

# The Stationary Boltzmann equation for a two component gas for soft forces in the slab.

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## Abstract

The stationary Boltzmann equation for weak forces in the context of a two component gas is considered in the slab. An existence theorem is proved when one component satisfies a given indata profile and the other component satisfies diffuse reflection at the boundaries in a renormalized sense. Weak  $L^1$  compactness is extracted from the control of the entropy production term. Trace at the boundaries are also controled.

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## 1 Introduction and setting of the problem.

This paper is devoted to the stationary Boltzmann equation for a two component gas in the slab in the situation of soft forces. Stationary solutions are of interest as candidates for the time asymptotics of evolutionary problems. The two components satisfy different boundary conditions. One component (the A component) is supposed to have a given indata profile and the other component (the B component) is supposed to boundary conditions of diffuse reflection type.

From a theoretical point of view, the problem of the existence of solutions in the situation of the Boltzmann equation for single component gas was studied in ([7], [8]). In ([7]), the authors prouve the existence of weak solutions for hard forces and of renormalized solutions in the situation of soft forces. The solutions are constructed for a given fixed mass for boundary conditions of given indata profiles and for the geometry of a slab. In ([8]), the existence of solutions is shown for hard forces, for boundary conditions

of diffuse reflection type and for the geometry of a slab. The solution is also constructed for a given weighted mass. The case of the Povzner equation for a one component gas with diffuse-reflection boundary conditions in the case of hard and soft forces is investigated in ([16]). For the geometry of two coaxial rotating cylinders an existence theorem is proved in ([9]) in the situation of hard forces. The problem of multi-component gases is investigated in [10], where the existence of weak solutions is performed for hard forces when the weighted mass of each component is fixed as in [7, 8]. The case of a mixture of two gases is also considered in [11] when the Knudsen number tends to 0. The solution of the system is obtained as a Hilbert expansion plus a rest term which is rigorously controlled.

From a physical point of view, the problems of evaporation condensation for multi-component gases was studied in ([17]). A binary mixture of vapor and non condensable gas is considered in contact with an infinite plane of a condensed vapor. The non condensable gas is supposed to be close to the condensed phase. The problem is solved numerically when the Boltzmann operator is replaced by the *BGK* operator. The physical context is described in those papers. In the situation where the Knudsen number tends to 0, this problem has been already studied in ([3, 1]) where two types of behaviour were pointed out. In a first situation the macroscopic velocity of the two gases is 0 ([3, 23]). That means physically that evaporation and condensation stop for the *A* component. But the Hilbert term of order 1 of the velocity of the *A* component keeps an influence at the hydrodynamical level. This is the ghost effect as defined for a one component gas in ([18]) and for a two component gas in ([3, 23, 22]). In a second case the *B* component becomes negligible and accumulates in a thin layer at the boundaries called Knudsen layer ([4]).

More precisely, we consider in this paper the stationary Boltzmann problem in a slab for a two component gas when the slab is represented by the interval  $[-1, 1]$

$$\begin{aligned} \xi \frac{\partial}{\partial x} f_A(x, v) &= Q(f_A, f_A + f_B)(x, v), \\ \xi \frac{\partial}{\partial x} f_B(x, v) &= Q(f_B, f_A + f_B)(x, v), \\ x &\in [-1, 1], v \in \mathbb{R}^3. \end{aligned} \tag{1.1}$$

The nonnegative functions represent the distribution functions  $f_A$  and  $f^B$  of the *A* and of the *B* component and  $\xi$  is the velocity component

The collision operator  $Q$  is the Boltzmann operator

$$\begin{aligned} Q(f, g)(x, v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \omega) [f' g'_* - f g_*] d\omega dv_*, \\ &= Q^+(f, g)(x, v) - Q^-(f, g)(x, v), \end{aligned}$$

where  $Q^+(f, g) - Q^-(f, g)$  is the splitting into gain and loss term,

$$\begin{aligned} f_* &= f(x, v_*), & f' &= f(x, v'), & f'_* &= f'(x, v_*) \\ v' &= v - \langle v - v_*, \omega \rangle \omega, & v'_* &= v_* + \langle v - v_*, \omega \rangle \omega. \end{aligned}$$

For a more general introduction to the Boltzmann equation for multi-component gases see ([12]).  $\langle v - v_*, \omega \rangle$  denotes the Euclidean product in  $\mathbb{R}^3$ . Let  $\omega$  be represented by the polar angle (with polar axis along  $v - v_*$ ) and the azimuthal angle  $\phi$ . The function  $B(v - v_*, \omega)$  is the kernel of the collision operator  $Q$  taken for weak forces as  $|v - v_*|^\beta b(\theta)$ , with

$$-3 < \beta < 0, \quad b \in L_+^1([0, 2\pi]), \quad b(\theta) \geq c > 0 \quad a.e.$$

Denote the collision frequency by

$$\nu(x, v) = \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}^2} B(v - v_*, \omega) f(x, v_*) dv_* d\omega.$$

In this paper, we study the case of soft forces for a two component gas in a slab with given indata profile on both side of the domain for the  $A$  component and diffuse reflection boundary conditions for the  $B$  component. The vapor will be called the  $A$  component and the noncondensable as will be called the  $B$  component. The boundary conditions for the  $A$  component are

$$f_A(-1, v) = kM_-(v), \quad \xi > 0, \quad f_A(1, v) = kM_+(v), \quad \xi < 0, \quad (1.2)$$

where  $k$  is a nonnegative constant which is a part of the unknown and will be determined during the resolution of the problem. The boundary conditions for the  $B$  component are

$$\begin{aligned} f_B(-1, v) &= \left( \int_{\xi' < 0} |\xi'| f_B(-1, v') dv' \right) M_-(v), \quad \xi > 0, \\ f_B(1, v) &= \left( \int_{\xi' > 0} \xi' f_B(1, v') dv' \right) M_+(v), \quad \xi < 0. \end{aligned} \quad (1.3)$$

