About viscous approximations of the bitemperature Euler system

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Abstract

This paper is devoted to the study of the construction of a viscous approximation of the nonconservative bitemperature Euler system. Starting from a BGK model coupled with Ampère and Poisson equations proposed in [1], we perform a Chapman-Enskog expansion up to order 1 leading to a Navier-Stokes system. Next, we prove that this system is compatible with the entropy of the bitemperature Euler system.

1 Introduction

This paper is devoted to a viscous approximation of the bitemperature Euler system that has been studied in [1]. This fluid model describes the interaction of a mixture of one species of ions and one species of electrons in thermal nonequilibrium, with applications in the field of Inertial Confinement Fusion where solutions with shocks occur. Quasineutrality being assumed, the electronic and ionic mass fractions are constant: subscripts e and i standing for electron and ions respectively,

$$\rho_e = m_e n_e = c_e \rho, \quad \rho_i = m_i n_i = c_i \rho, \quad c_e + c_i = 1$$

and the model consists of two conservation equations for mass and momentum and two nonconservative equations for each energy.

Moreover the pressure of each species is supposed to satisfy a gamma-law with its own γ constant:

$$p_e = (\gamma_e - 1)\rho_e \varepsilon_e = n_e k_B T_e, \quad p_i = (\gamma_i - 1)\rho_i \varepsilon_i = n_i k_B T_i, \tag{1.1}$$

where k_B is the Boltzmann constant, ε_α and T_α represent respectively the internal specific energy and the temperature of species α, $\alpha \in \{e, i\}$.

The total energies are given by $\mathcal{E}_\alpha = \rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha u^2$, $\alpha \in \{e, i\}$. We denote $\nu_{ei} \geq 0$ the interaction coefficient between electronic and ionic temperatures. The bitemperature Euler system is the following:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_e + p_i) = 0, \\ \partial_t \mathcal{E}_e + \partial_x(u(\mathcal{E}_e + p_e)) - u(c_i \partial_x p_e - c_e \partial_x p_i) = \nu_{ei}(T_i - T_e), \\ \partial_t \mathcal{E}_i + \partial_x(u(\mathcal{E}_i + p_i)) + u(c_i \partial_x p_e - c_e \partial_x p_i) = -\nu_{ei}(T_i - T_e). \end{cases} \tag{1.2}$$

A first step in the comprehension of this system is to suppose that $\gamma_e = \gamma_i = \gamma$. In this case, one can define a global internal energy $\varepsilon = c_e\varepsilon_e + c_i\varepsilon_i$ which satisfies $p_e + p_i = (\gamma - 1)\rho\varepsilon$. Denoting $\mathcal{E} = \mathcal{E}_e + \mathcal{E}_i$ the total energy one has

$$\mathcal{E} = \rho\varepsilon + \frac{1}{2}\rho u^2$$

and

$$\partial_t \mathcal{E} + \partial_x (u(\mathcal{E} + p)) = 0$$

so that $(\rho, \rho u, \mathcal{E})$ satisfies the usual Euler 3×3 system with γ law. Nevertheless, even in this case one needs to solve a nonconservative equation in order to get T_e and T_i separately. In our context, the nonconservativity is not only due to source terms but especially to terms multiplying u by pressure gradients, making delicate the definition of admissible shocks. In order to define nonconservative products, Dal Maso, Le Floch and Murat proposed in [10] a new theory based on the definition of family of paths. In [8], the authors consider the bitemperature Euler system with diffusive terms. By assuming that the electrons are isentropic, the system is transformed into a conservative model. In [14], the authors consider a kinetic system for sprays and derive a nonconservative hyperbolic system that is studied in [12].

In [1], the Euler bitemperature system has been derived by hydrodynamic limit from an underlying kinetic model which consists of a BGK model coupled with Poisson equation in the quasi-neutral regime. Moreover the obtained fluid system has been proved to be entropy dissipative by a direct approach and also by using the Boltzmann entropy. In particular, the nonconservative terms are obtained from the definition of the electric field according to a generalized Ohm's law. Moreover, changing the present BGK operator into a Landau-Fokker-Planck operator does not change the Euler limit and does not modify the structure of the Navier-Stokes asymptotics. Numerical simulations have been performed with a finite volume scheme designed from this kinetic model, and have been proved to be physically realistic. In [6], the kinetic model is discretized by using a DVM method with an asymptotic preserving scheme. Relaxation schemes based on the Suliciu approach have also been developped in [1] for this sytem and in [5] in the case of a transverse magnetic configuration. In [11], the authors perform a Chapman-Enskog expansion by introducing a small parameter representing the ratio between electronic and ionic molecular masses. They obtain an hyperbolic system with a parabolic regularisation on the electrons.

One of the motivation of the derivation of viscous terms is the construction of travelling waves for the determination of shock profiles. In particular, in the context of a nonconservative system for two-phase flows, the authors construct in ([12], [13]) travelling waves solutions. The construction of travelling waves is also considered in [16] by using a diffusive equation on the electrons. Therefore this work is a first step before the construction of travelling waves for the bitemperature Euler system.

More precisely, in the present paper, we perform a Chapman-Enskog expansion of our kinetic model up to order one in order to get rigorously a viscous, Navier-Stokes type approximation of the bitemperature Euler system in the case $\gamma_e = \gamma_i$. As a result, we obtain conservative and nonconservative second order terms. To go into details, let us denote $\mathcal{U} = (\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i)$. The Euler bitemperature system (1.2) being written in condensed form as

$$\partial_t \mathcal{U} + A(\mathcal{U})\partial_x \mathcal{U} = S(\mathcal{U}),$$

for a fixed relaxation parameter $\tau > 0$ the obtained second order system can be written under the form

$$\partial_t \mathcal{U}^\tau + A(\mathcal{U}^\tau)\partial_x \mathcal{U}^\tau = S(\mathcal{U}^\tau) + \tau \left(u^\tau \partial_x \left(J(\mathcal{U}^\tau)\partial_x \mathcal{U}^\tau\right) + \partial_x \left(D(\mathcal{U}^\tau)\partial_x \mathcal{U}^\tau\right)\right). \tag{1.3}$$

Here $J(\mathcal{U}^\tau)$ and $D(\mathcal{U}^\tau)$ are 4×4 matrices, while u^τ is the velocity. This result completes known models such as the one studied by C. Chalons and F. Coquel in [7], by constructing rigorously some second order terms to their system.

