0^\infty Q_{ee}^0 \zeta^2 d\zeta &= -2 \int_0^\infty \int_0^\infty \zeta^2 \zeta'^2 \tilde{J}(\zeta, \zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta F^0(\zeta) \right] d\zeta' d\zeta \\ &\quad + 2 \int_0^\infty \int_0^\infty \zeta^2 \zeta'^2 \tilde{J}(\zeta, \zeta') \left[ F^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] d\zeta' d\zeta, \end{aligned}$$

we get from Fubini theorem

$$\int_0^\infty Q_{ee}^0 \zeta^2 d\zeta = 0.$$

Besides, we have  $\langle Q_{ei}(f) \rangle = 0$ . Therefore the conservation properties follow.

Now we prove the entropy dissipation property of the whole collision operator  $Q$ . By using a Green formula we obtain easily that  $Q_{ei}$  dissipates entropy. Moreover

$$\begin{aligned} \left\langle \int_0^\infty \zeta^2 Q_{ee}(f) \log f d\zeta \right\rangle &= - \left\langle \int_0^\infty \int_0^\infty \zeta'^2 \zeta \tilde{J}(\zeta, \zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \right. \\ &\quad \left. \frac{1}{f} \partial_\zeta f d\zeta d\zeta' \right\rangle. \end{aligned} \quad (4.5)$$

Equation (4.5) can be rewritten in terms of  $f$  as

$$\begin{aligned} \left\langle \int_0^\infty \zeta^2 Q_{ee}(f) \log f d\zeta \right\rangle &= - \int_{-1}^1 \int_{-1}^1 \int_0^\infty \int_0^\infty \zeta'^2 \zeta \tilde{J}(\zeta, \zeta') \left[ f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} f(\zeta') \right] \\ &\quad \frac{1}{f} \partial_\zeta f d\zeta d\zeta' d\mu d\mu' \\ &= - \int_{-1}^1 \int_{-1}^1 \int_0^\infty \int_0^\infty \zeta'^2 \zeta^2 \tilde{J}(\zeta, \zeta') f(\zeta) f(\zeta') \left[ \frac{1}{\zeta} \partial_\zeta (\log f(\zeta)) - \frac{1}{\zeta'} \partial_{\zeta'} (\log f(\zeta')) \right] \\ &\quad \frac{1}{\zeta} \partial_\zeta \log f(\zeta) d\zeta d\zeta' d\mu d\mu'. \end{aligned}$$



Moreover the changing of variables  $(\zeta, \zeta') \mapsto (\zeta', \zeta)$  leads to

$$\begin{aligned} \left\langle \int_0^\infty \zeta^2 Q_{ee}(f) \log f d\zeta \right\rangle &= -\frac{1}{2} \int_{-1}^1 \int_{-1}^1 \int_0^\infty \int_0^\infty \zeta'^2 \zeta^2 \tilde{J}(\zeta, \zeta') f(\zeta) f(\zeta') \\ &\quad \left[ \frac{1}{\zeta} \partial_\zeta (\log f(\zeta)) - \frac{1}{\zeta'} \partial_{\zeta'} (\log f(\zeta')) \right]^2 d\zeta d\zeta' d\mu d\mu', \end{aligned} \quad (4.6)$$

and the entropy dissipation property follows.  $\square$

**Remark 2.** *The collision operator  $Q_{ee}$  does not preserve the impulsion. However, as the whole collision operator  $C$  does not preserve also the impulsion, the conservation properties of  $C$  are not affected.*

**5. Semi-discretized kinetic equation.** In this section we propose an energy discretization for the new continuous model defined in section 4 such that the fundamental properties of collision operator are conserved. We firstly discretize the kinetic equation and then take moments in section 7. The key point of the following scheme is the approximation of the distribution function  $f$  on the dual mesh, through an entropic average. This average leads to the entropy dissipation property for the discretized collision operator.

**5.1. Energy discretisation.** Let us define the primal mesh  $\mathcal{M}$ , for the energy variable  $\zeta$ , decomposed into a family of rectangles

$$\mathcal{M}_{j-\frac{1}{2}} = ]\zeta_{j-1}, \zeta_j[, \quad j \in \{1, m\},$$

and  $m \in \mathbb{N}$  corresponds to the number of points which discretize the energy domain.  $\Delta\zeta_j = \zeta_j - \zeta_{j-1}$  represents the discretization step, which can be variable.

We denote by  $\mathcal{D}$  its associated dual mesh consisting of cells

$$\mathcal{D}_j = ]\zeta_{j-\frac{1}{2}}, \zeta_{j+\frac{1}{2}}[, \quad \zeta_{j-\frac{1}{2}} = (j - \frac{1}{2}) \Delta\zeta_j, \quad j \in \{1; m\}, \quad \zeta_{-\frac{1}{2}} = 0.$$

The step  $\Delta\zeta_{j+\frac{1}{2}}$  of  $\mathcal{D}_j$  writes  $\Delta\zeta_{j+\frac{1}{2}} = \zeta_{j+\frac{1}{2}} - \zeta_{j-\frac{1}{2}}$ .

Let  $h_j$  be an approximation of  $h(\zeta_j)$  for all distribution function  $h$  and  $h_{j+\frac{1}{2}}$  an approximation of  $h(\zeta_{j+\frac{1}{2}})$ .

The discrete form of (4.3) reads for any  $j \in \{1; m\}$ ,

$$\partial_t f_j + \zeta_j \mu \partial_x f_j = Q_j, \quad Q_j = Q_{ee,j} + Q_{ei,j}, \quad (5.1)$$

where the expressions of  $Q_{ee,j}$  and  $Q_{ei,j}$  are given in Definition 1.

**Definition 1.** *The collision operators  $(Q_{ee,j})_{j \in \{1; m\}}$  and  $(Q_{ei,j})_{j \in \{1; m\}}$  are respectively defined by*

$$\begin{cases} Q_{ee,j} = \frac{G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}}{\zeta_j^2 \Delta\zeta_j}, \\ Q_{ei,j} = \frac{1}{\zeta_j^3} \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial f_j}{\partial \mu} \right), \end{cases} \quad (5.2)$$

with

$$G_{j+\frac{1}{2}} = \zeta_{j+\frac{1}{2}} \sum_{k=0}^m \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[ F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{f_{j+1} - f_j}{\Delta\zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} f_{j+\frac{1}{2}} \frac{F_{k+1}^0 - F_k^0}{\Delta\zeta_{k+\frac{1}{2}}} \right] \zeta_{k+\frac{1}{2}}^2 \Delta\zeta_{k+\frac{1}{2}}, \quad (5.3)$$

where  $\tilde{J}$  has been defined in (3.8) and  $f_{j+\frac{1}{2}}$  is defined by the following entropic average

$$\begin{cases} f_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{\log |f_{j+1}| - \log |f_j|} & \text{if } f_{j+1} \neq f_j, \\ f_{j+\frac{1}{2}} = f_{j+1} & \text{else.} \end{cases} \quad (5.4)$$

**Remark 3.** This entropic average has already been considered for an isotropic distribution function by Dellacherie ([15]) to construct an entropic scheme. Next, in [31], this approach has been generalized to the  $M_1$  model.

**Remark 4.**  $G_{j+\frac{1}{2}}$  can be simplified as

$$G_{j+\frac{1}{2}} = A_{j+\frac{1}{2}} \frac{f_{j+1} - f_j}{\Delta \zeta_{j+\frac{1}{2}}} + B_{j+\frac{1}{2}} f_{j+\frac{1}{2}}$$

where

$$A_{j+\frac{1}{2}} = \zeta_{j+\frac{1}{2}}^2 \sum_{k=0}^m \min \left( \frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3} \right) F_{k+\frac{1}{2}}^0 \zeta_{k+\frac{1}{2}}^4 \Delta \zeta_{k+\frac{1}{2}}, \quad (5.5)$$

$$B_{j+\frac{1}{2}} = -\zeta_{j+\frac{1}{2}}^3 \sum_{k=0}^m \min \left( \frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3} \right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0). \quad (5.6)$$

**Lemma 1.**  $B_{j+\frac{1}{2}}$  defined in (5.6) can be simplified into

$$B_{j+\frac{1}{2}} = \zeta_{j+\frac{1}{2}}^3 \sum_{k=0}^j (\zeta_{k+\frac{1}{2}}^3 - \zeta_{k-\frac{1}{2}}^3) F_k^0.$$

The property is left to appendix D.