$M_+$  and  $M_-$  are given normalized Maxwellians

$$M_-(v) = \frac{1}{2\pi T_-^2} e^{-\frac{|v|^2}{2T_-}} \quad \text{and} \quad M_+(v) = \frac{1}{2\pi T_+^2} e^{-\frac{|v|^2}{2T_+}}.$$

Consider the stationary Boltzmann problem in a slab for a two component gas

In this paper, mild, weak and renormalized solutions  $(f_A, f_B)$  to the stationary problem (1.1, 1.2, 1.3) can be formulated as follows when  $\frac{Q^-(f^A, f)}{1+f^A}$ ,  $\frac{Q^+(f^A, f)}{1+f^A}$ ,  $\frac{Q^-(f^B, f)}{1+f^B}$ ,  $\frac{Q^+(f^B, f)}{1+f^B} \in L^1_{loc}$ .

**Definition 1.1.** Let  $M_A$  and  $M_B$  be given nonnegative real numbers.  $(f_A, f_B)$  is a mild solution to the stationary Boltzmann problem with the  $\beta$ -norms  $M_A$  and  $M_B$ , if  $f_A$  and  $f_B \in L^1_{loc}((-1, 1) \times \mathbb{R}^3)$ ,  $\int (1 + |v|)^\beta f_A(x, v) dx dv = M_A$ ,  $\int (1 + |v|)^\beta f_B(x, v) dx dv = M_B$ , and there is a constant  $k > 0$  such that

$$f_A(1 + s\xi, v) = kM_-(v) + \int_{-\frac{2}{\xi}}^s Q(f^A, f)(1 + \tau\xi, v) d\tau, \quad \xi > 0, \quad s \in ]-\frac{2}{\xi}, 0],$$

$$f_A(-1 + s\xi, v) = kM_+(v) + \int_{-\frac{2}{\xi}}^s Q(f^A, f)(-1 + \tau\xi, v) d\tau, \quad \xi < 0, \quad s \in ]\frac{2}{\xi}, 0],$$

$$\begin{aligned} f_B(-1 + s\xi, v) &= \left( \int_{\xi' < 0} f_B(-1, v') dv' \right) M_-(v) \\ &+ \int_{\frac{2}{\xi}}^s Q(f^B, f)(-1 + \tau\xi, v) d\tau, \quad \xi < 0, \quad s \in ]\frac{2}{\xi}, 0], \end{aligned}$$

$$\begin{aligned} f_B(1 + s\xi, v) &= \left( \int_{\xi' > 0} \xi' f_B(1, v') dv' \right) M_+(v) \\ &+ \int_{-\frac{2}{\xi}}^s Q(f^B, f)(1 + \tau\xi, v) d\tau, \quad \xi > 0, \quad s \in ]-\frac{2}{\xi}, 0]. \end{aligned}$$

**Definition 1.2.** Let  $M_A$  and  $M_B$  be given nonnegative real numbers.  $(f_A, f_B)$  is a weak solution to the stationary Boltzmann problem with the  $\beta$ -norms  $M_A$  and  $M_B$ , if  $f_A$  and  $f_B \in L^1_{loc}((-1, 1) \times \mathbb{R}^3)$ ,  $\nu \in L^1_{loc}((-1, 1) \times \mathbb{R}^3)$ ,  $\int (1 + |v|)^\beta f_A(x, v) dx dv = M_A$ ,  $\int (1 + |v|)^\beta f_B(x, v) dx dv = M_B$ , and there is a constant  $k > 0$  such that for every test function  $\varphi \in C^1_c([-1, 1] \times \mathbb{R}^3)$  such

that  $\varphi$  vanishes in a neighborhood of  $\xi = 0$ , and on  $\{(-1, v); \xi < 0\} \cup \{(1, v); \xi > 0\}$ ,

$$\begin{aligned}
& \int_{-1}^1 \int_{\mathbb{R}^3} (\xi f_A \frac{\partial \varphi}{\partial x} + Q(f_A, f_A + f_B) \varphi)(x, v) dx dv \\
&= k \int_{\mathbb{R}^3, \xi < 0} \xi M_+(v) \varphi(1, v) dv - k \int_{\mathbb{R}^3, \xi > 0} \xi M_-(v) \varphi(-1, v) dv, \\
& \int_{-1}^1 \int_{\mathbb{R}^3} (\xi f_B \frac{\partial \varphi}{\partial x} + Q(f_B, f_A + f_B) \varphi)(x, v) dx dv, \\
&= \int_{\xi' < 0} |\xi| M_+(v) \varphi(1, v) dv \left( \int_{\xi' > 0} \xi' f_B(1, v') dv' \right) \\
&- \int_{\xi' > 0} \xi M_-(v) \varphi(-1, v) dv \left( \int_{\xi' < 0} \xi' f_B(-1, v') dv' \right).
\end{aligned}$$