Next we prove the compatibility of the entropy of the bitemperature Euler system with the diffusive terms. We recall that a dissipative strictly convex entropy η exists for (1.2), namely

$$\eta(\mathcal{U}) = \overline{\eta}_e(\rho c_e, \varepsilon_e) + \overline{\eta}_i(\rho c_i, \varepsilon_i), \quad \overline{\eta}_\alpha(\rho_\alpha, \varepsilon_\alpha) = -\frac{\rho_\alpha}{m_\alpha(\gamma_\alpha - 1)} \ln\left(\frac{p_\alpha}{\rho_\alpha^{\gamma_\alpha}}\right), \quad \alpha \in \{e, i\} \tag{1.4}$$

with flux $Q(\mathcal{U}) = u\eta(\mathcal{U})$ [1]. However, as $A(\mathcal{U})$ is not a gradient, known results for systems of conservation laws do not apply here. In particular, a lengthy but straightforward calculation shows that the change of unknown $\mathcal{V} = \eta'(\mathcal{U})$ does not symmetrize the system. Also, in the case of BGK approximations of systems of conservation laws, the entropy dissipativity of the Chapman-Enskog expansion is wellknown, (see [4], [9]). As a matter of fact if $\mathcal{F}$ is the flux function of such a system, the Chapman-Enskog expansion is

$$\partial_t U^\tau + \partial_x \mathcal{F}(U^\tau) = \tau \partial_x \left(D(U^\tau)\partial_x U^\tau\right) \tag{1.5}$$

with

$$D(U) = \Psi'(U) - \mathcal{F}'(u) \circ \mathcal{F}'(u),$$

the matrix $\Psi'(U)$ depending on the chosen BGK framework. Then one multiplies (1.5) by $\eta'(U^\tau)$ and uses the equality

$$\eta'(U)\partial_x \left(D(U)\partial_x U\right) = \partial_x \left(\eta'(U)D(U)\partial_x U\right) - \eta''(U)\left(\partial_x U, D(U)\partial_x U\right).$$

In this case, ${}^tD(U)\eta''(U)$ is symmetric nonnegative so that

$$\partial_t \eta(\mathcal{U}^\tau) + \partial_x Q(\mathcal{U}^\tau) \leq \tau \partial_x \left(\eta'(U^\tau)D(U^\tau)\partial_x U^\tau\right).$$

In the case of the present paper, we had to perform a direct calculation to investigate the dissipation property. We prove here that the solutions $\mathcal{U}^\tau$ of (1.3) formally satisfy the following inequality:

$$\partial_t \eta(\mathcal{U}^\tau) + \partial_x(u\eta(\mathcal{U}^\tau)) \leq -\frac{\nu_{ei}}{k_B T_i T_e}(T_i - T_e)^2 - \tau \frac{5k_B}{2m_\alpha} \sum_{\alpha = e,i} \partial_x \left(n_\alpha \partial_x T_\alpha\right).$$

As a consequence, if $(\mathcal{U}^\tau)_\tau$ is smooth and decreases fast enough at infinity one has

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(\mathcal{U}^\tau(x,t))dx \leq -\frac{\nu_{ei}}{k_B}\int_{\mathbb{R}} \frac{(T_i^\tau - T_e^\tau)^2}{T_i^\tau T_e^\tau} dx.$$

This is a first step to prove that $\mathcal{U}^\tau$ owns a limit $\mathcal{U}$ which is a weak entropy solution of the Euler bitemperature system, see [15] for detailed argumentation. In the limit we have

$$\partial_t \eta(\mathcal{U}) + \partial_x Q(\mathcal{U}) \leq -\frac{\nu_{ei}}{k_B T_i T_e}(T_i - T_e)^2 \,. \tag{1.6}$$

This inequality is the one satisfied formally by a limit of the moments of the solution of the BGK system [1].

The paper is organized as follows. The Section 2 deals with the derivation of a Navier-Stokes system starting from the kinetic system proposed in [1]. In section 3, the diffusive terms are shown to be dissipative w.r.t. the entropy of the bitemperature Euler system. Finally, section 4 gives conclusions to this work.

2 Derivation of the Navier-Stokes system

2.1 Notations

Kinetic models are described by the distribution function f_α of each species depending on the time variable $t \in \mathbb{R}_+$, on the position $x \in \mathbb{R}^3$ and on the velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The macroscopic quantities can be obtained by extracting moments on these distribution functions w.r.t the velocity variable. Indeed density, velocity and total energy of the species α can be defined as

$$n_\alpha = \int_{\mathbb{R}^3} f_\alpha dv, \quad u_\alpha = \frac{1}{n_\alpha}\int_{\mathbb{R}^3} v_1 f_\alpha dv, \quad \mathcal{E}_\alpha = \frac{3}{2}\rho_\alpha \frac{k_B}{m_\alpha} T_\alpha + \frac{1}{2}\rho_\alpha u_\alpha^2 = \int_{\mathbb{R}^3} m_\alpha \frac{v^2}{2} f_\alpha dv. \tag{2.7}$$

The present model is monoatomic ($\gamma = \frac{5}{3}$). Hence, the internal specific energy of species α writes

$$\varepsilon_\alpha = \frac{3}{2m_\alpha} k_B T_\alpha.$$

In the following we shall use the moment operator P_α defined by

$$P_\alpha(f_\alpha) = m_\alpha \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v_1 \\ \frac{v^2}{2} \end{pmatrix} f_\alpha dv. \tag{2.8}$$

We denote $P_\alpha(f_\alpha) = U_\alpha$:

$$U_\alpha = \begin{pmatrix} \rho_\alpha \\ \rho_\alpha u_\alpha \\ \mathcal{E}_\alpha \end{pmatrix}. \tag{2.9}$$

Usually the velocity and the temperature of the mixture are defined by

$$u = \frac{\rho_e u_e + \rho_i u_i}{\rho_e + \rho_i}, \quad n k_B T = \sum_\alpha \left(\frac{1}{2}\rho_\alpha (u_\alpha^2 - u^2)\right) + \sum_\alpha (n_\alpha k_B T_\alpha), \tag{2.10}$$

where $n = n_e + n_i$.