## 5.2. Properties of the semi-discretized scheme.

**Proposition 2.**  $(Q_j)_{j \in \{1; m\}}$  satisfies the following fundamental properties.

1) The operator  $(Q_j)_{j \in \{1; m\}}$  satisfies mass and energy conservations property i.e.

$$\left\langle \sum_{j=0}^m \zeta_j^2 Q_j \left( \frac{1}{\zeta^2} \right) \Delta \zeta_j \right\rangle = 0.$$

2) By defining  $f_{j+\frac{1}{2}}$  through the entropic average given by (5.4), the operator  $(Q_j)_{j \in \{1; m\}}$  satisfies the entropy dissipation property

$$\left\langle \sum_{j=0}^m \zeta_j^2 Q_j \log f_j \Delta \zeta_j \right\rangle \leq 0.$$

*Proof.* First we aim to prove the mass conservation. A simple computation gives

$$\left\langle \sum_{j=0}^m \zeta_j^2 Q_j \Delta \zeta_j \right\rangle = G_{m+\frac{1}{2}} - G_{-\frac{1}{2}}.$$

We suppose  $G$  as a compact support function, so  $G_{m+\frac{1}{2}} = 0$ . Besides  $G_{-\frac{1}{2}} = 0$  because  $\zeta_{-\frac{1}{2}} = 0$ . Therefore we get

$$\left\langle \sum_{j=0}^m \zeta_j^2 Q_j \Delta \zeta_j \right\rangle = 0.$$

Then we will show the conservation of energy which can be rewritten into

$$\left\langle \sum_{j=0}^m \zeta_j^2 Q_j \zeta_j^2 \Delta \zeta_j \right\rangle = \left\langle - \sum_{j=0}^{m-1} G_{j+\frac{1}{2}} (\zeta_{j+1}^2 - \zeta_j^2) \right\rangle. \quad (5.7)$$

By using the expression of  $G_{j+\frac{1}{2}}$  given in (5.3), equation (5.7) can be simplified into

$$\begin{aligned} \left\langle \sum_{j=0}^m \zeta_j^2 Q_j \zeta_j^2 \Delta \zeta_j \right\rangle = & \left\langle - \sum_{j=0}^{m-1} \zeta_{j+\frac{1}{2}} \sum_{k=0}^m \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[ F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{f_{j+1} - f_j}{\Delta \zeta_{j+\frac{1}{2}}} \right. \right. \\ & \left. \left. - \frac{1}{\zeta_{k+\frac{1}{2}}} f_{j+\frac{1}{2}} \frac{F_{k+1}^0 - F_k^0}{\Delta \zeta_{k+\frac{1}{2}}} \right] \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{k+\frac{1}{2}} (\zeta_{j+1}^2 - \zeta_j^2) \right\rangle. \quad (5.8) \end{aligned}$$

Since  $\zeta_{j+1}^2 - \zeta_j^2 = 2\zeta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}}$ , equation (5.8) can be rewritten into

$$\begin{aligned} \left\langle \sum_{j=0}^m \zeta_j^2 Q_j \zeta_j^2 \Delta \zeta_j \right\rangle = & - \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f_{k+\frac{1}{2}} \frac{1}{\zeta_{j+\frac{1}{2}}} \\ & \frac{f_{j+1} - f_j}{\Delta \zeta_{j+\frac{1}{2}}} d\mu d\mu' \\ & + \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f_{j+\frac{1}{2}} \frac{1}{\zeta_{k+\frac{1}{2}}} \\ & \frac{f_{k+1} - f_k}{\Delta \zeta_{k+\frac{1}{2}}} d\mu d\mu', \end{aligned}$$

and the conservation of energy follows by exchanging indexes  $k$  and  $j$ .

Next, we prove that the numerical operator  $Q_j$  is entropic. Firstly, by arguing like for the continuous case, we get that  $Q_{ei,j}$  dissipates the entropy. Moreover

$$\left\langle \sum_{j=0}^m \zeta_j^2 Q_{ee,j} \log f_j \Delta \zeta_j \right\rangle = \left\langle - \sum_{j=0}^{m-1} \log \left( \frac{f_{j+1}}{f_j} \right) G_{j+\frac{1}{2}} \right\rangle. \quad (5.9)$$

From the definition of  $G_{j+\frac{1}{2}}$ , equation (5.9) can be written as

$$\begin{aligned} \left\langle \sum_{j=0}^m \zeta_j^2 Q_{ee,j} \log f_j \Delta \zeta_j \right\rangle = & - \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) f_{k+\frac{1}{2}} f_{j+\frac{1}{2}} \\ & \left[ \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{f_{j+1} - f_j}{f_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} \frac{f_{k+1} - f_k}{f_{k+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}}} \right] \frac{\log f_{j+1} - \log f_j}{\zeta_{j+\frac{1}{2}} \Delta \zeta_{j+\frac{1}{2}}} d\mu d\mu'. \quad (5.10) \end{aligned}$$

By using the change of variables  $(\mu, \mu') \mapsto (\mu', \mu)$ , the change of index  $(j, k) \mapsto (k, j)$  and the entropic average (5.4) defining  $f_{j+\frac{1}{2}}$ , equation (5.10) can be rewritten as

$$\begin{aligned} \left\langle \sum_{j=0}^m \zeta_j^2 Q_{ee,j} \log f_j \Delta \zeta_j \right\rangle = & - \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \sum_{j,k=0}^m \zeta_{j+\frac{1}{2}}^2 \zeta_{k+\frac{1}{2}}^2 \Delta \zeta_{j+\frac{1}{2}} \Delta \zeta_{k+\frac{1}{2}} \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \\ & f_{k+\frac{1}{2}} f_{j+\frac{1}{2}} \left[ \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{\log f_{j+1} - \log f_j}{\Delta \zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} \frac{\log f_{k+1} - \log f_k}{\Delta \zeta_{k+\frac{1}{2}}} \right]^2 d\mu d\mu', \end{aligned}$$

and the entropic dissipation property follows.  $\square$

**6. Continuous moment model.** In this section, we explain the construction of the  $M_N$  model obtained from the kinetic model given in section 4. Besides collision operator properties are shown to be still preserved by the moments extraction.

**6.1. N-moment model for the new kinetic equation.** Firstly the  $M_N$  model constructed from the previous kinetic equation (4.3) is presented. This system is established by using a minimum entropy principle for the angle variable, keeping the energy of particles as a kinetic variable and a moments extraction.

The expression  $Q_{ee}^i$  and  $Q_{ei}^i$  are computed in the following property.

**Property 2.** *The moments of  $Q_{ee}$  are given by*

$$Q_{ee}^i(f) = \partial_\zeta \left( \zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta F^i(\zeta) - F^i(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right), \quad (6.1)$$

whereas the moments of  $Q_{ei}$  are expressed as

$$Q_{ei}^i(f) = \frac{i}{\zeta^3} ((i-1)f^{i-2} - (i+1)f^i), \quad i \geq 1, \quad Q_{ei}^0 = 0. \quad (6.2)$$

The expression of  $Q_{ee}^i$  comes from the linearity of  $Q_{ee}$  whereas the moment extraction  $Q_{ei}^i$  for the electron-ion collision is computed in Appendix C.

By setting

$$\bar{f} = \begin{pmatrix} f^0 \\ \vdots \\ f^N \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^{N+1} \end{pmatrix} \quad \text{and} \quad \bar{Q} = \begin{pmatrix} Q^0 \\ \vdots \\ Q^N \end{pmatrix}, \quad (6.3)$$

the moments extraction of 4.3 leads to the system

$$\partial_t \bar{f} + \zeta \partial_x \tilde{f} = \bar{Q}. \quad (6.4)$$

**6.2. Conservation of the realizability domain.** We show that the collision operator  $Q(f) = Q_{ee}(f) + Q_{ei}(f)$  conserves the realizability domain  $\mathcal{A}$ . That is why we prove in Proposition 3 that the space homogeneous version of (6.1, 6.3, 6.2, 6.4) preserves  $\mathcal{A}$ .

**Proposition 3.** *The continuous problem (6.1, 6.3, 6.2, 6.4) considered in the space homogeneous case preserves the realizability domain  $\mathcal{A}$  i.e if  $\bar{f}(t=0, \zeta) \in \mathcal{A}$  then  $\bar{f}(t, \zeta) \in \mathcal{A}$  for any  $t \geq 0$ .*

The proof of Proposition 3 is shown by using the following lemma.