Let  $g$  be defined for  $x > 0$  by

$$g(x) = \ln(1 + x).$$

**Definition 1.3.** Let  $M_A$  and  $M_B$  be given nonnegative real numbers.  $(f_A, f_B)$  is a renormalized solution to the stationary Boltzmann problem with the  $\beta$ -norms  $M_A$  and  $M_B$ , if  $f_A$  and  $f_B \in L^1_{loc}((-1, 1) \times \mathbb{R}^3)$ ,  $\nu \in L^1_{loc}((-1, 1) \times \mathbb{R}^3)$ ,  $\int (1 + |v|)^\beta f_A(x, v) dx dv = M_A$ ,  $\int (1 + |v|)^\beta f_B(x, v) dx dv = M_B$ , and there is a constant  $k > 0$  such that for every test function  $\varphi \in C^1_c([-1, 1] \times \mathbb{R}^3)$  such that  $\varphi$  vanishes in a neighborhood of  $\xi = 0$  and on  $\{(-1, v); \xi < 0\} \cup \{(1, v); \xi > 0\}$ ,

$$\begin{aligned}
& \int_{-1}^1 \int_{\mathbb{R}^3} (\xi g(f_A) \frac{\partial \varphi}{\partial x} + \frac{Q(f_A, f_A + f_B)}{1 + f_A} \varphi)(x, v) dx dv \\
&= \int_{\mathbb{R}^3, \xi < 0} \xi g(k M_+(v)) \varphi(1, v) dv - \int_{\mathbb{R}^3, \xi > 0} g(\xi k M_-(v)) \varphi(-1, v) dv, \\
& \int_{-1}^1 \int_{\mathbb{R}^3} (\xi g(f_B) \frac{\partial \varphi}{\partial x} + \frac{Q(f_B, f_A + f_B)}{1 + f_B} \varphi)(x, v) dx dv, \\
&= \int_{\xi < 0} \xi g\left(\int_{\xi' > 0} \xi' f_B(1, v') dv'\right) M_+(v) \varphi(1, v) dv \\
&- \int_{\xi > 0} \xi g\left(\int_{\xi' < 0} \xi' f_B(-1, v') dv'\right) M_-(v) \varphi(-1, v) dv.
\end{aligned}$$

**Remark 1.** By arguing as in [14], it can be shown that the concepts of renormalized and mild solutions are equivalent.

The main result of this paper is the following

**Theorem 1.1.** *Given  $\beta$  with  $-3 < \beta < 0$ , there is a renormalized solution to the stationary problem with  $\beta$ -norms equal to one.*

This paper is organized as follows. The second section of this paper deals with a construction of approximate solutions to the problem and with the passage to the limit in the sequence of approximations. The passage to the limit in the traces is also performed. Denote that the proof of the weak compactness of the gain term is not obtained by arguing as in the situation of a one component gas ([7, 14]). It will be given by lemma 2.1. In section 3, some extensions of Theorem 1.1 are made. In particular, the case where  $M_A$  and  $M_B$  have any positive values is considered.

## 2 Approximations with fixed total masses

This section is devoted to the proof of Theorem 1.1. First a solution  $(f_A^{r,\mu}, f_B^{r,\mu})$  to an approached problem is constructed by arguing as in [10]. Next, the passage to the limit is performed in the renormalized form of the approached problem when  $r$  tends to 0 and  $\mu$  tends to infinity. The compactness of  $(f_A^{r,\mu}, f_B^{r,\mu})$  and of the loss terms is obtained by classical arguments. But the weak compactness of the gain term cannot be directly obtained (Lemma 2.1). Finally the passage to the limit in the traces isn't performed.

By reasoning as in ([10]), we can show that there are  $(f_A^{r,\mu}, f_B^{r,\mu})$  satisfying

$$\begin{aligned} \xi \frac{\partial}{\partial x} f_A^{r,\mu} &= \int_{\mathbb{R}_v^3 \times \mathbb{S}^2} \chi^r B_\mu(v - v_*, \omega) f_A^{r,\mu}(x, v') f^{r,\mu}(x, v'_*) dv_* d\omega \\ - f_A^{r,\mu} \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}^2} \chi^r B_\mu(v - v_*, \omega) f^{r,\mu}(x, v_*) dv_* d\omega, \quad (x, v) &\in (-1, 1) \times \mathbb{R}_v^3, \\ f_A^{r,\mu}(-1, v) &= k_A M_-(v), \quad \xi > 0, \quad f_A^{r,\mu}(1, v) = k_A M_+(v), \quad \xi < 0, \end{aligned} \quad (2.1)$$

$$\begin{aligned} \xi \frac{\partial}{\partial x} f_B^\mu &= \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}^2} B_\mu(v - v_*, \omega) f_B^\mu(x, v') f^\mu(x, v'_*) dv_* d\omega \\ - f_B^\mu \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}^2} B_\mu(v - v_*, \omega) f^\mu(x, v_*) dv_* d\omega, \quad (x, v) &\in (-1, 1) \times \mathbb{R}_v^3, \\ f_B^\mu(-1, v) &= M_-(v) \int_{\xi < 0} |\xi| f_B^\mu(-1, v) dv, \quad \xi > 0, \\ f_B^\mu(1, v) &= M_+(v) \int_{\xi > 0} \xi f_B^\mu(1, v) dv, \quad \xi < 0, \end{aligned} \quad (2.2)$$

with

$$\int \max\left(\frac{1}{\mu}, \min(\mu, (1 + |v|)^\beta)\right) f_A^\mu(x, v) dx dv = 1$$

and

$$\int \max\left(\frac{1}{\mu}, \min(\mu, (1 + |v|)^\beta)\right) f_B^\mu(x, v) dx dv = 1.$$

$\chi^r$  is a  $C_0^\infty$  function with range  $[0, 1]$  invariant under the transformation  $J$ ,

$$J(v, \omega, v_*) = (v', -\omega, v'_*)$$

and satisfying

$$\begin{aligned} \chi^r(v, v_*, \omega) &= 1 & \text{if } |\xi| > r, |\xi_*| > r, |\xi'| > r, |\xi'_*| > r, \\ \chi^r(v, v_*, \omega) &= 0 & \text{if } |\xi| < \frac{r}{2}, |\xi_*| < \frac{r}{2}, |\xi'| < \frac{r}{2}, |\xi'_*| < \frac{r}{2} \end{aligned}$$

and the modified collision kernel  $B_\mu$  is defined by  $\max(\frac{1}{\mu} \min(B, \mu))$ .