Moreover, the current of the plasma j and the total charge $\bar{\rho}$ are defined by

$$\bar{\rho} = \int_{\mathbb{R}^3} (q_e f_e + q_i f_i)\, dv = n_e q_e + n_i q_i, \quad j = \int_{\mathbb{R}^3} v_1 (q_e f_e + q_i f_i)\, dv = n_e q_e u_e + n_i q_i u_i. \tag{2.11}$$

2.2 Rescaled kinetic system

In this part, the kinetic model introduced and used in ([1], [6]) is describded. This model is a BGK model coupled with Ampère and Poisson equations.

Consider the Vlasov-BGK-Ampère model, for $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $v = (v_1, v_2, v_3) \in \mathbb{R}^3$:

$$\begin{cases} \partial_t f^\alpha + v_1 \partial_x f^\alpha + \dfrac{q^\alpha E}{m^\alpha} \partial_{v_1} f^\alpha = \dfrac{1}{\tau_\alpha}(\mathcal{M}_\alpha - f^\alpha) + \dfrac{1}{\tau_{\alpha\beta}}(\overline{\mathcal{M}_\alpha} - f^\alpha), & (2.12a) \\ \partial_t E = -\dfrac{j}{\varepsilon_0}, & (2.12b) \\ \partial_x E = \dfrac{\overline{\rho}}{\varepsilon_0}, & (2.12c) \end{cases}$$

where $\alpha = e, i$, depending on the population (electrons or ions) that is considered, τ_α and $\tau_{\alpha\beta}$ are, respectively, the relaxation rate towards intra-species and inter-species equilibria ($\tau_{\alpha\beta} = \tau_{\beta\alpha}$). ε_0 is the dielectric permittivity of vacuum.

$\mathcal{M}_\alpha$ and $\overline{\mathcal{M}_\alpha}$ are the two Maxwellian distribution functions

$$\mathcal{M}_\alpha(f_\alpha) = \frac{n_\alpha}{(2\pi k_B T_\alpha/m_\alpha)^{3/2}} exp\left(-\frac{(v_1 - u_\alpha)^2 + v_2^2 + v_3^2}{2k_B T_\alpha/m_\alpha}\right), \quad \alpha = e, i, \tag{2.13}$$

$$\overline{\mathcal{M}_\alpha}(f_e, f_i) = \frac{n_\alpha}{(2\pi k_B T/m_\alpha)^{3/2}} exp\left(-\frac{(v_1 - u)^2 + v_2^2 + v_3^2}{2k_B T/m_\alpha}\right), \quad \alpha = e, i. \tag{2.14}$$

In the present case, we consider for the sake of clarity that $\tau_{\alpha\beta} = \tau_{\beta\alpha}$. The case of $\tau_{\alpha\beta} \neq \tau_{\beta\alpha}$ can be overcome by introducing fictitious quantities inside the Maxwellian distributions (2.13, 2.14) as in [1]. Hence all along this paper, we set $\tau_{ei} = \tau_{\alpha\beta} = \tau_{\beta\alpha}$.

Denote by t_0, L, v_{th}, q, m the reference time, length, velocity, electric charge and particle mass. In order to study the system, we consider the scaled variables

$$\tilde{t} = \frac{t}{t_0}, \qquad \tilde{x} = \frac{x}{L}, \qquad \tilde{v_1} = \frac{v_1}{v_{th}}, \qquad \widetilde{q_\alpha} = \frac{q_\alpha}{q}, \qquad \widetilde{m_\alpha} = \frac{m_\alpha}{m}.$$

From this scaling we define the following dimensionless quantities

$$\tilde{f}_\alpha = \frac{v_{th}}{n_0} f_\alpha, \qquad \widetilde{E} = t_0 \frac{q}{m} \frac{E}{v_{th}}, \qquad \widetilde{j} = \frac{j}{q v_{th} n_0}, \qquad \widetilde{\overline{\rho}} = \frac{\overline{\rho}}{q n_0},$$

and the system becomes after removing the tildas

$$\begin{cases} \partial_t f^\alpha + v_1 \partial_x f^\alpha + \dfrac{q^\alpha E}{m^\alpha} \partial_{v_1} f^\alpha = \dfrac{1}{\epsilon^\alpha}(\mathcal{M}_\alpha - f^\alpha) + \dfrac{1}{\epsilon^{ei}}(\overline{\mathcal{M}_\alpha} - f^\alpha), & (2.15a) \\ \partial_t E = -\dfrac{j}{\beta^2}, & (2.15b) \\ \partial_x E = \dfrac{\overline{\rho}}{\beta^2}, & (2.15c) \end{cases}$$

where $\epsilon^\alpha = \frac{\tau_\alpha}{t_0}$, $\epsilon^{ei} = \frac{\tau_{ei}}{t_0}$, $\beta = \frac{\lambda_{De}}{L}$, $\lambda_{De} = \left(\frac{\epsilon_0 k_B T_0}{q^2 n_0}\right)^{\frac{1}{2}}$, with $T_0 = \frac{m v_{th}^2}{k_B}$ the reference temperature. λ_{De} is the Debye length corresponding to the characteristic length of charges separation in a

plasma. In the present regime, this length is supposed small compared to the reference length L of the domain. Moreover the mean free path corresponding to collisions between particles of the same species is assumed to be small.
Therefore, by setting $\tau = \beta\ \tau = \epsilon^e = \epsilon^i$ and by taking again $\epsilon^{ei} = \tau_{ei}$, we consider in this paper the following rescalled kinetic model

$$\left\{\begin{array}{rcl} \partial_t f_\alpha + v_1 \partial_x f_\alpha + \frac{q_\alpha}{m_\alpha} E \partial_{v_1} f_\alpha &=& \frac{1}{\tau}(\mathcal{M}_\alpha - f_\alpha) + \frac{1}{\tau_{ei}}(\overline{\mathcal{M}_\alpha} - f_\alpha), \\ \partial_t E &=& -\frac{j}{\tau^2}, \\ \partial_x E &=& \frac{\bar{\rho}}{\tau^2}. \end{array}\right. \tag{2.16}$$