**Lemma 2.** *The kinetic equation (4.1, 4.2, 4.3) in the space homogeneous setting preserves the nonnegativity of the solution ie if  $g(0, \zeta, \mu) \geq 0$  then  $g(t, \zeta, \mu) \geq 0$  for any  $t \geq 0$ .*

*Proof.* For any quantity  $\alpha$ , define by  $\alpha^+$  (resp.  $\alpha^-$ ) its positive part (resp. negative).

Let  $g$  be solution to

$$\frac{\partial g}{\partial t} = \tilde{Q}_{ee}(g) + Q_{ei}(g), \quad (6.5)$$

with

$$\tilde{Q}_{ee}(g) = \frac{2}{3\zeta^2} \partial_\zeta \left( \tilde{A}(G^{0+}, \zeta) \partial_\zeta g + \tilde{B}(G^{0+}, \zeta) g \right). \quad (6.6)$$

Here  $\tilde{A}$  and  $\tilde{B}$  are given by

$$\begin{aligned}\tilde{A}(G^{0+}, \zeta) &= \zeta^2 \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^4 G^0(w)^+ dw, \\ \tilde{B}(G^{0+}, \zeta) &= -\zeta^3 \int_0^\infty \min\left(\frac{1}{\zeta^3}, \frac{1}{w^3}\right) w^3 \partial_w \left( \frac{g^0(w)^+}{w^2} \right) dw = 3 \int_0^\zeta w^2 G^0(w)^+ dw,\end{aligned}$$

where  $G^0 = g^0/\zeta^2$  as usual.

For more details on the last relation concerning  $\tilde{B}$  we refer ([31]).

We proceed like in ([3], [31]). Let  $H$  be a convex positive function such as  $H' \leq 0$  and defined by

$$H(x) = \begin{cases} -Cx & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}, \quad C > 0. \quad (6.7)$$

By multiplying equation (6.5) by  $\zeta^2 H'(g)$  and integrating over  $[0, +\infty[$ , we get

$$\int_0^{+\infty} \zeta^2 \partial_t H(g) d\zeta = \int_0^{+\infty} \zeta^2 Q_{ee}(g) H'(g) d\zeta + \int_0^{+\infty} \zeta^2 Q_{ei}(g) H'(g) d\zeta. \quad (6.8)$$

We show firstly the dissipation property

$$\int_0^{+\infty} \int_{-1}^1 \zeta^2 Q_{ee}(g) H'(g) d\mu d\zeta \leq 0. \quad (6.9)$$

The first right-hand side term of (6.8) writes

$$\begin{aligned}4\pi \int_0^{+\infty} \int_{-1}^1 \zeta^2 Q_{ee}(g) H'(g) d\mu d\zeta &= 4\pi \left( \int_0^\infty \int_{-1}^1 \frac{2}{3} \partial_\zeta \left( \tilde{A}(F^{0+}, \zeta) \partial_\zeta g \right) H'(g) d\mu d\zeta \right. \\ &\quad \left. + \int_0^{+\infty} \int_{-1}^1 \frac{2}{3} \partial_\zeta \left( \tilde{B}(F^{0+}, \zeta) g \right) H'(g) d\mu d\zeta \right).\end{aligned} \quad (6.10)$$

Firstly, remark that  $H$  can be regularized into a  $\mathcal{C}^2$  function satisfying  $H'(0) = H''(0) = 0$ . In this way, the second derivative of  $H$  can be considered. Hence an integration by parts on the first right-hand side term of (6.10) and the convexity of  $H$  lead to

$$\int_0^\infty \int_{-1}^1 \frac{2}{3} \partial_\zeta \left( \tilde{A}(F^{0+}, \zeta) \partial_\zeta g \right) H'(g) d\mu d\zeta \leq 0.$$

Moreover the second term of (6.10) can be simplified as

$$\begin{aligned}\int_0^\infty \int_{-1}^1 \frac{2}{3} \partial_\zeta \left( \tilde{B}(F^{0+}, \zeta) g \right) H'(g) d\mu d\zeta &= \int_0^\infty 2H'(g) \partial_\zeta g \int_0^\zeta w^2 F^{0+}(w) dw d\zeta \\ &\quad + \int_0^\infty 2H'(g) \zeta^2 g F^{0+} d\mu d\zeta.\end{aligned}$$

An integration by part leads to

$$\int_0^{+\infty} \int_{-1}^1 \frac{2}{3} \partial_\zeta \left( \tilde{B}(F^{0+}, \zeta) g \right) H'(g) d\mu d\zeta = \int_0^{+\infty} \int_{-1}^1 2\zeta^2 F^{0+} (-H(g) + H'(g)g) d\mu d\zeta.$$

Moreover, by construction of  $H$ ,  $\forall x$ ,  $(-H(x) + H'(x)x) x = 0$ . So, (6.9) is satisfied. Next we show the same inequality as (6.9) for  $Q_{ei}$ . By using an integration by parts

we get

$$\int_0^{+\infty} \int_{-1}^1 \zeta^2 Q_{ei}(g) H'(g) d\mu d\zeta = - \int_0^{+\infty} \int_{-1}^1 \zeta^2 (1 - \mu^2) \left( \frac{\partial g}{\partial \mu} \right)^2 H''(g) d\mu d\zeta .$$

By convexity of  $H$ ,  $Q_{ei}$  satisfies the inequality (6.9). So

$$\int_0^{+\infty} \int_{-1}^1 \zeta^2 \partial_t H(g) d\mu d\zeta \leq 0 .$$

After integrating the previous inequality, we get

$$\int_0^{+\infty} \int_{-1}^1 \zeta^2 H(g(t, \zeta)) d\mu d\zeta \leq \int_0^{+\infty} \int_{-1}^1 \zeta^2 H(g(0, \zeta)) d\mu d\zeta .$$

As  $g(0, \zeta) \geq 0$ , the definition of  $H$  gives  $H(g(0, \zeta)) = 0$ . Then by nonnegativity of  $H$ , it comes that  $\int_0^{+\infty} \int_{-1}^1 \zeta^2 H(g) d\mu d\zeta = 0$ . So  $H(g) = 0$  a.e. i.e.  $g \geq 0$ . So  $G^{0+} = G^0$  and by uniqueness of the solution of (4.1, 4.2, 4.3) we get the result.  $\square$

*Proof.* (Proposition 3). Let us choose  $\bar{f}(0, \zeta) \in \mathcal{A}$ . Hence

$$\bar{f}(0, \zeta) = 2\pi\zeta^2 \int_{-1}^1 f(0, \zeta, \mu) \bar{\mu} d\mu, \quad f(0, \zeta, \mu) \geq 0.$$

Moreover the solution of (4.1, 4.2, 4.3) for initial condition  $f(0, \zeta, \mu)$  is nonnegative. Next the moment system has a solution belonging to  $\mathcal{A}$ . Therefore by uniqueness of the solution to (6.4), we get that  $\mathcal{A}$  is conserved.  $\square$

**6.3. H-theorem.** One of the main result of this paper is the following theorem which proves that the system (6.3, 6.4) is entropic.

**Theorem 6.1.**  $E = \langle f \ln f - f \rangle \zeta^2$  is an entropy for the system (6.3, 6.4). More precisely, we have  $\partial_t E + \partial_x F \leq 0$ , where  $F$  is the entropic flux given by  $F = \zeta \langle (f \ln f - f) \mu \rangle \zeta^2$ .

The proof is performed in the same spirit as in ([28]). So we refer to this reference for more details.

