Around the electronic M_1 model.

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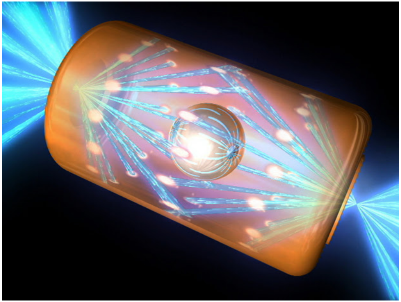

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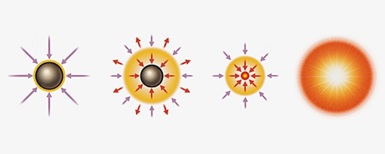
Context

Objectives: development of models and numerical methods

- robust
- fast
- accurate

↪ plasma physics applications (Inertial Confinement Fusion context)



↪ enable to study long time regimes (hydrodynamics scales)

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Introduction

Plasma: set of atoms totally ionized. Electronic transport, fixed ions

Kinetic description: electron distribution function $f(t,x,v)$,

$\hookrightarrow$ Resolution of the Vlasov or Fokker-Planck-Landau equation:

$$\underbrace{\frac{\partial f}{\partial t} + v.\nabla_{\mathbf{x}} f}_{advection\ \ term} + \underbrace{\frac{q}{m}(E + v \times B).\nabla_{\mathbf{v}} f}_{force\ \ \ term} = \underbrace{C_{e,e}(f,f) + C_{e,i}(f),}_{collisional\ \ \ terms}$$

- $C_{e,e}$: electron/electron collision operator,
- $C_{e,i}$: electron/ion collision operator.

$\hookrightarrow$ Accurate but numerically expensive

Hydrodynamic description: cheap but less accurate for far equilibrium regimes.

$\hookrightarrow$ Intermediate description, angular moment models.

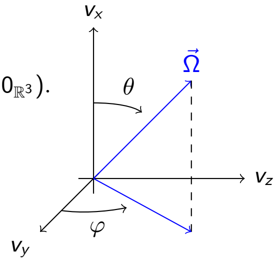
Electronic M_1 model

$\hookrightarrow$ Angular moments extraction: $v = \zeta\,\Omega$ with $\zeta = |v|$.

$$f_0(\zeta) = \zeta^2 \int_{S_2} f(v)d\Omega, \quad f_1(\zeta) = \zeta^2 \int_{S_2} f(v)\Omega d\Omega, \quad f_2(\zeta) = \zeta^2 \int_{S_2} f(v)\Omega\otimes\Omega d\Omega.$$

Set of realisable states

$$\mathcal{A} = \Big((f_0, f_1) \in \mathbb{R}\times\mathbb{R}^3,\ \ f_0 \geq 0, \quad |f_1| < f_0\Big) \cup (0_{\mathbb{R}}, 0_{\mathbb{R}^3}).$$



Collisionless electronic M_1 model

$$\begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \dfrac{q}{m}\partial_\zeta(f_1.E) = 0, \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \dfrac{q}{m}\partial_\zeta(f_2 E) - \dfrac{q}{m\zeta}(f_0 E - f_2 E) - \dfrac{q}{m}(f_1 \wedge B) = 0. \end{cases}$$

Closure

Different closures:

The P_N model : projection on Legendre polynomials.

Example: the P1 model

$$f = A_0(\zeta) + A_1(\zeta).\Omega, \qquad A_0(\zeta) = \frac{f_0(\zeta)}{4\pi},\ A_1(\zeta) = \frac{3\,f_1(\zeta)}{4\pi}.$$

The positivity of the distribution function is not ensured.

$\hookrightarrow$ Positive closure: P_N positive model[1]

$\hookrightarrow$ M_N model: entropy minimisation problem[2]

[1] Hauck, McLarreen, 2010

[2] G.N. Minerbo, J. Quant. Spectrosc. Radiat. Transfer, 1978.

The M_1 model

Determination of f_2 as a function of f_0 and f_1: Entropy minimisation problem[3,4].

$$\min_{f\geq 0}\left\{\ \mathcal{H}(f)\ \ /\ \ \forall\zeta\in\mathbb{R}^+,\ \ \zeta^2\int_{S^2}f(\Omega,\zeta)d\Omega=f_0(\zeta),\ \ \zeta^2\int_{S^2}f(\Omega,\zeta)\Omega d\Omega=f_1(\zeta)\ \right\}$$

$$\text{with } \mathcal{H}(f)=\int_{S^2}(f\ \ln\ f\ -\ f)d\Omega.$$

Entropy minimisation principle [5]:

$$f(\Omega,\zeta)=exp(a_0(\zeta)+a_1(\zeta).\Omega)\geq 0,\qquad a_1(\zeta)\in\mathbb{R}^3\qquad a_0(\zeta)\in\mathbb{R}.$$

Expression of f_2:

$$f_2=\Big(\frac{1-\chi(\alpha)}{2}Id+\frac{3\chi(\alpha)-1}{2}\frac{f_1}{|f_1|}\otimes\frac{f_1}{|f_1|}\Big)f_0$$

with

$$\chi(\alpha)=\frac{1+\alpha^2+\alpha^4}{3},\qquad \alpha=f_1/f_0,\qquad P1:\ \ \chi(\alpha)=1/3.$$

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[3] G.N. Minerbo, J. Quant. Spectrosc. Radiat. Transfer, 1978.
[4] D. Levermore, Moment Closure Hierarchies for Kinetic Theories, 1996.
[5] B. Dubroca and J.L. Feugeas. C. R. Acad. Sci. Paris Ser. I, 1999.

Limit of the M_1 model

$\hookrightarrow$ Accurate for isotropic configurations or configurations with one dominant direction[6].

Validity of the M_1 model for kinetic plasma studies?

Investigation of: particle beams interaction, Landau damping and laser-plasma absorption.

$\hookrightarrow$ Limit of the M_1 and M_2 models for collisionless regimes[7]

$\hookrightarrow$ Application to inertial confinement fusion

$\hookrightarrow$ collisional plasmas

[6] Dubroca, Feugeas and Frank. The European Phys. Journal 2010.

[7] Guisset, Moreau, Nuter, Brull, d'Humières, Dubroca, Tikhonchuk. J. Phys. A: Math. Theor. (2015).