Let  $(r_j)_{j \in \mathbb{N}}$  with  $\lim_{j \rightarrow +\infty} r_j = 0$  and  $(\mu_j)_{j \in \mathbb{N}}$  with  $\lim_{j \rightarrow +\infty} \mu_j = +\infty$ ,  
 $f_A^j = f_A^{r_j, \mu_j}$  and  $f_B^j = f_B^{r_j, \mu_j}$ .

The passage to the limit when  $j \rightarrow +\infty$  is now performed in the renormalized formulations of the equations (2.1, 2.2) satisfied by  $(f_A^j, f_B^j)$ .

A positive number  $\delta$  being fixed, let  $\varphi$  be a test function vanishing for  $|\delta| \leq \delta$  and for  $|v| \geq \frac{1}{\delta}$ . Since  $f^j = f_A^j + f_B^j$  satisfies the Boltzmann equation for a one component gas, it can be shown from [7] that  $f^j$  and  $\int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*|^\beta f^j dv_* d\omega$  are weakly compact in  $L^1([-1, 1] \times \{v \in \mathbb{R}^3; |\xi| \geq \delta, |v| \leq \frac{1}{\delta}\})$ . The weak compactness of  $f_A^j, f_B^j, Q_j^-(f_A^j, f^j), Q_j^-(f_B^j, f^j)$  follows from the inequalities

$$\begin{aligned} f_A \leq f, f_B \leq f, \frac{Q^-(f_A^j, f^j)}{1 + f_A^j} &\leq \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*|^\beta f^j dv_* d\omega \\ \frac{Q^-(f_B^j, f^j)}{1 + f_B^j} &\leq \int_{\mathbb{R}^3 \times \mathbb{S}^2} |v - v_*|^\beta f^j dv_* d\omega. \end{aligned}$$

Here, the weak compactness of  $\frac{Q^+(f_A^j, f^j)}{1 + f_A^j}$  and  $\frac{Q^+(f_B^j, f^j)}{1 + f_B^j}$  cannot be directly obtained from the weak compactness of the loss term in the renormalized form as in ([7, 14]). Hence we need to show the following lemma.

**Lemma 2.1.**  $\frac{Q^+(f_A^j, f^j)}{1 + f_A^j}$  and  $\frac{Q^+(f_B^j, f^j)}{1 + f_B^j}$  are weakly compact in

$$L^1([-1, 1] \times \{v \in \mathbb{R}^3; |\xi| \geq \delta, |v| \leq \frac{1}{\delta}\}).$$

Before giving the proof of Lemma 2.1 let us recall that

$$\begin{aligned} & \int_{\mathbb{R}^3} Q(f_A^j, f_A^j) \ln(f_A^j) dv + \int_{\mathbb{R}^3} Q(f_A^j, f_B^j) \ln(f_A^j) dv \\ & + \int_{\mathbb{R}^3} Q(f_B^j, f_A^j) \ln(f_B^j) dv + \int_{\mathbb{R}^3} Q(f_B^j, f_B^j) \ln(f_B^j) dv \leq 0. \end{aligned}$$

This estimate is shown in [2]. For the sake of clarity we will give a proof of this result. Indeed

$$\begin{aligned} & \int_{\mathbb{R}^3} Q(f_A^j, f_A^j) \ln(f_A^j) dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) \\ & (f_A^j(x, v') f_A^j(x, v'_*) - f_A^j(x, v) f_A^j(x, v_*)) \ln \left( \frac{f_A^j(x, v) f_A^j(x, v_*)}{f_A^j(x, v') f_A^j(x, v'_*)} \right) dv dv_* d\omega \\ & \int_{\mathbb{R}^3} Q(f_B^j, f_B^j) \ln(f_B^j) dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) \\ & (f_B^j(x, v') f_B^j(x, v'_*) - f_B^j(x, v) f_B^j(x, v_*)) \ln \left( \frac{f_B^j(x, v) f_B^j(x, v_*)}{f_B^j(x, v') f_B^j(x, v'_*)} \right) dv dv_* d\omega \\ & \int_{\mathbb{R}^3} Q(f_A^j, f_B^j) \ln(f_A^j) + Q(f_B^j, f_A^j) \ln(f_B^j) dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) \\ & (f_A^j(x, v') f_B^j(x, v'_*) - f_A^j(x, v) f_B^j(x, v_*)) \ln \left( \frac{f_A^j(x, v) f_B^j(x, v_*)}{f_A^j(x, v') f_B^j(x, v'_*)} \right) dv dv_* d\omega. \end{aligned}$$

By arguing as for the case of the one component gas, it follows that the terms

$$\begin{aligned} I_{AA}(f_A^j, f_A^j) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) (f_A^j(x, v') f_A^j(x, v'_*) - f_A^j(x, v) f_A^j(x, v_*)) \\ & \ln \left( \frac{f_A^j(x, v) f_A^j(x, v_*)}{f_A^j(x, v') f_A^j(x, v'_*)} \right) dv_* d\omega, \\ I_{BB}(f_B^j, f_B^j) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) (f_B^j(x, v') f_B^j(x, v'_*) - f_B^j(x, v) f_B^j(x, v_*)) \\ & \ln \left( \frac{f_B^j(x, v) f_B^j(x, v_*)}{f_B^j(x, v') f_B^j(x, v'_*)} \right) dv_* d\omega, \\ I_{AB}(f_A^j, f_B^j) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) (f_A^j(x, v') f_B^j(x, v'_*) - f_A^j(x, v) f_B^j(x, v_*)) \\ & \ln \left( \frac{f_A^j(x, v) f_B^j(x, v_*)}{f_A^j(x, v') f_B^j(x, v'_*)} \right) dv_* d\omega, \end{aligned}$$