2.3 Chapman-Enskog expansion

In this section, we perform a first order Chapman-Engskog expansion up to order 1. Hence the solution of the system (2.16, 2.13, 2.14) f_α is researched as the expansion

$$f_\alpha = \mathcal{M}_\alpha + \tau g_\alpha, \quad \alpha \in \{e, i\}, \tag{2.17}$$

with the constraints

$$\int_{\mathbb{R}^3} f_\alpha dv = \int_{\mathbb{R}^3} \mathcal{M}_\alpha dv,\ \int_{\mathbb{R}^3} v_1 f_\alpha\, dv = \int_{\mathbb{R}^3} v_1\, \mathcal{M}_\alpha\, dv,\ \int_{\mathbb{R}^3} v^2 f_\alpha\, dv = \int_{\mathbb{R}^3} v^2\, \mathcal{M}_\alpha\, dv. \tag{2.18}$$

So

$$\int_{\mathbb{R}^3} g_\alpha dv = 0,\ \int_{\mathbb{R}^3} v_1 g_\alpha\, dv = 0,\ \int_{\mathbb{R}^3} v^2 g_\alpha\, dv = 0. \tag{2.19}$$

2.4 Euler system

At this level, it is possible to compute the Euler system by arguing as in ([1]). Hence by plugging the expansion (2.17, 2.18, 2.19) into (2.16) and considering terms up to order 1, it holds that

$$\partial_t \mathcal{M}_\alpha + v_1 \partial_x \mathcal{M}_\alpha + \frac{q_\alpha}{m_\alpha} E \partial_{v_1} \mathcal{M}_\alpha = -g_\alpha + \frac{1}{\tau_{\alpha\beta}}(\overline{\mathcal{M}}_\alpha - \mathcal{M}_\alpha) + \mathcal{O}(\tau),$$

with $j = \mathcal{O}(\tau^2)$ and $\bar{\rho} = \mathcal{O}(\tau^2)$. Hence by using that

$$\int_{\mathbb{R}^3} m_\alpha\, \mathcal{M}_\alpha\, dv = \rho_\alpha, \quad \int_{\mathbb{R}^3} m_\alpha v_1\, \mathcal{M}_\alpha\, dv = \rho_\alpha u, \quad \int_{\mathbb{R}^3} m_\alpha v^2\, \mathcal{M}_\alpha\, dv = \rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha u^2,$$

we get

$$\left\{\begin{array}{rcl} \partial_t \rho + \partial_x(\rho u) &=& \mathcal{O}(\tau), \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_e + p_i) &=& \mathcal{O}(\tau), \\ \partial_t(\rho_e \varepsilon_e + \frac{1}{2}\rho_e u^2) + \partial_x(u(\rho_e \varepsilon_e + \frac{1}{2}\rho_e u^2 + p_e)) - u(c_i \partial_x p_e - c_e \partial_x p_i) &=& \nu_{ei}(T_i - T_e) + \mathcal{O}(\tau), \\ \partial_t(\rho_i \varepsilon_i + \frac{1}{2}\rho_i u^2) + \partial_x(u(\rho_i \varepsilon_i + \frac{1}{2}\rho_i u^2 + p_i)) + u(c_i \partial_x p_e - c_e \partial_x p_i) &=& -\nu_{ei}(T_i - T_e) + \mathcal{O}(\tau). \end{array}\right. \tag{2.20}$$

2.5 Obtention of the viscous fluid system

We expand f_α as in (2.17, 2.18, 2.19) and we extract the moments *w.r.t.* 1, v_1, v^2.

One important point to determine the viscous terms of the Navier-Stokes is to compute the term g_α of the expansion (2.17, 2.18, 2.19). The calculus is performed in the following proposition.

Proposition 1. *The first order terms g_e and g_i of the expansion (2.17, 2.18, 2.19) write*

$$
\begin{aligned}
g_e \;=\; & -\Bigg(\Bigg((v_1-u)\left(\frac{\partial_x n_e}{n_e}-\frac{3}{2}\frac{\partial_x T_e}{T_e}\right)+\partial_x u\left(\frac{(v_1-u)^2}{\frac{k_B}{m_e}T_e}-\frac{1}{3}\frac{(v_1-u)^2+v_2^2+v_3^2}{\frac{k_B}{m_e}T_e}\right)\\
+ & \frac{\nu_{ei}}{n_e k_B T_e}(T_i-T_e)\left(\frac{((v_1-u)^2+v_2^2+v_3^2)}{3\frac{k_B}{m_e}T_e}-1\right)-(v_1-u)\frac{(v_1-u)^2+v_2^2+v_3^2}{2\frac{k_B}{m_e}}\partial_x(\frac{1}{T_e})\\
- & \frac{(v_1-u)}{\frac{k_B}{m_e}T_e\rho}\partial_x(p_e+p_i)\Bigg)\mathcal{M}_e+\frac{q_e}{m_e}E\partial_{v_1}\mathcal{M}_e-\frac{1}{\tau_{ei}}\left(\overline{\mathcal{M}}_e-\mathcal{M}_e\right)\Bigg),
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
g_i \;=\; & -\Bigg(\Bigg((v_1-u)\left(\frac{\partial_x n_i}{n_i}-\frac{3}{2}\frac{\partial_x T_i}{T_i}\right)+\partial_x u\left(\frac{(v_1-u)^2}{\frac{k_B}{m_i}T_i}-\frac{1}{3}\frac{(v_1-u)^2+v_2^2+v_3^2}{\frac{k_B}{m_i}T_i}\right)\\
+ & \frac{\nu_{ei}}{n_i k_B T_i}(T_e-T_i)\left(\frac{(v_1-u)^2+v_2^2+v_3^2}{3\frac{k_B}{m_i}T_i}-1\right)-(v_1-u)\frac{((v_1-u)^2+v_2^2+v_3^2}{2\frac{k_B}{m_i}}\partial_x(\frac{1}{T_i})\\
- & \frac{(v_1-u)}{\frac{k_B}{m_i}T_i\rho}\partial_x(p_e+p_i)\Bigg)\mathcal{M}_i+\frac{q_i}{m_i}E\partial_{v_1}\mathcal{M}_i-\frac{1}{\tau_{ei}}\left(\overline{\mathcal{M}}_i-\mathcal{M}_i\right)\Bigg).
\end{aligned}
\tag{2.22}
$$

Proof. g_α is given by the relation

$$
g_\alpha=-\left(\partial_t\mathcal{M}_\alpha+v_1\partial_x\mathcal{M}_\alpha+\frac{q_\alpha}{m_\alpha}E\partial_{v_1}\mathcal{M}_\alpha-\frac{1}{\tau_{ei}}(\overline{\mathcal{M}_\alpha}-\mathcal{M}_\alpha)\right).
$$