Collisional operators

Collisional operators

The Fokker-Planck-Landau equation

$$\frac{\partial f}{\partial t} + v.\nabla_x f + \frac{q}{m}(E + v \times B).\nabla_v f = C_{e,e}(f,f) + C_{e,i}(f),$$

$$C_{ee}(f,f) = \alpha_{ee} div_v \Big(\int_{v' \in \mathbb{R}^3} S(v - v')[\nabla_v f(v) f(v') - f(v) \nabla_v f(v')] dv' \Big),$$

$$C_{ei}(f) = \alpha_{ei} div_v \Big[S(v) \nabla_v f(v) \Big], \quad S(u) = \frac{1}{|u|^3}(|u|^2 Id - u \otimes u).$$

$\hookrightarrow$ C_{ee} non-linear: complex angular moment extraction

Simplification

$$C_{ee}(f,f) \approx Q_{ee}(f_0) = C_{ee}(f_0, f_0)^{8},$$

Admissibility requirement

$$C_{ee}(f,f) \approx Q_{ee}(f) = C_{ee}(f, f_0)^{9}.$$

[8] Berezin, Khudick and Pekker (1987), Buet and Cordier (1998).
[9] J. Mallet, S. Brull and B. Dubroca. Kinetic and Related Models (2015).

Collisional operators

Not admissible M_1 model:

$$\begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta\left(\dfrac{qE}{m} f_1\right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta\left(\dfrac{qE}{m} f_2\right) - \dfrac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1). \end{cases}$$

Collisions operators

$$Q_0(f_0) = \alpha_{ee}\partial_\zeta\left(\zeta^2 A(\zeta)\partial_\zeta(\frac{f_0}{\zeta^2}) - \zeta B(\zeta) f_0\right), \qquad Q_1(f_1) = -\alpha_{ei}\frac{2f_1}{\zeta^3},$$

$$A(\zeta) = \int_0^\infty min(\frac{1}{\zeta^3}, \frac{1}{\mu^3})\mu^2 f_0(\mu) d\mu, \quad B(\zeta) = \int_0^\infty min(\frac{1}{\zeta^3}, \frac{1}{\mu^3})\mu^3 \partial_\mu(\frac{f_0(\mu)}{\mu^2}) d\mu.$$

Modification: admissible M_1 model[10]

$$\begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta\left(\dfrac{qE}{m} f_1\right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta\left(\dfrac{qE}{m} f_2\right) - \dfrac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1) + Q_0(f_1). \end{cases}$$

[10] J. Mallet, S. Brull and B. Dubroca. Kinetic and Related Models (2015).

Collisional operators

Fundamental properties of the M_1 collisional operators[11]:

- admissibility
- H-theorem (entropy dissipation)
- conservation properties
- caracterisation of the equilibrium states

$\hookrightarrow$ Long time behavior: derivation of the plasma transport coefficients.

Boltzmann $\rightarrow$ Chapman-Enskog expansion: Navier-Stokes

Fokker-Planck-Landau $\rightarrow$ Spitzer-Härm approximation: Electron collisional hydrodynamics

Electronic M_1 model $\rightarrow$ Spitzer-Härm approximation: Electron collisional hydrodynamics.

$\hookrightarrow$ different plasma transport coefficients

[11] J. Mallet, S. Brull and B. Dubroca. Kinetic and Related Models (2015)

Electron collisional hydrodynamics

Strongly collisional fully ionised hot plasma:

$$f(t,\vec{x},\zeta,\vec{\Omega}) = F_0(\zeta, T_e(t,\vec{x}), n_e(t,\vec{x})) + \vec{F}_1(t,\vec{x},\zeta).\vec{\Omega}$$

where

$$F_0(\zeta, T_e(t,\vec{x}), n_e(t,\vec{x})) = n_e(t,\vec{x})\Big(\frac{m_e}{2\pi T_e(t,\vec{x})}\Big)^{3/2} \exp\Big(-\frac{m_e\zeta^2}{2T_e(t,\vec{x})}\Big).$$

Density, momentum and energy conservation laws:

$$\begin{cases} \dfrac{\partial n_e}{\partial t} + \nabla_{\vec{x}}.(n_e\vec{u}_e) = 0, \\ \dfrac{\partial n_e\vec{u}_e}{\partial t} + \nabla_{\vec{x}}(n_e\vec{u}_e \otimes \vec{u}_e + \dfrac{n_e T_e}{m_e}\bar{\bar{I}}_d) = 0, \\ \dfrac{\partial T_e}{\partial t} + \vec{u}_e.\nabla_{\vec{x}}(T_e) + \dfrac{2}{3}T_e\nabla_{\vec{x}}.(\vec{u}_e) + \dfrac{2}{3n_e}\nabla_{\vec{x}}.(\vec{q}) = \dfrac{2}{3n_e}\vec{j}.\vec{E} \end{cases}$$

where

$$\vec{j} = -en_e\vec{u}_e = -\frac{4\pi e}{3}\int_0^{+\infty}\vec{F}_1\zeta^3 d\zeta, \qquad \vec{q} = \frac{2\pi}{3}\int_0^{+\infty}\vec{F}_1(m_e\zeta^2 - 5T_e)\zeta^3 d\zeta.$$

$\hookrightarrow$ Closure: derivation of $\vec{F}_1$

Plasma transport coefficients

Long time behavior

$$F_0\zeta\Big[\frac{e\vec{E}^*}{T_e}+\frac{1}{2T_e}\nabla_{\vec{x}}(T_e)(\frac{m_e\zeta^2}{T_e}-5)\Big]=-\frac{2\alpha_{ei}}{\zeta^3}\vec{F}_1+\frac{1}{\zeta^2}\vec{Q}_0(\zeta^2\vec{F}_1).$$

with

$$\vec{E}^*=\vec{E}+(1/en_e)\nabla_{\vec{x}}(n_eT_e).$$

$\hookrightarrow$ Solve an integro-differential equation.[12]