**7. Discretization of the N-moment model.** In this section, the moments procedure is applied on the semi-discrete scheme proposed in section 5 when the distribution function is obtained from the minimization entropy principle. So, in the present section we denote for any  $j \in \{1; m\}$ ,  $f_j$  by

$$f_j = \exp(\bar{\alpha}_j \cdot \bar{\mu}), \quad \bar{\alpha}_j = \begin{pmatrix} \alpha_{0,j} \\ \vdots \\ \alpha_{n,j} \end{pmatrix} \text{ and } \alpha_{i,j} = \alpha_i(\zeta_j). \quad (7.1)$$

**7.1. N-moment system for the semi-discretized kinetic equation.** By extracting moments on equation (5.1), we obtain the N-moment discretized system

$$\forall j \in \{1, m\} \partial_t \bar{f}_j + \zeta_j \partial_x \tilde{f}_j = \bar{Q}_j, \quad (7.2)$$

where

$$f_j^i = \zeta^2 \langle \mu^i f_j \rangle \bar{f}_j = \begin{pmatrix} f_j^0 \\ \vdots \\ f_j^n \end{pmatrix}, \quad \tilde{f}_j = \begin{pmatrix} f_j^1 \\ \vdots \\ f_j^{n+1} \end{pmatrix}, \quad \bar{Q}_j = \begin{pmatrix} Q_j^0 \\ \vdots \\ Q_j^n \end{pmatrix} = \langle \bar{\mu} Q_j \rangle .$$

**Definition 2.** The discrete collision operators  $(Q_{ee,j}^i)_{j \in \{1;m\}}$  and  $(Q_{ei,j}^i)_{j \in \{1;m\}}$  involved in (7.2) are given by

$$\begin{cases} Q_{ee,j}^i = \frac{G_{j+\frac{1}{2}}^i - G_{j-\frac{1}{2}}^i}{\Delta\zeta_j}, \\ Q_{ei,j}^i = \frac{i}{\zeta_j^3}((i-1)f_j^{i-2} - (i+1)f_j^i), \end{cases} \quad (7.3)$$

where

$$G_{j+\frac{1}{2}}^i = \zeta_{j+\frac{1}{2}} \sum_{k=0}^m \tilde{J}(\zeta_{j+\frac{1}{2}}, \zeta_{k+\frac{1}{2}}) \left[ F_{k+\frac{1}{2}}^0 \frac{1}{\zeta_{j+\frac{1}{2}}} \frac{F_{j+1}^i - F_j^i}{\Delta\zeta_{j+\frac{1}{2}}} - \frac{1}{\zeta_{k+\frac{1}{2}}} F_{j+\frac{1}{2}}^i \frac{F_{k+1}^0 - F_k^0}{\Delta\zeta_{k+\frac{1}{2}}} \right] \zeta_{k+\frac{1}{2}}^2 \Delta\zeta_{k+\frac{1}{2}}, \quad (7.4)$$

and

$$F_{j+\frac{1}{2}}^i = \int_{-1}^1 f_{j+\frac{1}{2}} \mu^i d\mu. \quad (7.5)$$

Recall that  $f_{j+\frac{1}{2}}$  is defined in (5.4).

**Remark 5.** From a computational point of view, an approximation of (7.5) can be obtained through a usual quadrature formula.

**7.2. Realizability domain.** In this section, we prove that the discretized collision operator defined in section 5 preserves the realizability domain  $\mathcal{A}$ . That is why we consider the moment system (7.1, 7.2, 7.3, 7.4, 7.5) in the space homogeneous context.

**Proposition 4.** The space homogeneous version of the semi-discretized problem (7.1, 7.2, 7.3, 7.4, 7.5) preserves the realizability domain  $\mathcal{A}$  i.e. if  $\overline{f_j}(t=0) \in \mathcal{A}$ , then  $\overline{f_j}(t) \in \mathcal{A}$  for any  $t \geq 0$ .

**Lemma 3.** The semi-discretized kinetic problem (5.1, 5.2) preserves the nonnegativity of the solution ie if  $f_j(0, \mu) \geq 0$  then  $f_j(t, \mu) \geq 0$ .

*Proof.* For any  $j \in \{1; m\}$ , let  $g_j$  be solution to

$$\frac{\partial g_j}{\partial t} = \frac{G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}}{\zeta_j^2 \Delta\zeta_j} + Q_{ei,j}, \quad (7.6)$$

where

$$\begin{aligned} G_{j+\frac{1}{2}} &= \tilde{A}_{j+\frac{1}{2}} \frac{g_j - g_{j+1}}{\Delta\zeta_{j+\frac{1}{2}}} + \tilde{B}_{j+\frac{1}{2}} g_{j+1}, \quad \tilde{B}_{j+\frac{1}{2}} = \left( B_{j+\frac{1}{2}} \right)^+ \frac{g_{j+\frac{1}{2}}^+}{g_{j+1}^+}, \\ \tilde{A}_{j+\frac{1}{2}} &= \zeta_{j+\frac{1}{2}}^2 \sum_{k=0}^m \min \left( \frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3} \right) \left( F_{k+\frac{1}{2}}^0 \right)^+ \zeta_{k+\frac{1}{2}}^4 \Delta\zeta_{k+\frac{1}{2}}. \end{aligned} \quad (7.7)$$

Let  $H$  be a convex positive function defined as in 6.7. Multiplying equation (7.6) by  $\zeta_j^2 H'(g_j)$  summing on  $j$  and integrating over  $\mu$  leads to

$$\begin{aligned} \int_{-1}^1 \sum_j \zeta_{j+\frac{1}{2}} \partial_t H(g_j) \Delta\zeta_j d\mu &= \int_{-1}^1 \sum_j (G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}) H'(g_j) d\mu \\ &+ \int_{-1}^1 \sum_j \zeta_{j+\frac{1}{2}}^2 Q_{ei,j} H'(g_j) \Delta\zeta_{j+\frac{1}{2}} d\mu. \end{aligned} \quad (7.8)$$

From (7.7) together to a shift of index, it comes that the first term of the right-hand side of equation (7.8) can be simplified into

$$\begin{aligned} \int_{-1}^1 \sum_j (G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}) H'(g_j) \Delta \zeta_j d\mu &= \int_{-1}^1 \sum_j \tilde{A}_{j+\frac{1}{2}} \frac{g_j - g_{j+1}}{\Delta \zeta_{j+\frac{1}{2}}} (H'(g_{j+1}) - H'(g_j)) d\mu \\ &+ \int_{-1}^1 \sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+1} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) d\mu. \end{aligned} \quad (7.9)$$

The second term of the right hand side of (7.9) can be rewritten as

$$\begin{aligned} \sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+1} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) &= \sum_j \left[ (\tilde{B}_{j+\frac{1}{2}} - \tilde{B}_{j-\frac{1}{2}}) g_j H'(g_j) \right. \\ &+ \left. \tilde{B}_{j+\frac{1}{2}} (g_{j+1} - g_j) H'(g_j) \right]. \end{aligned} \quad (7.10)$$

Moreover by convexity  $H$ , it holds that

$$H(g_{j+1}) \geq H(g_j) + (g_{j+1} - g_j) H'(g_j).$$

Then by non-negativity of  $\tilde{B}_{j+\frac{1}{2}}$ , equation (7.10) gives

$$\begin{aligned} \sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+1} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) &\leq \sum_j \left[ (\tilde{B}_{j+\frac{1}{2}} - \tilde{B}_{j-\frac{1}{2}}) g_j H'(g_j) \right. \\ &+ \left. \tilde{B}_{j+\frac{1}{2}} (H(g_{j+1}) - H(g_j)) \right]. \end{aligned}$$

By using a shift of index, we get

$$\sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+\frac{1}{2}} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) \leq \sum_j (\tilde{B}_{j+\frac{1}{2}} - \tilde{B}_{j-\frac{1}{2}}) (-H(g_j) + g_j H'(g_j)).$$

By construction of  $H$ , we get  $(-H(g_j) + g_j H'(g_j)) = 0$  and thus from the previous inequality it comes that

$$\sum_j (\tilde{B}_{j+\frac{1}{2}} g_{j+\frac{1}{2}} - \tilde{B}_{j-\frac{1}{2}} g_j) H'(g_j) \leq 0. \quad (7.11)$$

Moreover since  $H'$  is a non-decreasing function, the first right hand term side of (7.9) is negative. Therefore, from (7.11), we obtain the following inequality

$$\int_{-1}^1 \sum_j (G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}) H'(g_j) \Delta \zeta_j d\mu \leq 0.$$

Besides by arguing like for proof of lemma 2, the second term of the right-hand side of equation (7.8) is negative. Then by using definition of  $H$ , lemma 3 follows.  $\square$

*Proof.* (Proposition 4). We proceed like for the proof of Proposition 3. Consider  $\overline{f}_j(t=0) \in \mathcal{A}$ . Then

$$\overline{f}_j(0) = \int_{-1}^1 f_j(0, \mu) \overline{\mu} d\mu, \quad f_j(0, \mu) \geq 0.$$

From lemma 3, the solution of (5.1, 5.2) for the initial condition  $f_j(0, \mu)$  is nonnegative. So the solution of the moment system belongs to  $\mathcal{A}$  and we conclude like for the proof of Proposition 3.  $\square$



**7.3. Entropic property of the scheme.** Finally, this subsection is devoted to demonstrate the entropic property of the scheme.