are bounded in  $L^1$

*Proof.* (Lemma 2.1) By proceeding as for the case of a one component gas, it holds that

$$\begin{aligned}
Q^+(f_A^j, f^j) &\leq KQ^-(f_A^j, f^j) + \frac{1}{\ln K} I_{AA}(f_A^j, f_A^j) \\
&\quad + \frac{1}{\ln K} \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) (f_A^j(x, v') f_B^j(x, v'_*) - f_A^j(x, v) f_B^j(x, v_*)) \\
&\quad \quad \quad \ln \left( \frac{f_A^j(x, v)}{f_A^j(x, v')} \right) dv_* d\omega, \\
Q^+(f_B^j, f^j) &\leq KQ^-(f_B^j, f^j) + \frac{1}{\ln K} I_{BB}(f_B^j, f_B^j) \\
&\quad + \frac{1}{\ln K} \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_j(v - v_*, \omega) (f_A^j(x, v') f_B^j(x, v'_*) - f_A^j(x, v) f_B^j(x, v_*)) \\
&\quad \quad \quad \ln \left( \frac{f_A^j(x, v)}{f_A^j(x, v')} \right) dv_* d\omega d\omega.
\end{aligned}$$

The two previous inequalities lead to

$$\begin{aligned}
\frac{Q^+(f_A^j, f^j)}{1 + f_A^j} + \frac{Q^+(f_B^j, f^j)}{1 + f_B^j} &\leq K \frac{Q^-(f_A^j, f^j)}{1 + f_A^j} + K \frac{Q^-(f_B^j, f^j)}{1 + f_B^j} \\
&\quad + \frac{1}{\ln K} \left( I_{AA}(f_A^j, f_A^j) + I_{BB}(f_B^j, f_B^j) + I_{AB} f_A^j, f_B^j \right).
\end{aligned}$$

Hence the weak compactness of  $\frac{Q^+(f_A^j, f^j)}{1 + f_A^j}$  and  $\frac{Q^+(f_B^j, f^j)}{1 + f_B^j}$  follow by classical arguments as for the one component case ([7]).  $\square$

Denote by  $f^A$  and  $f^B$  the respective weak limits of  $f_A^j$  and  $f_B^j$  in  $L^1$ . Now, the aim is to pass to the limit in the boundary terms (1.3) i.e to prove the weak convergence in  $L^1(\{v \in \mathbb{R}_v^3, \xi > 0\})$  ( resp.  $L^1(\{v \in \mathbb{R}_v^3, \xi < 0\})$ ) of  $f_B^j(1, \cdot)$  ( resp.  $f_B^j(-1, \cdot)$ ) to  $f_B(1, \cdot)$  (resp.  $f_B(-1, \cdot)$ ). First, the fluxes  $\int_{\xi > 0} \xi f_B^j(1, v) dv$  and  $\int_{\xi < 0} |\xi| f_B^j(-1, v) dv$  are controlled in the following way. From (2.2) written in the exponential form, it holds that

$$\begin{aligned}
f_B^j(x, v) &\geq f_B^j(-1, v) e^{-\int_{-\frac{1+x}{\xi}}^0 \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}^2} B^j f^j(x+s\xi, v_*) dv_* d\omega ds}, \quad \xi > \frac{1}{2}, \\
f_B^j(x, v) &\geq f_B^j(1, v) e^{-\int_{\frac{1-x}{\xi}}^0 \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}^2} \chi^j B^j f^j(x+s\xi, v_*) dv_* d\omega ds}, \quad \xi < -\frac{1}{2}.
\end{aligned} \tag{2.3}$$

Recall that,

$$\nu^j(x, v) = \int_{\mathbb{R}_{v_*}^3 \times \mathbb{S}^2} \chi^j B^j f^j(x, v_*) dv_* d\omega.$$

For  $v$  satisfying  $\xi > \frac{1}{2}$  or  $\xi < -\frac{1}{2}$ ,  $\int_{-1}^1 \frac{\nu^j(z, v)}{|\xi|} dz$  is uniformly bounded from above. Hence, by using the definition of the boundary conditions (1.3) in (2.3), it holds that

$$\begin{aligned} f_B^j(x, v) &\geq cM_-(v) \int_{\xi < 0} |\xi| f_B^j(-1, v) dv \quad , \quad \xi > \frac{1}{2}, \\ f_B^j(x, v) &\geq cM_+(v) \int_{\xi > 0} \xi f_B^j(1, v) dv \quad , \quad \xi < -\frac{1}{2}. \end{aligned}$$

So

$$\begin{aligned} &\int_{\{\xi > \frac{1}{2}\} \cup \{\xi < -\frac{1}{2}\}} \max\left(\frac{1}{\mu}, \min(\mu, (1 + |v|)^\beta)\right) f_B^j(x, v) dx dv \\ &\geq \int_{\{\xi > \frac{1}{2}\}} \max\left(\frac{1}{\mu}, \min(\mu, (1 + |v|)^\beta)\right) M_+(v) dv \int_{\xi > 0} \xi f_B^j(1, v) dv \\ &+ \int_{\{\xi < -\frac{1}{2}\}} \max\left(\frac{1}{\mu}, \min(\mu, (1 + |v|)^\beta)\right) M_-(v) dv \int_{\xi < 0} |\xi| f_B^j(-1, v) dv. \end{aligned}$$

$f_B^j$  being non negative, we have for  $\mu$  big enough

$$\begin{aligned} &c \int_{-1}^1 \int_{\mathbb{R}_v^3} \max\left(\frac{1}{\mu}, \min(\mu, (1 + |v|)^\beta)\right) f_B^j(x, v) dx dv \\ &\geq \int_{\xi > 0} \xi f_B^j(1, v) dv + \int_{\xi < 0} |\xi| f_B^j(-1, v) dv. \end{aligned}$$

Since  $\int_{-1}^1 \int_{\mathbb{R}_v^3} \max\left(\frac{1}{\mu}, \min(\mu, (1 + |v|)^\beta)\right) f_B^j(x, v) dx dv = 1$ , the fluxes  $\int_{\xi > 0} \xi f_B^j(1, v) dv$  and  $\int_{\xi < 0} |\xi| f_B^j(-1, v) dv$  are bounded uniformly *w.r.t*  $j$ .