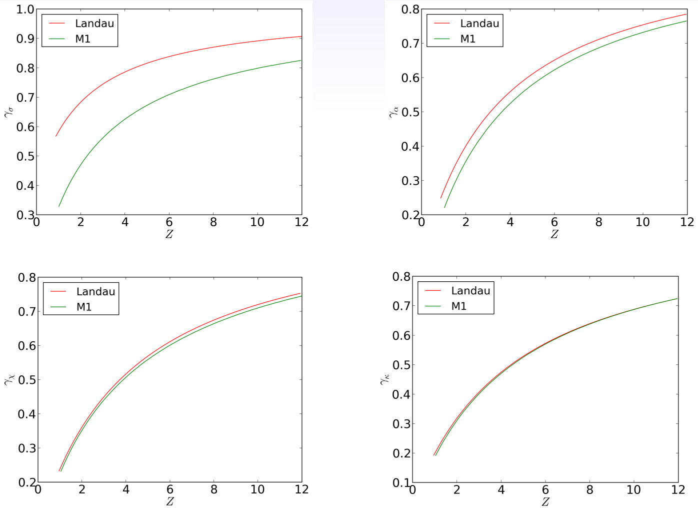
$\hookrightarrow$ Expansion[13,14] of $\vec{F}_1$ on the generalised Laguerre polynomials

Closure

$$\vec{j}=\sigma\vec{E}^*+\alpha\nabla_{\vec{x}}T_e,\quad \vec{q}=-\alpha T_e\vec{E}^*-\chi\nabla_{\vec{x}}T_e$$

[12] L. Spitzer and R. Harm. Phys. Rev. (1953).
[13] S.I. Braginskii. Reviews of Plasma Physics (1965).
[14] S. Chapman. Phil. Trans. Roy. Soc. (1916).

Plasma transport coefficients[15]



[15] Guisset, Brull, Dubroca, d'Humières, Tikhonchuck: Classical transport theory for the collisional electronic M1 model. Submitted.

Asymptotic limits and numerical schemes

Asymptotic limits and numerical schemes

$\hookrightarrow$ Study of the numerical schemes behaviors in long time regimes

$\hookrightarrow$ Severe constraints apply on classical schemes

Can not capture the asymptotic limit under acceptable conditions

Objectives: derivation of numerical methods which capture the correct asymptotic limit

$\hookrightarrow$ Asymptotic-Preserving (AP) schemes

Long time behavior and singular limit

- Different scales:

$$\lambda_{De}, \tau_{pe} \quad << \quad \lambda_{ei}, \tau_{ei} \quad << \quad L, T$$

Quasi-neutral limit ($t^* >> \tau_{pe}$)

$$\begin{cases} \dfrac{\partial f}{\partial t} + v.\nabla_x f + \dfrac{q}{m}(E + v \times B).\nabla_v f = C_{e,e}(f,f) + C_{e,i}(f), \\ \dfrac{\partial E}{\partial t} = -\dfrac{j}{\alpha^2} \end{cases}$$

with $\alpha = \tau_{pe}/t^*$.

Diffusive limit ($t^* >> \tau_{ei}$)

$$\begin{cases} \varepsilon\dfrac{\partial f}{\partial t} + v.\nabla_x f + \dfrac{q}{m}(E + v \times B).\nabla_v f = \dfrac{1}{\varepsilon}C_{e,e}(f,f) + \dfrac{1}{\varepsilon}C_{e,i}(f), \\ \dfrac{\partial E}{\partial t} = -\dfrac{j}{\varepsilon^3\alpha^2} \end{cases}$$

with $\varepsilon^2 = \tau_{ei}/t^*$.

$\hookrightarrow$ Constraints on the numerical schemes

Quasi-neutral limit

As $\alpha \to 0$, impossibility to compute E^{n+1}.

$\hookrightarrow$ Reformulation[16] of the M_1-Maxwell model[17]

Time semi-discretisation

$$\frac{f_1^{n+1} - f_1^n}{\Delta t} + \nabla_x(\zeta f_2^n) - \partial_\zeta(E^{n+1} f_2^n) + \frac{E^{n+1}}{\zeta}(f_0^n - f_2^n) = Q_0(f_1^n) + Q_1(f_1^n).$$

Electric current:
$$j^n = -\int_0^{+\infty} f_1^n \zeta d\zeta,$$

$$\begin{cases} \dfrac{j^{n+1} - j^n}{\Delta t} = \beta_1(f_0^n, f_1^n) E^{n+1} + \beta_2(f_0^n, f_1^n), \\ \dfrac{E^{n+1} - E^n}{\Delta t} = -\dfrac{j^{n+1}}{\alpha^2}. \end{cases} \quad E^{n+1} = \frac{-\dfrac{\alpha^2 E^n}{\Delta t^2} + \beta_2(f_0^n, f_1^n) + \dfrac{j^n}{\Delta t}}{-\dfrac{\alpha^2}{\Delta t^2} - \beta_1(f_0^n, f_1^n)}.$$

If $\alpha \to 0$ we can obtain E^{n+1}, Δt is not constrained by α (asymptotic stability).

$\hookrightarrow$ Realistic collision operators
$\hookrightarrow$ Application to Fokker-Planck-Landau

[16] Degond et al. (2012)
[17] Guisset, Brull, d'Humières, Dubroca, Karpov, Potapenko. To appear CICP.

Diffusive limit[18]

Diffusive scaling: $\tilde{t} = t/t^*, \quad \tilde{x} = x/x^*, \quad \tilde{v} = v/v_{th}$
such that $\tau_{ei}/t^* = \varepsilon^2$, $\lambda_{ei}/x^* = \varepsilon$.

Dimensionless system

$$\begin{cases} \varepsilon\partial_t f_0 + \zeta\partial_x f_1 + E\partial_\zeta f_1 = 0, \\ \varepsilon\partial_t f_1 + \zeta\partial_x f_2 + E\partial_\zeta f_2 - \dfrac{E}{\zeta}(f_0 - f_2) = -\dfrac{2\sigma}{\zeta^3}\dfrac{f_1}{\varepsilon}. \end{cases}$$

Hilbert expansion

$$f_0^\varepsilon = f_0^0 + \varepsilon f_0^1 + O(\varepsilon^2), \quad f_1^\varepsilon = f_1^0 + \varepsilon f_1^1 + O(\varepsilon^2).$$

Limit equation $\qquad f_1^0 = 0,$

$$\partial_t f_0^0 + \zeta\partial_x\left(-\frac{\zeta^4}{6\sigma}\partial_x f_0^0 - \frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma} f_0^0\right) + E\partial_\zeta\left(-\frac{\zeta^4}{6\sigma}\partial_x f_0^0 - \frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma} f_0^0\right) = 0.$$

$\hookrightarrow$ Mixed x and ζ derivatives.