A direct computation gives

$$
\partial_t\mathcal{M}_e=\left((\frac{\partial_t n_e}{n_e}-\frac{3}{2}\frac{\partial_t T_e}{T_e})+(v_1-u)\frac{\partial_t u}{\frac{k_B}{m_e}T_e}-\frac{\left((v_1-u)^2+v_2^2+v_3^2\right)}{2\frac{k_B}{m_e}T_e}\partial_t\left(\frac{1}{T_e}\right)\right)\mathcal{M}_e \tag{2.23}
$$

and

$$
v_1\partial_x\mathcal{M}_e=\left((\frac{v_1\partial_x n_e}{n_e}-\frac{3}{2}\frac{v_1\partial_x T_e}{T_e})+(v_1-u)\frac{v_1\partial_x u}{\frac{k_B}{m_e}T_e}-\frac{\left((v_1-u)^2+v_2^2+v_3^2\right)}{2\frac{k_B}{m_e}T_e}v_1\partial_x\left(\frac{1}{T_e}\right)\right)\mathcal{M}_e.
$$

By using the non-conservative Euler system (2.20), the time derivatives of (2.23) are computed in function of the space derivatives up to $\mathcal{O}(\tau)$ terms, as follows

$$\begin{aligned}
\frac{\partial_t n_e}{n_e} - \frac{3}{2}\frac{\partial_t T_e}{T_e} &= -u\frac{\partial_x n_e}{n_e} + \frac{3}{2}u\frac{\partial_x T_e}{T_e} - \frac{\nu_{ei}}{n_e k_B T_e}(T_i - T_e) + \mathcal{O}(\tau),\\
\partial_t u &= -u\partial_x u - \frac{1}{\rho}\partial_x(p_e + p_i) + \mathcal{O}(\tau),\\
\frac{\partial_t n_i}{n_i} - \frac{3}{2}\frac{\partial_t T_i}{T_i} &= -u\frac{\partial_x n_i}{n_i} + \frac{3}{2}u\frac{\partial_x T_i}{T_i} - \frac{\nu_{ei}}{n_i k_B T_i}(T_e - T_i) + \mathcal{O}(\tau),\\
\frac{\partial_t T_e}{T_e} &= -u\frac{\partial_x T_e}{T_e} - \frac{2}{3}\partial_x u + \frac{2}{3}\frac{\nu_{ei}}{n_e k_B T_e}(T_i - T_e) + \mathcal{O}(\tau),\\
\frac{\partial_t T_i}{T_i} &= -u\frac{\partial_x T_i}{T_i} - \frac{2}{3}\partial_x u + \frac{2}{3}\frac{\nu_{ei}}{n_i k_B T_i}(T_e - T_i) + \mathcal{O}(\tau).
\end{aligned}$$

Hence up to $\mathcal{O}(\tau)$ order terms, we get

$$\begin{aligned}
\partial_t \mathcal{M}_e + v_1\partial_x \mathcal{M}_e &= \Bigg(-u\frac{\partial_x n_e}{n_e} + \frac{3}{2}u\frac{\partial_x T_e}{T_e} - \frac{\nu_{ei}}{n_e k_B T_e}(T_i - T_e) + \frac{(v_1 - u)}{\frac{k_B}{m_e}T_e}\left(-u\partial_x u - \frac{1}{\rho}\partial_x(p_e + p_i)\right)\\
&+ \frac{\left((v_1 - u)^2 + v_2^2 + v_3^2\right)}{2\frac{k_B}{m_e}T_e}\left(-u\frac{\partial_x T_e}{T_e} - \frac{2}{3}\partial_x u + \frac{2}{3}\frac{\nu_{ei}}{n_e k_B T_e}(T_i - T_e)\right)\Bigg)\mathcal{M}_e.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\partial_t \mathcal{M}_e + v_1\partial_x \mathcal{M}_e &= \Bigg((v_1 - u)\left(\frac{\partial_x n_e}{n_e} - \frac{3}{2}\frac{\partial_x T_e}{T_e}\right) + \partial_x u\left(\frac{(v_1 - u)^2}{\frac{k_B}{m_e}T_e} - \frac{\left((v_1 - u)^2 + v_2^2 + v_3^2\right)}{3\frac{k_B}{m_e}T_e}\right)\\
&+ \frac{\nu_{ei}}{n_e k_B T_e}(T_i - T_e)\left(\frac{\left((v_1 - u)^2 + v_2^2 + v_3^2\right)}{3\frac{k_B}{m_e}T_e} - 1\right)\\
&- (v_1 - u)\frac{\left((v_1 - u)^2 + v_2^2 + v_3^2\right)}{2\frac{k_B}{m_e}}\partial_x\left(\frac{1}{T_e}\right)\\
&- \frac{(v_1 - u)}{\frac{k_B}{m_e}\rho T_e}\partial_x(p_e + p_i)\Bigg)\mathcal{M}_e
\end{aligned}$$

and we recover (2.21). The same result holds for (2.22). □

Proposition 2. *The viscous approximation of the kinetic system (2.16, 2.13, 2.14) writes*

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{2.24}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p_e + p_i) - \frac{4}{3}\tau \partial_x(p\,\partial_x u) = 0, \tag{2.25}$$

$$\begin{aligned}\partial_t(\rho_e\varepsilon_e + \frac{1}{2}\rho_e u^2) + \partial_x(u(\rho_e\varepsilon_e + \frac{1}{2}\rho_e u^2 + p_e)) - u(c_i\partial_x p_e - c_e\partial_x p_i)\\ -u\left(\frac{4}{3}\tau c_e\partial_x(p_i\partial_x u) - \frac{4}{3}\tau c_i\partial_x(p_e\partial_x u)\right)\\ -\frac{4}{3}\tau\partial_x\left(u p_e\,\partial_x u\right) - \frac{5}{2}\tau\partial_x(\frac{k_B}{m_e}p_e\,\partial_x T_e) = \nu_{ei}(T_e - T_i),\end{aligned} \tag{2.26}$$