Furthermore, the energy fluxes are controlled. Indeed, by conservation of the energy for  $f^j$ , it holds that

$$\begin{aligned} &\int_{\xi > 0} \xi v^2 f_B^j(1, v) dv + \int_{\xi < 0} |\xi| v^2 f_B^j(-1, v) dv \\ &\leq \int_{\xi > 0} \xi v^2 f^j(-1, v) dv + \int_{\xi < 0} |\xi| v^2 f^j(1, v) dv. \end{aligned}$$

The definition of the boundary conditions (2.1) and (2.2) yield

$$\begin{aligned}
& \int_{\xi>0} \xi v^2 f_B^j(1, v) dv + \int_{\xi<0} |\xi| v^2 f_B^j(-1, v) dv \\
& \leq (k^j + \int_{\xi'<0} |\xi'| f_B^j(-1, v') dv') \int_{\xi>0} \xi v^2 M_-(v) dv \\
& \quad + (k^j + \int_{\xi'>0} \xi' f_B^j(1, v') dv') \int_{\xi<0} |\xi| v^2 M_+(v) dv.
\end{aligned} \tag{2.4}$$

The right-hand side of (2.4) being bounded, it follows that

$$\int_{\xi>0} \xi v^2 f_B^j(1, v) dv + \int_{\xi<0} |\xi| v^2 f_B^j(-1, v) dv \leq c.$$

Therefore the entropy fluxes are controlled. Indeed,  $f^j = f_A^j + f_B^j$  satisfies the following equation

$$\xi \frac{\partial}{\partial x} (f^j (\log(f^j) - 1)) = Q_j(f^j, f^j) \log(f^j). \tag{2.5}$$

By using a Green's formula and an entropy estimate in (2.5), leads to

$$\begin{aligned}
& \int_{\xi>0} \xi f_B^j(1, v) \log f_B^j(1, v) dv + \int_{\xi<0} |\xi| f_B^j(-1, v) \log f_B^j(-1, v) dv \\
& \leq \left( \int_{\xi'>0} \xi' f_B^j(1, v') dv' + k^j \right) \\
& \quad \int_{\xi<0} |\xi| M_+(v) \log(M_+(v)) \left( \int_{\xi'>0} \xi' f_B^j(1, v') dv' + k^j \right) dv \\
& \quad + \left( \int_{\xi'<0} |\xi'| f_B^j(-1, v') dv' + k^j \right) \\
& \quad \int_{\xi>0} M_-(v) \log(M_-(v)) \left( \int_{\xi'<0} |\xi'| f_B^j(-1, v') dv' + k^j \right) dv.
\end{aligned}$$

Hence the Dunford-Pettis criterion ([13]),  $f_B^j(1, \cdot)$  is weakly compact in  $L^1(\{v \in \mathbb{R}_v^3, \xi > 0\})$ . Let one of its subsequence still denoted by  $f_B^j(1, \cdot)$ , converging weakly to some  $g_+$  in  $L^1(\{v \in \mathbb{R}_v^3, \xi > 0\})$ . It remains to show that  $g_+ = f_B(1, \cdot)$ . Let  $g_d$  be defined by

$$g_d(x) = \frac{1}{d} \ln(1 + dx). \tag{2.6}$$

$f_B^j$  being a weak solution to (2.2), it comes that

$$\xi \frac{\partial}{\partial x} g_d(f_B^j) = \frac{Q(f_B^j, f^j)}{1 + d f_B^j}.$$

Denote that by weak compactness of  $\frac{Q(f_B^j, f^j)}{1 + f_B^j}$ ,  $\frac{Q(f_B^j, f^j)}{1 + d f_B^j}$  is also weakly compact. Consider a test function  $\varphi$  vanishing on  $\{|\xi| \leq \delta\} \cup \{|v| \geq \frac{1}{\delta}\}$  and satisfying  $\varphi(x, v) = \varphi_1(x)\varphi_2(v)$  with  $\varphi_1 = 1$  in a neighborhood of 1.

$$\int_{\xi > 0} \xi f_B^j \varphi(1, v) dv = \int_{\xi > 0} \xi g_d(f_B^j) \varphi(1, v) dv + \int_{\xi > 0} \xi (f_B^j - \beta_d(f_B^j)) \varphi(1, v) dv$$

The fluxes being controlled, it holds that

$$\lim_{d \rightarrow 0} \int_{\xi > 0} \xi |f_B^j(1, v) - g_d(f_B^j)(1, v)| dv = 0,$$

uniformly *w.r.t*  $j$ . So,  $\eta$  being given, we can chose  $d > 0$  such that uniformly *w.r.t*  $j$ ,

$$\left| \int_{\xi > 0} \xi (f_B^j - g_d(f_B^j)) \varphi(1, v) dv \right| \leq \eta, \quad \left| \int_{\xi > 0} \xi (f_B - g_d(f_B)) \varphi(1, v) dv \right| \leq \eta.$$