[18] Collaboration with R. Turpault

Derivation of the scheme: problem setting

- Simplified case: no electric field

Model and diffusive limit

$$\begin{cases} \varepsilon\partial_t f_0^\varepsilon + \zeta\partial_x f_1^\varepsilon = 0, \\ \varepsilon\partial_t f_1^\varepsilon + \zeta\partial_x f_2^\varepsilon = -\dfrac{2\sigma}{\zeta^3}\dfrac{f_1^\varepsilon}{\varepsilon}. \end{cases}$$

$$\partial_t f_0^0(t,x) - \zeta\partial_x\left(\frac{\zeta^4}{6\sigma(x)}\partial_x f_0^0(t,x)\right) = 0.$$

Limit of the HLL scheme:

$$\begin{cases} \varepsilon\dfrac{f_{0i}^{n+1,\varepsilon} - f_{0i}^{n,\varepsilon}}{\Delta t} + \zeta\dfrac{f_{1i+1}^{n,\varepsilon} - f_{1i-1}^{n,\varepsilon}}{2\Delta x} - \zeta\Delta x\dfrac{f_{0i+1}^{n,\varepsilon} - 2f_{0i}^{n,\varepsilon} + f_{0i-1}^{n,\varepsilon}}{\Delta x^2} = 0, \\ \varepsilon\dfrac{f_{1i}^{n+1,\varepsilon} - f_{1i}^{n,\varepsilon}}{\Delta t} + \zeta\dfrac{f_{2i+1}^{n,\varepsilon} - f_{2i-1}^{n,\varepsilon}}{2\Delta x} - \zeta\Delta x\dfrac{f_{1i+1}^{n,\varepsilon} - 2f_{1i}^{n,\varepsilon} + f_{1i-1}^{n,\varepsilon}}{\Delta x^2} = -\dfrac{2\sigma_i}{\zeta^3}\dfrac{f_{1i}^{n,\varepsilon}}{\varepsilon}. \end{cases}$$

$\hookrightarrow$ *HLL* numerical scheme: Unphysical numerical viscosity in $O(\frac{\Delta x}{\varepsilon})$.

Harten Lax and van Leer formalism

Riemann problem for hyperbolic system of conservation laws

$$\partial_t U + \partial_x F(U) = 0,$$

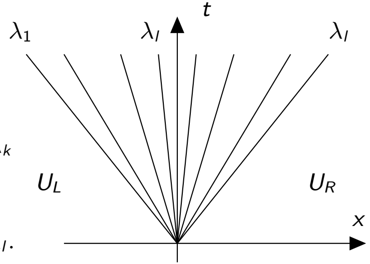
with $U \in \mathbb{R}^m, x \in \mathbb{R}, t > 0$. Initial conditions

$$U(x, t=0) = \begin{cases} U_L & \text{if} \quad x < 0, \\ U_R & \text{if} \quad x > 0. \end{cases}$$

$\hookrightarrow$ Self-similarity of the exact Riemann solution $U(x/t, U_L, U_R)$

Approximate Riemann solver

$$U_{RP}(x/t, U_L, U_R) = \begin{cases} U_1 = U_L & \text{if} \quad x/t < \lambda_1, \\ \vdots & \\ U_k & \text{if} \quad \lambda_{k-1} < x/t < \lambda_k \\ \vdots & \\ U_{l+1} = U_R & \text{if} \quad x/t > \lambda_l. \end{cases}$$



Harten Lax and van Leer formalism

Consistency with the integral form of the hyperbolic system

$$\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \int_0^{\Delta t} (\partial_t U + \partial_x F(U)) dxdt = 0,$$

$$\hookrightarrow \quad F(U_R) - F(U_L) = \sum_{k=1}^{l} \lambda_k (U_{k+1} - U_k).$$

Godunov-type scheme

$$U_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U^h(x, t^{n+1}) dx.$$

$$U^h(x, t^n + \Delta t) = U_{RP}\left(\frac{x - x_{i+1/2}}{t^n + \Delta t}, U_i, U_{i+1}\right) \text{ if } x \in [x_i, x_{i+1}].$$

Conservative form

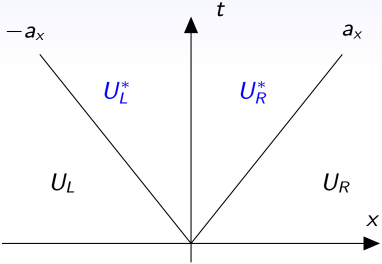
$$\begin{cases} U_i^{n+1} = U_i^n - \dfrac{\Delta t}{\Delta x}(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n), \\ F_{i+\frac{1}{2}}^n = F(U_i^n, U_{i+1}^n), \end{cases}$$

with $$F(U_L, U_R) = \frac{1}{2}\left[F(U_L) + F(U_R) - \sum_{k=1}^{l} |\lambda_k| (U_{k+1} - U_k)\right].$$

Derivation of the scheme

Source terms: approximate Riemann solvers which own a stationnary discontinuity[19,20] (0-contact discontinuity).

$$U_{\mathcal{R}}(x/t) = \begin{cases} U^L & \text{if } \; x/t < -a_x \\ U^{L*} & \text{if } \; -a_x < x/t < 0 \\ U^{R*} & \text{if } \; 0 < x/t < a_x \\ U^R & \text{if } \; a_x < x/t \end{cases}$$



$\hookrightarrow$ Two intermediate states $U^{L*} = {}^t(f_0^{L*}, f_1^*)$ and $U^{R*} = {}^t(f_0^{R*}, f_1^*)$.

CFL condition $\Delta t \leq \dfrac{\Delta x}{2a_x}$.