$$\begin{aligned}\partial_t(\rho_i\varepsilon_i + \frac{1}{2}\rho_i u^2) + \partial_x(u(\rho_i\varepsilon_i + \frac{1}{2}\rho_i u^2 + p_i)) + u(c_i\partial_x p_e - c_e\partial_x p_i)\\ +u\left(\frac{4}{3}\tau c_e\partial_x(p_i\partial_x u) - \frac{4}{3}\tau c_i\partial_x(p_e\partial_x u)\right)\\ -\frac{4}{3}\tau\partial_x\left(u p_i\,\partial_x u\right) - \frac{5}{2}\tau\partial_x(\frac{k_B}{m_i}p_i\,\partial_x T_i) = \nu_{ei}(T_i - T_e),\end{aligned} \tag{2.27}$$

where the electric field E is given by

$$\left(\frac{n_e q_e}{\rho_e} - \frac{n_i q_i}{\rho_i}\right)E = \frac{\rho}{\rho_e\rho_i}n_e q_e E = -\frac{\rho}{\rho_e\rho_i}n_i q_i E = \frac{\partial_x p_e}{\rho_e} - \frac{\partial_x p_i}{\rho_i} - \frac{4}{3}\frac{\tau}{\rho_e}\partial_x\left(p_e\,\partial_x u\right) + \frac{4}{3}\frac{\tau}{\rho_i}\partial_x\left(p_i\,\partial_x u\right). \tag{2.28}$$

Remark 1. *In Proposition 2, the kinetic model is considered in the monoatomic case. This means that $\gamma_e = \gamma_i = \frac{5}{3}$. The polyatomic case can be managed by considering a kinetic model with internal energy variable like in [2].*

Remark 2. *The relation (2.28) is an approximation at order τ of the Ohm law given in [1]. The nonconservative terms of the system (2.24, 2.25, 2.26, 2.27) are shown to appear from this relation defining E.*

Proof. By expanding f_α like in (2.17), we get

$$\partial_t(\mathcal{M}_\alpha + \tau g_\alpha) + v_1\partial_x(\mathcal{M}_\alpha + \tau g_\alpha) + \frac{q_\alpha}{m_\alpha}E\partial_{v_1}(\mathcal{M}_\alpha + \tau g_\alpha) = -g_\alpha + \frac{1}{\tau_{ei}}(\overline{\mathcal{M}}_\alpha - \mathcal{M}_\alpha - \tau g_\alpha) \tag{2.29}$$

and we extract the moments of (2.29) w.r.t 1, v_1, v^2. From the orthogonality conditions (2.19) the viscous part of the expansion will be given by the moments of $v_1\partial_x g_\alpha$ w.r.t 1, v_1, v^2.

Firstly, we obtain the same mass conservation equation as for the Euler system because from the relation (2.19).

In order to get the impulsion equation (2.25), we compute

$$\int_{\mathbb{R}^3} m_e v_1^2\, g_e dv = \int_{\mathbb{R}^3} m_e\left((v_1 - u)^2 + v_2^2 + v_3^2\right) g_e dv.$$

By using the expression of g_e (2.21) and some orthogonality relations, it holds that

$$\int_{\mathbb{R}^3} m_e(v_1-u)^2 g_e dv = -\partial_x u \int_{\mathbb{R}^3} \left(\frac{(v_1-u)^4}{\frac{k_B}{m_e}T_e} - \frac{1}{3}\frac{(v_1-u)^2\left((v_1-u)^2+v_2^2+v_3^2\right)}{\frac{k_B}{m_e}T_e} \right) \mathcal{M}_e\, m_e\, dv.$$

Moreover, by using

$$\int_{\mathbb{R}^3} \left((v_1-u)^2+v_2^2+v_3^2\right)^2 \mathcal{M}_e dv = n_e(\frac{k_B T_e}{m_e})^2,$$

we get finally

$$\int_{\mathbb{R}^3} m_e(v_1-u)^2 g_e dv = -\frac{4}{3}\, p_e\, \partial_x u. \tag{2.30}$$

Hence we get for species α

$$\partial_t(\rho_\alpha u) + \partial_x(\rho_\alpha u + p_\alpha) - q_\alpha n_\alpha E - \frac{4}{3}\tau \partial_x(p_\alpha \partial_x u) = 0. \tag{2.31}$$

Then by adding the previous equations for each species (2.25). Next in order to establish the relation defining E, (2.31) is rewritten in the nonconservative form as

$$\partial_t u + u\partial_x u + \frac{1}{\rho_\alpha}\partial_x p_\alpha - \frac{q_\alpha n_\alpha}{\rho_\alpha}E - \frac{4}{3}\frac{\tau}{\rho_\alpha}\partial_x(p_\alpha\, \partial_x u) = 0. \tag{2.32}$$

By substracting equation (2.32) for electrons and ions and by proceeding like in [1], we get the generalized Ohm's law (2.28).

By using the relation (2.19), we obtain

$$\int_{\mathbb{R}^3} \frac{m_e}{2} v_1 v^2 g_e\, dv = \int_{\mathbb{R}^3} \frac{m_e}{2}(v_1-u)\left((v_1-u)^2+v_2^2+v_3^2\right) g_e\, dv + u\int_{\mathbb{R}^3} m_e v_1^2 g_e\, dv.$$

By using orthogonality relations, it comes that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{m_e}{2}(v_1-u)\left((v_1-u)^2+v_2^2+v_3^2\right) g_e\, dv &= -5\rho_e \left(\frac{k_B}{m_e}T_e\right)^2 (\frac{\partial_x n_e}{n_e} - \frac{3}{2}\frac{\partial_x T_e}{T_e}) \\
&+ \partial_x(\frac{1}{T_e})\frac{35\rho_e}{2\frac{k_B}{m_e}}(\frac{k_B}{m_e}T_e)^3 \\
&+ \int_{\mathbb{R}^3} (v_1-u)^2 \frac{\left((v_1-u)^2+v_2^2+v_3^2\right)}{\frac{k_B}{m_e}\rho T_e}\, \partial_x(p_e+p_i)\mathcal{M}_e\, dv \\
&+ 5n_e\frac{k_B}{m_e}T_e\frac{q_e}{m_e}E.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{m_e}{2}(v_1-u)\left((v_1-u)^2+v_2^2+v_3^2\right) g_e\, dv &= -5\rho_e \left(\frac{k_B}{m_e}T_e\right)^2 (\frac{\partial_x n_e}{n_e} - \frac{3}{2}\frac{\partial_x T_e}{T_e}) \\
&- \partial_x T_e\, \frac{35\rho_e}{2\frac{k_B}{m_e}}(\frac{k_B}{m_e})^3 T_e \\
&+ 5\frac{\rho_e}{\frac{k_B}{m_e}\rho T_e}\left(\frac{k_B}{m_e}T_e\right)^2 \partial_x(p_e+p_i) + 5n_e\frac{k_B}{m_e}T_e q_e E.
\end{aligned}$$