**Theorem 7.1.**  $E = \sum_{j=0}^m \langle f_j \ln f_j - f_j \rangle \zeta_j^2$  is an entropy for the system (7.1, 7.2, 7.3, 7.4, 7.5). More precisely, we have  $\partial_t E + \partial_x F \leq 0$ , where

$$F = \sum_{j=0}^m \zeta_j \langle (f_j \ln f_j - f_j) \mu \rangle \zeta_j^2$$

is the entropic flux.

*Proof.* First, after multiplying equation (7.2) by  $\bar{\alpha}$ , we get

$$\bar{\alpha}_j \cdot \partial_t \bar{f}_j + \zeta_j \bar{\alpha}_j \cdot \partial_x \bar{f}_j = \bar{\alpha}_j \cdot \bar{Q}_j .$$

Hence, by using the expression  $f_j$ ,

$$\bar{\alpha}_j \cdot \partial_t \langle \bar{\mu} f_j \rangle \zeta_j^2 + \zeta_j \bar{\alpha}_j \cdot \partial_x \langle \bar{\mu} f_j \rangle \zeta_j^2 = \langle \bar{\alpha}_j \cdot \bar{\mu} Q_j \zeta_j^2 \rangle . \quad (7.12)$$

The first term of the left hand side term of (7.12) can be expressed as

$$\bar{\alpha}_j \cdot \partial_t \langle \bar{\mu} f_j \rangle = \partial_t \langle \bar{\alpha}_j \cdot \bar{\mu} f_j \rangle - \partial_t \langle \bar{\alpha}_j \cdot \bar{\mu} \rangle f_j ,$$

and simplified into

$$\bar{\alpha}_j \cdot \partial_t \langle \bar{\mu} f_j \rangle = \partial_t \langle \log f_j f_j \rangle - \langle \partial_t f_j \rangle . \quad (7.13)$$

In the same way, the second term of the right hand side can be rewritten into

$$\zeta_j \bar{\alpha}_j \cdot \partial_x \langle \bar{\mu} f_j \rangle = \zeta_j \partial_x \langle (\log f_j f_j \mu) \rangle - \langle \zeta_j \mu \partial_x f_j \rangle . \quad (7.14)$$

Then, from equations (7.13) and (7.14) and the definition of  $f_j$ , it comes

$$\partial_t \sum_{j=0}^m \langle f_j \ln f_j - f_j \rangle \zeta_j^2 \Delta \zeta_j + \partial_x \sum_{j=0}^m \zeta_j \langle (f_j \ln f_j - f_j) \mu \rangle \zeta_j^2 \Delta \zeta_j = \langle \log f_j Q_j \zeta_j^2 \rangle .$$

From the dissipative property of  $Q_j$  given in Proposition 2, Theorem 7.1 follows.  $\square$

**7.4. Numerical results.** We present in this section a test case, where the use of the  $M_2$  model is relevant compared to the  $M_1$  model.

However, using the  $M_2$  model is not so easy than using the  $M_1$  model. Indeed, the computation needs the knowledge of the third order moment  $f_3$  depending of the lower order moments. This relation cannot be computed explicitly and needs some numerical approximations. The Lagrange multipliers  $(\alpha_i)_{i=0,2}$  can be obtained from the moments  $f_i$  as follows

$$f(\mu) = \exp(\sum_{i=0}^2 \alpha_i \mu^i) \text{ where } f_i = \frac{1}{2} \int_{\mu=-1}^1 \mu^i f(\mu) d\mu .$$

In fact,  $(\alpha_i)_{i=0,2}$  are found by solving the following convex minimization problem:

$$\alpha = (\alpha_i)_{i=0,2} = \min_{\beta} \int_{\mu=-1}^1 \exp(\sum_{i=0}^2 \beta_i \mu^i) d\mu - \sum_{i=0}^2 \beta_i f_i, \quad \beta = (\beta_i)_{i=0,2} . \quad (7.15)$$

Once the  $(\alpha_i)_{i=0,2}$  have been computed, the moment  $f_3$  can be obtained directly from the underlying distribution function  $f$ . Generally, (7.15) is solved by computing the integrals numerically and minimizing the integral by some convex optimization solver (see [23]). In a first approach, we have used QUADPACK, an adaptative

numerical solver from netlib repository ([www.netlib.org](http://www.netlib.org)) and an unconstrained optimization solver HUMSL coming also from netlib. This method will be denoted by  $M_N^{opt}$ .

This method can be very expensive compared to a kinetic computation. Hence we have chosen another approach. Fortunately in the case of  $M_2$  the moments can be computed from the  $\alpha$  variables defined in (7.15):

$$\begin{aligned}
f_0 &= 1/2 \operatorname{erf} \left( 1/2 \frac{-2\alpha_2 + \alpha_1}{\sqrt{-\alpha_2}} \right) \sqrt{\pi} e^{1/4 \frac{4\alpha_0 \alpha_2 - \alpha_1^2}{\alpha_2}} \frac{1}{\sqrt{-\alpha_2}} \\
&\quad - 1/2 \operatorname{erf} \left( 1/2 \frac{2\alpha_2 + \alpha_1}{\sqrt{-\alpha_2}} \right) \sqrt{\pi} e^{1/4 \frac{4\alpha_0 \alpha_2 - \alpha_1^2}{\alpha_2}} \frac{1}{\sqrt{-\alpha_2}}, \\
f_1 &= 1/4 e^{\alpha_0 - \alpha_1} \alpha_1 \sqrt{\pi} e^{-1/4 \frac{\alpha_1(\alpha_1 - 4\alpha_2)}{\alpha_2}} \operatorname{erf} \left( 1/2 \frac{-2\alpha_2 + \alpha_1}{\sqrt{-\alpha_2}} \right) (-\alpha_2)^{-3/2} \\
&\quad - 1/4 e^{\alpha_0 - \alpha_1} \alpha_1 \sqrt{\pi} e^{-1/4 \frac{\alpha_1(\alpha_1 - 4\alpha_2)}{\alpha_2}} \operatorname{erf} \left( 1/2 \frac{2\alpha_2 + \alpha_1}{\sqrt{-\alpha_2}} \right) (-\alpha_2)^{-3/2} \\
&\quad - 1/2 \frac{e^{\alpha_0 - \alpha_1} (e^{\alpha_2} - e^{2\alpha_1 + \alpha_2})}{\alpha_2}, \\
f_2 &= 1/8 e^{\alpha_0 - \alpha_1} \sqrt{\pi} e^{-1/4 \frac{\alpha_1(\alpha_1 - 4\alpha_2)}{\alpha_2}} (-2\alpha_2 + \alpha_1^2) \operatorname{erf} \left( 1/2 \frac{-2\alpha_2 + \alpha_1}{\sqrt{-\alpha_2}} \right) (-\alpha_2)^{-5/2} \\
&\quad - 1/8 e^{\alpha_0 - \alpha_1} \sqrt{\pi} e^{-1/4 \frac{\alpha_1(\alpha_1 - 4\alpha_2)}{\alpha_2}} (-2\alpha_2 + \alpha_1^2) \operatorname{erf} \left( 1/2 \frac{2\alpha_2 + \alpha_1}{\sqrt{-\alpha_2}} \right) (-\alpha_2)^{-5/2} \\
&\quad + 1/4 \frac{e^{\alpha_0 - \alpha_1} (2e^{\alpha_2} \alpha_2 + \alpha_1 e^{\alpha_2} + 2e^{2\alpha_1 + \alpha_2} \alpha_2 - \alpha_1 e^{2\alpha_1 + \alpha_2})}{\alpha_2^2}.
\end{aligned}$$

In fact  $f^1/f^0$ ,  $f^2/f^0$  are function of  $\alpha_1$ ,  $\alpha_2$  variables on the domain of definition of  $M_2$ . The quantities  $\alpha_1$ ,  $\alpha_2$  can be computed on some grids points in the  $f^1/f^0$ ,  $f^2/f^0$  phase space and then the third reduced moment  $f^3/f^0$  can be deduced. Solving the problem at grid points is done by using a two-dimensional dichotomy. The process is rather expensive but used only once. The closure can be obtained by interpolating  $f^3/f^0$  on the  $f^1/f^0$ ,  $f^2/f^0$  grid space. This method is called  $M_2^{app}$ . A grid of about 100 points by direction was generated on  $f^1/f^0$ ,  $f^2/f^0$  phase space.

On the other hand, a same kind of process can be applied for  $M_3$ , leading to  $f^4/f^0$  interpolation over a  $100^3$  grid points in the  $f^1/f^0$ ,  $f^2/f^0$ ,  $f^3/f^0$  phase space. We called this method  $M_3^{app}$ .