Numerical scheme

$$U_i^{n+1} = \frac{a_x \Delta t}{\Delta x} U_{i-1/2}^{R*} + (1 - \frac{2a_x \Delta t}{\Delta x}) U_i^n + \frac{a_x \Delta t}{\Delta x} U_{i+1/2}^{L*}.$$

[19] F. Bouchut, Frontiers in Mathematics series (2004).
[20] L. Gosse, Math. Mod. Meth. Apl. Sci. (2001)

Derivation of f_1^*

Consistency condition for f_1^*

$$f_1^* = \frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x}(\zeta f_2^R - \zeta f_2^L) - \frac{2}{\zeta^3}\frac{1}{2a_x\Delta t}\int_{-a_x\Delta t}^{a_x\Delta t}\int_0^{\Delta t}\sigma(x) f_1(x,t)\,dt\,dx.$$

Approximation

$$\frac{1}{2a_x\Delta t}\int_{-a_x\Delta t}^{a_x\Delta t}\int_0^{\Delta t}\sigma(x) f_1(x,t)\,dt\,dx \approx \bar{\sigma}\Delta t f_1^*, \qquad \bar{\sigma} = \sigma(0).$$

Definition of f_1^*, [21]

$$f_1^* = \frac{2a_x\zeta^3}{2a_x\zeta^3 + 2\bar{\sigma}\Delta x}\left[\frac{f_1^L + f_1^R}{2} - \frac{1}{2a_x}(\zeta f_2^R - \zeta f_2^L)\right].$$

[21] C. Berthon and R.Turpault. Num. Meth. PDE. (2011).

Derivation of f_0^{L*} and f_0^{R*}

Consistency condition for f_0

$$\frac{f_0^{L*}+f_0^{R*}}{2}=\frac{f_0^{L}+f_0^{R}}{2}-\frac{1}{2a_x}[\zeta f_1^R-\zeta f_1^L].$$

Definition of f_0^{L*} and f_0^{R*}

$$\begin{cases} f_0^{L*}=\tilde{f}_0-\Gamma, \\ f_0^{R*}=\tilde{f}_0+\Gamma. \end{cases} \qquad \tilde{f}_0=\frac{f_0^{L}+f_0^{R}}{2}-\frac{1}{2a_x}[\zeta f_1^R-\zeta f_1^L].$$

The Rankine-Hugoniot conditions gives Γ

$$\begin{cases} f_0^{L*}=f_0^L-\dfrac{\zeta}{a_x}(f_1^*-f_1^L), \\ f_0^{R*}=f_0^R-\dfrac{\zeta}{a_x}(f_1^R-f_1^*). \end{cases} \qquad \Gamma=\frac{1}{2}[f_0^R-f_0^L-\frac{\zeta}{a_x}(f_1^L-2f_1^*+f_1^R)].$$

Admissibility conditions: modification of f_0^{L*} and f_0^{R*}

$$\begin{cases} f_0^{L*}=\tilde{f}_0-\Gamma\theta, \\ f_0^{R*}=\tilde{f}_0+\Gamma\theta. \end{cases} \qquad \tilde{\theta}=\frac{\tilde{f}_0-|f_1^*|}{|\Gamma|}\geq 0, \qquad \theta=\min(\tilde{\theta},1).$$

Asymptotic-preserving property

Theorem (AP Property)

When ε tends to zero, the unknown $f_{0i}^{n+1,0}$ satisfies the discrete equation

$$\frac{f_{0i}^{n+1,0}-f_{0i}^{n,0}}{\Delta t}-\frac{\zeta}{\Delta x}\left[\frac{\zeta^3}{6\bar{\sigma}_{i+1/2}\Delta x}\left[(\zeta f_{0i+1}^{n,0}-\zeta f_{0i}^{n,0})\right]-\frac{\zeta^3}{6\bar{\sigma}_{i-1/2}\Delta x}\left[(\zeta f_{0i}^{n,0}-\zeta f_{0i-1}^{n,0})\right]\right]=0.$$

Key ideas

Diffusive scaling[22]: $\varepsilon\dfrac{U_i^{n+1}-U_i^n}{\Delta t}=\dfrac{a_x}{\Delta x}U_{i+1/2}^{L*}-\dfrac{2a_x}{\Delta x}U_i^n+\dfrac{a_x}{\Delta x}U_{i-1/2}^{R*}.$

Intermediate state f_1^*

$$f_1^*=\frac{2a_x\zeta^3}{2a_x\zeta^3+2\bar{\sigma}\Delta x/\varepsilon}\left[\frac{f_1^L+f_1^R}{2}-\frac{1}{2a_x}(\zeta f_2^R-\zeta f_2^L)\right].$$

$\hookrightarrow$ No limitation is required $\theta=1$.

[22] C. Berthon, P. Charrier and B. Dubroca. J. Sci. Comput. (2007).

Homogeneous case with electric field

Model

$$\begin{cases} \partial_t f_0 + E\partial_\zeta f_1 = 0, \\ \partial_t f_1 + E\partial_\zeta f_2 - \frac{E}{\zeta}(f_0 - f_2) = -\frac{2\sigma f_1}{\zeta^3}. \end{cases}$$

Diffusive limit

$$\partial_t f_0^0(t,\zeta) - E\partial_\zeta\left(\frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0(t,\zeta) - \frac{E\zeta^2}{3\sigma} f_0^0(t,\zeta)\right) = 0.$$

↪ Limit of the relaxation approach

Intermediate state

$$f_1^* = \frac{2a_\zeta \zeta^3}{2a_\zeta\zeta^3 + 2\sigma\Delta\zeta}\left[\frac{f_1^L + f_1^R}{2} - \frac{1}{2a_\zeta}(Ef_2^R - Ef_2^L) + \frac{\Delta\zeta}{2a_\zeta} S_{L,R}\right],$$

with $$S_{L,R} = \frac{1}{2}\left[\frac{E}{\zeta_R}(f_0^R - f_2^R) + \frac{E}{\zeta_L}(f_0^L - f_2^L)\right].$$

↪ Asymptotic-preserving property[23]

[23] Guisset, Brull, d'Humières, Dubroca. Asymptotic-preserving well-balanced scheme for the electronic M1 model in the diffusive limit: particular cases. Submited

Well-balanced property

Equilibrium solution

$$\begin{cases} E\dfrac{\partial f_1}{\partial \zeta} = 0, \\ E\dfrac{\partial f_2}{\partial \zeta} - \dfrac{E}{\zeta}(f_0 - f_2) = -\dfrac{2\sigma f_1}{\zeta^3}. \end{cases} \qquad \begin{cases} f_0 = K\zeta^2, \quad K \in \mathbb{R}^+ \\ f_1 = 0. \end{cases}$$

Property: The numerical scheme preserves the equibrium solution.