Next, we use the generalized Ohm's law (2.28) that has been established previously. So neglecting the first order terms in τ in the expression of E, it comes that

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{m_e}{2}(v_1-u)\left((v_1-u)^2+v_2^2+v_3^2\right) g_e\,dv &= -5(\frac{k_B}{m_e}T_e)^2\partial_x n_e + \frac{15}{2}n_e\frac{k_B}{m_e}k_B\partial_x T_e \\ &- \frac{35}{2}k_B\frac{k_B}{m_e}T_e\partial_x T_e \\ &+ 5(\frac{k_B}{m_e}T_e)c_e\partial_x(p_e+p_i) + 5T_e\frac{k_B}{m_e}(c_i\partial_x p_e - c_e\partial_x p_i).\end{aligned}$$

So

$$\int_{\mathbb{R}^3} \frac{m_e}{2}(v_1-u)\left((v_1-u)^2+v_2^2+v_3^2\right) g_e\,dv = -\frac{5}{2}\frac{k_B}{m_e}p_e\,\partial_x T_e$$

and finally by using the relation (2.30), it holds that

$$\int_{\mathbb{R}^3} \frac{m_e}{2}v_1 v^2 g_e\,dv = -\frac{4}{3}up_e\partial_x u - \frac{5}{2}\frac{k_B}{m_e}p_e\,\partial_x T_e.$$

So we obtain (2.26). The same result holds for (2.27) □

2.6 Navier-Stokes viscous system written with internal energy

Next, in order to compare with ([3], [7]), the system (2.24, 2.25, 2.26, 2.27) is rewritten by using internal energies.

Proposition 3. *The system (2.24, 2.25, 2.26, 2.27) is equivalent to the system*

$$\partial_t\rho + \partial_x(\rho u) = 0, \tag{2.33}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p_e + p_i) - \frac{4}{3}\tau\partial_x(p\,\partial_x u) = 0, \tag{2.34}$$

$$\begin{aligned}\partial_t(\rho_e\varepsilon_e) + \partial_x(u(\rho_e\varepsilon_e)) + p_e\partial_x u + \frac{4}{3}\tau\partial_x\left(up_e\,\partial_x u\right) - \frac{5}{2}\tau\partial_x(\frac{k_B}{m_e}p_e\,\partial_x T_e)& \\ -\frac{4}{3}\tau p_e\,(\partial_x u)^2 = \nu_{ei}(T_e-T_i),&\end{aligned} \tag{2.35}$$

$$\begin{aligned}\partial_t(\rho_i\varepsilon_i) + \partial_x(u(\rho_i\varepsilon_i)) + p_i\partial_x u + \frac{4}{3}\tau\partial_x(up_i\partial_x u) - \frac{5}{2}\tau\partial_x(\frac{k_B}{m_i}p_i\,\partial_x T_i)& \\ -\frac{4}{3}\tau p_i\,(\partial_x u)^2 = \nu_{ei}(T_i-T_e).&\end{aligned} \tag{2.36}$$

Proof. Firstly, a straightforward computation gives

$$\begin{aligned}\frac{1}{2}\partial_t(\rho_i u^2) &= -\frac{1}{2}\partial_x(\rho u^3) - u\partial_x p_i + q_i n_i uE + \frac{4}{3}\tau u\partial_x(p_i\partial_x u) \\ &= -\partial_x(u(\frac{1}{2}\rho_i u^2 + p_i) + p_i\partial_x u + q_i n_i uE + \frac{4}{3}\tau u\partial_x(p_i\partial_x u).\end{aligned}$$

Hence, by using the viscous Ohm's law (2.28), we get the relation

$$un_iq_iE = -uc_i\partial_x p_e + uc_e\partial_x p_i + \frac{4}{3}c_i\tau u\partial_x(p_e\partial_x u) - \frac{4}{3}c_e\tau u\partial_x(p_i\partial_x u).$$

Therefore

$$\begin{aligned}\partial_t(\rho_i\varepsilon_i + \frac{1}{2}\rho_i u^2) + \partial_x(u(\rho_i\varepsilon_i + \frac{1}{2}\rho_i u^2 + p_i) &= \partial_t(\rho_i\varepsilon_i) + \partial_x(u\rho_i\varepsilon_i) + p_i\partial_x u \\ &- uc_i\partial_x p_e + uc_e\partial_x p_i + \frac{4}{3}c_i u\tau\partial_x(p\partial_x u).\end{aligned}$$

Hence from the relation

$$\partial_x(up_i\,\partial_x u) = u\partial_x(p_i\,\partial_x u) + p_i(\partial_x u)^2,$$

we obtain the equation (2.36). □

We firstly remark that the viscous system of ([3], [7]) contains the terms $p_\alpha\partial_x u$ and $-\frac{4}{3}\tau p_\alpha\,(\partial_x u)^2$, for $\alpha \in \{e;i\}$. However, the terms $\tau\frac{4}{3}\partial_x(up_\alpha\partial_x u)$ and $-\frac{5}{2}\tau\partial_x(\frac{k_B}{m_i}p_i\,\partial_x T_i)$ do not appear in ([3], [7]).

3 Dissipativity of the second order terms with respect to the entropy

This section is devoted to the proof of the entropy dissipativity of the viscous system (2.24-2.27).

Proposition 4. *We assume that $\gamma_e = \gamma_i = 5/3$. Let $\mathcal{U}^\tau$ be a solution of the second order system (2.24-2.27). Then $\mathcal{U}^\tau$ satisfies the following entropy inequality:*

$$\partial_t\eta(\mathcal{U}^\tau) + \partial_x(u^\tau\eta(\mathcal{U}^\tau)) \le -\frac{\nu_{ei}}{k_B T_i^\tau T_e^\tau}(T_i^\tau - T_e^\tau)^2 - \tau\frac{5k_B}{2}\sum_{\alpha=e,i}\frac{1}{m_\alpha}\partial_x\left(n_\alpha^\tau\partial_x T_\alpha^\tau\right) \tag{3.37}$$

where η is defined by (1.4).