Let us show that

$$\int_{\xi > 0} \xi g_d(f_B^j) \varphi(1, v) dv \rightarrow \int_{\xi > 0} \xi g_d(f_B) \varphi(1, v) dv.$$

By definition of the trace ([5], [10]),

$$g_d(f^j)(1, v) = \lim_{\varepsilon_0 \rightarrow 0} \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} g_d(f(1 - \varepsilon, v)) d\varepsilon$$

But  $(\varphi g_d(f_B^j))$  satisfies the equation

$$\xi \frac{\partial(\varphi g_d(f_B^j))}{\partial x} = \xi \frac{\partial \varphi}{\partial x} g_d(f_B^j) + \frac{Q_j(f_B^j, f^j)}{1 + d f_B^j} \varphi. \quad (2.7)$$

So, by integrating (2.7) on  $[1 - \varepsilon, 1] \times \mathbb{R}_v^3$  and by using a Green's formula, it holds that

$$\begin{aligned} & \left| \frac{1}{\varepsilon_0} \int_{\mathbb{R}_v^3} \int_0^{\varepsilon_0} (g_d(f_B^j)(1, v) - g_d(f_B^j)(1 - \varepsilon, v)) \varphi_2(v) dv d\varepsilon \right| \\ & \leq \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} \int_{\mathbb{R}_v^3} \int_{1 - \varepsilon_0}^1 \left| \frac{Q_j(f_B^j, f^j)}{1 + d f_B^j}(x, v) \varphi(x, v) \right| dx dv d\varepsilon \\ & \quad + \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} \int_{\mathbb{R}_v^3} \int_{1 - \varepsilon_0}^1 |g_d(f_B^j)(x, v) \xi \frac{\partial}{\partial x} \varphi(x, v)| dx dv d\varepsilon. \end{aligned} \quad (2.8)$$

By weak compactness of  $(f_B^j)$  and  $\frac{Q_j(f_B^j, f^j)}{1+d f_B^j}$  in  $L^1([-1, 1] \times \{|\xi| \geq \delta, |v| \geq \frac{1}{\delta}\})$ , there is  $\tilde{\varepsilon}_0 > 0$ , such that for  $\varepsilon_0 < \tilde{\varepsilon}_0$  and uniformly *w.r.t*  $j$ ,

$$\begin{aligned} \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} \int_{\mathbb{R}_v^3} \int_{1-\varepsilon_0}^1 \left| \frac{Q_j(f_B^j, f^j)}{1+d f_B^j}(x, v) \varphi(x, v) \right| dx dv d\varepsilon &\leq \eta, \\ \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} \int_{\mathbb{R}_v^3} \int_{1-\varepsilon_0}^1 \left| \beta_d(f_B^j)(x, v) \xi \frac{\partial}{\partial x} \varphi(x, v) \right| dx dv d\varepsilon &\leq \eta. \end{aligned}$$

But by weak compactness of  $g_d(f_B^j)(1, v)$ ,  $g_d(f_B^j)(1, v)$  is converging weakly in  $L^1$  to some  $\bar{g}_d$ . By weak compactness of  $g_d(f_B^j)(1 - \varepsilon, v)$  in  $L^1([-1, 1] \times \{v \in \mathbb{R}^3, |\xi| \geq \delta, |v| \leq \delta\})$

$$\int_{\mathbb{R}_v^3} \int_0^{\varepsilon_0} \xi g_d(f_B^j)(1 - \varepsilon, v) \varphi_2(v) dv d\varepsilon \rightarrow \int_{\mathbb{R}_v^3} \int_0^{\varepsilon_0} \xi g_d(f_B)(1 - \varepsilon, v) \varphi_2(v) dv d\varepsilon$$

So

$$\left| \frac{1}{\varepsilon_0} \int_0^{\varepsilon_0} \int_{\mathbb{R}_v^3} \xi (\bar{g}_d - g_d(f_B)) \varphi_2(v) dv d\varepsilon \right| \leq \eta.$$

Hence,  $\bar{g}_d$  and  $g_d(f_B)$  are equal on the sets  $\{v \in \mathbb{R}^3, |\xi| \geq \delta, |v| \leq \delta\}$  for all  $\delta > 0$  and so *a.e.*

From here, by arguing as in ([14]), we can prove that  $(f^A, f^B)$  satisfies the mild form or the renormalized form of (1.1, 1.2, 1.3).  $\square$

### 3 Some extensions.

This section is devoted to some extensions to Theorem [?] in particular in the situation of multi-component gases. The results are similar to those obtained in [10]. By reasoning as in [10], the following extensions can be proved.

**Corollary 3.1.** *Given  $\beta$  with  $-3 < \beta < 0$ ,  $M_A > 0$  and  $M_B > 0$ , there is a renormalized solution to the stationary problem with  $\beta$ -norms  $M_A$  and  $M_B$ .*

**Corollary 3.2.** *Given  $\beta$  with  $-3 < \beta < 0$ ,  $M_{A_1} \dots M_{A_{N_A}}$  and  $M_{B_1} \dots M_{B_{N_B}}$  there is a renormalized solution  $f_{A_1} \dots f_{B_{N_B}}$  to the stationary problem with respective  $\beta$ -norms  $M_{A_1}, \dots, M_{B_{N_B}}$ .*

Remark that the case of a one component gas with one boundary condition of the type (1.2) and another of the type (1.3) can also be solved. It comes

back to the diffuse-reflection problem solved in ([8]) in the case of soft forces. Furthermore, this problem can be generalized to several components by reasoning as in the proof of Corollary 3.2.