For the test case, we considered only elastic electron-ion collisions, so we take  $\alpha_{ee} = 0$ ,  $\alpha_{ei} = 1$ . In this case, the model has no  $\zeta$  derivative. Hence for the sake of simplicity we only consider one point in  $\zeta$ ,  $\zeta = 1$ . The space interval of study is  $[0, 5]$  and we set at  $x=0$ , the given incoming boundary conditions

$$f = 500 \exp(-200(1 - \mu)), \quad \mu \in [0, 1].$$

We have also consider  $P_N$  approximations which are based on polynomials underlying distribution functions for comparison. These kinds of approximations are very popular in plasma physic. In that case, the  $P_1$  closure is given by  $f_1 = 1/3 f_0$ , the  $P_2$  closure by  $f_3 = 2/5 f_1$ , and the  $P_3$  closure by  $f_4 = -3/35 f_0 + 6/7 f_2$ .

For the implementation, we perform a classical Euler scheme in time and we compute the steady state solution of the different equations. The kinetic computation need 180s on a single CPU of i7 Intel Mac computer with GFORTRAN compiler. The moments model  $M_N$  are rather expensive: 60s for  $M_1$ , 181s for  $M_2$ , and 310s

for  $M_3$ . On the other hand, the models  $M_N^{app}$  are very cheap in comparison (0.7s for  $M_1^{app}$ , 1s for  $M_2^{app}$ , and 2s for  $M_3^{app}$ ) without loss of accuracy. For example, Fig 3 shows the relative error between the two  $M_2$  approaches.

Remark that the most important differences are near the boundaries. To illustrate this fact, the parameters  $f_1/f_0$  and  $f_2/f_0$  computed from the kinetic reference solution are plotted on figure 4. We can note that more the solution is far from isotropy ( $f_1/f_0 \approx 0$  and  $f_2/f_0 \approx 1/3$ ), more the difference is important. In fact close to the isotropy area, the  $M_2$  closure is a relatively flat function and the interpolation is more accurate than the furthest isotropy area.

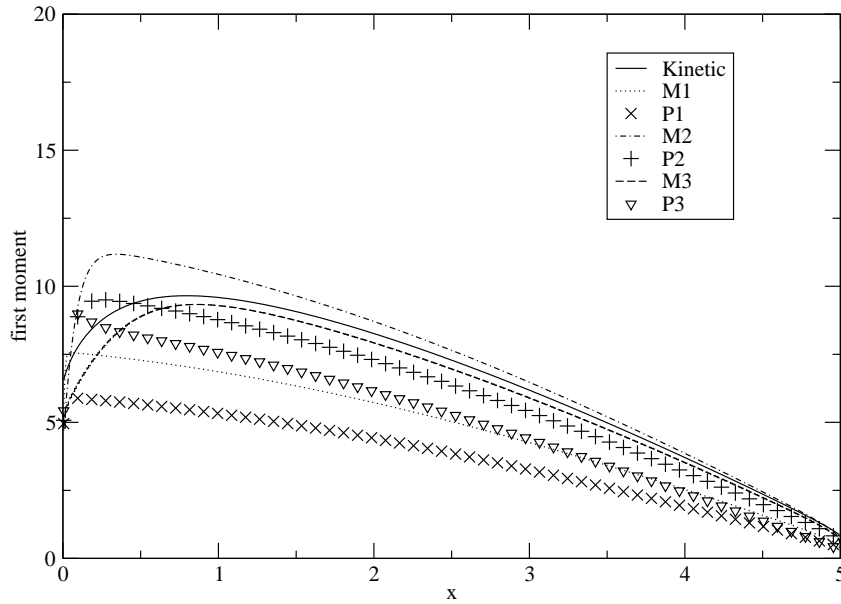


FIGURE 2. Representation of the steady state for the first moment  $f_1$  computed with the  $M_N$  model, the  $P_N$  model, for  $N = 1, 3$  and the kinetic one for the space interval  $[0, 5]$  and one group in energy,  $\zeta = 1$  with 500 points in space and 256 points in  $\mu$  for the kinetic model.

The distribution function converges to an isotropic distribution function, steady state of the electron-ion collision operator. So the first moment  $f_1$  should be close to 0. However, due to the anisotropic boundary conditions, we observed boundary layer which are represented by both moment models. However, Fig 2 shows that the result given by the  $M_2$  and  $M_3$  model is closer to the kinetic solution than the result given by the  $M_1$  model.

**8. Conclusion.** Firstly, we have proposed a model for plasma electrons transport with a new consistent collision operator for the electron-electron interactions. Its definition allows mainly to conserve fundamental properties such as the mass and energy conservation and entropy dissipation. Next we have motivated the development of a semi-discretized scheme in the energy variable for the new continuous model. This model still preserves fundamental properties by using a specific entropic average. This  $M_N$  model is based on an entropy minimization principle and

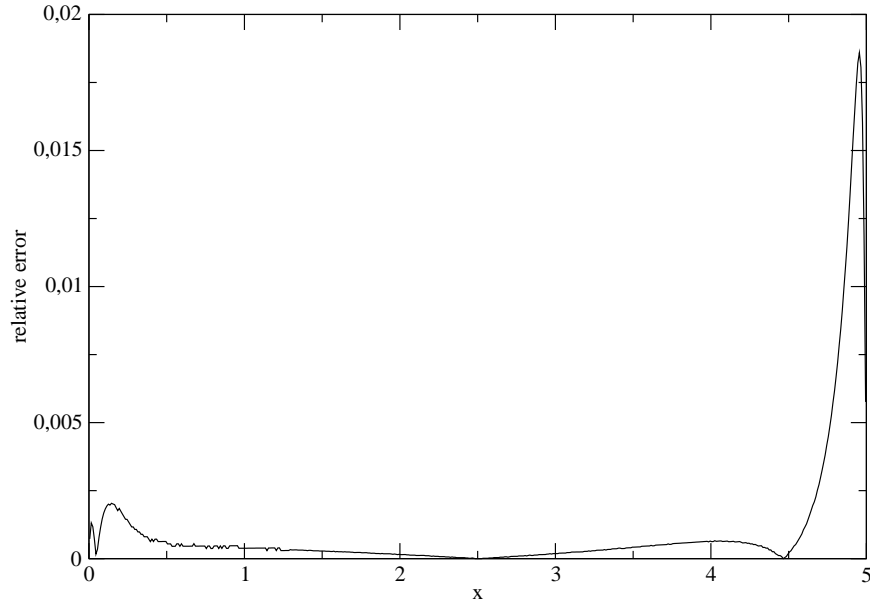


FIGURE 3. Representation of relative error between  $M_2$  and  $M_2^{app}$  about of the steady state for the moment  $f_1$ .

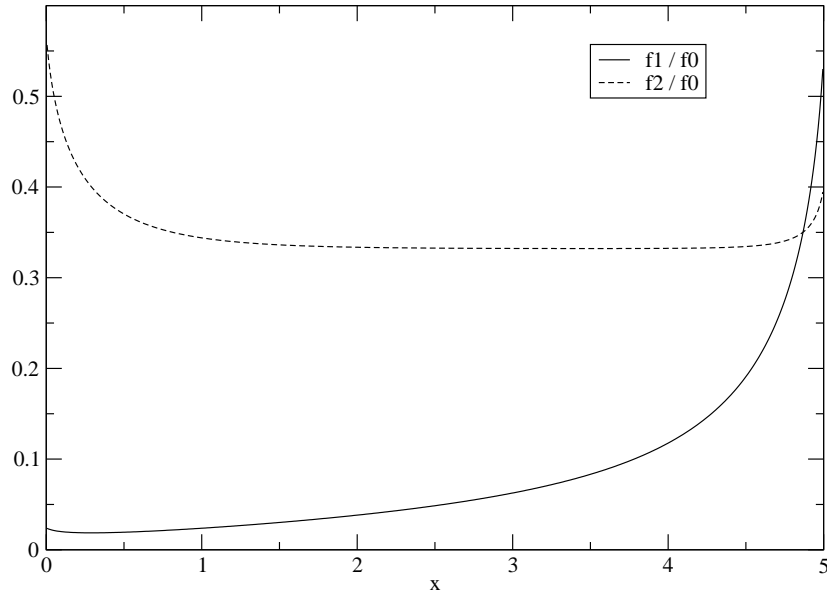


FIGURE 4. Representation of  $f_1/f_0$  and  $f_2/f_0$  parameters about the steady state for the kinetic computation.

an integration of the kinetic model w.r.t. the angle variable. Finally the discretized moment model constructed from the the semi-discretized kinetic model is proved to be entropic.