Proof. The result is obtained by multiplying (2.24)–(2.27) by $\eta'(\mathcal{U}^\tau)$. The system (2.24)–(2.27) being written in the synthetic form (1.3), we denote W the viscous terms

$$W = u^\tau\partial_x\left(J(\mathcal{U}^\tau)\partial_x\mathcal{U}^\tau\right) + \partial_x\left(D(\mathcal{U}^\tau)\partial_x\mathcal{U}^\tau\right).$$

In [1] we have shown that

$$\eta'(\mathcal{U}^\tau)\left[\partial_t\mathcal{U}^\tau + A(\mathcal{U}^\tau)\partial_x\mathcal{U}^\tau - S(\mathcal{U}^\tau)\right] = \frac{\nu_{ei}}{k_B T_i^\tau T_e^\tau}(T_i^\tau - T_e^\tau)^2.$$

It remains to prove that

$$\eta'(\mathcal{U}^\tau)W \le -\frac{5k_B}{2}\sum_{\alpha=e,i}\frac{1}{m_\alpha}\partial_x\left(n_\alpha^\tau\partial_x T_\alpha^\tau\right).$$

Note that the first component of W is equal to zero, so that $\partial_\rho \eta$ is not needed. We have

$$\begin{aligned}\eta'(\mathcal{U}) = \quad & (\partial_1 \overline{\eta}_e(\rho c_e, \varepsilon_e),\ \partial_2 \overline{\eta}_e(\rho c_e, \varepsilon_e)) \begin{pmatrix} c_e & 0 & 0 & 0 \\ -\dfrac{\mathcal{E}_e}{c_e \rho^2} + \dfrac{q^2}{\rho^3} & -\dfrac{q}{\rho^2} & \dfrac{1}{\rho c_e} & 0 \end{pmatrix} \\ & + (\partial_1 \overline{\eta}_i(\rho c_i, \varepsilon_i),\ \partial_2 \overline{\eta}_i(\rho c_i, \varepsilon_i)) \begin{pmatrix} c_i & 0 & 0 & 0 \\ -\dfrac{\mathcal{E}_i}{c_i \rho^2} + \dfrac{q^2}{\rho^3} & -\dfrac{q}{\rho^2} & 0 & \dfrac{1}{\rho c_i} \end{pmatrix}.\end{aligned}$$

Hence, denoting $q = \rho u$,

$$\partial_q \eta(\mathcal{U}) = \frac{u}{k_B} \left(\frac{c_e}{\varepsilon_e m_e (\gamma_e - 1)} + \frac{c_i}{\varepsilon_i m_i (\gamma_i - 1)} \right), \quad \partial_{\mathcal{E}_\alpha} \eta(\mathcal{U}) = \frac{-1}{\varepsilon_\alpha m_\alpha (\gamma_\alpha - 1)}, \ \alpha = e, i.$$

As $p_\alpha = n_\alpha k_B T_\alpha$, this can be written as

$$\partial_q \eta(\mathcal{U}) = \frac{u}{k_B} \left(\frac{c_e}{T_e} + \frac{c_i}{T_i} \right), \quad \partial_{\mathcal{E}_\alpha} \eta(\mathcal{U}) = \frac{-1}{k_B T_\alpha}, \quad \alpha = e, i.$$

Now we multiply $\eta'(\mathcal{U})$ by W, that is :

$$W = \begin{pmatrix} 0 \\ \dfrac{4}{3} \partial_x \left((p_e + p_i) \partial_x u \right) \\ \dfrac{4}{3} u c_e \partial_x \left((p_e + p_i) \partial_x u \right) + \dfrac{4}{3} p_e \left(\partial_x u \right)^2 + \dfrac{5}{2} \partial_x \left(\dfrac{k_B}{m_e} p_e \partial_x T_e \right) \\ \dfrac{4}{3} u c_i \partial_x \left((p_e + p_i) \partial_x u \right) + \dfrac{4}{3} p_i \left(\partial_x u \right)^2 + \dfrac{5}{2} \partial_x \left(\dfrac{k_B}{m_i} p_i \partial_x T_i \right) \end{pmatrix}.$$

Therefore

$$\begin{aligned}\eta'(\mathcal{U}) W &= -\frac{4}{3 k_B} \left(\partial_x u \right)^2 \left(\frac{p_e}{T_e} + \frac{p_i}{T_i} \right) - \frac{5}{2 k_B T_e} \partial_x \left(\frac{k_B}{m_e} p_e \partial_x T_e \right) - \frac{5}{2 k_B T_i} \partial_x \left(\frac{k_B}{m_i} p_i \partial_x T_i \right) \\ &= -\frac{4n}{3} \left(\partial_x u \right)^2 - \sum_{\alpha = e, i} \left(\frac{5}{2 m_\alpha T_\alpha} \partial_x \left(n_\alpha k_B T_\alpha \partial_x T_\alpha \right) \right).\end{aligned}$$

Using the fact that

$$T_\alpha^{-1} \partial_x \left(n_\alpha T_\alpha \partial_x T_\alpha \right) = \partial_x \left(n_\alpha \partial_x T_\alpha \right) + \frac{n_\alpha \left(\partial_x T_\alpha \right)^2}{T_\alpha},$$

we thus have

$$\eta'(\mathcal{U}) W = -\frac{4n}{3} \left(\partial_x u \right)^2 - \frac{5 k_B}{2} \sum_{\alpha = e, i} \frac{1}{m_\alpha} \left(\partial_x \left(n_\alpha \partial_x T_\alpha \right) + \frac{n_\alpha \left(\partial_x T_\alpha \right)^2}{T_\alpha} \right).$$

□

4 Conclusion

In this paper, starting from a kinetic model, we have derived a viscous approximation of the bitemperature Euler system from a Chapman-Enskog expansion. We have been able to compute explicitly all the viscous terms and we have obtained a generalization of the model proposed in [7]. Then we have proved an entropy inequality. These results support the approach taken in [1] where the same kinetic model is the basis of the numerical approximation of the system (1.2). The case $\gamma_e \neq \gamma_i$ can be handled by using a kinetic model with internal energy variable (see Remark 1). In a future work we plan to study the shocks obtained by limits of travelling waves constructed from this viscous model and to compare them to the ones numerically computed in our previous work [1].

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