Key ideas

$$f_1^* = \frac{2a_\zeta \zeta^3}{a_\zeta \zeta^3 + 2\sigma\Delta\zeta}\Big[-\frac{1}{3a_\zeta}(EK\zeta_R^2 - EK\zeta_L^2) + \frac{\Delta\zeta EK}{3a_\zeta}(\zeta_R + \zeta_L)\Big] = 0$$

since $(\zeta_R^2 - \zeta_L^2) = (\zeta_R + \zeta_L)(\zeta_R - \zeta_L) = (\zeta_R + \zeta_L)\Delta\zeta$.

The equilibrium condition implies $\theta = 1$ (Rankine-Hugoniot).

$$\Rightarrow \qquad f_{0i}^{n+1} = K\zeta_i^2 \quad \text{and} \quad f_{1i}^{n+1} = 0.$$

General model

Model

$$\begin{cases} \varepsilon\partial_t f_0 + \zeta\partial_x f_1 + E\partial_\zeta f_1 = 0, \\ \varepsilon\partial_t f_1 + \zeta\partial_x f_2 + E\partial_\zeta f_2 - \dfrac{E}{\zeta}(f_0 - f_2) = -\dfrac{2\sigma}{\zeta^3}\dfrac{f_1}{\varepsilon}. \end{cases}$$

Diffusive limit

$$\partial_t f_0^0 + \zeta\partial_x\left(-\frac{\zeta^4}{6\sigma}\partial_x f_0^0 - \frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma} f_0^0\right) + E\partial_\zeta\left(-\frac{\zeta^4}{6\sigma}\partial_x f_0^0 - \frac{E\zeta^3}{6\sigma}\partial_\zeta f_0^0 + \frac{E\zeta^2}{3\sigma} f_0^0\right) = 0.$$

Consider

$$\begin{aligned} \frac{U_{ij}^{n+1} - U_{ij}^n}{\Delta t} =& \frac{a_x}{\Delta x} U_{i-1/2j}^{R*} - \frac{2a_x}{\Delta x} U_{ij}^n + \frac{2a_x\Delta t}{\Delta x} U_{i+1/2j}^{L*} \\ &+ \frac{a_\zeta}{\Delta\zeta} U_{ij-1/2}^{R*} - \frac{2a_\zeta}{\Delta\zeta} U_{ij}^n + \frac{a_\zeta}{\Delta\zeta} U_{ij+1/2}^{L*} \end{aligned}$$

intermediate states

$$U_{i-1/2j}^{R*} = \begin{pmatrix} f_{0i-1/2j}^{R*} \\ f_{1i-1/2j}^{*} \end{pmatrix}, U_{i+1/2j}^{L*} = \begin{pmatrix} f_{0i+1/2j}^{L*} \\ f_{1i+1/2j}^{*} \end{pmatrix}, U_{ij-1/2}^{R*} = \begin{pmatrix} f_{0ij-1/2}^{R*} \\ f_{1ij-1/2j}^{*} \end{pmatrix}, U_{ij+1/2}^{L*} = \begin{pmatrix} f_{0ij+1/2}^{L*} \\ f_{1ij+1/2}^{*} \end{pmatrix}.$$

General model

Mixed derivatives: Modification of the intermediate state f_1^*

$$f^*_{1i+1/2j} = \alpha_{i+1/2j}\Big[\frac{f_{1i+1j}+f_{1ij}}{2} - \frac{1}{2a_x}(\zeta_j f_{2i+1j} - \zeta_j f_{2ij}) - c_{i+1/2j}\theta_{1i+1/2j}(\frac{\partial f_0}{\partial \zeta})_{i+1/2j}(1-\alpha_{i+1/2j})\Big]$$

$$f^*_{1ij+1/2} = \beta_{ij+1/2}\Big[\frac{f_{1ij+1}+f_{1ij}}{2} - \frac{1}{2a_\zeta}(E_i f_{2ij+1} - E_i f_{2ij}) - \bar{c}_{ij+1/2}\theta_{2ij+1/2}(\frac{\partial f_0}{\partial x})_{ij+1/2}(1-\beta_{ij+1/2})\Big]$$

with $$\alpha_{i+1/2j} = \frac{2a_x\zeta_j^3}{2a_x\zeta_j^3 + \sigma_{i+1/2}\Delta x}, \qquad \beta_{ij+1/2} = \frac{2a_\zeta\zeta_{j+1/2}^3}{2a_\zeta\zeta_{j+1/2}^3 + \sigma_i\Delta\zeta}.$$

$\hookrightarrow$ c and $\bar{c}$ are fixed to obtain the correct limit equation

$$c_{i+1/2j} = \frac{E_{i+1/2}\Delta x}{3a_x}, \qquad \bar{c}_{ij+1/2} = \frac{\zeta_{j+1/2}\Delta\zeta}{3a_\zeta}.$$

General model

Upwinding: the sign of $c_{i+1/2j}$ and $\bar{c}_{ij+1/2}$ gives

$$\bar{c}_{ij+1/2}(\frac{\partial f_0}{\partial x})_{ij+1/2} \approx \bar{c}_{ij+1/2}\frac{f_{0i+1j+1} - f_{0ij+1} + f_{0i+1j} - f_{0ij}}{2\Delta x},$$

$$c_{i+1/2j}(\frac{\partial f_0}{\partial \zeta})_{i+1/2j} \approx \begin{cases} c_{i+1/2j}\dfrac{f_{0i+1j} - f_{0i+1j-1} + f_{0ij} - f_{0ij-1}}{2\Delta\zeta} & \text{if } c_{i+1/2j} < 0, \\ c_{i+1/2j}\dfrac{f_{0i+1j+1} - f_{0i+1j} + f_{0ij+1} - f_{0ij}}{2\Delta\zeta} & \text{if } c_{i+1/2j} > 0. \end{cases}$$

$\hookrightarrow$ $\theta_{1i+1/2j}$ and $\theta_{2ij+1/2}$ fixed to ensure the admissibility conditions.

Theorem (Admissibility)

If for all $(i,j) \in \mathbb{N}^2$, $U^n_{i,j} \in \mathcal{A}$, then for all $(i,j) \in \mathbb{N}^2$, $U^{n+1}_{i,j} \in \mathcal{A}$ as soon as the CFL condition $\Delta t \leq \Delta\zeta\Delta x/(2a_x\Delta\zeta + 2a_\zeta\Delta x)$ holds.