Remark that for concrete applications, we prefer low order approach  $M_N$  models, because they do not use minimization problems which can be too costly.

To extend this model, the forthcoming work should be to consider the model with mobile ions.

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**Appendix A. Equation (3.7).** To establish this closure we consider 2.1 in the homogeneous case with the collision operator  $C = C_{ee}$ . For the sake of simplicity, the distribution function  $f$  is assumed to be isotropic, i.e.  $f = f(|v|) = f(\zeta)$ . Hence we have to deal with

$$\begin{aligned} \partial_t f &= C_{ee}(f, f) = \nabla_v \cdot \Gamma(v) \\ \Gamma(v) &= \int_{\mathbb{R}^3} \Phi(u) [f(v') \nabla_v f(v) - f(v) \nabla_{v'} f(v')] dv'. \end{aligned} \quad (\text{A.16})$$

Moreover if the distribution function  $f$  is isotropic,  $\Gamma(v)v$  is also isotropic. In order to perform a weak formulation of A.16, consider  $\varphi$  some regular test function. Hence, by setting  $\zeta = |v|$ , it comes that

$$\int_0^\infty \partial_t \left( \zeta^2 \int_{\mathbb{S}^2} f d\Omega \right) \varphi(\zeta) d\zeta + \int_0^\infty \frac{1}{\zeta} \varphi'(\zeta) \int_{\mathbb{S}^2} \Gamma(v) \cdot v d\Omega \zeta^2 d\zeta = 0.$$

Moreover by isotropy of  $\Gamma(v) \cdot v$ , it holds that

$$\int_0^\infty \partial_t f^0 \varphi(\zeta) d\zeta + \int_0^\infty \frac{1}{\zeta} \varphi'(\zeta) 4\pi \Gamma(v) \cdot v \zeta^2 d\zeta = 0.$$

Hence  $\partial_t f^0 = 4\pi \partial_\zeta (\zeta \Gamma(v) \cdot v)$ . Now to compute  $\Gamma(v) \cdot v$ , we introduce the notation  $\zeta' = |v'|$ ,  $v' = \zeta' \Omega'$ . For  $V = v' - v$ ,  $\Phi(V)$  has the following expression

$$\begin{aligned} \Phi(V) &= \frac{(v - \Omega' \zeta')^2 I - (v - \Omega' \zeta') \otimes (v - \Omega' \zeta')}{(v - \Omega' \zeta')^3}, \\ &= \frac{(\zeta^2 + \zeta'^2 - 2\zeta' v \cdot \Omega') I - (v \otimes v + \zeta'^2 \Omega' \otimes \Omega' - \zeta'(\Omega' \otimes v + v \otimes \Omega'))}{(\zeta^2 + \zeta'^2 - 2\zeta' v \cdot \Omega')^{\frac{3}{2}}}. \end{aligned} \quad (\text{A.17})$$

Hence from (A.17) we get

$$\begin{aligned}\Phi(V)v &= \frac{(\zeta^2 + \zeta'^2 - 2\zeta'v \cdot \Omega')v - (\zeta^2 v + \zeta'^2(\Omega' \cdot v)\Omega' - \zeta^2 \zeta' \Omega' - \zeta'(\Omega' \cdot v)v)}{(\zeta^2 + \zeta'^2 - 2\zeta'v \cdot \Omega')^{\frac{3}{2}}}, \\ &= \frac{(\zeta'^2 - \zeta'v \cdot \Omega')v - (\zeta'^2(\Omega' \cdot v) - \zeta^2 \zeta')\Omega'}{(\zeta^2 + \zeta'^2 - 2\zeta'v \cdot \Omega')^{\frac{3}{2}}}.\end{aligned}$$

To achieve the computation of  $\Gamma(v) \cdot v$ ,  $A_v = \Phi(V)v \cdot v$  and  $A_{\Omega'} = \Phi(V)v \cdot \Omega'$  have to be computed. We introduce  $\mu$  the cosine of that angle between the vector  $\Omega'$  and  $v$ , such that  $\Omega' \cdot v = \zeta\mu$ , and we obtain

$$\begin{aligned}A_v &= \Phi(V)v \cdot v = \frac{(\zeta'^2 - \zeta'v \cdot \Omega')\zeta^2 + \zeta'(\zeta^2 - \zeta'(\Omega' \cdot v))(\Omega' \cdot v)}{(\zeta^2 + \zeta'^2 - 2\zeta'v \cdot \Omega')^{\frac{3}{2}}}, \\ &= \frac{\zeta'(\zeta' - \zeta\mu)\zeta^2 + \zeta\zeta'\mu(\zeta^2 - \zeta\zeta'\mu)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} = \frac{\zeta^2\zeta'^2(1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}}, \quad (\text{A.18})\end{aligned}$$

$$\begin{aligned}A_{\Omega'} &= \Phi(V)v \cdot \Omega' = \frac{\zeta'(\zeta' - v \cdot \Omega')(\Omega' \cdot v) + \zeta'(\zeta^2 - \zeta'(\Omega' \cdot v))}{(\zeta^2 + \zeta'^2 - 2\zeta'v \cdot \Omega')^{\frac{3}{2}}}, \\ &= \frac{\zeta'(\zeta' - \zeta\mu)\zeta\mu + \zeta'(\zeta^2 - \zeta\zeta'\mu)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} = \frac{\zeta^2\zeta'(1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}}.\end{aligned} \quad (\text{A.19})$$

Hence by symmetry of  $\Phi(V)$  and  $d\Omega' = 2\pi d\mu$ ,  $\Gamma(v) \cdot v$  writes

$$\begin{aligned}\Gamma(v) \cdot v &= \int_0^\infty \int_{\mathbb{S}^2} \Phi(V)v \cdot \left[ f(\zeta') \frac{v}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \Omega' \partial_{\zeta'} f(\zeta') \right] d\Omega' \zeta'^2 d\zeta', \\ &= \int_0^\infty \int_{\mathbb{S}^2} \left[ f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) A_v - f(\zeta) \partial_{\zeta'} f(\zeta') A_{\Omega'} \right] d\Omega' \zeta'^2 d\zeta', \\ &= \int_0^\infty \int_{-1}^{+1} \frac{\zeta^2 \zeta'^2 (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} \left[ f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} f(\zeta') \right] 2\pi d\mu \zeta'^2 d\zeta' .\end{aligned}$$

As,

$$\int_{-1}^{+1} \frac{\zeta^2 \zeta'^2 (1 - \mu^2)}{(\zeta^2 + \zeta'^2 - 2\zeta\zeta'\mu)^{\frac{3}{2}}} d\mu = \frac{4}{3} \inf \left( \frac{1}{\zeta^3}, \frac{1}{\zeta'^3} \right),$$

$\Gamma(v) \cdot v$  can be rewritten as

$$\Gamma(v) \cdot v = \int_0^\infty 4\pi \tilde{J}(\zeta, \zeta') \left[ f(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} f(\zeta') \right] \zeta'^2 d\zeta'. \quad (\text{A.20})$$

As  $f$  is isotropic,  $f^0(\zeta) = \zeta^2 \int_{\mathbb{S}^2} f(\zeta) d\Omega = 4\pi\zeta^2 f(\zeta)$ . So

$$\partial_t f^0 = \partial_\zeta \left( \zeta \int_0^\infty \tilde{J}(\zeta, \zeta') \left[ \frac{f^0(\zeta')}{\zeta'^2} \frac{1}{\zeta} \partial_\zeta \left( \frac{f^0(\zeta)}{\zeta^2} \right) - \frac{f^0(\zeta)}{\zeta^2} \frac{1}{\zeta'} \partial_{\zeta'} \left( \frac{f^0(\zeta')}{\zeta'^2} \right) \right] \zeta'^2 d\zeta' \right).$$

We retrieve exactly the formula for the collision operator involved by the equation on  $f^0$ . We neglect the operator  $Q_{ee}$  for the equation on  $f^1$  because we retain only the isotropic part of this operator.