Theorem 1.1 can also be generalized to the case of a convex combination of boundary conditions of the type (1.2) and (1.3),

$$\begin{aligned} \xi \frac{\partial}{\partial x} f &= Q(f, f), \quad (x, v) \in (-1, 1) \times \mathbb{R}_v^3, \\ f(-1, v) &= a \left( \int_{\xi < 0} |\xi| f(-1, v) dv \right) M_-(v) + (1-a) k M_-(v) \quad , \quad \xi > 0, \\ f(1, v) &= a \left( \int_{\xi > 0} \xi f(1, v) dv \right) M_+(v) + (1-a) k M_+(v) \quad , \quad \xi < 0, \quad (3.1) \\ & \quad \quad \quad a \in [0, 1]. \end{aligned}$$

**Corollary 3.3.** *Given  $\beta$  with  $-3 < \beta < 0$ ,  $M > 0$  there is a renormalized solution to the stationary problem (3.1) with the  $\beta$ -norm  $M$ .*

## References

- [1] Aoki K. *The behaviour of a vapor-gas mixture in the continuum limit: Asymptotic analysis based on the Boltzman equation*, T.J.' Bartel, M.A.Gallis(Eds), Rarefied Gas Dynamic, AIP, Melville, 565-574, 2001.
- [2] Aoki K., Bardos C., Takata S. *Knudsen layer for a gas mixture.*, Journ.Stat.Phys, 112, 3/4, 2003.
- [3] Aoki K., Takata S., Kosuge S. *Vapor flows caused by evaporation and condensation on two parallel plane surfaces: Effect of the presence of a noncondensable gas*, Physics of Fluids, 10, 6, 1519-1532, 1998.
- [4] Aoki K., Takata S., Taguchi S. *Vapor flows with evaporation and condensation in the continuum limit: effect of a trace of non condensable gas*, European Journal of Mechanics B/Fluids, 22, 51-71, 2003.
- [5] Arkeryd, L.; Heintz, A. *'On the solvability and asymptotics of the Boltzmann equation in irregular domains.* Comm.Part.Diff.Eqs. 1997, 22, 2129-2152.
- [6] Arkeryd, L.; Maslova, N. *'On diffuse reflection at the boundary for the Boltzmann equation and related equations.* Journ.Stat.Phys. 1994, 77, 1051-1077.

- [7] Arkeryd, L.; Nouri, A. *'The stationary Boltzmann Equation in the Slab with Given Weighted Mass for Hard and Soft Forces.* Ann.Scuola.Norm.Sup.Pisa 1998, 27, 533-536.
- [8] Arkeryd, L.; Nouri, A. *'L<sup>1</sup> solutions to the stationary Boltzmann equation in a slab.* Annales de la Faculte des sciences de Toulouse 2000, 9, 375-413.
- [9] Arkeryd, L.; Nouri A. *'A Large data existence result for stationary Boltzmann equation in a cylindral geometry, Arkiv for Matematik,* 2005, vol 434, 29-50.
- [10] Brull, S. *'The Boltzmann equation for a two component gas.* To appear in MMAS.
- [11] Brull, S. *'Problem of evaporation-condensation for a two component gas in the slab.* Submitted.
- [12] Cercignani, C. *The Boltzmann Equation. The Boltzmann equation and its applications.* Scottish Academic Press, 1988, 40-103.
- [13] Cercignani, C.; Illner, R.; Pulvirenti, M. *Existence and Uniqueness Results. The mathematical theory of dilute gases.* Springer: New York, 1994, 133-163.
- [14] Di Perna, R.J.; Lions, P.L. *'On the Cauchy problem for the Boltzmann equations: Global existence and weak stability,* Ann. of Math. 1989. 130. 321-366.
- [15] Mischler, S. *'Kinetic equations with Maxwell boundary condition ,* Submitted.
- [16] Panferov, V.A. *On the interior boundary-value problem for the stationary Povzner equation with hard and soft interactions,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3 (2004), no. 4, 771–825.
- [17] Sone, Y.; Aoki, K.; Doi, T. *'Kinetic theory analysis of gas flows condensing on a plane condensed phase: Case of a mixture of a vapor and noncondensable gas. Transport theory and statistical physics,* 1992, 21 (4-6), 297-328.
- [18] Sone Y., Aoki K., Takata S., Sugimoto H., Bobylev A. *Inappropriateness of the heat-conduction equation for description of a temperature*

*field of a stationary gas in the continuum limit: examination by asymptotic and numerical computation of the Boltzmann equation*, Physics of Fluids, 8, 2, 628-638, 1996.

Erratum: Physics of Fluids 8, 841 1996.

- [19] Taguchi S., Aoki K., Takata S. *Vapor flows at incidence onto a plane condensed phase in the presence of a non condensable gas. II. Supersonic condensation*, Physics of Fluids, 16, 79, 2004.
- [20] Takata S. *Kinetic theory analysis of the two-surface problem of vapor-vapor mixture in the continuum limit*, Physics of Fluids, 16, 7, 2004.
- [21] Takata S., Aoki K. *Two-surface-problems of a multicomponent mixture of vapors and noncondensable gases in the continuum limit in the light of kinetic theory*, Physics of Fluids, 11, 9, 2743-2756, 1999.
- [22] Takata S., Aoki K. *The ghost effect in the continuum limit for a vapor-gas mixture around condensed phases: Asymptotic analysis of the Boltzmann equation* Transport Theory and Statistical Physic, 30, 205-237, 2001.  
Erratum: Transport Theory an Statistical Physic, 31, 289, 2001.
- [23] Takata S., Aoki K., Muraki T. *Behaviour of a vapor-gas mixture between two parallel plane condensed phases in the continuum limit*, Rarefied Gas Dynamic, edited by R. brun, R. Campargue, and J. C. Lengrand Cépaduès, Toulouse, 479, 1999.