Alternative: modified HLL scheme

Wrong behavior of the numerical viscosity in $O(\frac{\Delta x}{\varepsilon})$.

$\hookrightarrow$ Modification of the numerical viscosity [24].

Introduction of a correction θ^ε in $O(\varepsilon)$

$$\frac{f_{0i}^{n+1} - f_{0i}^{n}}{\Delta t} - \frac{\zeta}{\varepsilon}\frac{f_{1i+1}^{n+1} - f_{1i-1}^{n+1}}{2\Delta x} + \frac{\zeta \tilde{a} \theta^\varepsilon}{\varepsilon}\frac{f_{0i+1}^{n} - 2 f_{0i}^{n} + f_{0i-1}^{n}}{2\Delta x} = 0.$$

$\hookrightarrow$ Admissibility requirements?

Acceptable condition on the correction and the CFL condition.

[24] Collaboration with C. Chalons

Numerical test cases

Hot wall in the diffusive regime without electric field

Initial conditions

$$f_0(x,\zeta,0) = \sqrt{\frac{2}{\pi}}\frac{\zeta^2}{T_{ini}(x)^{3/2}}\exp\left(-\frac{\zeta^2}{2\,T_{ini}(x)}\right),\ f_1(x,\zeta,0) = 0, \quad T_{ini}(x) = 0.1.$$

Left boundary condition

$$f_0(0,\zeta,t) = \sqrt{\frac{2}{\pi}}\frac{\zeta^2}{T_{ext}(x)^{3/2}}\exp\left(-\frac{\zeta^2}{2\,T_{ext}(x)}\right),\ f_1(0,\zeta,t) = 0 \quad T_{ext}(x) = 1,\ \sigma = 10^4$$

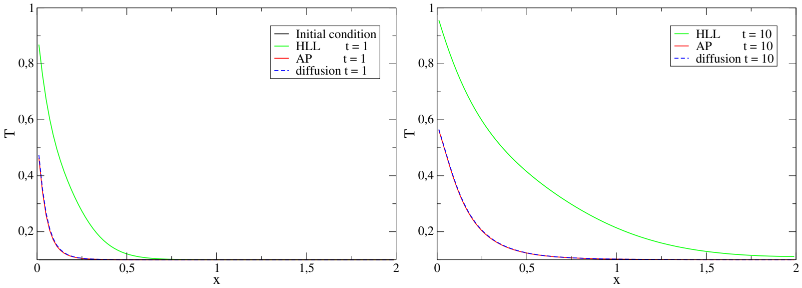


Figure: Temperature for AP scheme, HLL and diffusion at time t=1 and 10.

Numerical test cases: diffusive regime

Periodical boundary conditions and $\sigma = 10^4$. Initial conditions

$$f_0(x,\zeta,0) = \begin{cases} 1 \text{ if } x \le L/3, \\ 0 \text{ if } L/3 \le x \le 2L/3, \\ 1 \text{ if } L/3 \le x, \end{cases} \qquad f_1(x,\zeta,0) = 0.$$

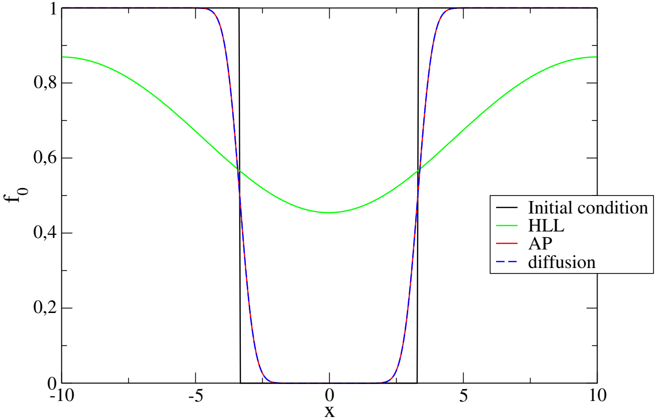


Figure: diffusive regime: f_0 profile at time t=200.

Numerical test cases: diffusive regime

Initial conditions

$$\begin{cases} f_0(t=0,x,\zeta) = \zeta^2 \exp(-x^2)\exp(2(\zeta-3)^2), \\ f_1(t=0,x,\zeta) = 0. \end{cases}$$

Periodical boundary conditions, $\sigma = 10^4$ and $E = 1$.

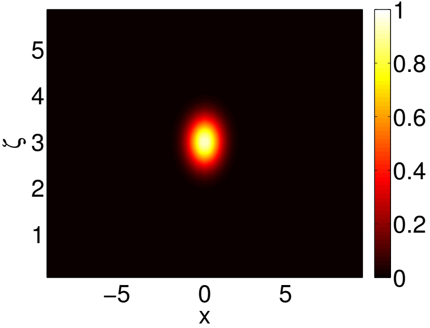


Figure: f_0 profile at the initial time.

Numerical test cases: diffusive regime

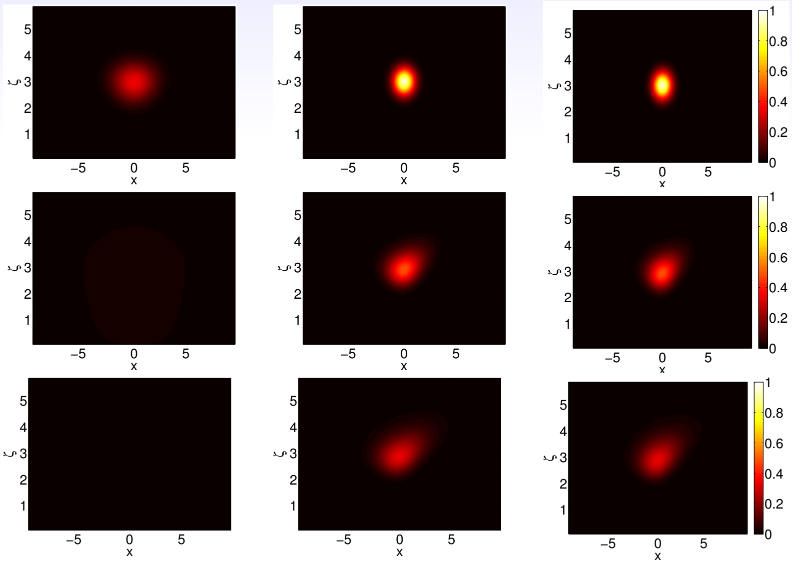
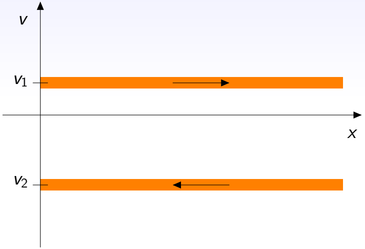


Figure: f_0 profile at time t=1 (top), t=50 (middle), t=100 (bottom), for the HLL scheme (left), the AP scheme (middle) and the diffusion equation (right), ($\sigma = 10^4$).