**Appendix B. Expression (4.2) of  $Q_{ei}$  in spherical coordinate  $(\zeta, \mu, \varphi)$ .** Consider 2.1 in the homogeneous case with  $C = C_{ei}$ . The direction of propagation  $\Omega$  and the velocity  $v$  write

$$\Omega = \begin{pmatrix} \frac{\mu}{\sqrt{1 - \mu^2}} \cos \varphi \\ \sqrt{1 - \mu^2} \sin \varphi \end{pmatrix}, \quad v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \zeta \Omega = \begin{pmatrix} \zeta \frac{\mu}{\sqrt{1 - \mu^2}} \cos \varphi \\ \zeta \sqrt{1 - \mu^2} \sin \varphi \end{pmatrix}.$$

The gradient  $\nabla_v f$  can be rewritten as

$$\nabla_v f = \frac{\partial f}{\partial \zeta} \nabla_v \zeta + \frac{\partial f}{\partial \mu} \nabla_v \mu + \frac{\partial f}{\partial \varphi} \nabla_v \varphi. \quad (\text{B.21})$$

Since we have

$$\begin{pmatrix} \nabla_v \zeta \\ \nabla_v \mu \\ \nabla_v \varphi \end{pmatrix} = \frac{D(\zeta, \mu, \varphi)}{D(v_x, v_y, v_z)} = \left( \frac{D(v_x, v_y, v_z)}{D(\zeta, \mu, \varphi)} \right)^{-1} = \begin{pmatrix} \mu & \sqrt{1-\mu^2} \cos \varphi & \sqrt{1-\mu^2} \sin \varphi \\ \frac{1-\mu^2}{\zeta} & \frac{-\mu \sqrt{1-\mu^2} \cos \varphi}{\zeta} & \frac{-\mu \sqrt{1-\mu^2} \sin \varphi}{\zeta} \\ 0 & \frac{-\sin \varphi}{\sqrt{1-\mu^2} \zeta} & \frac{\cos \varphi}{\sqrt{1-\mu^2} \zeta} \end{pmatrix},$$

the gradient vector can be simplified as

$$\nabla_v \zeta = \vec{\Omega}, \quad \nabla_v \mu = \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu \quad \text{and} \quad \nabla_v \varphi = \frac{1}{\sqrt{1-\mu^2} \zeta} \vec{e}_\varphi. \quad (\text{B.22})$$

By using (B.22), (B.21) writes

$$\nabla_v f = \frac{\partial f}{\partial \zeta} \vec{\Omega} + \frac{\partial f}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu + \frac{\partial f}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2} \zeta} \vec{e}_\varphi. \quad (\text{B.23})$$

Next we determine the expression of the electron/ion operator in its weak form. Consider a test function  $\Psi$ . Hence, by using the expression of the divergence term (B.23) and a Green formula, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla_v \cdot (\Phi(v) \nabla_v f) \Psi(v) dv &= - \int_{\mathbb{R}^3} \Phi(v) \left( \frac{\partial f}{\partial \zeta} \vec{\Omega} + \frac{\partial f}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu + \frac{\partial f}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2} \zeta} \vec{e}_\varphi \right) \cdot \\ &\quad \left( \frac{\partial \Psi}{\partial \zeta} \vec{\Omega} + \frac{\partial \Psi}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta} \vec{e}_\mu + \frac{\partial \Psi}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2} \zeta} \vec{e}_\varphi \right) dv, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla_v \cdot (\Phi(v) \nabla_v f) \Psi(v) dv &= - \int_{\mathbb{R}^3} \left( \frac{\partial f}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta^2} \vec{e}_\mu + \frac{\partial f}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2} \zeta^2} \vec{e}_\varphi \right) \cdot \\ &\quad \left( \frac{\partial \Psi}{\partial \mu} \frac{\sqrt{1-\mu^2}}{\zeta^2} \vec{e}_\mu + \frac{\partial \Psi}{\partial \varphi} \frac{1}{\sqrt{1-\mu^2} \zeta^2} \vec{e}_\varphi \right) dv, \end{aligned} \quad (\text{B.24})$$

because  $\Phi$  is an orthogonal projection on  $\vec{\Omega}$  ( $\Phi(v) \vec{\Omega} = 0$ ,  $\Phi(v) \vec{e}_\mu = \frac{1}{\zeta} \vec{e}_\mu$  and  $\Phi(v) \vec{e}_\varphi = \frac{1}{\zeta} \vec{e}_\varphi$ ). Besides  $(\vec{e}_\zeta, \vec{e}_\mu, \vec{e}_\varphi)$  is an orthonormal basis, so (B.24) writes

$$\int_{\mathbb{R}^3} \nabla_v \cdot (\Phi(v) \nabla_v f) \Psi(v) dv = - \int_{\mathbb{R}^3} \left( \frac{\partial f}{\partial \mu} \frac{\partial \Psi}{\partial \mu} \frac{1-\mu^2}{\zeta^3} + \frac{\partial f}{\partial \varphi} \frac{\partial \Psi}{\partial \varphi} \frac{1}{(1-\mu^2) \zeta^3} \right) dv. \quad (\text{B.25})$$

As  $dv = \zeta^2 d\zeta d\mu d\varphi$  and by using integration by parts, (B.25) writes

$$\int_{\mathbb{R}^3} \nabla_v \cdot (\Phi(v) \nabla_v f) \Psi(v) dv = \int_{\mathbb{R}^3} \frac{1}{\zeta^3} \left[ \frac{\partial}{\partial \mu} \left( (1-\mu^2) \frac{\partial f}{\partial \mu} \right) + \frac{\partial}{\partial \varphi} \left( \frac{1}{1-\mu^2} \frac{\partial f}{\partial \varphi} \right) \right] \Psi(v) dv.$$

Therefore we obtain (4.2) when  $\frac{\partial}{\partial \varphi} f = 0$ .



**Appendix C. Moment extraction for the electron-ion collision operator.**

In this appendix, we show formula 6.2 from (4.2) by using its weak formulation. By considering the  $i^{\text{th}}$  moment of  $Q_{ei}$  and by using successively two Green formula, we get

$$\int_0^\infty \langle Q_{ei}(f)\mu^i \zeta^2 \rangle \psi(\zeta) d\zeta = \int_0^\infty \frac{i}{\zeta^3} \left[ (i-1)\zeta^2 \int_{-1}^1 f \mu^{i-2} d\mu - (i+1)\zeta^2 \int_{-1}^1 f \mu^i d\mu \right] \psi(\zeta) d\zeta,$$

where  $\psi$  denotes some test function.

**Appendix D. Proof of Lemma 1.** By using the definition (5.6) of  $B_{j+\frac{1}{2}}$ , it holds that

$$\begin{aligned} B_{j+\frac{1}{2}} &= -\zeta_{j+\frac{1}{2}}^3 \sum_{k=0}^m \min\left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3}\right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0) \\ &= -\zeta_{j+\frac{1}{2}}^3 \sum_{k=0}^j \min\left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3}\right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0) \\ &\quad - \zeta_{j+\frac{1}{2}}^3 \sum_{k=j+1}^m \min\left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3}\right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0). \end{aligned}$$

But, as  $\min\left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3}\right) = \frac{1}{\zeta_{j+\frac{1}{2}}^3}$  for  $k \in \{0; j\}$ , it holds that

$$-\zeta_{j+\frac{1}{2}}^3 \sum_{k=0}^j \min\left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3}\right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0) = -\sum_{k=0}^j \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0).$$

Hence

$$\begin{aligned} &-\zeta_{j+\frac{1}{2}}^3 \sum_{k=0}^j \min\left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3}\right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0) \\ &= -\frac{1}{\zeta_{j+\frac{1}{2}}^3} \left( \zeta_{j+\frac{1}{2}}^3 F_{j+1}^0 + \sum_{k=0}^{j-1} \zeta_{k+\frac{1}{2}}^3 F_{k+1}^0 - \sum_{k=0}^j \zeta_{k+\frac{1}{2}}^3 F_k^0 \right) \\ &= -F_{j+1}^0 + \sum_{k=0}^j (\zeta_{k+\frac{1}{2}}^3 - \zeta_{k-\frac{1}{2}}^3) F_k^0. \end{aligned}$$

Moreover

$$\sum_{k=j+1}^m \min\left(\frac{1}{\zeta_{j+\frac{1}{2}}^3}, \frac{1}{\zeta_{k+\frac{1}{2}}^3}\right) \zeta_{k+\frac{1}{2}}^3 (F_{k+1}^0 - F_k^0) = -\sum_{k=j+\frac{1}{2}}^m (F_{k+1}^0 - F_k^0) = F_{j+1}^0.$$

So the expression of  $B_{j+\frac{1}{2}}$  follows.

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