Numerical test cases: Electron beams interaction

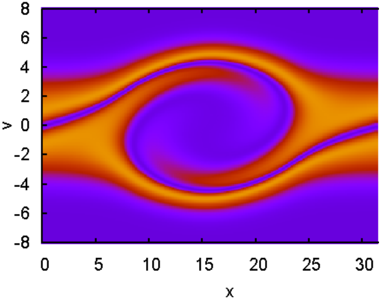


Relevance for laser-plasma interaction

Initial conditions:

$$f(t=0,x,v) = \frac{1}{2}(1 + A\cos(kx))\exp(-(v+v_1)^2) + \frac{1}{2}(1 - A\cos(kx))\exp(-(v+v_2)^2),$$

$$E(0,x) = 0.$$



Numerical test cases: two beams interaction

Initial conditions

$$f(t=0,x,v) = 0.5[(1 + A\cos(kx))M_{v_d}(v) + (1 - A\cos(kx))M_{-v_d}(v)],$$

with $M_{\pm v_d}(v) = \exp\left(-\frac{(v \mp v_d)^2}{2}\right)$, $v_d = 4$ and $A = 10^{-3}$.

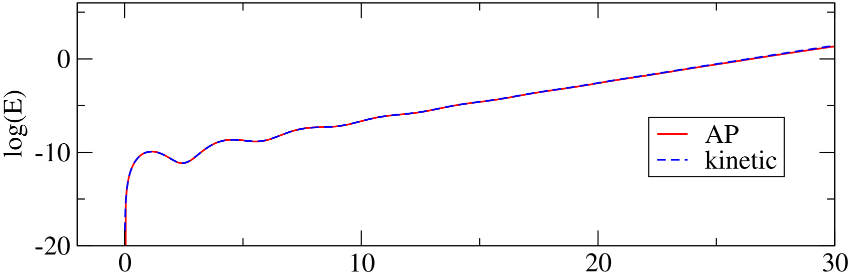


Figure: Temporal evolution of the electrostatic energy.

Numerical test cases: non-constant collisional parameter

Linear profile $\sigma(x) = ax + b$: $\sigma(x_{min} = -40) = 5.10^3$ and $\sigma(x_{max} = 40) = 10^5$

Self-consistent electric field: $E = -\partial_x T$

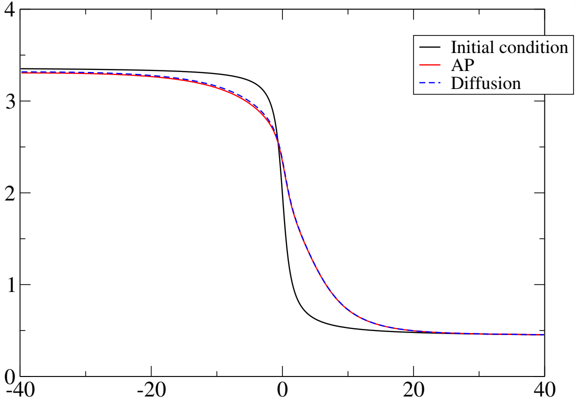


Figure: Temperature profile at time t=5000.

Conclusion

- Limit of the M_1 and M_2 model for collisionless plasma applications.
- Study of the collisional operators for the electronic M_1 model.
- Reformulation of the Maxwell-Ampere equation in the quasi-neutral regime.
- Study of numerical resolution of the electronic M_1 model in the diffusive limit.

Perspectives

- Coupling with the Maxwell equations in the quasi-neutral and diffusive regimes.
- Asymptotic-Preserving extension for the electron-electron collisional operator
- Multidimensional extension and magnetic fields

- Consider the motion of ions (collaboration with D. Aregba).

Perspective: M_1 moments model in a moving frame

The basis $(1, \dfrac{v}{|v|} = \Omega)$ is not galilean invariant [25].

$\hookrightarrow$ Model not galilean invariant

Galilean invariance of the moving frame $(1, \dfrac{v-u}{|v-u|} = \Omega)$.

Kinetic equation in a moving frame

$$\partial_t f + div_x((v+u)f) - div_v\Big[(\partial_t u + \partial_x u(v+u))f\Big] = C(f).$$

Angular moments extraction

$\hookrightarrow$ M_1 model in a moving frame

$\hookrightarrow$ Analysis and numerical scheme

[25] D. Levermore. Moment closure hierarchies for kinetic theories. J. Statist. Phys. (1996).

Thank you

Publications

Publications in referred journals

- S. Guisset, S. Brull, E. d'Humières, B. Dubroca, S. Karpov, I. Potapenko. Asymptotic-Preserving scheme for the M1-Maxwell system in the quasi-neutral regime. To appear in Communications in Computational Physics (CICP).
- S. Guisset, R. Nuter, J. Moreau, S. Brull, E. d'Humières, B. Dubroca, V. Tikhonchuk. Limits of the M1 and M2 angular moments models for kinetic plasma physics studies. J. Phys. A: Math. Theor. 48 (2015) 335501.

Articles submited

- S. Guisset, S. Brull, E. d'Humières, B. Dubroca. Asymptotic-preserving well-balanced scheme for the electronic M1 model in the diffusive limit: particular cases.
- S. Guisset, S. Brull, E. d'Humières, B. Dubroca, V. Tikhonchuk. Classical transport theory for the collisional electronic M1 model.

In preparation

- S. Guisset, S. Brull, E. d'Humières, B. Dubroca, R. Turpault. Asymptotic-preserving scheme for the electronic M1 model in the diffusive limit.